Ain Shams Engineering Journal (2016) xxx, xxx–xxx

Ain Shams University

Ain Shams Engineering Journal

www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

The homotopy analysis method for *q*-difference equations

Mourad S. Semary, Hany N. Hassan *

Department of Basic Engineering Sciences, Benha Faculty of Engineering, Benha University, Benha 13512, Egypt

Received 7 September 2015; revised 15 November 2015; accepted 7 February 2016

KEYWORDS

Quantum calculus;
q-difference equations;
q-derivative;
Homotopy analysis method

Abstract The q-difference equations are kind of important problems in q-calculus and applied mathematics. In this paper, the homotopy analysis method is extended to find approximate solution for some of q-differential equations. The q-diffusion equation and some examples are analytically investigated. The series solutions obtained by the proposed method are checked by reducing the solutions of q-calculus problems to h-calculus approximate solutions when $q \to 1$.

1. Introduction

At the last quarter of 20th century, q-calculus appears connections between physics and mathematics [1–9]. It has a lot of applications in different applied science and various mathematical areas, such as statistic physics fractal geometry, combinatorics, number theory, orthogonal polynomials, basic hypergeometric functions, quantum theory and theory of relativity. The q-differential equations are used to modelling the linear and nonlinear problems and played an important role in different fields of engineering and science. Recently, the semi-analytic techniques have been successfully employed to solve linear and nonlinear q-difference models, such as the differential transformation method [10–12], successive approximation method [13,14], variational iteration method [15,16] and homotopy perturbation method [17] and other methods. The homotopy analysis method (HAM) [18–27] is one of the semi-analytical techniques used most often for solving various differential equations in h-calculus; in this study, the homotopy analysis method is extended to solve q-differential equations. The solutions obtained by the proposed method are the semi-analytic solutions for the problems in h-calculus for the parameter $q \to 1$. We describe some definitions and properties of the q-calculus when $0 < q < 1$, let T_b be the timescales. $T_b = \{q^m : m \in Z\} \cup \{0\}$. The q-number corresponding to the ordinary number k is defined as [28],

$$[k]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}, \quad (1)$$

so that k is the limit of $[k]_q$ as $q \to 1$ and the q-binomial coefficients are defined by:

$$\begin{bmatrix} k \\ i \end{bmatrix}_q = \frac{[k]_q!}{[i]_q![k-i]_q!} \quad (2)$$

where $[k]_q! = [1]_q[2]_q \ldots [k]_q$.

* Corresponding author. Tel.: +20 1121135012.
E-mail addresses: mourad.semary@yahoo.com (M.S. Semary), h_nasr77@yahoo.com (H.N. Hassan).

Peer review under responsibility of Ain Shams University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.asej.2016.02.005
2090-4479 © 2016 Faculty of Engineering, Ain Shams University. Production and hosting by Elsevier B.V.

Please cite this article in press as: Semary MS, Hassan HN, The homotopy analysis method for q-difference equations, Ain Shams Eng J (2016), http://dx.doi.org/10.1016/j.asej.2016.02.005

Definition 1. The q-derivative of a real continuous function $f(x)$ is defined as follows [28]:

$$\frac{d_q}{d_q x} f(x) = \frac{f(qx) - f(x)}{(q-1)x}, x \neq 0, \quad q \in (0,1). \tag{3}$$

When $q \to 1$ it reduces to the standard derivative $\frac{d_q}{d_q x} f(x) \to f'(x)$. Using the Definition 1 one can easily see that the effect of q-derivative on the power function is as follows:

$$\frac{d_q}{d_q x}(x-a)^k = [k]_q (x-a)^{k-1}, \quad (x, a \in \mathbb{R}, k \in \mathbb{N}).$$

Definition 2. The partial q-derivative of a function $f(x_1; x_2)$ to a variable x_1 is defined by [10]

$$\frac{\partial_q}{\partial_q} f(x_1; x_2) = \frac{f(qx_1; x_2) - f(x_1; x_2)}{(q-1)x_1}. \tag{4}$$

Definition 3. The q-Leibniz rule for a q-derivative of a product of two functions [10] is

$$\frac{d_q^m}{d_q x^m}(g(x)f(x)) = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \frac{d_q^{m-k}}{d_q x^{m-k}} f(q^k x) \frac{d_q^k}{d_q x^k} g(x), \tag{5}$$

and

$$\left. \frac{d_q^{m-k}}{d_q x^{m-k}} f(q^k x) \right|_{x=0} = a_{k,m-k}(q) \left. \frac{d_q^{m-k}}{d_q x^{m-k}} f(x) \right|_{x=0} \tag{6}$$

where

$$a_{k,m-k}(q) = \sum_{i=0}^{k} \sum_{j=0}^{m-k} (-1)^i (1-q)^i [i]_q! \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} m-k \\ j \end{bmatrix}_q q^{\frac{i(i-1)}{2}+ij} \tag{7}$$

$a_{k,m-k}(q)$ is exists when $i+j-m+k=0$ and equal zero otherwise.

Definition 4. The q-integration is defined as [10]

$$\int_0^x f(t) d_q t = (1-q)x \sum_{i=0}^{\infty} q^i f(q^i x). \tag{8}$$

2. The homotopy preliminaries

Definition 5. Assume ψ be a function of homotopy-parameter p, then

$$D_q^m(\psi) = \frac{1}{[m]_q!} \left. \frac{\partial_q^m \psi}{\partial_q p^m} \right|_{p=0} \tag{9}$$

is called the mth-order q-homotopy derivative of ψ, where $m \geqslant 0$ is an integer.

Theorem 1. *For homotopy-series* $\psi(x,p) = \sum_{i=0}^{+\infty} u_i(x) p^i$, *then*

(a) $D_q^m(\psi) = u_m$ (10)

(b) $D_q^m(p\psi) = u_{m-1}$ (11)

where $m \geqslant 0$ *is an integer.*

Proof.

(a) According to Taylor's theorem [29,30], the series of ψ is given by $u_m = \frac{1}{[m]_q!} \left. \frac{\partial_q^m \psi}{\partial_q p^m} \right|_{p=0}$, which given (a) by means of the Definition 5 of $D_q^m(\psi)$.

(b) According to homotopy series $\psi(x,p) = \sum_{i=0}^{+\infty} u_i(x) p^i$, then

$$D_q^m(p\psi) = D_q^m \left(p \sum_{i=0}^{+\infty} u_i p^i \right) = D_q^m \sum_{i=0}^{+\infty} u_i p^{i+1} = \sum_{i=0}^{+\infty} u_i D_q^m(p^{i+1}),$$

$$D_q^m(p^{i+1}) = \begin{cases} 1 & \text{if } i+1-m=0 \\ 0 & \text{if } i+1-m \neq 0 \end{cases} \tag{12}$$

From (12), $D_q^m(p\psi)$ exists when $i = m-1$ and zero otherwise, then

$D_q^m(p\psi) = u_{m-1}$ □

Theorem 2. *Let L be a linear operator independent of the homotopy parameter p. For homotopy series*

$$\psi = \sum_{i=0}^{+\infty} u_i p^i, \tag{13}$$

then $D_q^m(L\psi) = L[D_q^m(\psi)]$

Proof. Since L is independent of p, it holds

$$L\psi = \sum_{k=0}^{+\infty} [Lu_i] p^i, \tag{14}$$

taking mth-order q-homotopy derivative on both sides of the above expression (14) and using Theorem (1a)

$$L.H.S = D_q^m(L\psi) = D_q^m \sum_{k=0}^{+\infty} [L(u_k)] p^k = L(u_m). \tag{15}$$

$$R.H.S = L\left[D_q^m(\psi)\right] = L(u_m). \tag{16}$$

From Eqs. (15) and (16) then $D_q^m(L\psi) = L[D_q^m(\psi)]$.

Theorem 3. *Let a homotopy-series be*

$$\psi(x,t,p) = \sum_{m=0}^{+\infty} u_m(x,t) p^m,$$

where $p \in [0,1]$, *L an auxiliary linear operator independent of p and* u_0 *initial solution. Then*

$$D_q^m\{(1-p)L[\psi - u_0]\} = L\left[u_m(x,t) - \chi_m q^{m-1} u_{m-1}(x,t)\right], \tag{17}$$

where the operator $D_q^m(\psi)$ *is defined by (9) and* χ_m *is defined by*

$$\chi_m = \begin{cases} 0, & m \leqslant 1 \\ 1, & m > 1 \end{cases} \tag{18}$$

Proof. Since L is a linear operator independent of p, then

$$(1-p)L[\psi - u_0] = L[\psi - p\psi + u_0 p - u_0].$$

Using Theorems 1 and 2, we have

$$D_q^m\{(1-p)L[\psi-u_0]\} = D_q^m\{L[\psi-p\psi+u_0p-u_0]\}$$
$$= L\Big\{D_q^m[\psi-p\psi+u_0p-u_0]\Big\}$$
$$= L\Big[D_q^m(\psi)-D_q^m(p\psi)+u_0D_q^m(p)\Big]$$
$$= L\Big[u_m-u_{m-1}+u_0D_q^m(p)\Big]$$

which equals to $L[u_m]$ when $m=1$, and $L[u_m-u_{m-1}]$ when $m>1$, respectively. Thus using the definition (18) of χ_m, it holds

$$D_q^m\{(1-p)L[\psi-u_0]\} = L[u_m(x,t)-\chi_m u_{m-1}(x,t)]. \qquad \square$$

Theorem 4. *Let a homotopy-series be*

$$\psi = \sum_{m=0}^{+\infty} u_m(x,t)p^m, \tag{19}$$

where $p\in[0,1]$, L is an auxiliary linear operator independent of p, N is a nonlinear operator, $u_0(x,t)$ is initial solution, $\hbar$ the control parameter and $H(x,t)$ an auxiliary function, respectively. The zeroth-order deformation equation is given by

$$(1-p)L[\psi-u_0] = p\hbar H(x,t)N[\psi]. \tag{20}$$

The corresponding mth-order deformation equation ($m\geqslant 1$) reads

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = \hbar H(x,t)D_q^{m-1}N[\psi], \tag{21}$$

where the operator D_q^{m-1} is given by (9) and χ_m is given by (18).

Proof. Using Theorem 3, we have

$$D_q^{m-1}\{(1-p)L[\psi-u_0]\} = D_q^{m-1}\{p\hbar H(x,t)N[\psi]\} \tag{22}$$

According to Theorem 3, it holds

$$D_m\{(1-p)L[\psi-u_0]\} = L[u_m(x,t)-\chi_m u_{m-1}(x,t)]. \tag{23}$$

According to Theorem (1b), one has

$$D_q^{m-1}\{p\hbar H(x,t)N[\psi]\} = \hbar H(x,t)D_q^{m-1}\{pN[\psi]\}$$
$$= \hbar H(x,t)D_q^{m-1}\{N[\psi]\} \tag{24}$$

Substituting (23) and (24) into (22), one has the mth-order deformation equation

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = \hbar H(x,t)D_q^{m-1}N[\psi]. \qquad \square$$

Theorem 5. *For an arbitrary homotopy-series (19), it holds*

$$D_q^m(\psi^2) = \sum_{k=0}^{m} a_{k,m-k}(q)u_{m-k}u_k, \tag{25}$$

where $a_{k,m-k}(q)$ is given by (7) and $m\geqslant 0$ is positive integer.

Proof. According to Definition 5, Theorem (1a) and the q-Leibniz Product Law (5), it holds

$$D_q^m(\psi^2) = \frac{1}{[m]_q!}\frac{\partial_q^m(\psi)^2}{\partial_q p^m}\bigg|_{p=0}$$
$$= \sum_{k=0}^{m}\frac{1}{[k]_q![m-k]_q!}D_q^{m-k}\big(\psi(x;pq^k)\big)D_q^k(\psi)$$
$$= \sum_{k=0}^{m}a_{k,m-k}(q)u_k u_{m-k}. \qquad \square$$

3. Analysis of method

Consider the nonlinear q-difference equation:

$$\frac{d_q^m}{d_qx^m}f(x) + g\left(f,\frac{d_q}{d_qx}f,\ldots,\frac{d_q^{m-1}}{d_qx^{m-1}}f\right) - y(x) = 0, \tag{26}$$

with boundary conditions

$$\mathcal{B}\left(u,\frac{\partial u}{\partial n}\right) = 0, \quad x\in\Gamma, \tag{27}$$

where $\mathcal{B}$ is a boundary operator and Γ is boundary of the domain Ω. The Eq. (26) can be written in the form

$$N[f(x)] = 0. \tag{28}$$

where N is a nonlinear operator, x denotes independent variable, and $f(x)$ is an unknown function. From Eq. (20), we can construct zeroth-order deformation equation as follows:

$$(1-p)\frac{d_q^m}{d_qx^m}[\psi(x,p)-f_0(x)] = p\hbar H(x)(N[\psi(x,p)]), \tag{29}$$

where $p\in[0,1]$ denotes the so-called embedding parameter. $\hbar\neq 0$ is a control parameter, $H(x)$ is a non-zero auxiliary function, $\psi(x,p)$ is an unknown function and $f_0(x)$ is an initial solution of $f(x)$. It is obvious that when $p=0$ and $p=1$, Eq. (29) becomes

$$\psi(x,0)=f_0(x), \psi(x,1)=f(x) \tag{30}$$

respectively. Thus as p increases from zero to one, the solution $\psi(x,p)$ varies from the initial solution $f_0(x)$ to the solution $f(x)$. The initial solution $f_0(x)$ satisfies the conditions (27). Expanding $\psi(x,p)$ in the Taylor series [29,30] then

$$\psi(x,p) = f_0(x) + \sum_{m=1}^{+\infty} f_m(x)p^m, \tag{31}$$

where

$$f_m(x) = D_q^m\psi(x,p)\Big|_{p=0}, \tag{32}$$

To determine the higher order terms $f_m(x)(m,1,2,\ldots)$ define the vector

$$\vec{f}_i(x) = \{f_0(x),f_1(x),\ldots,f_i(x)\} \tag{33}$$

From Theorem 4, the so-called mth-order deformation equation is

$$\frac{d_q^m}{d_qx^m}[f_m(x)-\chi_m f_{m-1}(x)] = \hbar H(x)R_m(\vec{f}_{m-1}(x)), \tag{34}$$

where

$$R_m(\vec{f}_{m-1}(x)) = \frac{1}{[m-1]_q!}\frac{\partial_q^{m-1}(N[\psi(x,p)])}{\partial_q p^{m-1}}\bigg|_{p=0}, \tag{35}$$

and χ_m is given by (18). Now the solution of the mth-order deformation Eq. (34) for $m \geqslant 1$ when $H(x) = 1$ becomes

$$f_m(x) = \chi_m f_{m-1}(x) + \underbrace{\iint \cdots \int}_{r} \left[\hbar R_m\left(\vec{f}_{m-1}(x)\right)\right] \underbrace{d_q x d_q x \ldots d_q x}_{r} + c_0 + c_1 x + \cdots + c_{r-1} x^{r-1} \quad (36)$$

where $c_0, c_1, \ldots, c_{r-1}$ are constants and can be calculated by

$$\left.\frac{\partial_q^m \mathcal{B}\left(\psi(x,p), \frac{\partial \psi(x,p)}{\partial n}\right)}{\partial_q p^m}\right|_{p=0} = 0, \quad (37)$$

Starting by $f_0(x)$, we obtain the functions $f_m(x)$ for $m, 1, 2, 3, \ldots$ form Eqs. (36) and (37) successively. Accordingly, the M-th order of approximate solution of the problem (26) and (27) is given by

$$f(x) \cong F_M(x) = \sum_{m=0}^{M} f_m(x). \quad (38)$$

4. Numerical examples

Example 1. Suppose the nonlinear q-difference equation is [15] in the following form:

$$\frac{d_q}{d_q x} f(x) + f^2(x) - 1 = 0, \quad (39)$$

and

$$f(0) = 0. \quad (40)$$

For q $\rightarrow$ 1, the problem (39) becomes

$$\frac{df(x)}{dx} + f^2(x) - 1 = 0, \quad (41)$$

and has the exact solution

$$f(x) = \tanh(x). \quad (42)$$

Now, apply the homotopy analysis method for the problems (39) and (40). Choosing the initial solution $f_0(x) = zero$, the mth-order deformation Eq. (36) for $m \geqslant 1$ becomes

$$f_m(x) = \chi_m f_{m-1}(x) + \int \hbar R_m(\vec{f}_{m-1}(x)) d_q x + c_0 \quad (43)$$

where c_0 can be calculated by

$$f_m(0) = 0 \quad (44)$$

and by using Theorems 1 and 5, $R_m(\vec{f}_{m-1}(x))$ is defined by

$$R_m(\vec{f}_{m-1}(x)) = \frac{d_q}{d_q x} f_{m-1}(x) + \sum_{k=0}^{m-1} a_{k,m-1-k}(q) f_k f_{m-1-k} - (1 - \chi_m), \quad (45)$$

Now giving the solution of mth-order deformation Eq. (43) as follows:

$$f_1(x) = -\hbar x,$$

$$f_2(x) = -\hbar(1+\hbar)x,$$

$$f_3(x) = -\hbar(1+\hbar)^2 x + \frac{\hbar^3 q x^3}{1+q+q^2},$$

$$f_4(x) = \hbar^2(1+\hbar)^2(2 - 3q^2 + q^3 + 2q^4 + q^5)x^2 - \frac{\hbar^4 q(1 - q^2 + q^3 + q^5)x^4}{1+q+q^2},$$

$$f_5(x) = -\hbar(1+\hbar)^4 x + \frac{\hbar^3(1+\hbar)^2(3 - 2q^2 + 2q^3 + 2q^4 + q^5)x^3}{1+q+q^2} - \frac{\hbar^5 q(1 - q^2 + q^3 + q^5)x^5}{(1+q+q^2)(1+q+q^2+q^3+q^4)},$$

and so on. The analytic approximation solution is given by

$$f(x) \cong F_M(x) = \sum_{m=0}^{M} f_m(x) \quad (46)$$

and $\left.\frac{d_q}{d_q x} f(x)\right|_{x=0}$ is given by [31]

$$\left.\frac{d_q}{d_q x} f(x)\right|_{x=0} \cong \left.\frac{d_q}{d_q x} F_m(x)\right|_{x=0} = \lim_{x \to 0} \frac{d_q}{d_q x} F_m(x) \quad (47)$$

The value of $\hbar$ must be found. To find the valid region of $\hbar$, the $\hbar$ curve given by $\left.\frac{d_q}{d_q x} F_5(x)\right|_{x=0}$ for different values of q is drawn in Fig. 1, which clearly indicates that the valid region of $\hbar$ is about $-1.5 \leqslant \hbar \leqslant -0.5$. Also, it is clear from Fig. 1 that the value did not change with the value of q that is consistent with the problem, and from (39) and (40), we find that $\frac{d_q}{d_q x} f(0) = 1$. For $q \rightarrow 1$ the q-difference Eq. (39) is converted to the ordinary differential Eq. (41). The analytic approximation solution for the problem (41) when $\hbar = -1$, becomes

$$F_M(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} + \cdots, \quad (48)$$

Eq. (48) is consistent with the exact solution (42) for problem (41). This shows the accuracy of the method used.

Example 2. Suppose the q-analogue of an oscillator equation [15,16] is in the following form:

$$\frac{d_q^2}{d_q x^2} f(x) - f(x) = 0 \quad (49)$$

and

$$f(0) = 1, \left.\frac{d_q}{d_q x} f(x)\right|_{x=0} = 1, \quad (50)$$

and has the exact solution e_q^x which is the q-exponential function [15,16]. For $q \rightarrow 1$, the problem (49) becomes

$$\frac{d^2 f(x)}{d^2 x} - f(x) = 0, \quad (51)$$

and has the exact solution e^x.

Now, applying HAM for the problem (49) and (50). By choosing the initial solution $f_0(x) = 1 + x$, the mth-order deformation Eq. (36) for $m \geqslant 1$ becomes

$$f_m(x) = \chi_m f_{m-1}(x) + \iint \hbar R_m(\vec{f}_{m-1}(x)) d_q x d_q x + c_0 + c_1 x \quad (52)$$

where c_0 and c_1 can be calculated by

$$f_m(0) = 0 \quad \frac{d_q}{d_q x} f(x)\Big|_{x=0} = 0 \tag{53}$$

And $R_m(\vec{f}_{m-1}(x))$ is defined by

$$R_m(\vec{f}_{m-1}(x)) = \frac{d_q^2}{d_q^2 x} f_{m-1}(x) - f_{m-1}(x) \tag{54}$$

Now giving the solution of mth-order deformation Eq. (52) as follows:

$$f_1(x) = -\frac{\hbar x^2}{1+q} - \frac{\hbar x^3}{(1+q)(1+q+q^2)},$$

$$f_2(x) = -\frac{\hbar(1+\hbar)x^2}{1+q} - \frac{x^3\hbar(1+\hbar)}{(1+q)(1+q+q^2)} + \frac{x^4\hbar^2}{(1+q)^2(1+q^2)(1+q+q^2)} + \frac{x^5\hbar^2}{(1+q)^2(1+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)}$$

$$f_3(x) = -\frac{x^2\hbar(1+\hbar)^2}{1+q} - \frac{x^3\hbar(1+\hbar)^2}{(1+q)(1+q+q^2)} + \frac{2x^4\hbar^2(1+\hbar)}{(1+q)^2(1+q^2)(1+q+q^2)} + \frac{2x^5\hbar^2(1+\hbar)}{(1+q)^2(1+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)} - \frac{x^6\hbar^3}{(1+q)^3(1+q+q^2)^2(1+2q^2+q^3+2q^4+q^5+2q^6+q^8)} - \frac{x^7\hbar^3}{(1+q)^3(1+q^2)(1-q+q^2)(1+q+q^2)^2(1+q+q^2+q^3+q^4)(1+q+q^2+q^3+q^4+q^5+q^6)}$$

and so on. The analytic approximation solution is given by (38). A proper value of $\hbar$ must be found. To find the valid region of $\hbar$, the $\hbar$ curve given by $\frac{d_q^2}{d_q x^2} F_6(x)\Big|_{x=0}$ for $q = \frac{1}{2}$ and $q \to 1$ is drawn in Fig. 2, which clearly indicates that the valid region of $\hbar$ is about $-1.5 \leqslant \hbar \leqslant -0.5$. When $\hbar = -1$, the analytic approximation solution becomes

$$F_M(x) = 1 + x + \frac{x^2}{1+q} + \frac{x^3}{(1+q)(1+q+q^2)} + \frac{x^4}{(1+q)^2(1+q^2)(1+q+q^2)} + \frac{x^5}{(1+q)^2(1+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)} + \cdots,$$

$$F_M(x) = 1 + \frac{x}{[1]_q!} + \frac{x^2}{[2]_q!} + \frac{x^3}{[3]_q!} + \frac{x^4}{[4]_q!} + \frac{x^5}{[5]_q!} + \cdots = \sum_{i=0}^{M} \frac{x^i}{[i]_q!} \tag{55}$$

When $f(x) = \lim_{M\to\infty} F_M(x) = e_q^x$ is an exact solution of (49). The same result can be obtained by using the variational iteration method [15,16]. For $q \to 1, [i]_q! = i!$, then Eq. (55) becomes, $F_M(x) = \sum_{i=0}^{M} \frac{x^i}{i!}$. The limit $f(x) = \lim_{M\to\infty} \sum_{i=0}^{M} \frac{x^i}{i!} = e^x$ is an exact solution of (51).

Example 3. Suppose q-diffusion equation [10] is in the following form:

$$\frac{\partial_q}{\partial_q t} f(x,t) = \frac{\partial_q^2}{\partial_q x^2} f(x,t) \tag{56}$$

subjecting to the initial condition

$$f(x,0) = e_q^x, \tag{57}$$

Eq. (56) allows the q-exponential distribution which describes a non-equilibrium system. Applying HAM for the problem (56) and (57) using the initial solution $f_0(x,t) = e_q^x$, the mth-order deformation Eq. (36) for $m \geqslant 1$ becomes

$$f_m(x,t) = \chi_m f_{m-1}(x,t) + \int \hbar R_m(\vec{f}_{m-1}(x,t)) d_q t + c_0 \tag{58}$$

where c_0 can be calculated by

$$f_m(x,0) = 0 \tag{59}$$

and

$$R_m(\vec{f}_{m-1}(x,t)) = \frac{\partial_q}{\partial_q t} f_{m-1}(x,t) - \frac{\partial_q^2}{\partial_q x^2} f_{m-1}(x,t) \tag{60}$$

The solutions of mth-order deformation Eq. (58) are as follows:

$$f_1(x,t) = -e_q^x t\hbar,$$

$$f_2(x,t) = e_q^x \left(-t\hbar(1+\hbar) + \frac{t^2\hbar^2}{1+q}\right),$$

$$f_3(x,t) = e_q^x \left(-t\hbar(1+\hbar)^2 + \frac{2t^2\hbar^2(1+\hbar)}{1+q} - \frac{t^3\hbar^3}{(1+q)(1+q+q^2)}\right),$$

$$f_4(x,t) = e_q^x \left(-t\hbar(1+\hbar)^3 + \frac{3t^2\hbar^2(1+\hbar)^2}{1+q} - \frac{3t^3\hbar^3(1+\hbar)}{(1+q)(1+q+q^2)} + \frac{t^4\hbar^4}{(1+q)^2(1+q^2)(1+q+q^2)}\right)$$

Please cite this article in press as: Semary MS, Hassan HN, The homotopy analysis method for q-difference equations, Ain Shams Eng J (2016), http://dx.doi.org/10.1016/j.asej.2016.02.005

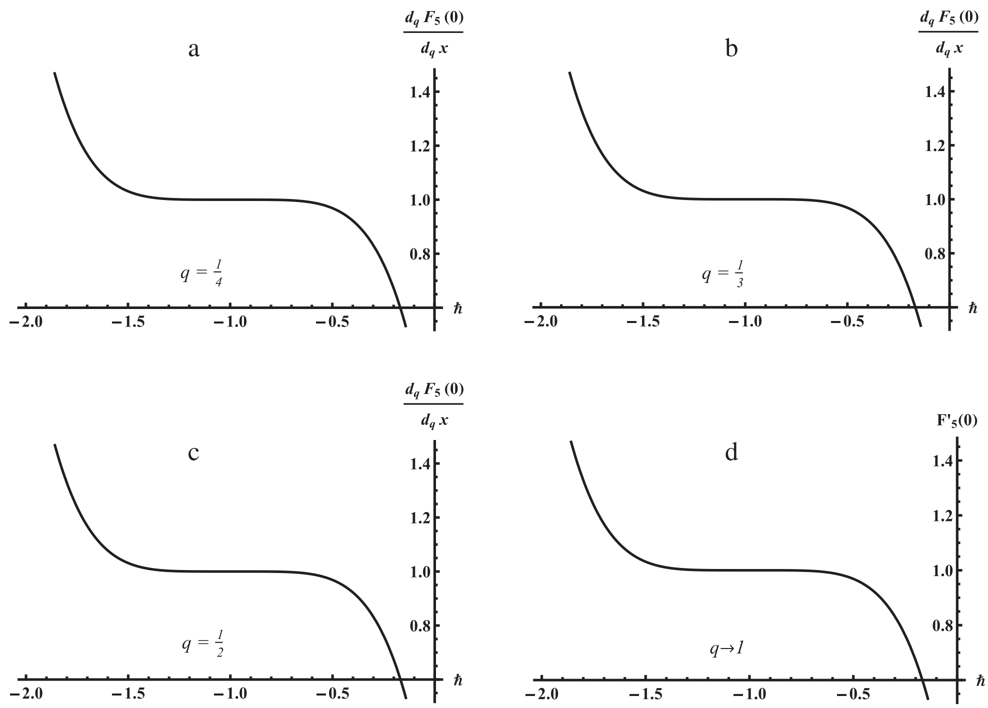

Figure 1 $\hbar$-curves of $\left.\frac{d_q}{d_q x}F_5\right|_{x=0}$ for different values of q of Example 1.

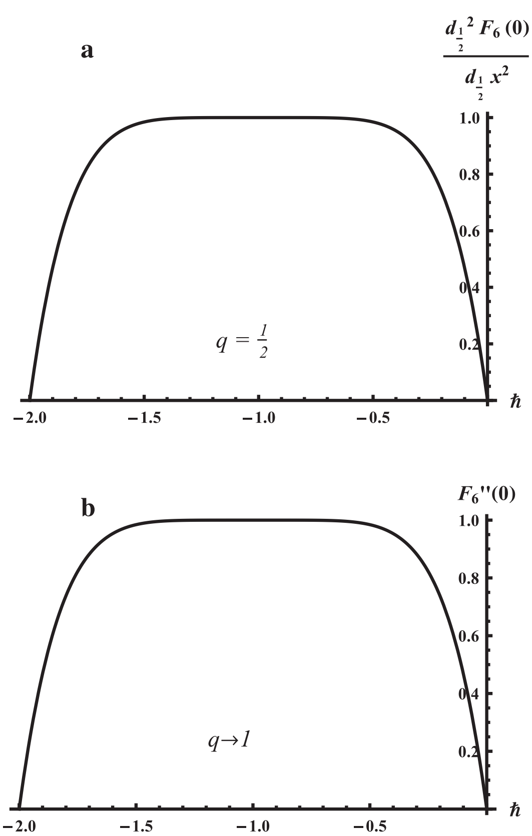

Figure 2 $\hbar$-curves of $\left.\frac{d_q^2}{d_q x^2}F_6(x)\right|_{x=0}$ for different values of q of Example 2.

and so on. The analytic approximation solution is

$$F_M(x,t)=\sum_{m=0}^{M} f_m(x,t)=e_q^x-e_q^x t\hbar+e_q^x\left(-t\hbar(1+\hbar)+\frac{t^2\hbar^2}{1+q}\right)+e_q^x\left(-t\hbar(1+\hbar)^2+\frac{2t^2\hbar^2(1+\hbar)}{1+q}-\frac{t^3\hbar^3}{(1+q)(1+q+q^2)}\right)+\cdots \quad (61)$$

For $\hbar=-1$, Eq. (61) becomes

$$F_M(x,t)=e_q^x\left(1+t+\frac{t^2}{1+q}+\frac{t^3}{(1+q)(1+q+q^2)}+\frac{t^4}{(1+q)^2(1+q^2)(1+q+q^2)}+\cdots\right)=e_q^x\left(1+\frac{t}{[1]_q!}+\frac{t^2}{[2]_q!}+\frac{t^3}{[3]_q!}+\frac{t^4}{[4]_q!}+\cdots\right)=e_q^x\sum_{i=0}^{M}\frac{t^i}{[i]_q!} \quad (62)$$

The limit $f(x,t)=\lim_{M\to\infty}F_M(x,t)=e_q^x e_q^t$ is an exact solution of (56). The same result can be obtained by using the q-differential transform [10].

5. Conclusion

In this study, the homotopy analysis method is successfully extended to q-differential equations. The q-diffusion equation and some examples are analytically investigated to show the efficiency and the importance of the proposed method. The results demonstrate reliability and efficiency of the algorithm developed.

References

[1] Ernst T. A new notation for q-Calculus a new q-Taylor's formula, U.U.D.M. Report 1999:25, Uppsala University; 1999.

[2] Jing SC, Fan HY. q-Taylor's formula with its q-remainder. Commun Theor Phys 1995;23(1):117–20.

[3] Koekoek R, Swarttow RF. The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue. Report 98-17, Delft University of Technology; 1998.

[4] Koelink E. Eight lectures on quantum groups and q-special functions. Revista Colomb Matemaric 1996;30:93–180.

[5] Koornwinder TH, Swarttow RF. On q-analogues of the Fourier and Hankel transforms. Trans Am Math Soc 1992;333:445–61.

[6] Tsallis C. Possible generalization of Boltzmann–Gibbs statistics. J Stat Phys 1988;52(1–2):479–87.

[7] Erzan A. Finite q-differences and the discrete renormalization group. Phys Lett A 1997;225(4–6):235–8.

[8] Erzan A, Eckmann JP. q-Analysis of fractal sets. Phys Rev Lett 1997;78(17):3245–8.

[9] Eryılmaz A. Spectral analysis of q-sturm-liouville problem with the spectral parameter in the boundary condition. J Funct Spaces Appl 2012;2012:17. http://dx.doi.org/10.1155/2012/736437 736437.

[10] El-Shahed M, Gaber M. Two-dimensional q-differential transformation and its application. Appl Math Comput 2011;217(22):9165–72.

[11] Liu HK. Application of a differential transformation method to strongly nonlinear damped q-difference equations. Comput Math Appl 2011;61(9):2555–61.

[12] El-Shahed M, Gaber M, Al-Yami M. The fractional q-differential transformation and its application. Commun Nonlinear Sci Numer Simul 2013;18:42–55.

[13] Abdeljawad T, Baleanu D. Caputo q-fractional initial value problems and a q-analogue Mittag–Leffler function. Commun Nonlinear Sci Numer Simul 2011;16(12):4682–8.

[14] Salahshour S, Ahmadian A, Chan CS. Successive approximation method for Caputo q-fractional IVPs. Commun Nonlinear Sci Numer Simul 2015;24(1–3):153–8.

[15] Zeng YX, Zeng Y, Wu G-C. Application of the variational iteration method to the initial value problems of q-difference equations-some examples. Commun Numer Anal 2013. http://dx.doi.org/10.5899/2013/cna-00180.

[16] Wu GC. Variational iteration method for q-difference equations of second order. J Appl Math 2012. http://dx.doi.org/10.1155/2012/102850.

[17] Qin YM, Zeng DQ. Homotopy perturbation method for the q-diffusion equation with a source term. Commun Fract Calculus 2012;3(1):34–7.

[18] Liao S. Homotopy analysis method in nonlinear differential equations. Heidelberg Dordrecht London., New York: Springer; 2012.

[19] Liao S. Beyond perturbation: introduction to homotopy analysis method. Boca Raton: Chapman & Hall/CRC Press; 2003.

[20] Seshadri R, Munjam SR. Mixed convection flow due to a vertical plate in the presence of heat source and chemical reaction. Ain Shams Eng J, in press. doi:http://dx.doi.org/10.1016/j.asej.2015.05.008.

[21] Srinivas S, Reddy AS, Ramamohan TR, Shukla AK. Influence of heat transfer on MHD flow in a pipe with expanding or contracting permeable wall. Ain Shams Eng J 2014;5:817–30.

[22] Semary MS, Hassan HN. The homotopy analysis method for strongly nonlinear initial/boundary value problems. Int J Modern Math Sci 2014;9(3):154–72.

[23] Khader MM, Sweilam NH, EL-Sehrawy ZI, Ghwail SA. Analytical study for the nonlinear vibrations of multiwalled carbon nanotubes using homotopy analysis method. Appl Math Inform Sci 2014;8(4):1675–84.

[24] Hassan HN, Semary MS. An analytic solution to a parameterized problems arising in heat transfer equations by optimal homotopy analysis method. Walailak J Sci Technol 2014;11(8):659–77.

[25] Hassan HN, Semary MS. Analytic approximate solution for the Bratu's problem by optimal homotopy analysis method. Commun Numer Anal 2013:1–14.

[26] Eryılmaz A, Atay MT, Coşkun SB, Başbük M. Buckling of euler columns with a continuous elastic restraint via homotopy analysis method. J Appl Math 2013:8. http://dx.doi.org/10.1155/2013/341063 341063.

[27] Karakoç SBG, Eryılmaz A, Başbük M. The approximate solutions of fredholm integrodifferential-difference equations with variable coefficients via homotopy analysis method. Math Problems Eng 2013:7. http://dx.doi.org/10.1155/2013/248740 261645.

[28] Kac V, Cheung P. Quantum calculus. New York: Springer; 2002.

[29] Rajkovic PM, Stankovic MS, Marinkovi SD. On q-iterative methods for solving equations and systems. Novi Sad J Math 2003;33(2):127–37.

[30] Ernst T. q-Taylor formulas with q-integral remainder, U.U.D.M. Report 2009:19, Uppsala University; 2009.

[31] Ahmad B, Alsaedi A, K Ntouyas S. A study of second-order q-difference equations with boundary conditions. Advances in difference equations 2012:35. http://dx.doi.org/10.1186/1687-1847-2012-35.

Mourad Samir Semary received his B.Sc. from the Department of Electrical Engineering, Benha Faculty of Engineering, Benha University, Egypt, in 2006. His M.Sc. in Engineering Mathematics was from Benha University in 2014. He is Lecturer in Department of Basic Science, Benha Faculty of Engineering. His research interests are in Engineering Mathematics, q-Difference Equations, Fractional-order circuit analysis, Numerical and Semi-analytic techniques for solving differential equations and published several papers in various international journals.

Hany Nasr Hassan received his Ph.D. from Department of Engineering Mathematics and Physics, Engineering Faculty, Cairo University (Egypt) in January 2011. He is an assistant professor in Department of Engineering Basic Sciences, Benha Faculty of Engineering, Benha University, Egypt. His research interests are in nonlinear analysis, engineering mathematics, mathematical modelling, fractional-order differential equations, mathematical methods and numerical techniques for solving deterministic and stochastic differential equations. He published several papers in various international refereed journals. He is referee of several international journals, in the frame of pure and applied mathematics. He received the Best Thesis Award from Cairo University for his PhD. thesis.