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Letter to the Editor

Global existence and decay properties for a degenerate Keller–Segel model with a power factor in drift term

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Abstract

The following degenerate parabolic system modelling chemotaxis is considered:

$$\begin{cases} u_t = \nabla \cdot (\nabla u^m - u^{q-1}\nabla v), & x \in \mathbb{R}^N,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), \qquad \tau v(x,0) = \tau v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{KS}$$

where $m \geqslant 1$, $q \geqslant 2$, $\tau = 0$ or 1, and $N \geqslant 1$. The aim of this paper is to prove the existence of a time global weak solution (u, v) of (KS) with the $L^\infty(0, \infty; L^\infty(\mathbb{R}^N))$ bound. Such a global bound is obtained in the case of (i) $m > q - \frac{2}{N}$ for large initial data and (ii) $1 \leqslant m \leqslant q - \frac{2}{N}$ for small initial data. In the case of (ii), the decay properties of the solution (u, v) are also discussed.

Keywords: Degenerate parabolic system; Chemotaxis; Keller–Segel model; Global existence; Decay property; Asymptotic behavior; Porous medium equation; Drift term

1. Introduction

We consider the following parabolic system of degenerate quasilinear type:

$$\begin{cases} u_t = \nabla \cdot (\nabla u^m - u^{q-1}\nabla v), & x \in \mathbb{R}^N,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), \qquad \tau v(x,0) = \tau v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{KS}$$

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where $m \geqslant 1$, $q \geqslant 2$, $\tau = 0$ or 1, and $N \geqslant 1$. The initial data (u_0, v_0) is a non-negative function and in $L^1 \cap L^\infty(\mathbb{R}^N) \times L^1 \cap H^1 \cap W^{1,\infty}(\mathbb{R}^N)$ with $u_0^m \in H^1(\mathbb{R}^N)$.

In this paper, we shall give a complete proof of our previous announcement in [25].

Keller and Segel [12] proposed the mathematical model describing the aggregation process of amoebae by chemotaxis and nowadays it is called Keller–Segel model. We consider the Keller–Segel model of degenerate type (KS).

Concerning the existence of a weak solution (u, v) of (KS), for example, in our previous papers [24,26], we restricted to the case $q = 2$ and established the systematic construction of a weak solution in the following cases:

(1) When $m \geqslant 2$ and $\tau = 1$, (KS) is globally solvable without any restriction on the size of the initial data.
(2) When $m > 2 - \frac{2}{N}$ and $\tau = 0$, (KS) is globally solvable without any restriction on the size of the initial data.
(3) When $1 < m \leqslant 2 - \frac{2}{N}$ and $\tau = 0$, (KS) is globally solvable for small initial data. Furthermore, the decay of solution in $L^p(\mathbb{R}^N)$ was obtained.

(Recently, another degenerate case is treated by Laurencot and Wrzosek [16].)

In this paper, we consider the case of $q \geqslant 2$ and prove that:

(i) When $m \geqslant q$ and $\tau = 0, 1$, (KS) is globally solvable without any restriction on the size of the initial data.
(ii) When $m > q - \frac{2}{N}$ and $\tau = 0$, (KS) is globally solvable without any restriction on the size of the initial data.
(iii) When $1 \leqslant m \leqslant q - \frac{2}{N}$ and $\tau = 0$, (KS) is globally solvable for small initial data. Furthermore, the decay of solution in $L^p(\mathbb{R}^N)$ is shown.

Throughout this paper, we deal with a weak solution of (KS). Our definition of a weak solution now reads:

Definition 1. Let $m \geqslant 1$, $q \geqslant 2$ and let $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$ with $u_0^m \in H^1(\mathbb{R}^N)$ and $\tau v_0 \in L^1 \cap H^1 \cap W^{1,\infty}(\mathbb{R}^N)$. A pair (u, v) of non-negative functions defined in $\mathbb{R}^N \times [0, T)$ is called a weak solution of (KS) on $[0, T)$ if

(i) $u \in L^\infty(0, T; L^2(\mathbb{R}^N))$, $u^m \in L^2(0, T; H^1(\mathbb{R}^N))$,
(ii) $v \in L^\infty(0, T; H^1(\mathbb{R}^N))$,
(iii) (u, v) satisfies the equations in the sense of distribution, i.e., that

$$\int_0^T \int_{\mathbb{R}^N} \left(\nabla u^m \cdot \nabla \varphi - u^{q-1} \nabla v \cdot \nabla \varphi - u \varphi_t\right) dx\, dt = \int_{\mathbb{R}^N} u_0(x) \varphi(x, 0)\, dx,$$

$$\int_0^T \int_{\mathbb{R}^N} (\nabla v \cdot \nabla \varphi + v\varphi - u\varphi - \tau v \varphi_t)\, dx\, dt = \tau \int_{\mathbb{R}^N} v_0(x) \varphi(x, 0)\, dx$$

for any continuously differentiable function φ with compact support in $\mathbb{R}^N \times [0, T)$.

The first theorem gives the existence of a time global weak solution to (KS) with $\tau = 1$ and the uniform bound of the weak solution when $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$ and $v_0 \in L^1 \cap H^1 \cap W^{1,\infty}(\mathbb{R}^N)$. We note that the initial data is not assumed to be small.

Theorem 1 *(Time global existence of $\tau = 1$ case). Let $N \geqslant 1$, $\tau = 1$, $2 \leqslant q \leqslant m$ and $T > 0$. Suppose that u_0 and v_0 are non-negative everywhere with the property in Definition* 1. *Then,* (KS) *has a weak solution (u, v) on $[0, T)$. Moreover, $u^m \in C((0,T); L^2_{\mathrm{loc}}(\mathbb{R}^N))$ and (u, v) satisfies a uniform estimate, i.e., that there exists a constant $K_1 = K_1(\|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, \|v_0\|_{L^1(\mathbb{R}^N)}, \|v_0\|_{H^1(\mathbb{R}^N)}, \|v_0\|_{W^{1,\infty}(\mathbb{R}^N)}, m, q, N, T) > 0$ such that*

$$\sup_{0<t<T} \left(\left\|u(t)\right\|_{L^r(\mathbb{R}^N)} + \left\|v(t)\right\|_{L^r(\mathbb{R}^N)}\right) \leqslant K_1 \quad \text{for all } r \in [1, \infty]. \tag{1.1}$$

In addition, there exists a positive constant $K_2 = K_2(\|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^2(\mathbb{R}^N)}, \|u_0\|_{L^{m-q+2}(\mathbb{R}^N)}, \|v_0\|_{H^1(\mathbb{R}^N)}, m, q, N)$ independent of T such that

$$\|v_t\|_{L^2(0,T;L^2(\mathbb{R}^N))} + \left\|v(t)\right\|_{L^2(0,T;H^2(\mathbb{R}^N))} \leqslant K_2. \tag{1.2}$$

We next consider the case when $\tau = 0$ and $m \geqslant 1$, which includes degenerate versions of "the Nagai model" for the semilinear Keller–Segel system.

Theorem 2 *(Time global existence of $\tau = 0$ case). Let $N \geqslant 1$, $\tau = 0$, $m \geqslant 1$, $q \geqslant 2$, $m > q - \frac{2}{N}$ and suppose that u_0 is non-negative with the property in Definition* 1. *In addition, let $m > q - 1$ in the case of $N = 1$. Then,* (KS) *has a weak solution (u, v) on $[0, \infty)$. Moreover, it satisfies a uniform estimate, i.e., that there exists $K_1 = K_1(\|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, m, q, N)$ such that*

$$\sup_{0<t<\infty} \left(\left\|u(t)\right\|_{L^r(\mathbb{R}^N)} + \left\|v(t)\right\|_{L^r(\mathbb{R}^N)}\right) \leqslant K_1 \quad \text{for all } r \in [1, \infty]. \tag{1.3}$$

In addition, there exists a positive constant $K_2 = K_2(\|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, m, q, N)$ such that

$$\sup_{0<t<\infty} \left\|v(t)\right\|_{W^{2,r}(\mathbb{R}^N)} \leqslant K_2 \quad \text{for all } r \in (1, \infty]. \tag{1.4}$$

Remark 1. It is known that the semilinear case ($m = 1$ and $\tau = 0$), the following functional $W_0(u(t), v(t))$ becomes the Lyapunov function (see Nagai, Senba and Yoshida [18]):

$$W_0(t) = \int_{\mathbb{R}^N} u(t) \log u(t)\, dx - \int_{\mathbb{R}^N} u(t)v(t)\, dx + \frac{1}{2}\left(\left\|\nabla v(t)\right\|_{L^2}^2 + \left\|v(t)\right\|_{L^2}^2\right).$$

In the quasilinear case: $m > 1$, the analogous functional $W(u(t), v(t))$ can be introduced as follows:

$$\begin{aligned} W(t) = {} & \frac{m}{(m-q+1)(m-q+2)} \int u(t)^{m-q+2}\, dx \\ & - \int u^{q-1}(t)v(t)\, dx + \frac{1}{2}\left(\left\|\nabla v(t)\right\|_{L^2}^2 + \left\|v(t)\right\|_{L^2}^2\right). \end{aligned}$$

We show that the above functional $W(t)$ plays a substitutional role as the Lyapunov function for the quasi-linear case. As a result, the functional $W(t)$ yields the uniform $L^{m-q+2}(\mathbb{R}^N)$ bounds independently of ε, T under the condition $m > q - \frac{2}{N+2}$ for any $N \geqslant 1$ (see Appendix A).

Finally, we present the decay property for the weak solution of (KS) in the $\tau = 0$ case under the smallness assumption on $\|u_0\|_{L^{N(q-m)/2}(\mathbb{R}^N)}$.

Theorem 3 *(Decay property). Let $1 \leqslant p < \infty$, $N \geqslant 1$, $\tau = 0$, $q \geqslant 2$, $1 \leqslant m \leqslant q - \frac{2}{N}$ and suppose that the initial data u_0 is non-negative everywhere. Then, there exist an absolute constant M and a positive number ε depending only on M, p, N, m such that if $u_0 \in L^{\frac{N(q-m)}{2}}(\mathbb{R}^N)$ satisfies that*

$$\|u_0\|_{L^1(\mathbb{R}^N)} = M, \qquad \|u_0\|_{L^{N(q-m)/2}(\mathbb{R}^N)} \leqslant \varepsilon, \tag{1.5}$$

then (KS) *has a weak solution* (u, v) *on* $[0, \infty)$ *with the following decay property*:

$$(1+t)^d \left(\|u(t)\|_{L^p(\mathbb{R}^N)} + \|v(t)\|_{L^p(\mathbb{R}^N)} \right) < \infty, \tag{1.6}$$

where

$$d = \sigma\left(1 - \frac{1}{p}\right), \qquad \sigma = \frac{N}{N(m-1)+2}.$$

Remark 2. Our decay rate in Theorem 3 seems to be optimal. In fact, when $m = 1$, our decay rate coincides with the L^1–L^r estimate for the linear heat equation.

When we substitute the second equation: $\Delta v = v - u$ into the first equation in (KS), it holds that

$$u_t = \Delta u^m - \nabla u^{q-1} \cdot \nabla v - u^{q-1}\Delta v = \Delta u^m + u^q - \nabla u \cdot \nabla v - u^{q-1} v. \tag{E}$$

The above equation (E) includes the terms u_t, Δu^m and u^q. Therefore, we observe that the following equation is analogous to (E):

$$\begin{cases} u_t = \Delta u^m + u^q, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{PS}$$

We are interested in finding similarities and differences between (KS) and (PS) in order to observe the effects of the reaction term $-\nabla(u^{q-1}\nabla v)$ in the first equation of (KS).

As for the remarkable difference between (KS) and (PS), we easily find the mass conservation law ($\|u(t)\|_{L^1(\mathbb{R}^N)} = \|u_0\|_{L^1(\mathbb{R}^N)}$ for all $t \geqslant 0$), which holds for (KS) but not for (PS). In order to mention similarities and another differences between (KS) and (PS), we divide the situation into three cases: the first one is of the case $1 \leqslant m \leqslant q - \frac{2}{N}$; the second one is of the case $q - \frac{2}{N} < m < q$; the final one is of the case $m \geqslant q$. Concerning the first case: $1 \leqslant m \leqslant q - \frac{2}{N}$, in this paper we prove that a solution of (KS) exists globally in time for small initial data. On the other hand, the solution of (PS) with small initial data exists globally in time, too (see, for example, Samarskii et al. [23]). Thus, we see that (KS) and (PS) are similar to each other in the first case of $1 \leqslant m \leqslant q - \frac{2}{N}$. In the second case: $q - \frac{2}{N} < m < q$, we prove that a solution of (KS) exists globally in time without any restriction on the size of the initial data. On the other hand, the solution of (PS) blows up in a finite time. Also for the third case of $m \geqslant q$, the solution of (PS) blows up in a finite time, which occurs in some region (see [23] for details.) However, by virtue of the mass conservation law for (KS), if the solution of (KS) blows up in a finite time, then singular points cannot have positive measure. Thus, we are led to the following problem: in the case of $m \geqslant q$, whether solutions of (KS) blow up in discrete points or exist globally in time? In this paper, we show that (KS) is solvable globally in time without any restriction on the size of the initial data in the case of $m \geqslant q$. Thus, we make it clear that (KS) and (PS) differ from each other in the second and third cases, i.e., when $m > q - \frac{2}{N}$.

In the semilinear case: $m = 1$, we refer to Corrias, Perthame and Zaag [5], Diaz, Nagai and Rakotoson [7], Herrero and Velázquez [9], Jäger and Luckhaus [11], Nagai, Senba and Yoshida [18], Nagai [19]. We also refer to Horstmann [10] which summarized various aspects and results for Keller–Segel models.

Concerning the Keller–Segel system of quasilinear type, we refer to Biler, Nadzieja and Stanczy [2], Bonami, Hilhorst, Logak and Mimura [3], Calvez and Carrillo [6], Kowalczyk [17] and Luckhaus and Sugiyama [15]. In those papers, essentially, the problem of the following type was considered:

$$\text{(QKS)} \quad \begin{cases} u_t = \nabla \cdot (u \nabla h(u) - u \cdot \nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega,\ t > 0. \end{cases} \tag{1.7}$$

The finite time blow-up was first formally obtained by [2] for Neumann problem. They [2] consider the second equation as $0 = \Delta v + u$ and gave a proof by using Riesz potential. On the other hand, we [27] gave a rigorous complete proof for the Cauchy problem (KS) (with the absorption term in the second equation) using the Bessel potential. Those results were obtained independently each other. In [17], the time global L^∞ bound was obtained for (QKS) of nondegenerate type and the existence of a solution was not considered.

In the following section, we shall prepare several lemmas which will be often used in this paper. In Section 3, we organize the proof of the existence of a time global solution of the approximated problem of (KS). In Section 4, we give a proof of Theorems 1 and 2. Moreover, we discuss the decay property of the weak solution (u, v) in Section 5.

In what follows, we will use the simplified notations:

(1) $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij}^2 = \partial_i \partial_j$, $\nabla^2 u = (\partial_{11}^2, \partial_{12}^2, \ldots)$, $\|\cdot\|_{L^r} = \|\cdot\|_{L^r(\mathbb{R}^N)}$, $1 \leqslant r \leqslant \infty$, $\int \cdot\, dx := \int_{\mathbb{R}^N} \cdot\, dx$.
(2) $Q_T := \mathbb{R}^N \times (0, T)$.
(3) When the weak derivatives $\nabla u, \nabla^2 u$ and u_t are in $L^p(Q_T)$ for some $p \geqslant 1$, we say that $u \in W_p^{2,1}(Q_T)$, i.e.,

$$\begin{aligned} W_p^{2,1}(Q_T) := \bigl\{ u \in L^p\bigl(0, T; W^{2,p}\bigl(\mathbb{R}^N\bigr)\bigr) \cap W^{1,p}\bigl(0, T; L^p\bigl(\mathbb{R}^N\bigr)\bigr):& \\ \|u\|_{W_p^{2,1}(Q_T)} := \|u\|_{L^p(Q_T)} + \|\nabla u\|_{L^p(Q_T)} + \bigl\|\nabla^2 u\bigr\|_{L^p(Q_T)}& \\ + \|u_t\|_{L^p(Q_T)} < \infty \bigr\}.& \end{aligned}$$

2. Some lemmas

In this section, we shall prepare several lemmas which will be used often in the next section.

The following inequalities are easily ensured by Duoandikoetxea [8, p. 110] and Brezis [4, IX.12].

Lemma 4. *Let $w \in W^{2,r}(\mathbb{R}^N)$. Then the following inequalities hold*:

$$\|\nabla^2 w\|_{L^r(\mathbb{R}^N)} \leqslant C\left(\frac{r^2}{r-1}\right)^2 \|\Delta w\|_{L^r(\mathbb{R}^N)} \quad \textit{for } 1 < r < \infty, \tag{2.1}$$

$$\|w\|_{L^\infty(\mathbb{R}^N)} \leqslant \frac{2r}{r-N} \cdot \|w\|_{W^{1,r}(\mathbb{R}^N)} \quad \textit{for } r > N, \tag{2.2}$$

where C is a positive constant depending only on N.

We consider the following Cauchy problem:

$$\begin{cases} z_t = \Delta z - z + f, & x \in \mathbb{R}^N,\ t > 0, \\ z(x,0) = z_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{P}$$

The following definition is a standard one from semigroup theory. (For instance, see Pazy [22].)

Definition 2. Let X be a Banach space, z_0 belong to X, and $f \in L^1(0,T;X)$. The function $z(x,t) \in C([0,T];X)$ given by

$$z(t) = e^{-t}e^{t\Delta}z_0 + \int_0^t e^{-(t-s)} \cdot e^{(t-s)\Delta} f(s)\,ds, \quad 0 \leqslant t \leqslant T, \tag{P}$$

is the *mild solution* of (P) on $[0,T]$, where $(e^{t\Delta}f)(x,t) = \int_{\mathbb{R}^N} G(x-y,t)f(y)\,dy$ and $G(x,t)$ is the heat kernel by $G(x,t) = \frac{1}{(4\pi t)^{N/2}}\exp(-\frac{|x|^2}{4t})$.

It is easily shown that the following L^p estimates hold:

$$\|e^{t\Delta} f\|_{L^p(\mathbb{R}^N)} \leqslant Ct^{-(\frac{1}{q}-\frac{1}{p})\cdot\frac{N}{2}} \|f\|_{L^q(\mathbb{R}^N)} \quad \text{for } 1 \leqslant q \leqslant p \leqslant \infty, \tag{2.3}$$

$$\|\nabla e^{t\Delta} f\|_{L^p(\mathbb{R}^N)} \leqslant Ct^{-\frac{1}{2}-(\frac{1}{q}-\frac{1}{p})\cdot\frac{N}{2}} \|f\|_{L^q(\mathbb{R}^N)} \quad \text{for } 1 \leqslant q \leqslant p \leqslant \infty, \tag{2.4}$$

where C is a positive constant depending only on p, q and N.

The following lemma is an immediate consequence from (2.3) and (2.4). It plays an important role in establishing the a priori estimates of solution v in (KS).

Lemma 5. *Let $1 \leqslant q \leqslant p \leqslant \infty$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{N}$ and suppose that z is the function given by* (P) *and $z_0 \in W^{1,p}(\mathbb{R}^N)$. If $f \in L^\infty(0,\infty;L^q(\mathbb{R}^N))$, then*

$$\|z(t)\|_{L^p(\mathbb{R}^N)} \leqslant \|z_0\|_{L^p(\mathbb{R}^N)} + C \cdot \Gamma(\gamma) \sup_{0<s<t} \|f(s)\|_{L^q(\mathbb{R}^N)}, \tag{2.5}$$

$$\|\nabla z(t)\|_{L^p(\mathbb{R}^N)} \leqslant \|\nabla z_0\|_{L^p(\mathbb{R}^N)} + C \cdot \Gamma(\tilde{\gamma}) \sup_{0<s<t} \|f(s)\|_{L^q(\mathbb{R}^N)} \tag{2.6}$$

for $t \in [0,\infty)$, where C is a positive constant independent of p, $\Gamma(\cdot)$ is the gamma function, and $\gamma = 1 - (\frac{1}{q} - \frac{1}{p}) \cdot \frac{N}{2}$, $\tilde{\gamma} = \frac{1}{2} - (\frac{1}{q} - \frac{1}{p}) \cdot \frac{N}{2}$.

In addition, let $|\nabla^i z_0| \in L^p(\mathbb{R}^N)$, *and* $f \in L^2(0,T; W^{i-1,p}(\mathbb{R}^N))$ *for* $i = 1, 2, 3$. *Then, it holds that*

$$\left\|\nabla^i z(t)\right\|^2_{L^p(\mathbb{R}^N)} \leqslant \left\|\nabla^i z_0\right\|^2_{L^p(\mathbb{R}^N)} + 2(p+N-2)\int_0^t \left\|\nabla^{i-1} f(s)\right\|^2_{L^p(\mathbb{R}^N)}\,ds \quad \textit{for } t \in [0,\infty). \tag{2.7}$$

The following lemma gives us a variant of Gagliardo–Nirenberg inequality, which was obtained in [27, Lemma 2.4]. See also [20]. It will be often used in the following sections, as the main part of our arguments.

Lemma 6. [27, Lemma 2.4] *Let* $N \geqslant 1$, $m \geqslant 1$, $a > 2$, $u \in L^{q_1}(\mathbb{R}^N)$ *with* $q_1 \geqslant 1$ *and* $u^{\frac{r+m-1}{2}} \in H^1(\mathbb{R}^N)$ *with* $r > 0$. *If* $q_1 \in [1, r+m-1]$, $q_2 \in [\frac{r+m-1}{2}, \frac{a(r+m-1)}{2}]$ *and*

$$\begin{cases} 1 \leqslant q_1 \leqslant q_2 \leqslant \infty & \textit{when } N = 1, \\ 1 \leqslant q_1 \leqslant q_2 < \infty & \textit{when } N = 2, \\ 1 \leqslant q_1 \leqslant q_2 \leqslant \frac{(r+m-1)N}{N-2} & \textit{when } N \geqslant 3, \end{cases} \tag{2.8}$$

then, it holds that

$$\|u\|_{L^{q_2}(\mathbb{R}^N)} \leqslant C^{\frac{2}{r+m-1}} \|u\|^{1-\Theta}_{L^{q_1}(\mathbb{R}^N)} \left\|\nabla u^{\frac{r+m-1}{2}}\right\|^{\frac{2\Theta}{r+m-1}}_{L^2(\mathbb{R}^N)} \quad \textit{with} \tag{2.9}$$

$$\Theta = \frac{r+m-1}{2}\left(\frac{1}{q_1} - \frac{1}{q_2}\right)\left(\frac{1}{N} - \frac{1}{2} + \frac{r+m-1}{2q_1}\right)^{-1}, \tag{2.10}$$

where

$$\begin{cases} C \textit{ depends only on } N \textit{ and } a & \textit{when } q_1 \geqslant \frac{r+m-1}{2}, \\ C = c_0^{1/\beta} \textit{ with } c_0 \textit{ depending only on } N \textit{ and } a & \textit{when } 1 \leqslant q_1 < \frac{r+m-1}{2}, \end{cases} \tag{2.11}$$

and

$$\beta = \frac{q_2 - \frac{r+m-1}{2}}{q_2 - q_1}\left[\frac{2q_1}{r+m-1} + \left(1 - \frac{2q_1}{r+m-1}\right)\frac{2N}{N+2}\right]. \tag{2.12}$$

3. Approximated problem

The first equation of (KS) is a quasi-linear parabolic equation of degenerate type with $m > 1$. Therefore, we cannot expect the system (KS) to have a classical solution at the point where the first solution u vanishes. In order to justify all the formal arguments, we need to introduce the following approximated equation of (KS):

$$(\mathrm{KS})_\varepsilon \quad \begin{cases} u_{\varepsilon t}(x,t) = \nabla \cdot (\nabla (u_\varepsilon + \varepsilon)^m - (u_\varepsilon + \varepsilon)^{q-2} u_\varepsilon \cdot \nabla v_\varepsilon), \quad (x,t) \in \mathbb{R}^N \times (0,T), & (1) \\ \tau v_{\varepsilon t}(x,t) = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, \quad (x,t) \in \mathbb{R}^N \times (0,T), & (2) \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \qquad \tau v_\varepsilon(x,0) = \tau v_{0\varepsilon}(x), \quad x \in \mathbb{R}^N, \end{cases}$$

where $m \geqslant 1$, $q \geqslant 2$, $\tau = 0, 1$ and ε is a positive parameter.

In the case of $N \geqslant 2$, $(u_{0\varepsilon}, v_{0\varepsilon})$ is an approximation for the initial data (u_0, v_0) such that

(A.1) $0 \leqslant u_{0\varepsilon} \in L^1 \cap W^{2,p}(\mathbb{R}^N)$ for all $p \in [\frac{N}{N-1}, N+3]$ for all $\varepsilon \in (0,1]$, $0 \leqslant \tau v_{0\varepsilon} \in L^1 \cap W^{3,p}(\mathbb{R}^N)$ for all $p \in [\frac{N}{N-1}, \infty]$, for all $\varepsilon \in (0,1]$,
(A.2) $\|u_{0\varepsilon}\|_{L^p} \leqslant \|u_0\|_{L^p}$, $\tau\|v_{0\varepsilon}\|_{W^{1,p}} \leqslant \tau\|v_0\|_{W^{1,p}}$ for all $p \in [1,\infty]$, for all $\varepsilon \in (0,1]$,
(A.3) $\|\nabla u_{0\varepsilon}\|_{L^2} \leqslant \|\nabla u_0\|_{L^2}$ for all $\varepsilon \in (0,1]$,
(A.4) $u_{0\varepsilon} \to u_0$, $\tau v_{0\varepsilon} \to \tau v_0$ strongly in $L^p(\mathbb{R}^N)$, as $\varepsilon \to 0$ for some $p \in [\frac{N}{N-1}, \infty)$.

As for the case of $N = 1$, $(u_{0\varepsilon}, v_{0\varepsilon})$ is an approximation for the initial data (u_0, v_0) such that

(A.1)′ $0 \leqslant u_{0\varepsilon} \in L^1 \cap W^{2,p}(\mathbb{R})$, $0 \leqslant \tau v_{0\varepsilon} \in L^1 \cap W^{3,p}(\mathbb{R})$ for all $p \in [2,3]$, for all $\varepsilon \in (0,1]$,
(A.2)′ $\|u_{0\varepsilon}\|_{L^p} \leqslant \|u_0\|_{L^p}$, $\tau\|v_{0\varepsilon}\|_{W^{1,p}} \leqslant \tau\|v_0\|_{W^{1,p}}$ for all $p \in [1,\infty]$, for all $\varepsilon \in (0,1]$,
(A.3)′ $\|\nabla u_{0\varepsilon}\|_{L^2} \leqslant \|\nabla u_0\|_{L^2}$ for all $\varepsilon \in (0,1]$,
(A.4)′ $u_{0\varepsilon} \to u_0$, $\tau v_{0\varepsilon} \to \tau v_0$ strongly in $L^p(\mathbb{R})$, as $\varepsilon \to 0$ for some $p \in [1,\infty)$.

We call $(u_\varepsilon, v_\varepsilon)$ a strong solution of $(\mathrm{KS})_\varepsilon$ if it belongs to $W_p^{2,1} \times W_p^{2,1}(Q_T)$ for some $p \geqslant 1$ and Eqs. (1), (2) in $(\mathrm{KS})_\varepsilon$ are satisfied almost everywhere. The strong solution v_ε coincides with the mild solution defined in Definition 2 if $u_\varepsilon \in L^1(0,T; L^p(\mathbb{R}^N))$ with $p \geqslant 1$.

We denote $\mathbf{W}(Q_T)$ by

$$\mathbf{W}(Q_T) := \mathbf{W}_1(Q_T) \times \mathbf{W}_2(Q_T) := \begin{cases} (W^{2,1}_{N/(N-1)} \cap W^{2,1}_{N+3}(Q_T)) \times W^{2,1}_{N+2}(Q_T) & \text{for } N \geqslant 2, \\ W^{2,1}_3(Q_T) \times W^{2,1}_3(Q_T) & \text{for } N = 1. \end{cases} \tag{3.1}$$

In the definition of $W(Q_T)$, the exponent $\frac{N}{N-1}$ stems from validity of mass conservation law. Indeed, L^1-summability can be obtained by integration by parts of $(\mathrm{KS})_\varepsilon$ multiplied by some cut-off function. In such procedure, we need to control the behavior of boundary integral at ∞. This is the reason why the exponent $\frac{N}{N-1}$ appears in $W(Q_T)$ (see Kozono [13, Lemma 2.1]). Moreover, in order to justify the energy estimate in the proof of Lemma 13 (which gives L^{N+2}-a priori bound for Δu_ε), we require the class $W^{2,1}_{N+3}(Q_T)$.

The main purpose of this section is to construct the time global strong solution of $(\mathrm{KS})_\varepsilon$, which reads:

Theorem 7 *(Time global strong solution). Let $N \geqslant 1$, $\tau = 0, 1$, $m \geqslant 1$, $q \geqslant 2$. Suppose that* (A.1) (*respectively* (A.1)′) *is satisfied in the case of $N \geqslant 2$* (*respectively $N = 1$*)*. Then,* $(\mathrm{KS})_\varepsilon$ *has the unique strong solution in the class $\mathbf{W}(Q_T)$ for all $T > 0$.*

For the proof of Theorem 7, it suffices to show the following three propositions. We first show the local existence theorem and then its extension criterion. Finally, we will carry out our local solution satisfies such a criterion for extension.

Proposition 8 *(Time local existence). Let the same assumptions as that in Theorem 7 hold. Then, there exists a number $T_1 = T_1(\varepsilon, \|u_{0\varepsilon}\|_{W^{2,N+2}(\mathbb{R}^N)}, \tau\|v_{0\varepsilon}\|_{W^{3,\infty}(\mathbb{R}^N)}, m, q, N) > 0$ such that* $(\mathrm{KS})_\varepsilon$ *has the unique non-negative strong solution $(u_\varepsilon, v_\varepsilon)$ belonging to $\mathbf{W}(Q_{T_1})$.*

Proposition 9 *(Extension criterion). Let the same assumptions as that in Theorem 7 hold and let $T > 0$. Suppose that $(u_\varepsilon, v_\varepsilon)$ is a strong solution of $(\mathrm{KS})_\varepsilon$ in the class $\mathbf{W}(Q_T)$. If it holds that*

$$\sup_{0<t<T} \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} < \infty,$$

then there is $T^{'} > T$ such that $(u_\varepsilon, v_\varepsilon)$ can be a strong solution of $(\mathrm{KS})_\varepsilon$ in $\mathbf{W}(Q_{T'})$.

Proposition 10 *(A priori estimate in L^∞). Let the same assumptions as that in Theorem 7 hold and let $T > 0$. We assume that $(u_\varepsilon, v_\varepsilon)$ is the non-negative strong solution of $(\mathrm{KS})_\varepsilon$ in $\mathbf{W}(Q_T)$. Then, the mass conservation law holds, i.e., that*

$$\|u_\varepsilon(t)\|_{L^1(\mathbb{R}^N)} = \|u_{0\varepsilon}\|_{L^1(\mathbb{R}^N)} \quad \text{for all } t \in [0, T).$$

Moreover, u_ε satisfies the following estimates:

(i) *For $\tau = 1$, $m \geqslant q$, it holds that*

$$\sup_{0<t<T} \|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)} \leqslant M_u^T \quad \text{for all } r \in [1, \infty], \tag{3.2}$$

where M_u^T is a constant depending on $\|u_{0\varepsilon}\|_{L^1\cap L^\infty(\mathbb{R}^N)}$, $\|\nabla v_{0\varepsilon}\|_{L^2\cap L^\infty(\mathbb{R}^N)}$, m, q, N, T but not on ε.

(ii) *For $\tau = 0$, $m > q - \frac{2}{N}$, it holds that*

$$\sup_{0<t<T} \|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)} \leqslant M_u \quad \text{for all } r \in [1, \infty]. \tag{3.3}$$

(iii) *For $\tau = 0$, $1 < m \leqslant q - \frac{2}{N}$, if $\|u_{0\varepsilon}\|_{L^{\frac{N(q-m)}{2}}}$ is small enough, then (3.3) holds, where M_u is a constant depending on $\|u_{0\varepsilon}\|_{L^1\cap L^\infty(\mathbb{R}^N)}$, $\|\nabla v_{0\varepsilon}\|_{L^2\cap L^\infty(\mathbb{R}^N)}$, m, q, N but not on ε, T.*

In the following Sections 3.1–3.3, we prove Propositions 8–10.

3.1. Local existence; proof of Proposition 8

To prove Proposition 8, we introduce the following problem (which is not a system):

$$^h(\mathrm{KS})_\varepsilon \quad \begin{cases} u_{\varepsilon t}(x,t) = \nabla\cdot\big(m(u_\varepsilon+\varepsilon)^{m-1}\nabla u_\varepsilon - (u_\varepsilon+\varepsilon)^{q-2}u_\varepsilon\nabla h\big), \\ \qquad\qquad (x,t)\in\mathbb{R}^N\times(0,T), & (1)_h \\ v_{\varepsilon t}(x,t) = \Delta v_\varepsilon - v_\varepsilon + f, \quad (x,t)\in\mathbb{R}^N\times(0,T), & (2)_f \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \qquad \tau v_\varepsilon(x,0) = \tau v_{0\varepsilon}(x), \quad x\in\mathbb{R}^N, \end{cases}$$

where $f \in L^p(Q_T)$ is a non-negative function and $h \in L^\infty(0,T; W^{3,\infty}(\mathbb{R}^N))$.

In order to prove the existence of a strong solution u_ε of $(1)_h$ in ${}^h(\mathrm{KS})_\varepsilon$, we consider the following equation:

$$\begin{aligned} u_{\varepsilon t}(x,t) &= \nabla\cdot\big(m(g+\varepsilon)^{m-1}\nabla u_\varepsilon\big) - \nabla\big((g+\varepsilon)^{q-2}u_\varepsilon\big)\cdot\nabla h - (g+\varepsilon)^{q-2}u_\varepsilon\Delta h \\ &= m(g+\varepsilon)^{m-1}\Delta u_\varepsilon + \big(m\nabla(g+\varepsilon)^{m-1} - (g+\varepsilon)^{q-2}\nabla h\big)\cdot\nabla u_\varepsilon \\ &\quad - \big(\nabla(g+\varepsilon)^{q-2}\cdot\nabla h + (g_\varepsilon+\varepsilon)^{q-2}\Delta h\big)u_\varepsilon. \end{aligned} \tag{3.4}$$

Here, we denote X_T by

$$\begin{aligned} X_T := \big\{ & g\in L^\infty\big(0,T;W^{2,N+2}\big(\mathbb{R}^N\big)\big),\ g_t\in L^{N+2}(Q_T)\colon g\geqslant 0 \text{ in } Q_T, \\ & \|g_t\|_{L^{N+2}(Q_T)} + \|g\|_{L^\infty(0,T;W^{2,N+2}(\mathbb{R}^N))} \leqslant 2\|u_{0\varepsilon}\|_{W^{2,N+2}(\mathbb{R}^N)} + 1\big\}. \end{aligned} \tag{3.5}$$

Then, Morrey inequality assures that if $g\in X_T$, then g satisfies that

$$\big|g(x,t)-g(y,s)\big| \leqslant c\big(\|\nabla g\|_{L^{N+2}(Q_T)} + \|g_t\|_{L^{N+2}(Q_T)}\big)\big|(x,t)-(y,s)\big|^{1-\frac{N+1}{N+2}}. \tag{3.6}$$

We now remark that it is not difficult to generalize Theorem 9.1 in Ladyzhenskaya, Solonnikov and Ural'ceva [14] for Cauchy problem. The modern treatment such as maximal regularity theorem in L^p can be found in Amann [1, Chapter IV, Theorem 1.5.1]. By virtue of [1, Theorem 1.5.1] or [14, Theorem 9.1], we have the following lemma:

Lemma 11. *Let the same assumption as that in Proposition* 8 *hold. We assume that* $g\in X_T$, *and* h *satisfies that*

$$\|\nabla h\|_{L^\infty(Q_T)} + \|\Delta h\|_{L^\infty(Q_T)} \leqslant B_0 \tag{3.7}$$

for some positive constant B_0. *Then, Eq.* (3.4) *corresponding to the initial data* $u_{0\varepsilon}$ *has the unique non-negative strong solution* u_ε^g *belonging to* $\mathbf{W}(Q_T)$.

By virtue of Lemma 11, we can define an operator S by

$$S\colon g\in X_T \mapsto u_\varepsilon^g \in \mathbf{W}(Q_T).$$

We find that the strong solution u_ε^g is a non-negative function as follows.

We multiply (3.4) by $|u_\varepsilon|^{r-2}u_\varepsilon$ with $r>1$ and integrate it over $\mathbb{R}^N$. Then, we have

$$\begin{aligned} &\frac{1}{r}\frac{d}{dt}\int |u_\varepsilon|^r\,dx \\ &\quad\leqslant -\frac{m(r-1)}{2}\int (g+\varepsilon)^{m-1}|u_\varepsilon|^{r-2}|\nabla u_\varepsilon|^2\,dx \\ &\qquad + \frac{1}{m(r-1)}\left(\big(\|g\|_{L^\infty(Q_T)}+\varepsilon\big)^{2q-m-3} + \frac{1}{\varepsilon^{-2q+m+3}}\right)\|\nabla h\|^2_{L^\infty(Q_T)}\int |u_\varepsilon|^r\,dx \\ &\qquad + \big(\big\|\nabla(g+\varepsilon)^{q-2}\cdot\nabla h\big\|_{L^\infty(Q_T)} + \big\|(g+\varepsilon)^{q-2}\Delta h\big\|_{L^\infty(Q_T)}\big)\int |u_\varepsilon|^r\,dx \end{aligned}$$

for all $\varepsilon\in(0,1]$ and $r>1$.

Whence follows

$$
\begin{aligned}
&\|u_\varepsilon(t)\|_{L^r} \\
&\leqslant \|u_{0\varepsilon}\|_{L^r} + \frac{1}{m(r-1)}\left(\left(\|g\|_{L^\infty(Q_T)}+\varepsilon\right)^{2q-m-3} + \frac{1}{\varepsilon^{-2q+m+3}}\right)\|\nabla h\|^2_{L^\infty(Q_T)} \\
&\quad \times \int_0^t \|u_\varepsilon(s)\|_{L^r}\,ds + \left(\left\|\nabla(g+\varepsilon)^{q-2}\cdot\nabla h\right\|_{L^\infty(Q_T)} + \left\|(g+\varepsilon)^{q-2}\Delta h\right\|_{L^\infty(Q_T)}\right) \\
&\quad \times \int_0^t \|u_\varepsilon(s)\|_{L^r}\,ds \quad \text{for all } \varepsilon\in(0,1] \text{ and } r>1.
\end{aligned}
$$

Letting $r\to\infty$, we derive

$$
\begin{aligned}
\|u_\varepsilon(t)\|_{L^\infty} &\leqslant \|u_{0\varepsilon}\|_{L^\infty} + \left(\left\|\nabla(g+\varepsilon)^{q-2}\nabla h\right\|_{L^\infty(Q_T)} + \left\|(g+\varepsilon)^{q-2}\Delta h\right\|_{L^\infty(Q_T)}\right) \\
&\quad \times \int_0^t \|u_\varepsilon(s)\|_{L^\infty}\,ds.
\end{aligned}
$$

Using the Gronwall inequality, we have

$$
\sup_{0<t<T}\|u_\varepsilon(t)\|_{L^\infty} \leqslant \|u_{0\varepsilon}\|_{L^\infty}\exp\left\{\left(\left\|\nabla(g+\varepsilon)^{q-2}\cdot\nabla h\right\|_{L^\infty(Q_T)} + \left\|(g+\varepsilon)^{q-2}\Delta h\right\|_{L^\infty(Q_T)}\right)T\right\}.
$$

We multiply (3.4) by $u_\varepsilon^- := -\min(0,u_\varepsilon)$ and integrate it over $\mathbb{R}^N$. Then, it holds that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |u_\varepsilon^-|^2\,dx \\
&\quad = -m\int (g+\varepsilon)^{m-1}|\nabla u_\varepsilon^-|^2\,dx + \int (g+\varepsilon)^{q-2}u_\varepsilon\nabla h\cdot\nabla u_\varepsilon^-\,dx \\
&\quad \leqslant \frac{1}{m}\int (g+\varepsilon)^{2q-m-3}|\nabla h|^2|u_\varepsilon^-|^2\,dx \\
&\quad \leqslant \frac{1}{m}\left(\left(\|g\|_{L^\infty(Q_T)}+\varepsilon\right)^{2q-m-3} + \frac{1}{\varepsilon^{-2q+m+3}}\right)\|\nabla h\|^2_{L^\infty(Q_T)}\int |u_\varepsilon^-|^2\,dx.
\end{aligned}
$$

Again using the Gronwall inequality, we obtain

$$
\begin{aligned}
&\sup_{0<t<T}\|u_\varepsilon^-(\cdot,t)\|_{L^2} \\
&\quad \leqslant \exp\left\{\frac{1}{m}\left(\left(\|g\|_{L^\infty(Q_T)}+\varepsilon\right)^{2q-m-3} + \frac{1}{\varepsilon^{-2q+m+3}}\right)\|\nabla h\|^2_{L^\infty(Q_T)}T\right\}\|u_\varepsilon^-(\cdot,0)\|_{L^2} = 0,
\end{aligned}
$$

which assures that

$$u_\varepsilon(x,t) \geqslant 0 \quad \text{for a.a. } x \in \mathbb{R}^N, \text{ for all } 0 \leqslant t < T. \tag{3.8}$$

Moreover, by the standard argument, we find that there exists $T_* = T_*(\varepsilon, \|h\|_{W^{3,\infty}(Q_T)}, \|u_{0\varepsilon}\|_{W^{2,N+2}}, m, q, N, T)$ such that the above operator S maps X_{T_*} into itself.

We now introduce the metric on X_T by

$$d(g_1, g_2) = \|g_1 - g_2\|_{L^\infty(0,T;L^{N+2}(\mathbb{R}^N))}. \tag{3.9}$$

Then, it is easily seen that (X_T, d) is the complete metric space. With this metric, we see that S becomes a contraction from X_{T_*} into itself. Consequently, S has a fixed point $\bar{g} = S(\bar{g}) = u^{\bar{g}} \in X_{T_*}$. Thus, we assure the existence of a strong solution $u_\varepsilon^{\bar{g}}$ of (3.4) on $[0, T_*]$ corresponding to the initial data $u_{0\varepsilon}$, where $T_* = T_*(\varepsilon, \|h\|_{W^{3,\infty}(Q_T)}, \|u_{0\varepsilon}\|_{W^{2,N+2}}, m, N, q, T)$. In addition, by the comparison principle, we see that for any $f \geqslant 0$ in Q_T, it holds that

$$v_\varepsilon(x,t) \geqslant 0 \quad \text{for a.a. } x \in \mathbb{R}^N,\ t \in (0,T). \tag{3.10}$$

By (3.8) and (3.10), we observe that $u_\varepsilon, v_\varepsilon \geqslant 0$ as long as the strong solution $(u_\varepsilon, v_\varepsilon)$ of ${}^h(\mathrm{KS})_\varepsilon$ exists.

Lemma 12. *Let the same assumptions as that in Proposition* 8 *hold. We assume that non-negative functions $f \in L^{N+2}(Q_T)$ and h satisfy that*

$$\|\nabla h\|_{L^\infty(Q_T)} + \|\Delta h\|_{L^\infty(Q_T)} \leqslant B_1, \tag{3.11}$$

$$\|f\|_{L^{N+2}(Q_T)} \leqslant B_2 \tag{3.12}$$

for some positive constants B_1 and B_2. Then, there exists a positive number T_* depending on ε, $\|u_{0\varepsilon}\|_{W^{2,N+2}}, \|h\|_{W^{3,\infty}(Q_T)}, m, q, N, T$ such that ${}^h(\mathrm{KS})_\varepsilon$ has the unique non-negative strong solution $(u_\varepsilon, v_\varepsilon)$ belonging to $\mathbf{W}(Q_{T_*})$.

Moreover, $(u_\varepsilon, v_\varepsilon)$ satisfies the following estimates with $\frac{N}{N-1} \leqslant p \leqslant N+3$ for $N \geqslant 2$ and $p = 3$ for $N = 1$:

$$\|u_\varepsilon\|_{W_p^{2,1}(Q_{T_*})} \leqslant c_1 T_*^{\frac{1}{p}} \|u_{0\varepsilon}\|_{W^{2,p}(\mathbb{R}^N)}, \tag{3.13}$$

$$\|v_\varepsilon\|_{L^{N+2}(0,T_*;W^{2,N+2}(\mathbb{R}^N))} \leqslant c_2\big(\tau \cdot T_*^{\frac{1}{N+2}} \|v_{0\varepsilon}\|_{W^{2,N+2}(\mathbb{R}^N)} + \|f\|_{L^{N+2}(Q_{T_*})}\big) \tag{3.14}$$

for some positive constants $c_1 = c_1(\varepsilon, \|u_{0\varepsilon}\|_{W^{2,N+2}(\mathbb{R}^N)}, B_1, m, q, N)$ and $c_2 = c_2(B_2)$.

Proof of Proposition 8. From Lemma 12, we see that:

(i) if $u_{0\varepsilon} \in W^{\frac{N}{N-1}} \cap W^{N+3}(R^N)$ (respectively $W^{2,3}(\mathbb{R}^N)$) for $N \geqslant 2$ (respectively $N = 1$) and if h belongs to $L^\infty(0,T; W^{2,\infty}(\mathbb{R}^N))$, then $(1)_h$ in ${}^h(\mathrm{KS})_\varepsilon$ has a unique strong solution $\bar{u}^h \in \mathbf{W}_1(Q_{T_*})$ defined in (3.1). Moreover, $\bar{u}^h$ satisfies the following estimate with $\frac{N}{N-1} \leqslant p \leqslant N+3$ (respectively $p = 3$) for $N \geqslant 2$ (respectively $N = 1$):

$$\big\|\bar{u}^h\big\|_{W_p^{2,1}(Q_{T_*})} \leqslant T_*^{\frac{1}{p}} \cdot c_3 \tag{3.15}$$

for some constant $c_3 = c_3(\varepsilon, \|u_{0\varepsilon}\|_{W^{2,p}(\mathbb{R}^N)}, \|h\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{R}^N))}, m, q, N)$; and

(ii) if $v_{0\varepsilon} \in W^{3,\infty}(\mathbb{R}^N)$ and if f belongs to $L^{N+2}(0,T;W^{2,N+2}(\mathbb{R}^N))$, then by Lemma 12, there exists a unique strong solution $\bar{v}^f \in W^{2,1}_{N+2}(Q_T)$ of $(2)_f$ in ${}^h(\mathrm{KS})_\varepsilon$. Moreover, in both cases $\tau = 0$ and $\tau = 1$, $\bar{v}^f$ satisfies the following estimate:

$$\begin{aligned}
&\|\bar{v}^f\|_{L^\infty(0,T;W^{3,\infty}(\mathbb{R}^N))} \\
&\quad \leqslant \tau\|\bar{v}^f\|_{L^\infty(0,T;W^{3,\infty}(\mathbb{R}^N))} + c_4(1-\tau)\|\bar{v}^f\|_{L^\infty(0,T;W^{4,N+2}(\mathbb{R}^N))} \\
&\quad \leqslant c_4\big(\tau\|v_{0\varepsilon}\|_{W^{3,\infty}(\mathbb{R}^N)} + (1-\tau)\|\bar{v}^f\|_{L^\infty(0,T;W^{2,N+2}(\mathbb{R}^N))} + \|f\|_{L^\infty(0,T;W^{2,N+2}(\mathbb{R}^N))}\big) \\
&\quad \leqslant c_4\big(\tau\|v_{0\varepsilon}\|_{W^{3,\infty}(\mathbb{R}^N)} + (1-\tau)\|\bar{v}^f\|_{L^\infty(0,T;L^{N+2}(\mathbb{R}^N))} + 2\|f\|_{L^\infty(0,T;W^{2,N+2}(\mathbb{R}^N))}\big) \\
&\quad \leqslant c_4\big(\tau\|v_{0\varepsilon}\|_{W^{3,\infty}(\mathbb{R}^N)} + 3\|f\|_{L^\infty(0,T;W^{2,N+2}(\mathbb{R}^N))}\big),
\end{aligned} \tag{3.16}$$

where c_4 depends only on N.

By using this strong solution $\bar{v}^f$, ${}^h(\mathrm{KS})_\varepsilon$ with $h = \bar{v}^f$ (denoted by ${}^{h=\bar{v}^f}(\mathrm{KS})_\varepsilon$) has a unique strong solution $(\bar{u}^{h=\bar{v}^f}, \bar{v}^f) \in \mathbf{W}(Q_{T_*})$.

Here, we recall the Banach space X_T with the metric defined in (3.5). From (i) and (ii), we can define an operator Φ by

$$\Phi : f \in X_{T_*} \mapsto \bar{u}^{h=v^f} \in \mathbf{W}_1(Q_{T_*}),$$

where T_* is the existence time of $\bar{u}^{h=v^f}$ obtained from Lemma 12. Moreover, by (3.15) and (3.16), we find that there exists $T_0 = T_0(\varepsilon, \|u_{0\varepsilon}\|_{W^{2,N+2}(\mathbb{R}^N)}, \tau\|v_{0\varepsilon}\|_{W^{3,\infty}(\mathbb{R}^N)}, m, q, N, T) \leqslant T_*$ such that the above operator Φ maps X_{T_0} into itself. To apply the Banach fixed point theorem, we now recall the complete metric space (X_T, d) defined in (3.9). We denote w by

$$w := \bar{u}^{h=v^{f_1}} - \bar{u}^{h=v^{f_2}}.$$

Then, the multiplication $((1)_{h=\bar{v}^{f_1}} - (1)_{h=\bar{v}^{f_2}})$ by $|w|^N w$ and the integration over $\mathbb{R}^N$ give that

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^{N+2}(\mathbb{R}^N)} \leqslant c_5\|w(t)\|^2_{L^{N+2}(\mathbb{R}^N)} + c_5\|\nabla v^{f_1}(t) - \nabla v^{f_2}(t)\|^2_{L^{N+2}(\mathbb{R}^N)}. \tag{3.17}$$

Here and in what follows c_5 will denote a general constant (not necessarily the same at different occurrences) but which depends only on ε, $\|u_{0\varepsilon}\|_{W^{2,N+2}(\mathbb{R}^N)}$, $\tau\|\nabla v_{0\varepsilon}\|_{L^\infty(\mathbb{R}^N)}$, m, q and N. By (2.6) in Lemma 5, (3.17) and Gronwall's inequality, it holds that

$$\sup_{0<t<T_0}\|w(t)\|^2_{L^{N+2}(\mathbb{R}^N)} \leqslant c_5\|f_1 - f_2\|^2_{L^2(0,T_0;L^{N+2}(\mathbb{R}^N))}\exp\{c_5T_0\}.$$

Therefore, there exists $T^0 = T^0(\varepsilon, \|u_{0\varepsilon}\|_{W^{2,N+2}(\mathbb{R}^N)}, \tau\|\nabla v_{0\varepsilon}\|_{L^\infty(\mathbb{R}^N)}, m, q, N, T) \leqslant T_*$ such that

$$\begin{aligned}
\sup_{0<t<T^0}\|w(t)\|_{L^{N+2}} &\leqslant (c_5T^0)^{\frac{1}{2}} \cdot \|f_1 - f_2\|_{L^\infty(0,T^0;L^{N+2}(\mathbb{R}^N))}\exp\{c_5T\} \\
&\leqslant \frac{1}{2}\|f_1 - f_2\|_{L^\infty(0,T^0;L^{N+2}(\mathbb{R}^N))}.
\end{aligned}$$

Now we apply Banach's fixed point theorem to find that there exists a positive number $T_1 = \min(T_0, T^0) \leqslant T_*$ depending on $\varepsilon, \|u_{0\varepsilon}\|_{W^{2,N+2}(\mathbb{R}^N)}, \tau\|v_{0\varepsilon}\|_{W^{3,\infty}(\mathbb{R}^N)}, m, q, N, T$ such that Φ becomes a contraction from X_{T_1} into itself. Consequently Φ has a fixed point $\bar{f} = \Phi(\bar{f}) = u^{h=\bar{v}^{\bar{f}}} \in X_{T_1}$. Thus we obtain the desired solution $(u^{h=v^{\bar{f}}}, v^{\bar{f}})$ in Proposition 8. □

3.2. Extension criterion; proof of Proposition 9

To extend the local solution which is constructed in Proposition 8, it is sufficient to show the following lemma.

Lemma 13. *Let the same assumption as that in Proposition* 9 *hold. Then, it holds that*

$$\sup_{0<t<T} \left\| \nabla u_\varepsilon(t) \right\|_{L^\infty(\mathbb{R}^N)} \leqslant M_{\nabla u}, \tag{3.18}$$

$$\sup_{0<t<T} \left\| \Delta u_\varepsilon(t) \right\|_{L^{N+2}(\mathbb{R}^N)} \leqslant M_{\Delta u}, \tag{3.19}$$

where

$$M_{\nabla u} = M_{\nabla u}\bigl(\varepsilon, M, \|\nabla u_{0\varepsilon}\|_{L^2\cap L^\infty}, \tau\|v_{0\varepsilon}\|_{W^{2,\infty}}, m, q, N, T\bigr),$$
$$M_{\Delta u} = M_{\Delta u}\bigl(M_{\nabla u}, \|\Delta u_{0\varepsilon}\|_{L^2\cap L^\infty}, \tau\|v_{0\varepsilon}\|_{W^{3,\infty}}\bigr).$$

Proof. To establish (3.18), we develop the method by Bernstein [21]. Concretely, we introduce the decomposition of the domain as follows.

Let $\omega \in \mathbb{R}$ be chosen sufficiently small. Define

$$\Omega_k(t) := \bigl\{x \in \mathbb{R}^N\colon (k-1)\omega \leqslant u_\varepsilon(x,t) < k\omega\bigr\} \quad \text{for } k \in \mathbb{N}.$$

Then, we find that there exists a constant $k_0 \in \mathbb{N}$ such that

$$\Omega_k(t) \cap \Omega_j(t) = \emptyset \quad \text{for all } j,k = 1,2,\ldots,k_0 \quad \text{and} \quad \mathbb{R}^N = \bigcup_{k=1}^{k_0} \Omega_k(t).$$

For any fixed $t \in [0,T]$, define the operator $\psi_k(\bar{u}_\varepsilon)$ which is given by

$$u_\varepsilon(x,t) = \psi_k\bigl(\bar{u}_\varepsilon(x,t)\bigr) := (k-3)\omega + 4e\omega \int_0^{\bar{u}_\varepsilon(x,t)} e^{-s^2}\,ds \quad \text{in } x \in \bigcup_{i=k-1}^{k+1} \Omega_i(t).$$

By virtue of this decomposition, we can obtain the Bochner type inequality for $|\nabla \bar{u}_\varepsilon|^2 \cdot \eta$, where η is a cut-off function defined in $\bigcup_{i=k-1}^{k+1} \Omega_i(t)$. Using the boundedness of $\psi_k'(\bar{u}_\varepsilon)$, we establish the boundedness of $\sup_{t>0} \|\nabla u_\varepsilon(t)\|_{L^\infty(\Omega_k)}$ for all $k = 1,2,\ldots,k_0$. Consequently, we prove (3.18) (see [26, Appendix A] for more details).

We are now going to show (3.19). For the rigorous proof, we should multiply $\Delta(1)_h$ by $|\Delta u_\varepsilon|^N \Delta u_\varepsilon \psi_\ell$, where ψ_ℓ is a standard cut-off function. If $u_\varepsilon(t)$ belongs to $\mathbf{W}(Q_{T_1})$, by the limiting process, we can justify the following formal calculation.

For the sake of simplicity, we multiply $\Delta(1)_h$ by $|\Delta u_\varepsilon|^N \Delta u_\varepsilon$ and integrate it over $\mathbb{R}^N$. Then, there exists a positive constant $\mathbf{M}_{\nabla u}$ depending only on $\varepsilon, m, q, N, M_u, M_u^T, M_{\nabla u}$ such that

$$\begin{aligned} \frac{d}{dt}\|\Delta u_\varepsilon(t)\|_{L^{N+2}}^{N+2} \leqslant \mathbf{M}_{\nabla u}\big(1+\|\nabla v_\varepsilon(t)\|_{L^\infty}^2\big)\cdot\|\Delta u_\varepsilon(t)\|_{L^{N+2}}^{N+2} + \big(\mathbf{M}_{\nabla u}\big(1+\|\Delta v_\varepsilon(t)\|_{L^{N+2}}^2\big) \\ + \mathbf{M}_{\nabla u}\|\nabla\Delta v_\varepsilon(t)\|_{L^{N+2}}^2\big)\cdot\|\Delta u_\varepsilon(t)\|_{L^{N+2}}^N. \end{aligned} \tag{3.20}$$

The same letter $\mathbf{M}_{\nabla u}$ will be used to denote different constants depending on $\varepsilon, m, q, N, M_u, M_u^T, M_{\nabla u}$.

By (2.6) in Lemma 5 and Gronwall's inequality, it is seen that

$$\sup_{0<t<T}\|\Delta u_\varepsilon(t)\|_{L^{N+2}}^2 \leqslant \big(\|\Delta u_{0\varepsilon}\|_{L^{N+2}}^2 + \mathbf{M}_{\nabla u}\big(1+T+\|v_{0\varepsilon}\|_{W^{3,N+2}}^2\big)T\big)e^{\mathbf{M}_{\nabla u}(1+T+\|\nabla v_{0\varepsilon}\|_{L^\infty}^2)T}.$$

Thus, we complete the proof of Lemma 13. □

Proof of Proposition 9. We are now in a position to prove Proposition 9. By Proposition 8, we see that the local existence time interval can be characterized in terms of $\|u_{0\varepsilon}\|_{W^{2,N+2}(\mathbb{R}^N)}$, $\tau\|v_{0\varepsilon}\|_{W^{3,\infty}(\mathbb{R}^N)}$. Hence for the extension of the strong solution $(u_\varepsilon, v_\varepsilon)$ on $(0,T)$ onto $(0,T')$ with $T' > T$, it suffices to show that

$$\sup_{0<t<T}\|u_\varepsilon(t)\|_{W^{2,N+2}} < \infty, \qquad \sup_{0<t<T}\|v_\varepsilon(t)\|_{W^{3,\infty}} < \infty,$$

which was given by Lemma 13. □

3.3. Proof of Proposition 10

For the sake of simplicity, throughout this subsection, we denote $(u_\varepsilon, v_\varepsilon)$ by (u, v) and give formal calculations. We denote by c a positive number depending only on m, q, N. The same letter c will denote a general constant greater than 1 (not necessarily the same at different occurrences) but which depends only on m, q, N throughout this paper.

Case (i): $\tau = 1$ and $m \geqslant q$. In this case, we have divided the proof into two steps. The first one gives L^r bounds for all r with $1 \leqslant r < \infty$, and the second one gives L^∞ bound for the strong solution u_ε of Eq. (1) in $(\mathrm{KS})_\varepsilon$.

Lemma 14. *Let the same assumptions as that in Proposition* 10 *hold. In addition, let* $\tau = 1$ *and* $m \geqslant q$. *Then, there exist positive numbers* $M_{u,r}^T$ *and* $M_{\nabla v}^T$ *depending on the initial data* $(u_{0\varepsilon}, v_{0\varepsilon})$, T *but not* ε *such that the strong solution* $(u_\varepsilon, v_\varepsilon)$ *of* $(\mathrm{KS})_\varepsilon$ *satisfies*

$$\sup_{0<t<T}\|u_\varepsilon(t)\|_{L^r(\mathbb{R}^N)} \leqslant M_{u,r}^T \quad \text{for all } r \in [1,\infty), \tag{3.21}$$

$$\sup_{0<t<T}\big(\|v_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} + \|\nabla v_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)}\big) \leqslant M_{\nabla v}^T. \tag{3.22}$$

Proof. We multiply (1) in $(\mathrm{KS})_\varepsilon$ by u^{r-1}, where $r > 1$, and integrate it over $\mathbb{R}^N$. Then, we have

$$\frac{1}{r}\frac{d}{dt}\|u\|_{L^r}^r \leqslant -\frac{4m(r-1)}{(r+m-1)^2}\big\|\nabla u^{\frac{r+m-1}{2}}\big\|_{L^2}^2 + (r-1)\int u^{r+q-3}\nabla v\cdot\nabla u\,dx =: -I + II. \tag{3.23}$$

By $m\ (> q-\frac{2}{N}) \geqslant q-2$ and by taking $r \geqslant m-2q+3$, we have

$$\begin{aligned} II &= \frac{2(r-1)}{r+m-1}\int \nabla u^{\frac{r+m-1}{2}} u^{\frac{r-m+2q-3}{2}}\cdot\nabla v\,dx \\ &\leqslant \frac{2m(r-1)}{(r+m-1)^2}\big\|\nabla u^{\frac{r+m-1}{2}}\big\|_{L^2}^2 + \frac{r-1}{2m}\int u^{r-m+2q-3}|\nabla v|^2\,dx \\ &= \frac{1}{2}I + \frac{r-1}{2m}\|u\|_{L^{r+q-1}}^{r-m+2q-3}\|\nabla v\|_{L^{2(r+q-1)/(m-q+2)}}^2. \end{aligned} \tag{3.24}$$

On the other hand, by Hölder's inequality, it holds that

$$\|u\|_{L^{2(r+q-1)/(m-q+2)}} \leqslant \|u_{0\varepsilon}\|_{L^1}^{\frac{m-q}{2(r+q-2)}}\|u\|_{L^{r+q-1}}^{1-\frac{m-q}{2(r+q-2)}}. \tag{3.25}$$

Combining (3.25) with (2.7) in Lemma 5, we have

$$\begin{aligned} \big\|\nabla v(t)\big\|_{L^{2(r+q-1)/(m-q+2)}}^2 &\leqslant \|\nabla v_{0\varepsilon}\|_{L^{2(r+q-1)/(m-q+2)}}^2 + cr\int_0^t \big\|u(s)\big\|_{L^{2(r+q-1)/(m-q+2)}}^2\,ds \\ &\leqslant \|\nabla v_{0\varepsilon}\|_{L^{2(r+q-1)/(m-q+2)}}^2 + cr\|u_{0\varepsilon}\|_{L^1}^{\frac{m-q}{r+q-2}}\int_0^t \big\|u(s)\big\|_{L^{r+q-1}}^{2(1-\frac{m-q}{2(r+q-2)})}\,ds. \end{aligned} \tag{3.26}$$

Substituting (3.26) into (3.24), we have

$$\begin{aligned} II &\leqslant \frac{1}{2}I + (r-1)\big\|u(t)\big\|_{L^{r+q-1}}^{r-m+2q-3}\|\nabla v_{0\varepsilon}\|_{L^{2(r+q-1)/(m-q+2)}}^2 \\ &\quad + cr(r-1)\big\|u(t)\big\|_{L^{r+q-1}}^{r-m+2q-3}\|u_{0\varepsilon}\|_{L^1}^{\frac{m-q}{r+q-2}}\int_0^t\big\|u(s)\big\|_{L^{r+q-1}}^{2(1-\frac{m-q}{2(r+q-2)})}\,ds \\ &\leqslant \frac{1}{2}I + (r-1)^{\frac{r+q-1}{r-m+2q-3}}\big\|u(t)\big\|_{L^{r+q-1}}^{r+q-1} + \|\nabla v_{0\varepsilon}\|_{L^{2(r+q-1)/(m-q+2)}}^{\frac{2(r+q-1)}{m-q+2}} \\ &\quad + cr(r-1)\|u_{0\varepsilon}\|_{L^1}^{\frac{m-q}{r+q-2}}\big\|u(t)\big\|_{L^{r+q-1}}^{r-m+2q-3}\int_0^t\big\|u(s)\big\|_{L^{r+q-1}}^{2(1-\frac{m-q}{2(r+q-2)})}\,ds. \end{aligned} \tag{3.27}$$

By (3.23) and (3.27),

$$\frac{1}{r}\frac{d}{dt}\|u(t)\|_{L^r}^r \leqslant -\frac{2m(r-1)}{(r+m-1)^2}\|\nabla u^{\frac{r+m-1}{2}}(t)\|_{L^2}^2 + (r-1)^{\frac{r+q-1}{r-m+2q-3}}\|u(t)\|_{L^{r+q-1}}^{r+q-1} + \|\nabla v_{0\varepsilon}\|_{L^{2(r+q-1)/(m-q+2)}}^{\frac{2(r+q-1)}{m-q+2}} + cr(r-1)\|u_{0\varepsilon}\|_{L^1}^{\frac{m-q}{r+q-2}}\|u(t)\|_{L^{r+q-1}}^{r-m+2q-3}\int_0^t \|u(s)\|_{L^{r+q-1}}^{2(1-\frac{m-q}{2(r+q-2)})}\,ds. \tag{3.28}$$

By integrating (3.28) from 0 to t,

$$\frac{1}{r}\|u(t)\|_{L^r}^r \leqslant \frac{1}{r}\|u_{0\varepsilon}\|_{L^r}^r - \frac{2m(r-1)}{(r+m-1)^2}\int_0^t \|\nabla u^{\frac{r+m-1}{2}}(s)\|_{L^2}^2\,ds + (r-1)^{\frac{r+q-1}{r-m+2q-3}}\int_0^t \|u(s)\|_{L^{r+q-1}}^{r+q-1}\,ds + t\|\nabla v_{0\varepsilon}\|_{L^{2(r+q-1)/(m-q+2)}}^{\frac{2(r+q-1)}{m-q+2}} + cr(r-1)\|u_{0\varepsilon}\|_{L^1}^{\frac{m-q}{r+q-2}} \times \int_0^t \|u(s)\|_{L^{r+q-1}}^{r-m+2q-3}\,ds \int_0^t \|u(s)\|_{L^{r+q-1}}^{2(1-\frac{m-q}{2(r+q-2)})}\,ds. \tag{3.29}$$

Here, we note that

$$\int_0^t \|u(s)\|_{L^{r+q-1}}^{r-m+2q-3}\,ds \int_0^t \|u(s)\|_{L^{r+q-1}}^{2(1-\frac{m-q}{2(r+q-2)})}\,ds \leqslant \left(\int_0^t \|u(s)\|_{L^{r+q-1}}^{r+q-1}\,ds\right)^{\frac{r-m+2q-3}{r+q-1}} t^{\frac{m-q+2}{r+q-1}} \left(\int_0^t \|u(s)\|_{L^{r+q-1}}^{r+q-1}\,ds\right)^{\frac{2(r+q-2)-m+q}{(r+q-1)(r+q-2)}} t^{1-\frac{2(r+q-2)-m+q}{(r+q-1)(r+q-2)}} = \left(\int_0^t \|u(s)\|_{L^{r+q-1}}^{r+q-1}\,ds\right)^{\frac{r-m+2q-2}{r+q-2}} t^{\frac{r+m-2}{r+q-2}} \quad \text{for } r \geqslant 2.$$

Using the above inequality, for $m \geqslant q$ and $r \geqslant \max\{2, m-2q+3\}$, we have

$$cr(r-1)\|u_{0\varepsilon}\|_{L^1}^{\frac{m-q}{r+q-2}}\int_0^t \|u(s)\|_{L^{r+q-1}}^{r-m+2q-3}\,ds \int_0^t \|u(s)\|_{L^{r+q-1}}^{2(1-\frac{m-q}{2(r+q-2)})}\,ds$$

$$\leqslant cr(r-1)\|u_{0\varepsilon}\|_{L^1}^{\frac{m-q}{r+q-2}}\left(\int_0^t \|u(s)\|_{L^{r+q-1}}^{r+q-1}\,ds\right)^{\frac{r-m+2q-2}{r+q-2}} t^{\frac{r+m-2}{r+q-2}}$$

$$\leqslant \|u_{0\varepsilon}\|_{L^1}+\bigl(cr(r-1)\bigr)^{\frac{r+q-2}{r-m+2q-2}}\cdot t^{\frac{r+m-2}{r+2q-m-2}}\int_0^t \|u(s)\|_{L^{r+q-1}}^{r+q-1}\,ds. \tag{3.30}$$

By (3.29) and (3.30), we consequently obtain

$$\begin{aligned}
\frac{1}{r}\|u(t)\|_{L^r}^r &\leqslant \frac{1}{r}\|u_{0\varepsilon}\|_{L^r}^r-\frac{2m(r-1)}{(r+m-1)^2}\int_0^t \bigl\|\nabla u^{\frac{r+m-1}{2}}(s)\bigr\|_{L^2}^2\,ds\\
&\quad+(r-1)^{\frac{r+q-1}{r-m+2q-3}}\int_0^t \|u(s)\|_{L^{r+q-1}}^{r+q-1}\,ds+t\|\nabla v_{0\varepsilon}\|_{L^{2(r+q-1)/(m-q+2)}}^{2(r+q-1)/(m-q+2)}\\
&\quad+\|u_{0\varepsilon}\|_{L^1}+\bigl(cr(r-1)\bigr)^{\frac{r+q-2}{r-m+2q-2}}\cdot t^{\frac{r+m-2}{r+2q-m-2}}\int_0^t \|u(s)\|_{L^{r+q-1}}^{r+q-1}\,ds\\
&\leqslant -\frac{2m(r-1)}{(r+m-1)^2}\int_0^t \bigl\|\nabla u^{\frac{r+m-1}{2}}(s)\bigr\|_{L^2}^2\,ds\\
&\quad+\frac{1}{r}\|u_{0\varepsilon}\|_{L^r}^r+\|u_{0\varepsilon}\|_{L^1}+T\|\nabla v_{0\varepsilon}\|_{L^{2(r+1)/m}}^{\frac{2(r+1)}{m}}\\
&\quad+(r-1)\Bigl[r^{\frac{m-q+2}{r-m+2q-3}}+r^{\frac{m-q}{r-m+2q-2}}(cr)^{\frac{r+q-2}{r-m+2q-2}}\cdot T^{\frac{r+m-2}{r-m+2q-2}}\Bigr]\int_0^t \|u(s)\|_{L^{r+q-1}}^{r+q-1}\,ds
\end{aligned}$$

for $m \geqslant q,\ r \geqslant \max\{2, m-2q+3\}$ and $t \in [0,T)$. (3.31)

From Lemma 6 with $a=3$, it holds that

$$\|u_\varepsilon\|_{L^{r+q-1}} \leqslant c^{\frac{1}{\beta_1}\cdot\frac{2}{r+m-1}}\|u\|_{L^1}^{1-\theta_1}\bigl\|\nabla u^{\frac{r+m-1}{2}}\bigr\|_{L^2}^{\frac{2\theta_1}{r+m-1}} \tag{3.32}$$

for $r \geqslant m-2q+1$, where

$$\begin{aligned}
\beta_1 &:= \frac{2N}{N+2}\cdot\frac{r-m+2q-1}{(r+q-2)(r+m-1)}\left(\frac{2}{N}+r+m-2\right),\\
\theta_1 &:= \frac{r+m-1}{2}\cdot\left(1-\frac{1}{r+q-1}\right)\frac{1}{\frac{1}{N}-\frac{1}{2}+\frac{r+m-1}{2}}.
\end{aligned}$$

It is easy to verify that $\frac{2\theta_1(r+q-1)}{r+m-1}<2$ if $m>q-\frac{2}{N}$ and $\frac{1}{\beta_1}\leqslant\frac{N+2}{N}$ if $r\geqslant\max\{2,2(m-q)\}$. Therefore, by Young inequality, it holds that

$$\begin{aligned}&\left[r^{\frac{m-q+2}{r-m+2q+3}} + r^{\frac{m-q}{r-m+2q-3}} (cr)^{\frac{r+q-2}{r-m+2q-2}} \cdot T^{\frac{r+m-2}{r+2q-m-2}}\right] \left\| u(t) \right\|_{L^{r+q-1}}^{r+q-1} \\ &\quad \leqslant C_r + \frac{2m}{(r+m-1)^2} \left\| \nabla u^{\frac{r+m-1}{2}}(t) \right\|_{L^2}^2 \\ &\quad \text{for any } r \in \left[\max\left\{2, 2(m-q)\right\}, \infty\right) \text{ and } m > q - \frac{2}{N}, \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} C_r &= (r+m-1)^{\frac{2(r+q-2)}{m-q+2/N}} \\ &\quad \times \Bigg(\left[r^{\frac{m-q+2}{r-m+2q-3}} + r^{\frac{m-q}{r-m+2q-2}} (cr)^{\frac{r+q-2}{r-m+2q-2}} \cdot T^{\frac{r+m-2}{r+2q-m-2}} \right] \\ &\quad \times c^{\frac{2(N+2)}{N} \cdot \frac{2(r+q-1)}{r+m-1}} \| u \|_{L^1}^{\frac{2r/N+m-2+2/N}{r+m-2+2/N} + (q-2)(2/N-1)} \Bigg)^{\frac{r+m-2+2/N}{m-q+2/N}}. \end{aligned} \tag{3.34}$$

From (3.31) and (3.33), we have

$$\sup_{0<t<T} \left\| u(t) \right\|_{L^r} \leqslant \left(\| u_{0\varepsilon} \|_{L^r} + r \| u_{0\varepsilon} \|_{L^1} + rT \| \nabla v_{0\varepsilon} \|_{L^{2(r+1)/m}}^{\frac{2(r+1)}{m}} + r(r-1) C_r \cdot T \right)^{\frac{1}{r}} =: M_{u,r}^T \tag{3.35}$$

for any $r \geqslant [\max\{2, m-2q+3, 2(m-q)\}, \infty)$, and $m > q - \frac{2}{N}$, and for C_r in (3.34). Combining the mass conservation law for $u_\varepsilon(t)$ with (3.35), we establish (3.21) in Lemma 14.

By Lemma 5 and (3.35), we have

$$\begin{aligned} \sup_{0<t<T} \left\| v(t) \right\|_{L^\infty} &\leqslant \| v_{0\varepsilon} \|_{L^\infty} + c \cdot \Gamma(\gamma) \sup_{0<t<T} \left\| u(t) \right\|_{L^{r_0}}, \\ \sup_{0<t<T} \left\| \nabla v(t) \right\|_{L^\infty} &\leqslant \| \nabla v_{0\varepsilon} \|_{L^\infty} + c \cdot \Gamma(\tilde{\gamma}) \sup_{0<t<T} \left\| u(t) \right\|_{L^{r_0}} \end{aligned}$$

with $r_0 = \max\{2, m-2q+3, 2(m-q)\}$. Hence, we deduce that

$$\begin{aligned} &\sup_{0<t<T} \left(\left\| v(t) \right\|_{L^\infty} + \left\| \nabla v(t) \right\|_{L^\infty} \right) \\ &\quad \leqslant \| v_{0\varepsilon} \|_{L^\infty} + \| \nabla v_{0\varepsilon} \|_{L^\infty} + c \left(\Gamma(\gamma) + \Gamma(\tilde{\gamma}) \right) M_{u,r_0}^T =: M_{\nabla v}^T. \end{aligned}$$

Thus, we obtain (3.22) and complete the proof of Lemma 14. □

We are now in a position to prove the uniform $L^\infty(\mathbb{R}^N)$ bound for $(u_\varepsilon, v_\varepsilon)$.

Lemma 15. *Let the same assumptions as that in Proposition* 10 *hold. We assume that there exist positive numbers $M_{u,r}^T$ and $M_{\nabla v}^T$ such that the strong solution $(u_\varepsilon, v_\varepsilon)$ of* $(\mathrm{KS})_\varepsilon$ *satisfies*

$$\sup_{0<t<T} \left\| u_\varepsilon(t) \right\|_{L^r(\mathbb{R}^N)} \leqslant M_{u,r}^T \quad \text{for all } r \in [1, \infty), \tag{3.36}$$

$$\sup_{0<t<T} \left\| \nabla v_\varepsilon(t) \right\|_{L^\infty(\mathbb{R}^N)} \leqslant M_{\nabla v}^T. \tag{3.37}$$

Then, u_ε satisfies the following estimate:

$$\sup_{0<t<T} \left\| u_\varepsilon(t) \right\|_{L^\infty(\mathbb{R}^N)} \leqslant M^T_{u,\infty} < \infty, \tag{3.38}$$

where $M^T_{u,\infty}$ is a constant depending on $\|u_{0\varepsilon}\|_{L^1 \cap L^\infty}, M^T_{u,r_1}$ (with $r_1 = r_1(m,q,N)$ large enough), $M^T_{\nabla v}, m, q, N, T$ but not ε.

We multiply (1) in $(\mathrm{KS})_\varepsilon$ by u^{r-1}, where $r > 1$, and integrate it over $\mathbb{R}^N$. Then, we have

$$\begin{aligned} \frac{1}{r}\frac{d}{dt}\|u\|^r_{L^r} &\leqslant -m(r-1)\int u^{m-1}u^{r-2}|\nabla u|^2\,dx + (r-1)\int u^{q-1}\nabla v \cdot u^{r-2}\nabla u\,dx \\ &\leqslant -\frac{2m(r-1)}{(r+m-1)^2}\left\|\nabla u^{\frac{r+m-1}{2}}\right\|^2_{L^2} + (r-1)\left(M^T_{\nabla v}\right)^2\|u\|^{r-m+2q-3}_{L^{r-m+2q-3}}. \end{aligned} \tag{3.39}$$

We divide the cases into two: one is $q > 3 - \frac{2}{N}$; the other one is $2 < q \leqslant 3 - \frac{2}{N}$. Firstly, we treat the case of $q > 3 - \frac{2}{N}$.

(a) Case of $q > 3 - \frac{2}{N}$. Let ℓ be a natural number which is chosen later. By Lemma 6 with $a = 3$, there exists a positive constant r_* depending only on m, q, N, ℓ such that

$$\|u\|_{L^{r-m+2q-3}} \leqslant c^{\frac{1}{\beta_2}\cdot\frac{2}{r+m-1}}\|u\|^{1-\theta_2}_{L^{r/\ell}}\left\|\nabla u^{\frac{r+m-1}{2}}\right\|^{\frac{2\theta_2}{r+m-1}}_{L^2} \tag{3.40}$$

for any $r \geqslant r_*$, where

$$\begin{aligned} \beta_2 &= \frac{r-m+2q-3-\frac{r+m-1}{2}}{r-m+2q-3-\frac{r}{\ell}}\left[\frac{2}{r+m-1}\cdot\frac{r}{\ell} + \left(1 - \frac{2}{r+m-1}\cdot\frac{r}{\ell}\right)\frac{2N}{N+2}\right], \\ \theta_2 &= \frac{r+m-1}{2}\cdot\left(\frac{\ell}{r} - \frac{1}{r-m+2q-3}\right)\cdot\frac{1}{\frac{1}{N}-\frac{1}{2}+\frac{r+m-1}{2}\cdot\frac{\ell}{r}}. \end{aligned}$$

The same letter r_* will denote a general constant (not necessarily the same at different occurrences) but which depends only on m, q, N, ℓ in what follows.

It is easy to verify that $\frac{2\theta_2(r-m+2q-3)}{r+m-1} < 2$ and $\frac{1}{\beta_2} \leqslant 6$ for $r \geqslant r_*$. Therefore, the Young inequality and (3.40) yield that

$$\begin{aligned} &(r-1)\left(M^T_{\nabla v}\right)^2\|u\|^{r-m+2q-3}_{L^{r-m+2q-3}} \\ &\quad\leqslant (r-1)\left(M^T_{\nabla v}\right)^2 c^{\frac{12(r-m+2q-3)}{r+m-1}}\|u\|^{(1-\theta_2)(r-m+2q-3)}_{L^{r/\ell}}\left\|\nabla u^{\frac{r+m-1}{2}}\right\|^{\frac{2\theta_2(r-m+2q-3)}{r+m-1}}_{L^2} \\ &\quad\leqslant \frac{m(r-1)}{(r+m-1)^2}\left\|\nabla u^{\frac{r+m-1}{2}}\right\|^2_{L^2} \\ &\qquad + (r-1)(cr)^{k_1}\left(\left(M^T_{\nabla v}\right)^2 c\right)^{k_2}\|u\|^{(1-\theta_2)(r-m+2k-3)\cdot k_2}_{L^{r/\ell}} \quad \text{for any } r \geqslant r_*, \end{aligned} \tag{3.41}$$

where

$$k_1 := \frac{\theta_2(r-m+2q-3)}{r+m-1-\theta_2(r-m+2q-3)}, \qquad k_2 := \frac{r+m-1}{r+m-1-\theta_2(r-m+2q-3)}.$$

We set a by $a := \frac{q+2/N-3}{m-1}$. Then, this constant a becomes a positive number by the assumption $q > 3 - \frac{2}{N}$. In addition, it holds that

$$\begin{cases} 1-a>0 & \text{if } m > q - \frac{2}{N} \text{ when } N=2, \\ 1-a>0 & \text{if } m \geqslant q - \frac{2}{N} \text{ when } N \geqslant 3. \end{cases}$$

From the above, it is easily seen that there exists a natural number ℓ depending only on m, q, N such that

$$\frac{\frac{1}{2}(\ell-1)}{\frac{1}{2}(\ell-1)+\frac{1}{N}} > 1 - \frac{1}{2}(1-a). \tag{3.42}$$

We choose the constant ℓ used in (3.40) such that it satisfies (3.42). Then, it holds that

$$\theta_2 = \frac{r+m-1}{2} \cdot \left(\frac{\ell}{r} - \frac{1}{r-m+2q-3}\right) \cdot \frac{1}{\frac{1}{N} - \frac{1}{2} + \frac{r+m-1}{2} \cdot \frac{\ell}{r}}$$
$$\longrightarrow \frac{\frac{1}{2}(\ell-1)}{\frac{1}{2}(\ell-1)+\frac{1}{N}} \quad \text{as } r \to \infty.$$

This assures that

$$\frac{\frac{1}{2}(\ell-1)-\frac{1}{2N}}{\frac{1}{2}(\ell-1)+\frac{1}{N}} \leqslant \theta_2 \leqslant \frac{\frac{1}{2}(\ell-1)+\frac{1}{2N}}{\frac{1}{2}(\ell-1)+\frac{1}{N}} \quad \text{for } r \geqslant r_*. \tag{3.43}$$

Using (3.43), we obtain the following upper bound for k_1, k_2:

$$k_1 \leqslant N\ell, \qquad k_2 \leqslant N\ell + 2 \quad \text{for all } r \geqslant r_*. \tag{3.44}$$

Combining (3.41) with (3.44), we see that

$$(r-1)\left(M^T_{\nabla v}\right)^2 \|u\|^{r-m+2q-3}_{L^{r-m+2q-3}}$$
$$\leqslant \frac{m(r-1)}{(r+m-1)^2} \left\|\nabla u^{\frac{r+m-1}{2}}\right\|^2_{L^2} + r^C C \|u\|^{(1-\theta_2)(r-m+2k-3)\cdot k_2}_{L^{r/\ell}} \quad \text{for any } r \geqslant r_*, \tag{3.45}$$

where C will denote a general constant (not necessarily the same at different occurrences) but which depends only on $M^T_{\nabla v}, m, q, N, \ell$ throughout this Section 3.3.

We apply the Young inequality again for (3.45). To this end, we set

$$k_3 := (1-\theta_2)k_2 \cdot \frac{r-m+2q-3}{r} = (1-\theta_2) \cdot \frac{r-m+2q-3}{r} \cdot \frac{r+m-1}{r+m-1-\theta_2 r}.$$

From (3.42), we see that

$$\frac{q+\frac{2}{N}-3}{m-1} =: a < 2\cdot\frac{\frac{1}{2}(\ell-1)}{\frac{1}{2}(\ell-1)+\frac{1}{N}} - 1 = \frac{\frac{1}{2}(\ell-1)-\frac{1}{N}}{\frac{1}{2}(\ell-1)+\frac{1}{N}} \leqslant \theta_2. \tag{3.46}$$

By virtue of (3.46), we see that $\frac{1}{k_3} \geqslant 1$. In addition, by taking r_* larger if necessary, we have $\frac{1}{k_3} \leqslant N\ell+2$ for $r \geqslant r_*$. Now applying Young's inequality with the exponent $\frac{1}{k_3}$ to (3.45), we have

$$(r-1)\left(M_{\nabla v}^T\right)^2 \|u\|_{L^{r-m+2q-3}}^{r-m+2q-3} \leqslant \frac{m(r-1)}{(r+m-1)^2}\left\|\nabla u^{\frac{r+m-1}{2}}\right\|_{L^2}^2 + 1 + r^C C\|u\|_{L^{r/\ell}}^r \tag{3.47}$$

for any $r \geqslant r_*$. Substituting (3.47) into (3.39), we have

$$\frac{1}{r}\frac{d}{dt}\|u\|_{L^r}^r \leqslant -\frac{m(r-1)}{(r+m-1)^2}\left\|\nabla u^{\frac{r+m-1}{2}}\right\|_{L^2}^2 + 1 + r^C C\|u\|_{L^{r/\ell}}^r \tag{3.48}$$

for any $r \geqslant r_*$. By the similar argument to that from (3.40) to (3.47), we have

$$\frac{1}{r}\|u\|_{L^r}^r \leqslant \frac{m(r-1)}{(r+m-1)^2}\left\|\nabla u^{\frac{r+m-1}{2}}\right\|_{L^2}^2 + 1 + r^C C\|u\|_{L^{r/\ell}}^r \tag{3.49}$$

for $r \geqslant r_*$. Combining (3.48) with (3.49), we have

$$\frac{1}{r}\frac{d}{dt}\|u\|_{L^r}^r + \frac{1}{r}\|u\|_{L^r}^r \leqslant 2 + r^C C\|u\|_{L^{r/\ell}}^r \tag{3.50}$$

for any $r \geqslant r_*$. Integrating (3.50) from 0 to t, we obtain

$$\begin{aligned}\sup_{0<t<T}\left\|u(t)\right\|_{L^r}^r &\leqslant \|u_{0\varepsilon}\|_{L^r}^r + 2r + r^C C \sup_{0<t<T}\|u\|_{L^{r/\ell}}^r \\ &\leqslant \max\Big\{\|u_{0\varepsilon}\|_{L^1}, \|u_{0\varepsilon}\|_{L^\infty}, 1, \sup_{0<t<T}\|u\|_{L^{r/\ell}}\Big\}^r \times r^C C\end{aligned} \tag{3.51}$$

for any $r \geqslant r_*$. Hence, applying the Moser's iteration technique, we establish

$$\sup_{0<t<T}\left\|u(t)\right\|_{L^\infty} \leqslant C\max\Big(\|u_{0\varepsilon}\|_{L^1}, \|u_{0\varepsilon}\|_{L^\infty}, 1, \sup_{0<t<T}\left\|u(t)\right\|_{L^{r_*}}\Big) =: M_{u,\infty}^T. \tag{3.52}$$

Here, we remark that the upper bound of $\sup_{0<t<T}\|u(t)\|_{L^\infty}$ is depending on T since $\sup_{0<t<T}\|u(t)\|_{L^{r_*}}$ depends on T by Lemma 14. Thus, we complete the proof of Lemma 15 in case of (i)-(a).

Next, we treat the case of $2 \leqslant q \leqslant 3-\frac{2}{N}$.

(b) Case of $2 \leqslant q \leqslant 3-\frac{2}{N}$. From $q \leqslant m+\frac{2}{N}$ and $q \leqslant 3-\frac{2}{N}$, it holds that $2q \leqslant m+3$. Therefore,

$$\|u\|_{L^{r-m+2q-3}}^{r-m+2q-3} \leqslant \|u_{0\varepsilon}\|_{L^1} + \|u\|_{L^r}^r. \tag{3.53}$$

Substituting (3.53) into (3.39), we have

$$\frac{1}{r}\frac{d}{dt}\|u\|_{L^r}^r \leqslant -\frac{2m(r-1)}{(r+m-1)^2}\left\|\nabla u^{\frac{r+m-1}{2}}\right\|_{L^2}^2 + (r-1)\left(M_{\nabla v}^T\right)^2\|u_{0\varepsilon}\|_{L^1} + (r-1)\left(M_{\nabla v}^T\right)^2\|u\|_{L^r}^r. \tag{3.54}$$

By Lemma 6 with $a=3$, there exists a positive constant r_{**} depending only on m, N such that

$$\|u\|_{L^r} \leqslant c^{\frac{1}{\beta_3}\cdot\frac{2}{r+m-1}}\|u\|_{L^{r/4}}^{1-\theta_3}\left\|\nabla u^{\frac{r+m-1}{2}}\right\|_{L^2}^{\frac{2\theta_3}{r+m-1}} \quad \text{for any } r \geqslant r_{**},$$

where

$$\beta_3 = \frac{2(r-m+1)}{3(r+m-1)r}\left[\frac{r}{2} + \frac{N}{N+2}(r+2m-2)\right],$$
$$\theta_3 = \frac{r+m-1}{2}\cdot\left(\frac{4}{r}-\frac{1}{r}\right)\cdot\frac{1}{\frac{1}{N}-\frac{1}{2}+\frac{2(r+m-1)}{r}}.$$

It is easy to verify that $\frac{2\theta_2\cdot r}{r+m-1} < 2$ by $m \geqslant 1$, and $\frac{1}{\beta_2} \leqslant 6$ for $r \geqslant r_{**}$.
Therefore, from the Young inequality, it holds that

$$(r-1)\left(M_{\nabla v}^T\right)^2\|u\|_{L^r}^r \leqslant \frac{m(r-1)}{(r+m-1)^2}\left\|\nabla u^{\frac{r+m-1}{2}}\right\|_{L^2}^2 + 1 + r^C C\|u\|_{L^{r/4}}^r \quad \text{for } r \geqslant r_{**}. \tag{3.55}$$

Substituting (3.55) into (3.54), we have

$$\frac{d}{dt}\|u\|_{L^r}^r \leqslant -\frac{m(r-1)}{(r+m-1)^2}\left\|\nabla u^{\frac{r+m-1}{2}}\right\|_{L^2}^2 + \frac{r-1}{2m}\left(M_{\nabla v}^T\right)^2\|u_{0\varepsilon}\|_{L^1} + 1 + r^C C\|u\|_{L^{r/4}}^r \quad \text{for } r \geqslant r_{**}. \tag{3.56}$$

Since the above (3.55) means that

$$\|u_\varepsilon\|_{L^r}^r \leqslant \frac{m(r-1)}{(r+m-1)^2}\left\|\nabla u^{\frac{r+m-1}{2}}\right\|_{L^2}^2 + 1 + r^C C\|u\|_{L^{r/4}}^r \tag{3.57}$$

for any $r \geqslant r_{**}$, combining (3.56) with (3.57), we have

$$\frac{1}{r}\frac{d}{dt}\|u\|_{L^r}^r + \frac{1}{r}\|u\|_{L^r}^r \leqslant (r-1)\left(M_{\nabla v}^T\right)^2\|u_{0\varepsilon}\|_{L^1} + 2 + 2r^C C\|u\|_{L^{r/4}}^r \quad \text{for } r \geqslant r_{**}.$$

By the similar argument to that from (3.50) to (3.52), we prove Lemma 15 in case of (i)-(b). Finally, denoting $\max\{r_*, r_{**}\}$ by r_1 and combining the case of (i)-(a) with (i)-(b), we complete the proof of Lemma 15. □

Moreover, we define M_u^T as follows:

$$\sup_{0<t<T} \left\| u_\varepsilon(t) \right\|_{L^r(\mathbb{R}^N)} \leqslant M_{u,r}^T \leqslant \sup_{1\leqslant r\leqslant\infty} M_{u,r}^T =: M_u^T \quad \text{for } r \in [1,\infty].$$

Then, collecting Lemmas 14 and 15, we obtain (3.2) in Proposition 10.

Case (ii): $\tau = 0$ and $m > q - \frac{2}{N}$. We multiply Eq. (1) in $(\mathrm{KS})_\varepsilon$ by u^{r-1} and integrate it over $\mathbb{R}^N$. Then, we have

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\|u\|_{L^r}^r &\leqslant -m(r-1)\int u^{r+m-3}|\nabla u|^2\,dx + (r-1)\int u^{q-1}\nabla v\cdot u^{r-2}\nabla u\,dx \\ &= -\frac{4m(r-1)}{(r+m-1)^2}\int \left|\nabla u^{\frac{r+m-1}{2}}\right|^2 dx - \frac{r-1}{r+q-2}\int u^{r+q-2}\Delta v\,dx.\end{aligned}$$

We substitute (2) of $(\mathrm{KS})_\varepsilon$: $\Delta v = v - u$ into the above inequality and use the Young inequality. Then, we obtain

$$\frac{1}{r}\frac{d}{dt}\|u\|_{L^r}^r \leqslant -\frac{2m(r-1)}{(r+m-1)^2}\int \left|\nabla u^{\frac{r+m-1}{2}}\right|^2 dx + \frac{(r-1)}{r+q-2}\int u^{r+q-1}\,dx.$$

Integrating with respect to t, we have

$$\begin{aligned}\frac{1}{r}\|u\|_{L^r}^r \leqslant{}& -\frac{2m(r-1)}{(r+m-1)^2}\int_0^t \left\|\nabla u^{\frac{r+m-1}{2}}(s)\right\|_{L^2}^2 ds + \frac{1}{r}\|u_{0\varepsilon}\|_{L^r}^r \\ &+ \frac{(r-1)}{r+q-2}\int_0^t \left\|u(s)\right\|_{r+q-1}^{r+q-1} ds \quad \text{for } m \geqslant 1,\ r > 1. \end{aligned} \tag{3.58}$$

Thus, we obtain the similar inequality to (3.31). By repeating the process from (3.32) to (3.35), we establish the following lemma.

Lemma 16. *Let the same assumptions as that in Proposition* 10 *hold. In addition, let* $\tau = 0$ *and* $m > q - \frac{2}{N}$. *Then, there exist positive numbers* $M_{u,r}$ *and* $M_{\nabla v}$ *depending on the initial data* $(u_{0\varepsilon}, v_{0\varepsilon})$ *but not* ε, T *such that the strong solution* $(u_\varepsilon, v_\varepsilon)$ *of* $(\mathrm{KS})_\varepsilon$ *satisfies*

$$\sup_{0<t<T} \left\| u_\varepsilon(t) \right\|_{L^r(\mathbb{R}^N)} \leqslant M_{u,r} \quad \text{for all } r \in [1,\infty), \tag{3.59}$$

$$\sup_{0<t<T} \left(\left\| v_\varepsilon(t) \right\|_{L^\infty(\mathbb{R}^N)} + \left\| \nabla v_\varepsilon(t) \right\|_{L^\infty(\mathbb{R}^N)} \right) \leqslant M_{\nabla v}. \tag{3.60}$$

Lemma 15 holds true for the case of $\tau = 0$ and $m > q - \frac{2}{N}$. Therefore, collecting Lemmas 15 and 16, we establish (3.3) in Proposition 10.

Case (iii): $\tau = 0$ and $1 \leqslant m \leqslant q - \frac{2}{N}$. Taking $q_1 = \frac{N(q-m)}{2}$, $q_2 = r + q - 1$, $a = 2 + \frac{2}{N}$ in Lemma 6, we have

$$\|u\|_{L^{r+q-1}}^{r+q-1} \leqslant c_0^{\frac{2(N+2)}{N} \cdot \frac{r+q-1}{r+m-1}} \|u\|_{L^{N(q-m)/2}}^{q-m} \cdot \left\|\nabla u^{\frac{r+m-1}{2}}\right\|_{L^2}^2 \quad \text{for } r \geqslant \frac{N(q-m)}{2} \tag{3.61}$$

for some absolute constant c_0, where we used

$$\frac{1}{\beta} = \frac{N+2}{N} \cdot \frac{r+m-1}{r-m+2q-1} \leqslant \frac{N+2}{N}$$

by m $(\leqslant q - \frac{2}{N}) < q$.

Combining (3.58) with (3.61), we obtain

$$\frac{d}{dt}\|u\|_{L^r}^r \leqslant \left[\frac{r(r-1)}{r+q-2} c_0^{\frac{2(N+2)}{N} \cdot \frac{r+q-1}{r+m-1}} \|u\|_{L^{N(q-m)/2}}^{q-m} - \frac{4mr(r-1)}{(r+m-1)^2}\right] \left\|\nabla u^{\frac{r+m-1}{2}}\right\|_{L^2}^2. \tag{3.62}$$

Since $\|u(t)\|_{L^{N(q-m)/2}} \in C([0,T])$, using this continuity and (3.62) with $r = \frac{N(q-m)}{2}$, we find that there exists a short interval $[0, t_1]$ such that $\frac{d}{dt}\|u(t)\|_{L^r} \leqslant 0$ for $t \in [0, t_1]$, and $\|u(t)\|_{L^{N(q-m)/2}} \leqslant \|u_{0\varepsilon}\|_{L^{N(q-m)/2}}$ for $t \in [0, t_1]$. Since this implies that $\|u(t_1)\|_{L^{N(q-m)/2}} \leqslant \|u_{0\varepsilon}\|_{L^{N(q-m)/2}}$, we can repeat this procedure. In consequence, we obtain

$$\left\|u(t)\right\|_{L^{N(q-m)/2}} \leqslant \|u_{0\varepsilon}\|_{L^{N(q-m)/2}} \quad \text{for } t \in [0, T]. \tag{3.63}$$

Substituting (3.63) into (3.62), we have

$$\begin{aligned} \frac{d}{dt}\|u\|_{L^r}^r &\leqslant \left(\frac{r(r-1)}{r+q-2} c_0^{\frac{2(N+2)}{N} \cdot \frac{r+q-1}{r+m-1}} \|u_{0\varepsilon}\|_{L^{N(q-m)/2}}^{q-m} - \frac{4mr(r-1)}{(r+m-1)^2}\right) \left\|\nabla u^{\frac{r+m-1}{2}}\right\|_{L^2}^2 \\ &\leqslant -\frac{2mr(r-1)}{(r+m-1)^2} \left\|\nabla u^{\frac{r+m-1}{2}}\right\|_{L^2}^2 \leqslant 0 \quad \text{for } r \in \left[\frac{N(q-m)}{2}, \infty\right). \end{aligned} \tag{3.64}$$

By the similar argument to that in Lemma 15, we obtain (3.3) for the case of (iii) in Proposition 10, which completes the proof of Proposition 10. □

4. Proof of Theorems 1 and 2

In this section, we give a proof of Theorems 1 and 2.

By virtue of Proposition 10, we can extract a subsequence $\{u_{\varepsilon_n}\}$ which converges in L^p $(1 < p \leqslant \infty)$ such that

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{weakly in } L^p\big(0, T; L^p(\mathbb{R}^N)\big). \tag{4.1}$$

Moreover, we obtain a subsequence, still denoted by $\{u_{\varepsilon_n}\}$ such that for any $2 \leqslant p < \infty$

$$u_{\varepsilon_n} \to u \quad \text{strongly in } C\big((0,T); L^p_{\mathrm{loc}}(\mathbb{R}^N)\big), \tag{4.2}$$

$$\nabla u_{\varepsilon_n}^m \rightharpoonup \nabla u^m \quad \text{weakly star in } L^\infty\big(0, T; L^2(\mathbb{R}^N)\big). \tag{4.3}$$

The above relations (4.2), (4.3) are shown as follows. We multiply Eq. (1) in $(\mathrm{KS})_\varepsilon$ by $\frac{\partial (u_\varepsilon+\varepsilon)^m}{\partial t}$ and integrate with respect to the space variable over $\mathbb{R}^N$. Then, we obtain

$$\begin{aligned}
&\frac{4m}{(m+1)^2}\int \left|\left((u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right)_t\right|^2 dx \\
&\quad= -\frac{1}{2}\frac{d}{dt}\int \left|\nabla (u_\varepsilon+\varepsilon)^m\right|^2 dx + \frac{2m}{(m+1)^2}\int \left|\left((u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right)_t\right|^2 dx \\
&\qquad+ \frac{4m(q-1)^2}{(m+1)^2}\|\nabla v_\varepsilon\|^2_{L^\infty(Q_T)}\left(\|u_\varepsilon\|_{L^\infty(Q_T)}+\varepsilon\right)^{2(q-2)}\int \left|\nabla (u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right|^2 dx \\
&\qquad+ m\int (u_\varepsilon+\varepsilon)^{m+2q-3}|\Delta v_\varepsilon|^2\,dx.
\end{aligned} \tag{4.4}$$

Integrating (4.4) with respect to time variable, we have

$$\begin{aligned}
&\frac{2m}{(m+1)^2}\int_0^T\!\!\int \left|\left((u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right)_t\right|^2 dx\,dt + \frac{1}{2}\sup_{0<t<T}\int \left|\nabla (u_\varepsilon+\varepsilon)^m\right|^2 dx \\
&\quad= \frac{1}{2}\int \left|\nabla (u_{0\varepsilon}+\varepsilon)^m\right|^2 dx + m\left(\|u_\varepsilon\|_{L^\infty(Q_T)}+\varepsilon\right)^{m+2q-3}\int_0^T\!\!\int |\Delta v_\varepsilon|^2\,dx\,dt \\
&\qquad+ \frac{4m(q-1)^2}{(m+1)^2}\|\nabla v_\varepsilon\|^2_{L^\infty(Q_T)}\left(\|u_\varepsilon\|_{L^\infty(Q_T)}+\varepsilon\right)^{2(q-2)}\int_0^T\!\!\int \left|\nabla (u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right|^2 dx\,dt.
\end{aligned} \tag{4.5}$$

On the other hand, by the multiplication Eq. (1) in $(\mathrm{KS})_\varepsilon$ by u_ε and the integration with respect to x and t, we have

$$\begin{aligned}
&\int_0^T\!\!\int \left|\nabla (u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right|^2 dx\,dt \\
&\quad\leqslant \frac{(m+1)^2}{8m}\|u_{0\varepsilon}\|^2_{L^2} + \frac{(m+1)^2}{8m}\left(\int_0^T\!\!\int \frac{1}{q^2}u_\varepsilon^{2q} + \frac{\varepsilon^2}{(q-1)^2}u_\varepsilon^{2q-2} + 2|\Delta v_\varepsilon|^2\,dx\,dt\right).
\end{aligned} \tag{4.6}$$

From (2.6) in Lemma 5, Proposition 10, (4.5) and (4.6), we see that for $q \geqslant 2$ there exists a positive constant C which is independent of ε such that

$$\begin{aligned}
&\int_0^T\!\!\int \left|\left(u_\varepsilon^m\right)_t\right|^2 dx\,dt + \sup_{0<t<T}\int \left|\nabla u_\varepsilon^m\right|^2 dx \\
&\quad\leqslant \int_0^T\!\!\int \left|\left((u_\varepsilon+\varepsilon)^m\right)_t\right|^2 dx\,dt + \sup_{0<t<T}\int \left|\nabla (u_\varepsilon+\varepsilon)^m\right|^2 dx
\end{aligned}$$

$$\leqslant \frac{4m^2}{(m+1)^2}\big(\|u_\varepsilon\|_{L^\infty(Q_T)}+\varepsilon\big)^{m-1}\int_0^T\int\big|\big((u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\big)_t\big|^2\,dx\,dt$$

$$+\sup_{0<t<T}\int\big|\nabla(u_\varepsilon+\varepsilon)^m\big|^2\,dx\leqslant C.$$

Thus, we find that $u_\varepsilon^m\in L^\infty(0,T;H^1(\mathbb{R}^N))\cap H^1(0,T;L^2(\mathbb{R}^N))$. Hence, we can extract a subsequence such that

$$u_{\varepsilon_n}^m\to\xi\quad\text{strongly in } C\big((0,T);L^2_{\mathrm{loc}}\big(\mathbb{R}^N\big)\big). \tag{4.7}$$

This gives

$$u_{\varepsilon_n}^m(x,t)\to\xi(x,t)\quad\text{a.a. } x\in\mathbb{R}^N,\ t\in(0,T).$$

A function $g(u)=u^{\frac{1}{m}}$ is continuous with respect to u. Thus, we see that

$$u_{\varepsilon_n}(x,t)\to\xi^{\frac{1}{m}}(x,t)\quad\text{a.a. } x\in\mathbb{R}^N,\ t\in(0,T), \tag{4.8}$$

On the other hand, by Proposition 10, independently of ε such that

$$\sup_{0<t<T}\big\|u_{\varepsilon_n}(t)\big\|_{L^\infty(\mathbb{R}^N)}\leqslant\max\big\{M_u^T,M_u\big\}=:M,$$

which yields that

$$\begin{gathered}\big|u_{\varepsilon_n}(x,t)\big|\leqslant M\quad\text{a.a. } x\in\mathbb{R}^N,\ t\in(0,T),\\ M_{u,\infty}^T\in L^p\big(0,T;L^p_{\mathrm{loc}}\big(\mathbb{R}^N\big)\big)\quad\text{for any } 1<p<\infty.\end{gathered} \tag{4.9}$$

From Lebesgue dominated convergence theorem, (4.1) and (4.8), we find that

$$u_{\varepsilon_n}\to\xi^{\frac{1}{m}}=u\quad\text{strongly in } L^p\big(0,T;L^p_{\mathrm{loc}}\big(\mathbb{R}^N\big)\big) \tag{4.10}$$

for any $1<p<\infty$. From (4.10), we observe that

$$u_{\varepsilon_n}(x,t)\to\xi^{\frac{1}{m}}(x,t)=u(x,t)\quad\text{a.a. } x\in\mathbb{R}^N,\ \text{all } t\in(0,T). \tag{4.11}$$

By (4.7) and (4.11),

$$u_{\varepsilon_n}^m\to u^m\quad\text{strongly in } C\big((0,T);L^2_{\mathrm{loc}}\big(\mathbb{R}^N\big)\big). \tag{4.12}$$

In addition, since $|b-a|^m\leqslant|b^m-a^m|$ for $0\leqslant a\leqslant b$ and $m\geqslant 1$, from (4.2) we see that

$$u_{\varepsilon_n}\to u\quad\text{strongly in } C\big((0,T);L^{2m}_{\mathrm{loc}}\big(\mathbb{R}^N\big)\big). \tag{4.13}$$

By Hölder inequality and (4.13), in all cases of $2 \leqslant p < \infty$, it holds that

$$u_{\varepsilon_n} \to u \quad \text{strongly in } C\big((0,T); L^p_{\mathrm{loc}}\big(\mathbb{R}^N\big)\big), \tag{4.14}$$

which prove (4.2).

From (4.7) and (4.12), we obtain (4.3).

Using Proposition 10, for both cases $\tau = 0$ and $\tau = 1$ we have

$$\tau \int_0^T \int \big|(v_\varepsilon)_t\big|^2 \, dx\, dt + \sup_{0<t<T} \int |\nabla v_\varepsilon|^2 \, dx \leqslant C$$

for some constant C independent of ε. Hence, we can extract a subsequence $\{v_{\varepsilon_n}\}$ such that

$$v_{\varepsilon_n} \to v \quad \text{weakly in } L^2\big(0,T; L^2\big(\mathbb{R}^N\big)\big), \tag{4.15}$$

$$\nabla v_{\varepsilon_n} \rightharpoonup \chi = \nabla v \quad \text{weakly in } L^2\big(0,T; L^2\big(\mathbb{R}^N\big)\big). \tag{4.16}$$

Integrating (1) and (2) in $(\mathrm{KS})_\varepsilon$ with respect to x and t, we see that $(u_{\varepsilon_n}, v_{\varepsilon_n})$ satisfies

$$\int_0^T \int_{\mathbb{R}^N} \big(\nabla (u_{\varepsilon_n} + \varepsilon_n)^m \cdot \nabla\varphi - (u_{\varepsilon_n} + \varepsilon)^{q-2} u_\varepsilon \cdot \nabla v_{\varepsilon_n} \cdot \nabla\varphi - u_{\varepsilon_n} \cdot \varphi_t\big)\, dx\, dt$$
$$= \int_{\mathbb{R}^N} u_{0\varepsilon_n}(x) \cdot \varphi(x,0)\, dx,$$

$$\int_0^T \int_{\mathbb{R}^N} (\nabla v_{\varepsilon_n} \cdot \nabla\varphi + v_{\varepsilon_n} \cdot \varphi - u_{\varepsilon_n} \cdot \varphi - \tau v_{\varepsilon_n} \cdot \varphi_t)\, dx\, dt = \int_{\mathbb{R}^N} \tau v_{0\varepsilon_n}(x) \cdot \varphi(x,0)\, dx$$

for any continuously differentiable function φ with compact support in $\mathbb{R}^N \times [0,T)$.

Using (4.2), (4.3), (4.15), (4.16) and (A.4), by the standard convergence argument, we obtain

$$\int_0^T \int_{\mathbb{R}^N} \big(\nabla u^m \cdot \nabla\varphi - u^{q-1} \nabla v \cdot \nabla\varphi - u \cdot \varphi_t\big)\, dx\, dt = \int_{\mathbb{R}^N} u_0(x) \cdot \varphi(x,0)\, dx$$

for the case of $m > 1$, $q \geqslant 2$. Similarly, we find that

$$\int_0^T \int_{\mathbb{R}^N} (\nabla v \cdot \nabla\varphi + v \cdot \varphi - u \cdot \varphi - \tau v \cdot \varphi_t)\, dx\, dt = \int_{\mathbb{R}^N} \tau v_0(x) \cdot \varphi(x,0)\, dx$$

for the case of $m > 1$, $q \geqslant 2$. Thus, we construct the desired weak solution (u, v) of (KS).

Consequently, we complete the proof of Theorems 1 and 2.

5. Proof of Theorem 3

Combining (3.63), (3.64) with Proposition 9, we prove the following lemma.

Lemma 17. *Let* $N \geqslant 1$, $m \geqslant 1$, $q \geqslant \max\{m + \frac{2}{N}, 2\}$ *and let* $T > 0$ *and suppose that* (A.1) *and* (A.2) *are satisfied. Then, there exist an absolute constant* M *and a positive number* δ *depending only on* M, N, m *such that if* u_0 *satifies*:

$$\|u_0\|_{L^1(\mathbb{R}^N)} = M, \qquad \|u_0\|_{L^{N(m-q)/2}(\mathbb{R}^N)} \leqslant \delta, \tag{5.1}$$

then $(\mathrm{KS})_\varepsilon$ *has the strong solution* $(u_\varepsilon, v_\varepsilon)$ *in the class obtained from Proposition* 9 *with the following property*:

$$\frac{d}{dt}\|u_\varepsilon\|_{L^r}^r + \frac{2mr(r-1)}{(r+m-1)^2}\left\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\right\|_{L^2}^2 \leqslant 0 \quad \text{for all } t \in (0,T) \text{ and } r \in \left[\frac{N(q-m)}{2}, \infty\right). \tag{5.2}$$

Using the above Lemma 17, we are going to show Theorem 3. Lemma 6 with $a = 3$ and (A.2) give that

$$\|u_\varepsilon\|_{L^r} \leqslant c^{\frac{1}{\beta_4}\cdot\frac{2}{r+m-1}}\|u_0\|_{L^1}^{1-\theta_4} \cdot \left\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\right\|_{L^2}^{\frac{2\theta_4}{r+m-1}}$$

for any $r \in [2, \infty)$, where

$$\beta_4 := \frac{N}{N+2}\frac{(r+m-2+\frac{2}{N})(r-m+1)}{(r-1)(r+m-1)},$$
$$\theta_4 := \frac{r+m-1}{2}\cdot\left(1-\frac{1}{r}\right)\cdot\frac{1}{\frac{1}{N}-\frac{1}{2}+\frac{r+m-1}{2}}.$$

Noting that

$$\frac{1}{\theta_4} \leqslant 2, \qquad \frac{1-\theta_4}{\theta_4}(r+m-1) \leqslant \frac{N+2}{N} \quad \text{for } r \in [q, \infty),$$
$$\frac{1}{\beta_4} \leqslant \frac{2(N+2)}{N} \quad \text{for } r \in \big[3(m-1), \infty\big),$$

we obtain

$$\|u_\varepsilon\|_{L^r}^{\frac{r+m-1}{\theta_4}} \leqslant \big(c\|u_0\|_{L^1}\big)^{\frac{N+2}{N}} \cdot \left\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\right\|_{L^2}^2 \quad \text{for any } r \in \big[\max\{q, 3(m-1)\}, \infty\big). \tag{5.3}$$

By (5.3), we easily see that

$$C_r\|u_\varepsilon\|_{L^r}^{r\cdot\lambda} \leqslant \frac{2mr(r-1)}{(r+m-1)^2}\left\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\right\|_{L^2}^2 \quad \text{for } m > 1 - \frac{2}{N}, \tag{5.4}$$

where

$$\lambda := \frac{r+m-1}{\theta_4 \cdot r} = 1 + \frac{m-1+\frac{2}{N}}{r-1} > 1, \qquad C_r := \frac{2mr(r-1)}{(r+m-1)^2} \cdot \left(c\|u_0\|_{L^1}\right)^{-\frac{N+2}{N}}.$$

By combining (5.4) with (5.2) in Lemma 17,

$$\frac{d}{dt}\left\|u_\varepsilon(t)\right\|_{L^r}^r + C_r\left\|u_\varepsilon(t)\right\|_{L^r}^{r\cdot\lambda} \leqslant 0 \quad \text{for } r \in \left[(N+2)q, \infty\right). \tag{5.5}$$

Let us denote $\|u_\varepsilon(t)\|_{L^r}^r$ by $X(t)$. Then, (5.5) gives

$$\frac{X(t)'}{X(t)^\lambda} + C_r = \frac{1}{1-\lambda} \cdot \left(X(t)^{-\lambda+1}\right)' + C_r \leqslant 0. \tag{5.6}$$

From (5.6), we obtain

$$\begin{aligned} X(t) &\leqslant \frac{1}{((\lambda-1)C_r \cdot t + X(0)^{-\lambda+1})^{\frac{1}{\lambda-1}}} \leqslant \frac{1}{\min\{(\lambda-1)C_r, \|u_0\|_{L^r}^{r(-\lambda+1)}\}^{\frac{1}{\lambda-1}}} \cdot \frac{1}{(1+t)^{\frac{1}{\lambda-1}}} \\ &= \max\left\{\left((\lambda-1)C_r\right)^{-\frac{1}{\lambda-1}}, \|u_0\|_{L^r}^r\right\} \cdot (1+t)^{-\frac{1}{\lambda-1}}. \end{aligned}$$

This means that

$$\left\|u_\varepsilon(t)\right\|_{L^r} \leqslant \max\left\{\left((\lambda-1)C_r\right)^{-\frac{1}{\lambda-1}\cdot\frac{1}{r}}, \|u_0\|_{L^r}\right\} \cdot (1+t)^{-\frac{1}{\lambda-1}\cdot\frac{1}{r}} \leqslant \tilde{C}_r(1+t)^{-\frac{N}{(m-1)N+2}\cdot(1-\frac{1}{r})},$$

where

$$\tilde{C}_r := \max\left\{\left[\frac{(r+m-1)^2}{r} \cdot \frac{1}{2m(m-1+\frac{2}{N})}\left(c\|u_0\|_{L^1}\right)^{\frac{N+2}{N}}\right]^{\frac{N}{(m-1)N+2}\cdot(1-\frac{1}{r})}, \|u_0\|_{L^r}\right\}.$$

We thus establish the decay estimate for $r \in [(N+2)q, \infty)$. On the other hand, by the Hölder inequality and the mass conservation law,

$$\|u_\varepsilon\|_{L^p} \leqslant \|u_0\|_{L^1}^{1-\frac{p-1}{p}\cdot\frac{r}{r-1}} \|u_\varepsilon\|_{L^r}^{\frac{p-1}{p}\cdot\frac{r}{r-1}} \quad \text{for } p \in [1, r].$$

Therefore, we have the L^p-decay estimates for all $p \in [1, \infty)$ as follows:

$$\left\|u_\varepsilon(t)\right\|_{L^p} \leqslant \|u_0\|_{L^1}^{1-\frac{p-1}{p}\cdot\frac{r}{r-1}} \cdot \tilde{C}_r^{\frac{p-1}{p}\cdot\frac{r}{r-1}} \cdot (1+t)^{\frac{N}{(m-1)N+2}\cdot(1-\frac{1}{p})} \quad \text{for } p \in [1, \infty). \tag{5.7}$$

In addition, a solution v_ε of Eq. (2) in $(\mathrm{KS})_\varepsilon$ can be expressed by the Bessel potential. Therefore, we obtain the same decay estimate as (5.7) for v_ε. Furthermore, by the lower semi-continuity of the norm for $p \in (1, \infty)$ and Fatou lemma for $p = 1$, we obtain the decay estimate (1.6) in Theorem 3. In consequence, we complete the proof of Theorem 3.

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Appendix A

In [26] we investigated the case of $q = 2$ in (KS). By the similar argument to that in there, we can obtain the following Lyapunov function for the case of $q \geqslant 2$:

$$W(t) = \frac{m}{(m-q+1)(m-q+2)} \|u(t)\|_{L^{m-q+2}}^{m-q+2} - \int u(t)v(t)\,dx + \frac{1}{2}\left(\|\nabla v(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2\right).$$

Moreover, we obtain the following lemma:

Lemma 18. *Let $N \geqslant 1$, $\tau = 0, 1$, $q \geqslant 2$, $m > q - \frac{2}{N+2}$ and the non-negative functions $(u_0, \tau v_0) \in L^1 \cap L^m(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. We assume that (u, v) is a weak solution of* (KS). *In the case of $\tau = 0$, v_0 denotes $G * u_0$ with the Bessel potential G. Then, there exists a positive number $R = R(u_0, v_0) > 0$ such that the weak solution (u, v) of* (KS) *satisfies*

$$\sup_{0<t<\infty} \|u(t)\|_{L^{m-q+2}(\mathbb{R}^N)}^{m-q+2} \leqslant R,$$

$$\tau \|v_t\|_{L^2(0,\infty;L^2(\mathbb{R}^N))}^2 \leqslant R,$$

$$\sup_{0<t<\infty} \left(\|v(t)\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla v(t)\|_{L^2(\mathbb{R}^N)}^2\right) \leqslant R.$$

More precisely,

$$R = 4\left\{ \frac{m}{(m-q+1)(m-q+2)} \|u_0\|_{L^{m-q+2}(\mathbb{R}^N)}^{m-q+2} - \int_{\mathbb{R}^N} u_0 v_0\,dx + \frac{1}{2}\left(\|\nabla v_0\|_{L^2(\mathbb{R}^N)}^2 + \|v_0\|_{L^2(\mathbb{R}^N)}^2\right) + C_m\left(\|v_0\|_{L^1(\mathbb{R}^N)} + \|u_0\|_{L^1(\mathbb{R}^N)}\right)^{\gamma_m} \right\},$$

where C_m and γ_m are positive numbers depending only on m, q and N.

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