Noise suppresses exponential growth under regime switching

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1. Introduction

It has been known that a given ordinary differential equation and its corresponding stochastic perturbed equation may have significant differences. The pioneering work was due to Hasminskii [11, p. 229], who stabilised an unstable system by using two white noise sources, and his work opened a new chapter in the study of stochastic stabilisation. There is an extensive literature concerned with the stabilisation by noise and we here mention [1–6,8,9,13–16,20,22–24]. It is now well known that noise can be used to stabilise a given unstable system or to make a system even more stable when it is already stable. A few years ago, Mao et al. [18] showed another important fact that the environmental noise can suppress explosions (in a finite time) in population dynamics and this paper made an important impact on the study of stochastic population systems (see e.g. [7,12]). Recently, Deng et al. [10] reveal one more important feature that noise can suppress or expresses exponential growth.

However, most of the papers mentioned above consider only the perturbation by white noise but not colour noise yet. In this paper, we will develop the theory presented in [10] to cope with much more general systems where they are subject to not only white noise but also colour noise. More precisely, we will consider a given system under regime switching (colour noise) whose solution grows exponentially, and suppose that the system is subject to environmental noise (white noise) in some regimes. Can the regime switching and the environmental noise work together to make the system change significantly? To explain this feature more clearly, let us consider a simple linear scalar differential equation with Markovian switching of the form

\[
\frac{dx(t)}{dt} = a(r(t)) + b(r(t))x(t) \quad \text{on } t \geq 0
\]
with initial value \( x(0) = x_0 > 0 \). Here \( f(t) \) is a Markov chain on the state space \( S = \{1, 2\} \) with generator
\[
\Gamma = \begin{pmatrix}
-\gamma_{12} & \gamma_{12} \\
\gamma_{21} & -\gamma_{21}
\end{pmatrix},
\]
while \( a(i) = a_i \) and \( b(i) = b_i \) (\( i = 1, 2 \)) are all positive constants. Eq. (1.1) has the explicit solution
\[
x(t) = e^{\int_0^t b(r(s)) \, ds} \left( x_0 + \int_0^t a(r(s)) e^{-\int_0^s b(r(u)) \, du} \, ds \right).
\]
This implies
\[
x(t) \geq y_0 e^{\hat{b}t},
\]
where \( \hat{b} = b_1 \wedge b_2 > 0 \). Hence, the solution of Eq. (1.1) will tend to infinity exponentially with probability one. Eq. (1.1) may be regarded as a system which switches between two operation modes, say mode 1 and mode 2, and the switching obeys the law of the Markov chain, where in mode 1, the system evolves according to the differential equation
\[
\frac{dx(t)}{dt} = a_1 + b_1 x(t)
\]
while in mode 2, according to the other differential equation
\[
\frac{dx(t)}{dt} = a_2 + b_2 x(t).
\]
We observe that in both modes, the solution will grow exponentially whence it is not surprising that the solution of the switching system (1.1) will grow exponentially too. Let us now suppose that in mode 1, the system is subject to an environmental noise and the stochastically perturbed system is described by a stochastic differential equation (SDE)
\[
dx(t) = \left[ a_1 + b_1 x(t) \right] \, dt + \sigma_1 x(t) \, dB(t),
\]
while there is no environmental noise in mode 2, where \( B(t) \) is a scalar Brownian motion independent of the Markov chain \( r(t) \). As the result, the system switches between Eqs. (1.4) and (1.3) according to the law of the Markov chain, whence this system becomes an SDE with Markovian switching
\[
dx(t) = \left[ a(r(t)) + b(r(t)) x(t) \right] \, dt + \sigma(r(t)) x(t) \, dB(t),
\]
where we set \( \sigma(1) = \sigma_1 \) and \( \sigma(2) = 0 \). We shall see from Example 3.2 below that if \( \gamma_{21} \) and \( \sigma_1 \) are both sufficiently large for
\[
\gamma_{21} > 0.5 b_2 \quad \text{and} \quad (0.5 \sigma^2 - 2 b_1)(\gamma_{21} - 0.5 b_2) > 2 b_2 \gamma_{12},
\]
then the solution of Eq. (1.6) obeys
\[
\limsup_{t \to \infty} \frac{\log(x(t))}{\log t} \leq 2 \quad \text{a.s.}
\]
This shows that for any \( \varepsilon > 0 \), there is a positive random variable \( T_\varepsilon \) such that, with probability one,
\[
x(t) \leq t^{2+\varepsilon} \quad \forall t \geq T_\varepsilon.
\]
In other words, the solution will grow at most polynomially with order \( 2 + \varepsilon \). Comparing this polynomial growth with the exponential growth of the solution to Eq. (1.1), we see the important fact that the noise suppresses the exponential growth, and switching makes this happened even in the case when noise exists only for some modes. The main aim of this paper is to develop this idea for general nonlinear SDEs. We will then consider a nonlinear system described by a differential equation with Markovian switching
\[
\frac{dx(t)}{dt} = f(x(t), r(t), t),
\]
whose coefficient obeys the one-side linear growth condition
\[
\langle x, f(x, i, t) \rangle \leq K_1 + K_2 |y|^2, \quad (x, i, t) \in \mathbb{R}^n \times S \times \mathbb{R}_+.
\]
Clearly, the solution of this equation may grow exponentially. However, we will show that we can always design a linear stochastic feedback control \( A(r(t)) x(t) dB(t) \) (i.e. choose square matrices \( A(i) \)) so that the stochastically controlled system
\[
dx(t) = f(x(t), r(t), t) \, dt + A(r(t)) x(t) \, dB(t)
\]
will grow at most polynomially with probability one.
2. Polynomial growth of SDEs

Throughout the paper, unless otherwise specified, we will employ the following notation. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a complete probability space with a filtration \((\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions (i.e., it is increasing and right continuous while \(\mathcal{F}(0)\) contains all \(\mathbb{P}\)-null sets). Let \(B(t) = (B_1(t), \ldots, B_m(t))^T, t \geq 0\), be an \(m\)-dimensional Brownian motion defined on the probability space, where \(T\) denotes the transpose of a vector or matrix. If \(x, y\) are real numbers, then \(x \vee y\) denotes the maximum of \(x\) and \(y\), and \(x \wedge y\) denotes the minimum of \(x\) and \(y\). Let \(|x|\) be the Euclidean norm of a vector \(x \in \mathbb{R}^n\) and \((x, y)\) be the inner product of vectors \(x, y \in \mathbb{R}^n\). Vectors \(x \in \mathbb{R}^n\) are thought as column ones so to get row vectors we use \(x^T\). The space of \(n \times m\) matrices with real entries is denoted by \(\mathbb{R}^{n \times m}\). If \(A = (a_{ij})\) is an \(n \times m\) matrix, we denote its Frobenius or trace norm by

\[
|A| = \sqrt{\sum_{i=1}^n \sum_{j=1}^m a_{ij}^2}
\]

while its operator norm by \(\|A\| = \sup\{|Ax|: x \in \mathbb{R}^m, |x| = 1\}\). If \(A \in \mathbb{R}^{n \times n}\) is symmetric, its largest and smallest eigenvalues are denoted by \(\lambda_{\max}(A)\) and \(\lambda_{\min}(A)\), respectively.

Let \(r(t), t \geq 0\), be a right-continuous Markov chain on the probability space taking values in a finite state space \(\mathbb{S} = \{1, 2, \ldots, N\}\) with generator \(\Gamma = (\gamma_{ij})_{N \times N}\) given by

\[
\mathbb{P}\{r(t + \Delta) = j \mid r(t) = i\} = \begin{cases} 
\gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\
1 + \gamma_{ij}\Delta + o(\Delta) & \text{if } i = j,
\end{cases}
\]

where \(\Delta > 0\). Here \(\gamma_{ij} \geq 0\) is a transition rate from \(i\) to \(j\) if \(i \neq j\) while

\[
\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}.
\]

We assume that the Markov chain \(r(\cdot)\) is independent of the Brownian motion \(B(\cdot)\). It is well known that almost every sample path of \(r(t)\) is a right-continuous step function.

Let us consider an \(n\)-dimensional stochastic differential equation (SDE) with Markovian switching of the form

\[
\begin{align*}
\dot{x}(t) &= f(x(t), r(t), t) \, dt + g(x(t), r(t), t) \, dB(t) \\
it &\geq 0 \text{ with the initial data } x(0) = x_0 \in \mathbb{R}^n \text{ and } r(0) = r_0 \in \mathbb{S},
\end{align*}
\]

on \(t \geq 0\) with the initial data \(x(0) = x_0 \in \mathbb{R}^n\) and \(r(0) = r_0 \in \mathbb{S}\), where

\[
f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n \times m}.
\]

We impose the following assumption as a standing hypothesis.

**Assumption 2.1.** Assume that both coefficients \(f\) and \(g\) are locally Lipschitz continuous, that is, for each \(k = 1, 2, \ldots\), there is a positive number \(H_k\) such that

\[
|f(x, i, t) - f(y, i, t)| \vee |g(x, i, t) - g(y, i, t)| \leq H_k |x - y|
\]

for all \(i \in \mathbb{S}, t \geq 0\) and those \(x, y \in \mathbb{R}^n\) with \(|x| \vee |y| \leq k\). Assume also that both coefficients \(f\) and \(g\) obey the linear growth condition, that is, there is a positive constant \(H\) such that

\[
|f(x, i, t)|^2 \vee |g(x, i, t)|^2 \leq H(1 + |x|^2)
\]

for all \((x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+\).

It is known (see e.g., [19, Theorem 3.17 on page 93]) that under Assumption 2.1, the SDE (2.1) has a unique global solution \(x(t)\) on \(t \in \mathbb{R}_+\). We also observe from [19, Theorem 3.26 on page 102] that under Assumption 2.1 the solution obeys

\[
\limsup_{t \to \infty} \frac{1}{t} \log(|x(t)|) \leq \sqrt{H} + \frac{H}{2} \quad \text{a.s.}
\]

That is, the solution will grow at most exponentially with probability one. The following theorem shows that if the noise is sufficiently large, it will suppress this potentially exponential growth and make the solution grow at most polynomially.

**Theorem 2.2.** Let Assumption 2.1 hold. Assume that for each \(i \in \mathbb{S}\), there is a pair of constants \(\beta_i\) and \(\rho_i\), as well as a positive constant \(\alpha\), such that

\[
2|x, f(x, i, t)| + |g(x, i, t)|^2 \leq \alpha + \beta_i |x|^2
\]

and

\[
|x^T g(x, i, t)|^2 \geq \rho_i |x|^4 - \alpha
\]

(2.2)
Define \( \varepsilon > 0 \) is a nonsingular M-matrix. Then the solution of Eq. (2.1) obeys

\[
\limsup_{t \to \infty} \frac{\log(\|x(t)\|)}{\log t} \leq \frac{1}{2\theta} \text{ a.s.}
\]  

(2.5)

**Proof.** The proof is rather technical so we divide it into three steps.

**Step 1.** By the theory of M-matrices (see e.g. [19, Theorem 2.10 on page 68]), there are positive numbers \( q_1, \ldots, q_n \) such that

\[
\tilde{q}_i := \theta q_i \left[ 2(1 - \theta) \rho_i - \beta_i \right] - \sum_{j=1}^{N} y_{ij} q_j > 0, \quad i \in \mathbb{S}.
\]

(2.6)

Define

\[
V(x, i) = q_i (1 + |x|^2)^\theta, \quad (x, i) \in \mathbb{R}^n \times \mathbb{S}.
\]

By the generalized Itô formula (see e.g. [19, Lemma 1.9 on page 49]),

\[
E[V(x(t), r(t))] = V(x_0, r_0) + \int_0^t LV(x(s), r(s), s) \, ds,
\]

where \( LV : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R} \) is defined by

\[
LV(x, i, t) = q_i (\theta (1 + |x|^2)^{\theta - 1} [2x_i f(x, i, t)] + |g(x, i, t)|^2) + 2\theta (\theta - 1) (1 + |x|^2)^{\theta - 2} |x|^2 g(x, i, t)^2 + \sum_{j=1}^{N} y_{ij} q_j (1 + |x|^2)^\theta.
\]

By conditions (2.2) and (2.3), we compute

\[
LV(x, i, t) \leq q_i (\theta (1 + |x|^2)^{\theta - 2} \left[ (1 + |x|^2)[\alpha + \beta_i |x|^2] - 2(1 - \theta) (\rho_i |x|^4 - \alpha) \right] + \sum_{j=1}^{N} y_{ij} q_j (1 + |x|^2)^\theta
\]

\[
= q_i \theta (1 + |x|^2)^{\theta - 2} \left( 3\alpha + (\alpha + \beta_i) |x|^2 - 2\rho_i (1 - \theta) - \beta_i |x|^4 \right) + \sum_{j=1}^{N} y_{ij} q_j (1 + |x|^2)^{\theta - 2} \left( 1 + 2|x|^2 + |x|^4 \right)
\]

\[
= \left( 1 + |x|^2 \right)^{\theta - 2} \left( q_i \theta \left[ 3\alpha + (\alpha + \beta_i) |x|^2 \right] + \sum_{j=1}^{N} y_{ij} q_j (1 + 2|x|^2) - \tilde{q}_i |x|^4 \right).
\]

(2.7)

Choose \( \varepsilon > 0 \) sufficiently small for

\[
0 < \min_{i \in \mathbb{S}} \tilde{q}_i.
\]

(2.8)

Then, by the generalised Itô formula again,

\[
E[e^{\varepsilon t} V(x(t), r(t))] = V(x_0, r_0) + \int_0^t e^{\varepsilon s} \left[ \varepsilon V(x(s), r(s)) + LV(x(s), r(s), s) \right] \, ds.
\]

(2.9)

But, by (2.7) and (2.8), we estimate that, for \( (x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \),

\[
\varepsilon V(x, i) + LV(x, i, t) \leq (1 + |x|^2)^{\theta - 2} \left( q_i \theta \left[ 3\alpha + (\alpha + \beta_i) |x|^2 \right] + \left[\varepsilon q_i + \sum_{j=1}^{N} y_{ij} q_j \right] (1 + 2|x|^2) - (\tilde{q}_i - \varepsilon q_i) |x|^4 \right)
\]

\[
\leq C_1,
\]

where \( C_1 \) is a positive constant. We therefore derive from (2.9) that

\[
\tilde{q} \int_{0}^{t} e^{\varepsilon s} ds \leq V(x_0, r_0) + \int_{0}^{t} C_1 e^{\varepsilon s} \, ds \leq V(x_0, r_0) + \frac{C_1}{\varepsilon} e^{\varepsilon t}.
\]
where \( \hat{q} = \min_{i \in \mathcal{S}} q_i \). This implies that

\[
\mathbb{E} \left[ (1 + |x(t)|^2)^\theta \right] \leq \frac{1}{\hat{q}} \left( V(x_0, r_0) + \frac{C_1}{k} \right) := C_2 \quad \forall t \geq 0.
\]  

(2.10)

**Step 2.** Choose \( \delta > 0 \) so small for

\[
\mathbb{E} \left[ (\delta^2 H)^\theta + C_{2\theta}(\delta H)^\theta \right] \leq \frac{1}{2},
\]  

(2.11)

where \( C_{2\theta} \) is the constant given by the well-known Burkholder–Davis–Gundy inequality (see e.g. [17,19]). Let \( k = 1, 2, \ldots \).

For \( k \delta \in [k\delta, (k + 1)\delta) \), we clearly have

\[
1 + |x(t)|^2 \leq 3(1 + |x(k\delta)|^2) + 3 \left| \int_{k\delta}^t f(x(s), r(s), s) \, ds \right|^2 + 3 \left| \int_{k\delta}^t g(x(s), r(s), s) \, dB(s) \right|^2.
\]

Noting that for any \( a, b, c \geq 0 \)

\[
(a + b + c)^\theta \leq 3(a \vee b \vee c)^\theta \leq 3\left[ a^\theta + b^\theta + c^\theta \right],
\]

and using (2.10), we then have

\[
\mathbb{E} \left( \sup_{k\delta \leq t \leq (k+1)\delta} (1 + |x(t)|^2)^\theta \right) \leq 9C_2 + 9\mathbb{E} \left( \sup_{k\delta \leq t \leq (k+1)\delta} \left| \int_{k\delta}^t f(x(s), r(s), s) \, ds \right|^2 \right) + 9\mathbb{E} \left( \sup_{k\delta \leq t \leq (k+1)\delta} \left| \int_{k\delta}^t g(x(s), r(s), s) \, dB(s) \right|^2 \right).
\]

(2.12)

Compute, by Assumption 2.1,

\[
\mathbb{E} \left( \sup_{k\delta \leq t \leq (k+1)\delta} \left| \int_{k\delta}^t f(x(s), r(s), s) \, ds \right|^2 \right) \leq \mathbb{E} \left( \int_{k\delta}^{(k+1)\delta} \delta \left| f(x(s), r(s), s) \right| \, ds \right)^2 \leq \mathbb{E} \left( \delta \sup_{k\delta \leq s \leq (k+1)\delta} \left| f(x(s), r(s), s) \right| \right)^2 \leq (\delta^2 H)^\theta \mathbb{E} \left( \sup_{k\delta \leq s \leq (k+1)\delta} (1 + |x(s)|^2) \right).
\]

(2.13)

Also, by the well-known Burkholder–Davis–Gundy inequality (see e.g. [17,19]), we compute

\[
\mathbb{E} \left( \sup_{k\delta \leq t \leq (k+1)\delta} \left| \int_{k\delta}^t g(x(s), r(s), s) \, dB(s) \right|^2 \right) \leq C_{2\delta} \mathbb{E} \left( \int_{k\delta}^{(k+1)\delta} \left| g(x(s), r(s), s) \right|^2 \, ds \right)^\theta \leq C_{2\delta}(\delta H)^\theta \mathbb{E} \left( \sup_{k\delta \leq s \leq (k+1)\delta} (1 + |x(s)|) \right)^2.
\]

(2.14)

Substituting this into (2.12) gives

\[
\mathbb{E} \left( \sup_{k\delta \leq t \leq (k+1)\delta} (1 + |x(t)|^2)^\theta \right) \leq 9C_2 + 9\left[ (\delta^2 H)^\theta + C_{2\delta}(\delta H)^\theta \right] \mathbb{E} \left( \sup_{k\delta \leq t \leq (k+1)\delta} (1 + |x(t)|)^2 \right).
\]

Recalling (2.11), we get

\[
\mathbb{E} \left( \sup_{k\delta \leq t \leq (k+1)\delta} (1 + |x(t)|^2)^\theta \right) \leq 18C_2,
\]

whence

\[
\mathbb{E} \left( \sup_{k\delta \leq t \leq (k+1)\delta} |x(t)|^2 \right) \leq 18C_2 \quad \forall k \geq 1.
\]

(2.15)
Step 3. Let $\tilde{\epsilon} > 0$ be arbitrary. Then, by the well-known Chebyshev inequality, we have
\[
P \left\{ \sup_{k\delta \leq t \leq (k+1)\delta} |x(t)|^{2^\theta} > (k\delta)^{1+\tilde{\epsilon}} \right\} \leq \frac{18C}{(k\delta)^{1+\tilde{\epsilon}}}, \quad k = 1, 2, \ldots.
\]
Applying the well-known Borel–Cantelli lemma (see e.g. [17]), we obtain that for almost all $\omega \in \Omega$,
\[
\sup_{k\delta \leq t \leq (k+1)\delta} |x(t)|^{2^\theta} \leq (k\delta)^{1+\tilde{\epsilon}}
\]  
holds for all but finitely many $k$. Hence, there exists a $k_0(\omega)$, for almost all $\omega \in \Omega$, for which (2.16) holds whenever $k \geq k_0$.

Consequently, for almost all $\omega \in \Omega$, if $k \geq k_0$ and $k\delta \leq t \leq (k+1)\delta$,
\[
\frac{\log(|x(t)|^{2^\theta})}{\log t} \leq \frac{(1+\tilde{\epsilon}) \log(k\delta)}{\log(k\delta)} = 1 + \tilde{\epsilon}.
\]
Therefore
\[
\limsup_{t \to \infty} \frac{\log(|x(t)|^{2^\theta})}{\log t} \leq \frac{1 + \tilde{\epsilon}}{2^\theta} \quad \text{a.s.}
\]
Letting $\tilde{\epsilon} \to 0$ we obtain that
\[
\limsup_{t \to \infty} \frac{\log(|x(t)|^{2^\theta})}{\log t} \leq \frac{1}{2^\theta} \quad \text{a.s.},
\]
which is the desired assertion (2.5). The proof is therefore complete. \(\square\)

The following theorem is particularly useful although a slightly stronger condition is imposed.

**Theorem 2.3.** Let Assumption 2.1 as well as (2.2) and (2.3) hold. If
\[
2\rho_i > \beta_i, \quad i \in S,
\]  
then the solution of Eq. (2.1) obeys
\[
\limsup_{t \to \infty} \frac{\log(|x(t)|^{2^\theta})}{\log t} \leq \max_{i \in S} \frac{\rho_i}{2\rho_i - \beta_i} \quad \text{a.s.}
\]  
(2.18)

To prove this theorem, let us state a classical result by Minkowski [21] as a lemma which will be used later as well.

**Lemma 2.4.** Let $A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$. If $a_{ij} \leq 0$ for all $i \neq j$ and
\[
\sum_{j=1}^n a_{ij} > 0 \quad \text{for all } 1 \leq i \leq n,
\]  
then $\det A > 0$.

Let us now prove Theorem 2.3.

**Proof of Theorem 2.3.** For any
\[
0 < \theta < \min_{i \in S} \frac{2\rho_i - \beta_i}{2\rho_i},
\]  
(2.19)
we have
\[
2(1 - \theta)\rho_i - \beta_i > 0, \quad i \in S.
\]
Recalling the fact that
\[
-\gamma_i = \sum_{j \neq i} \gamma_{ij}, \quad i \in S,
\]  
we have
\[
2(1 - \theta)\rho_i - \beta_i - \gamma_i > \sum_{j \neq i} \gamma_{ij}, \quad i \in S.
\]
By Lemma 2.4, all the principal minors of the matrix 
\[ \text{diag}(\theta [2(1-\theta)\rho_1 - \beta_1], \ldots, \theta [2(1-\theta)\rho_N - \beta_N]) - \Gamma \]
are positive. Hence, by [19, Theorem 2.10 on page 68], the matrix above is a nonsingular M-matrix. By Theorem 2.2, we then have 
\[ \limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \leq \frac{1}{2\theta} \quad \text{a.s.} \]  
(2.20)
But this holds for any \( \theta \) obeys (2.19), we must therefore have the assertion (2.18). \( \square \)

3. Examples and further motivation

Before we proceed to develop further theory, let us discuss two examples to explain what the theorems above tell us and to show further motivation.

Example 3.1. Consider a scalar linear differential equation with Markovian switching of the form
\[ \frac{dx(t)}{dt} = a(r(t)) + b(r(t))x(t) \]  
(3.1)
on \( t \geq 0 \) with initial value \( x(0) = x_0 > 0 \), where \( a, b \) are both mappings from \( S \rightarrow (0, \infty) \). It has the explicit solution
\[ x(t) = e^{\int_0^t b(r(s)) \, ds} \left( x_0 + \int_0^t a(r(s)) e^{\int_0^s b(r(u)) \, du} \, ds \right). \]

This implies
\[ x(t) \geq x(0) e^{bt}, \]
where \( b = \min_{i \in S} b(i) > 0 \). In other words, the solution of Eq. (3.1) will tend to infinity exponentially with probability one.

Let us now perturb Eq. (3.1) by a scalar Brownian motion \( B(t) \) and suppose the stochastic perturbed system is described by
\[ dx(t) = \left[ a(r(t)) + b(r(t))x(t) \right] dt + \sigma(r(t))x(t) dB(t), \]  
(3.2)
where \( \sigma \) is a mapping from \( S \rightarrow (0, \infty) \). We shall write \( \sigma(i) = \sigma_i \), etc. We suppose that the noise is sufficient large in the sense that
\[ \sigma_i^2 > 2b_i, \quad i \in S. \]  
(3.3)
If we define
\[ f(x, i, t) = a_i + b_i x \quad \text{and} \quad g(x, i, t) = \sigma_i x, \quad \langle x, i, t \rangle \in \mathbb{R} \times S \times \mathbb{R}_+, \]
then Eq. (3.2) can be written as Eq. (2.1). In this case, we clearly have
\[ |xg(x, i, t)|^2 = \sigma_i^2 x^4. \]
Moreover, for any \( \varepsilon > 0 \) sufficiently small for
\[ \varepsilon < \min_{i \in S} (\sigma_i^2 - 2b_i), \]
we have
\[ 2xf(x, i, t) + |g(x, i, t)|^2 = 2a_i x + 2b_i x^2 + \sigma_i^2 x^2 \leq \frac{\sigma_i^2}{\varepsilon} + (2b_i + \sigma_i^2 + \varepsilon) x^2. \]
Applying Theorem 2.3 with \( \rho_i = \sigma_i^2 \) and \( \beta_i = 2b_i + \sigma_i^2 + \varepsilon \), we see that the solution of Eq. (3.2) obeys
\[ \limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \leq \max_{i \in S} \frac{\sigma_i^2}{\sigma_i^2 - 2b_i - \varepsilon} \quad \text{a.s.} \]
Letting \( \varepsilon \to 0 \) yields
\[ \limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \leq \max_{i \in S} \frac{\sigma_i^2}{\sigma_i^2 - 2b_i} \quad \text{a.s.} \]
This means that the solution of Eq. (3.2) will grow at most polynomially with probability one. This shows clearly that noise suppresses exponential growth.
Example 3.2. In the previous example, we suppose $\sigma_i^2 > 2b_i$ for all $i \in \mathbb{S}$, namely the noise needs to be sufficiently large for every mode $i \in \mathbb{S}$. In this example, we will explain that this is unnecessary. To explain this more clearly, let us assume that $\mathbb{S}$ contains only two states, namely $\mathbb{S} = \{1, 2\}$ and of course assume that $\gamma_{12} > 0$ and $\gamma_{21} > 0$. We also assume that $\sigma_i^2 < 2b_i$ but $\sigma_1$ is sufficiently large. To apply Theorem 2.2, choose $\theta \in (0, 1)$ sufficiently small for
\begin{equation}
\theta [2(1 - \theta)\sigma_i^2 - 2b_i - \sigma_i^2] + \gamma_{21} = \theta [(1 - \theta)\sigma_i^2 - 2b_i] + \gamma_{21} > 0.
\end{equation}
Then choose $\sigma_1$ sufficiently large for
\begin{equation}
[(1 - \theta)\sigma_i^2 - 2b_i] + \gamma_{21} > \gamma_{21}[2b_2 - (1 - \theta)\sigma_i^2].
\end{equation}
These two conditions guarantee
\begin{equation*}
\begin{bmatrix}
\theta[(1 - \theta)\sigma_i^2 - 2b_i] & 0 \\
0 & \theta[(1 - \theta)\sigma_i^2 - 2b_i]
\end{bmatrix}
\begin{bmatrix}
-\gamma_{12} & \gamma_{12} \\
\gamma_{21} & -\gamma_{21}
\end{bmatrix}
\end{equation*}
is a nonsingular M-matrix. Now, choose $\varepsilon > 0$ sufficiently small for
\begin{equation*}
\begin{bmatrix}
\theta[(1 - \theta)\sigma_i^2 - 2b_i - \varepsilon] & 0 \\
0 & \theta[(1 - \theta)\sigma_i^2 - 2b_i - \varepsilon]
\end{bmatrix}
\begin{bmatrix}
-\gamma_{12} & \gamma_{12} \\
\gamma_{21} & -\gamma_{21}
\end{bmatrix}
\end{equation*}
to be a nonsingular M-matrix. Applying Theorem 2.2, we see that the solution of Eq. (3.2) obeys
\begin{equation*}
\limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \leq \frac{1}{2\theta} \text{ a.s.}
\end{equation*}
In other words, under conditions (3.4) and (3.5), the solution of Eq. (3.2) will still grow at most polynomially with probability one.

Applying this result to Eq. (1.5) with $\theta = 1/4$ we see that conditions (3.4) and (3.5) reduce to (1.6) whence the assertion (1.7) follows.

4. Noise suppresses exponential growth

The two examples discussed in the previous section show that noise can suppress exponential growth. Let us now develop the idea to cope with more general situations. Consider a nonlinear system described by a differential equation with Markovian switching of the form
\begin{equation}
k(t) = f(x(t), r(t), t).
\end{equation}
Here $f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$ is locally Lipschitz continuous and obeys the linear growth condition (see Assumption 2.1). Clearly, the solution of this equation may grow exponentially with probability one. The question is: can we design a linear stochastic feedback control of the form
\begin{equation}
\sum_{k=1}^m A_k(r(t))x(t)dB_k(t)
\end{equation}
(i.e. choose square matrices $A_k(i) \in \mathbb{R}^{n \times n}$) so that the stochastically controlled system
\begin{equation}
dx(t) = f(x(t), r(t), t)dt + \sum_{k=1}^m A_k(r(t))x(t)dB_k(t)
\end{equation}
will grow at most polynomially with probability one? Let us first establish a corollary from Theorem 2.3. Based on this corollary, we will then answer the question very positively. For convenience, we will write $A_k(i) = A_{ki}$.

Corollary 4.1. Assume that there are constants $\alpha, \xi_i, \delta_i, \rho_i (i \in \mathbb{S})$ such that
\begin{equation}
2|x, f(x, i, t)| \leq \alpha + \xi_i|x|^2
\end{equation}
and
\begin{equation}
\sum_{k=1}^m |A_{ki}||x|^2 \leq \delta_i|x|^2, \quad \sum_{k=1}^m |x^TA_{ki}|^2 \geq \rho_i|x|^4
\end{equation}
for $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$. If
\begin{equation}
2\rho_i > \xi_i + \delta_i, \quad i \in \mathbb{S},
\end{equation}
then, for any initial value $x(0) = x_0 \in \mathbb{R}^n$, the solution of the stochastically controlled system (4.2) obeys
\begin{equation}
\limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \leq \max_{i \in \mathbb{S}} \frac{\rho_i}{2\rho_i - \xi_i - \delta_i} \text{ a.s.}
\end{equation}
By Corollary 4.1, it is straightforward to show that the solution of system (4.9) obeys

\[ (4.10) \text{ or } (4.12) \]

This is based on the assumption that in every regime Theorem 4.2.

Summarizing these cases we obtain the following result.

The conclusion hence follows from Theorem 2.3 directly. □

Let us now assume that (4.3) holds and show that there are many matrices \( A_{ki} \) that satisfy conditions (4.4) and (4.5). First of all, let \( A_{ki} = \sigma_{ki} I \) for \( 1 \leq k \leq m \) and \( i \in \mathbb{S} \), where \( I \) is the \( n \times n \) identity matrix and \( \sigma_{ki} \)'s are non-negative real numbers which represent the intensity of the noise. In this case, the stochastically controlled system becomes

\[
dx(t) = f(x(t), t) \, dt + \sum_{k=1}^{m} \sigma_{k,r(t)} x(t) \, dB_k(t). \tag{4.9}
\]

By Corollary 4.1, it is straightforward to show that the solution of system (4.9) obeys

\[
\limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \leq \max_{i \in \mathbb{S}} \frac{\sum_{k=1}^{m} \sigma_{ki}^2}{\sum_{k=1}^{m} \sigma_{ki}^2 - \xi_i} \text{ a.s.}
\]

if we choose \( \sigma_{ki} \) sufficiently large for

\[ \sum_{k=1}^{m} \sigma_{ki}^2 > \xi_i, \quad i \in \mathbb{S}. \tag{4.10} \]

Hence, the solution will grow at most polynomially with probability one.

Let us consider a more general case. For each pair of \( k = 1, \ldots, m \) and \( i \in \mathbb{S} \), choose a positive-definite matrix \( D_{ki} \) such that

\[ x^T D_{ki} x \geq \frac{\sqrt{3}}{2} \|D_{ki}\| |x|^2 \quad \forall x \in \mathbb{R}^n. \tag{4.11} \]

Obviously, there are many such matrices. Then, for each \( i \in \mathbb{S} \), choose \( \sigma_i > 0 \) large enough for

\[ \sigma_i^2 > \frac{2\xi_i}{\sum_{k=1}^{m} \|D_{ki}\|^2}. \tag{4.12} \]

Set \( A_{ki} = \sigma_i D_{ki} \). Then

\[ \sum_{k=1}^{m} |A_{ki}|^2 \leq \sigma_i^2 \sum_{k=1}^{m} \|D_{ki}\|^2 |x|^2 \quad \text{and} \quad \sum_{k=1}^{m} |x^T A_{ki} x|^2 \geq \frac{3\sigma_i^2}{4} \sum_{k=1}^{m} \|D_{ki}\|^2 |x|^4. \tag{4.13} \]

Thus, by Corollary 4.1, we can conclude that the solution of the stochastically controlled system (4.3) obeys

\[
\limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \leq \max_{i \in \mathbb{S}} \frac{\frac{3\sigma_i^2}{2} \sum_{k=1}^{m} \|D_{ki}\|^2}{\sum_{k=1}^{m} \|D_{ki}\|^2 - \xi_i} \quad \text{a.s.}
\]

Summarizing these cases we obtain the following result.

**Theorem 4.2.** The potentially exponential growth of the solution to a nonlinear switching system \( x(t) = f(x(t), r(t), t) \) can be suppressed by Brownian motions provided (4.3) is satisfied. Moreover, one can even use only a scalar Brownian motion to suppress the exponential growth.

In Corollary 4.1, we require condition (4.5) hold for all \( i \in \mathbb{S} \). As the result, we need to design \( A_{ki} \) to obey, for example, (4.10) or (4.12). This is based on the assumption that in every regime \( i \) (or mode \( i \)), we can design our feedback control \( \sum_{k=1}^{m} A_{ki} x(t) \, dB_k(t) \). However, in many practical situations, the state \( x(t) \) is not observable in some regimes whence the state feedback control cannot be designed. To model such a situation, we decompose \( \mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2 \), where \( \mathbb{S}_1 = \{1, \ldots, N\} \) and \( \mathbb{S}_2 = \{N+1, \ldots, m\} \).
and \( S_2 = \{ N + 1, \ldots, N \} \). We assume that the system is not observable in any regime \( i \in S_1 \) but observable in every regime \( i \in S_2 \). Accordingly, in the controlled system (4.2) we are forced to set

\[
A_{ki} = 0, \quad 1 \leq k \leq m, \quad i \in S_1.
\]

The question is: can we design matrices \( A_{ki} \) for \( 1 \leq k \leq m \) and \( i \in S_2 \) only so that the solution of the controlled system (4.2) will grow at most polynomially with probability one?

To answer the question positively, let us establish another corollary from Theorem 2.2.

**Corollary 4.3.** Assume that there are constants \( \alpha \) and \( \xi_i (i \in S) \) such that

\[
2|x, f(x, i, t)| \leq \alpha + \xi_i |x|^2
\]

for \((x, i, t) \in \mathbb{R}^n \times S \times \mathbb{R}_+ \). Assume also that there are constants \( \delta_i \) and \( \rho_i (i \in S_2) \) such that

\[
\sum_{k=1}^{m} |A_{ki}|^2 \leq \delta_i |x|^2, \quad \sum_{k=1}^{m} |x^T A_{ki}|^2 \geq \rho_i |x|^4
\]

for \((x, i, t) \in \mathbb{R}^n \times S_1 \times \mathbb{R}_+ \). Assume moreover that there is a constant \( \bar{\theta} \in (0, 1) \) such that

\[
\text{diag}(\ldots, -\theta \xi_N, \ldots, \theta[2(1-\theta)\rho_N - \xi_N - \delta_N]) - \Gamma
\]

is a nonsingular M-matrix. Then, for any initial value \( x(0) = x_0 \in \mathbb{R}^n \), the solution of the stochastically controlled system (4.2) obeys

\[
\limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \leq \frac{1}{2\delta} \quad \text{a.s.}
\]

**Proof.** Define \( g : \mathbb{R}^n \times S \times \mathbb{R}_+ \to \mathbb{R}^n \times m \) as in the proof of Corollary 4.1, namely

\[
g(x, i, t) = (A_{1i}, \ldots, A_{mi}x).
\]

Then system (4.2) can be written as Eq. (2.1). For \( i \in S_2 \), we still have (4.7) and (4.8). But for \( i \in S_1 \), recalling (4.14), we have

\[
2|x, f(x, i, t)| + |g(x, i, t)|^2 \leq \alpha + \xi_i |x|^2
\]

and

\[
|x^T g(x, i, t)|^2 = 0.
\]

The conclusion hence follows from Theorem 2.2 directly. \( \Box \)

Making use of this corollary, we can now answer the question above very positively. For this purpose, we impose a simple hypothesis.

**Assumption 4.4.** For each \( i \in S_1 \), there is at least one \( j \in S_2 \) such that \( \gamma_{ij} > 0 \).

This assumption means that the system will be able to switch from any unobservable regime \( i \in S_1 \) to an observable regime \( j \in S_2 \). This is certainly reasonable; otherwise the system may be absorbed in unobservable regimes where we cannot control it by feedback control.

Under Assumption 4.4 and (4.15), we can now show that we can always design matrices \( A_{ki} \) for \( 1 \leq k \leq m \) and \( i \in S_2 \) only so that the solution of the controlled system (4.2) will grow at most polynomially with probability one.

First of all, let \( A_{ki} = \sigma_{ki} I \) for \( 1 \leq k \leq m \) and \( i \in S_2 \), where \( I \) is the \( n \times n \) identity matrix and \( \sigma_{ki} \)'s are non-negative real numbers. So the parameters \( \delta_i \) and \( \rho_i \) in (4.16) become

\[
\delta_i = \rho_i = \sum_{k=1}^{n} \sigma_{ki}^2, \quad i \in S_2.
\]

Consequently, the matrix defined by (4.17) becomes

\[
\text{diag}(\ldots, -\theta \xi_N, \ldots, \theta(1-\theta)\rho_N - \xi_N - \delta_N) - \Gamma.
\]

To explain, let us denote this matrix by \( Q = (q_{ij})_{N \times N} \), namely

\[
q_{ij} = -\gamma_{ij}, \quad i \neq j,
\]

\[
q_{ii} = -\theta \xi_i - \gamma_i, \quad 1 \leq i \leq \bar{N},
\]

\[
q_{ii} = \theta(1-\theta)\rho_i - \xi_i - \gamma_i, \quad \bar{N} + 1 \leq i \leq N.
\]
By Assumption 4.4, we can choose \( \theta \in (0, 0.5) \) sufficiently small for
\[
\theta \xi_i < \sum_{j=N+1}^{N+\Delta} \gamma_j, \quad i \in S_1.
\]  
(4.20)

For this chosen \( \theta \), we can then choose \( \sigma_{ki} \) (hence \( \rho_i \)) sufficiently large for
\[
(1 - 2\theta)\rho_i - \xi_i > 0, \quad i \in S_2.
\]  
(4.21)

Recalling that
\[
-\gamma_i = \sum_{j \neq i} \gamma_{ij}, \quad i \in S,
\]
we observe, by Lemma 2.4, that all the principal minors of the matrix \( Q \) are positive, namely
\[
\begin{vmatrix}
q_{11} & q_{12} & \cdots & q_{1u} \\
q_{11} & q_{12} & \cdots & q_{2u} \\
\vdots & \vdots & \ddots & \vdots \\
q_{u1} & q_{u2} & \cdots & q_{uu}
\end{vmatrix} > 0, \quad 1 \leq u \leq N,
\]
because \( q_{ij} = -\gamma_{ij} \leq 0 \) for \( i \neq j \) and, by (4.20) and (4.21),
\[
\sum_{j=1}^{\Delta} q_{ij} > 0, \quad 1 \leq i \leq u.
\]

Hence, by [19, Theorem 2.10 on page 68], the matrix defined by (4.19) is a nonsingular M-matrix. We therefore conclude by Corollary 4.3 that for any initial value \( x(0) = x_0 \in \mathbb{R}^n \), the solution of the stochastically controlled system (4.2) obeys (4.18).

Let us consider a more general example. For each pair of \( k = 1, \ldots, m \) and \( i \in S_2 \), choose a positive-definite matrix \( D_{ki} \) such that
\[
x^T D_{ki} x \geq \frac{\sqrt{3}}{2} \| D_{ki} \| |x|^2 \quad \forall x \in \mathbb{R}^n.
\]

Set \( A_{ki} = \sigma_i D_{ki} \), where \( \sigma_i > 0 \). It is then easy to show that the parameters \( \delta_i \) and \( \rho_i \) in (4.16) become
\[
\delta_i = \sigma_i^2 \sum_{k=1}^{m} \| D_{ki} \|^2 \quad \text{and} \quad \rho_i = \frac{3\delta_i}{4}, \quad i \in S_2.
\]  
(4.22)

Consequently, the matrix defined by (4.17) becomes
\[
\text{diag}(-\theta \xi_1, \ldots, -\theta \xi_N, \theta [0.5(1 - 3\theta)\delta_{N+1} - \xi_{N+1}], \ldots, \theta [0.5(1 - 3\theta)\delta_N - \xi_N]) - \Gamma.
\]  
(4.23)

By Assumption 4.4, we can first choose \( \theta \in (0, 1/3) \) sufficiently small for (4.20) to hold, and then choose \( \sigma_i \) (hence \( \rho_i \)) sufficiently large for
\[
0.5(1 - 3\theta)\delta_i - \xi_i > 0, \quad i \in S_2.
\]  
(4.24)

In the same way as above, we can then show by Lemma 2.4 that all the principal minors of the matrix defined by (4.23) are positive, whence the matrix is a nonsingular M-matrix. We therefore conclude by Corollary 4.3 that for any initial value \( x(0) = x_0 \in \mathbb{R}^n \), the solution of the stochastically controlled system (4.2) obeys (4.18).

Summarizing these cases we obtain the following result:

**Theorem 4.5.** Under condition (4.3) and Assumption 4.4, we can always design matrices \( A_{ki} \) for \( 1 \leq k \leq m \) and \( i \in S_2 \) only so that the solution of the controlled system (4.2) will grow at most polynomially with probability one. In other words, the potentially exponential growth of the solution to a nonlinear switching system \( \dot{y}(t) = f(y(t), r(t), t) \) can be suppressed by Brownian motions even if the system is not observable in some regimes.

5. **Linear systems**

In particular, let us consider a linear switching differential equation
\[
\frac{dx(t)}{dt} = u(r(t)) + U(r(t))x(t), \quad t > 0,
\]
where \( u : S \to \mathbb{R}^n \) and \( U : S \to \mathbb{R}^{n \times n} \), and as before we will write \( u(i) = u_i \) and \( U(i) = U_i \). In the case when the system is observable in every regime \( i \in S \), we may stochastically perturb it into the linear switching SDE

\[
dx(t) = (u(r(t)) + U(r(t))x(t)) dt + \sum_{k=1}^{m} A_k(r(t))x(t) dB_k(t). \tag{5.1}\]

where \( A_{kl} = \sigma_l D_{kl} \), \( D_{kl} \)'s obey (4.11) and \( \sigma_l > 0 \) is large enough for

\[
\sigma_l^2 > \frac{4\|U_l\|}{\sum_{k=1}^{m} \|D_{kl}\|^2}. \tag{5.2}\]

Note that for any sufficiently small \( \epsilon > 0 \),

\[
2\langle x, u_i + U_i x \rangle \leq 2|u_i| |x| + 2\|U_i\| \|x\|^2 \leq \frac{|u_i|^2}{\epsilon} + (2\|U_i\| + \epsilon)|x|^2.
\]

Using this and (4.13) we observe, by Corollary 4.1, that the solution of the linear controlled system (5.1) obeys

\[
\limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \leq \max_{i \in S} \frac{3\sigma_l^2}{4\epsilon} \sum_{k=1}^{m} \|D_{kl}\|^2 - (2\|U_l\| + \epsilon) a.s.
\]

Since \( \epsilon > 0 \) is arbitrary, we can therefore conclude that under conditions (4.11) and (5.2), the solution of the linear controlled system (5.1) obeys

\[
\limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \leq \max_{i \in S} \frac{3\sigma_l^2}{4\epsilon} \sum_{k=1}^{m} \|D_{kl}\|^2 - 2\|U_l\| a.s.
\]

In the case when the system is not observable in any regime \( i \in S_1 \) but observable in every regime \( i \in S_2 \), we are forced to set \( A_{kl} = 0 \) for \( 1 \leq k \leq m \) and \( i \in S_1 \) in Eq. (5.1). Of course, we need Assumption 4.4 in this case. We first choose \( \theta \in (0, 1/3) \) sufficiently small for

\[
2\theta\|U_i\| < \sum_{j=N+1}^{N} \gamma_j, \quad i \in S_1. \tag{5.3}\]

We then choose \( \sigma_l \) sufficiently large for

\[
0.5(1 - 3\theta)\delta_l - 2\|U_l\| > 0, \quad i \in S_2, \tag{5.4}\]

where \( \delta_l = \sigma_l^2 \sum_{k=1}^{m} \|D_{kl}\|^2 \). In this way, the matrix (defined by (4.17))

\[
\text{diag}(-2\theta\|U_1\|, \ldots, -2\theta\|U_{N+1}\|, \theta\big[0.5(1 - 3\theta)\delta_{N+1} - 2\|U_{N+1}\|, \ldots, \theta\big[0.5(1 - 3\theta)\delta_N - 2\|U_N\|\big]\big] - \Gamma
\]

is a nonsingular M-matrix. We therefore conclude by Corollary 4.3 that for any initial value \( x(0) = x_0 \in \mathbb{R}^n \), the solution of the stochastically controlled system (5.1) obeys (4.18).

6. Conclusions

In this paper, we consider a system described by a differential equation under regime switching, whose solution may grow exponentially. We suppose that the system is subject to environmental noise in some regimes, or we can stochastically perturb the system in some regimes. We then show that the regime switching and the environmental noise will make the original system whose solution grows exponentially become a new system whose solutions will grow at most polynomially. In other words, we reveal that the regime switching and the environmental noise will suppress the exponential growth.

However, everything has two sides. To close our paper we would like to point out that the regime switching and the environmental noise may also express exponential growth. But, due to the page limit here, we will report these results elsewhere.

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