On the asymptotic analysis of the Boltzmann equation tending towards the Stokes equations

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Abstract

This work deals with the analysis of the asymptotic limit for the Boltzmann equation tending towards the linearized Navier–Stokes equations when the Knudsen number $\varepsilon$ tends to zero. Global existence and uniqueness theorems are proven for regular initial fluctuations. As $\varepsilon$ tends to zero, the solution converges strongly to the solution of the linearized Navier–Stokes systems.

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1. Introduction

This work deals with the analysis of the asymptotic limit, when the Knudsen number tends to zero, of the solution of the Boltzmann equation tending towards the linearized Navier–Stokes equations. The asymptotic theory for small Knudsen numbers related to models of the kinetic theory of gases means, as is known [1], analyzing the macroscopic description delivered by the kinetic equations when the distance between particles tends to zero. In this way, one obtains in alternative to the purely phenomenological derivation, a macroscopic description delivered by the microscopic one. The method applies to various kinetic equations. Different scalings generate different macroscopic equations as documented by the
formal expansions proposed [2]. On the other hand, the derivation of fluid dynamical equations from kinetic theory is well understood at the formal level; however, its full mathematical justification is still not properly developed. Indeed, the justification of the formal approximation of the classical Boltzmann equation is a difficult mathematical problem, considering that various basic regularity properties needed in the proof still have to be properly treated.

Several analytic approaches are documented in the pertinent literature; see among others Bardos and Ukai [3], Bardos, Golse and Levermore [4,5], Caflish [6], De Masi, Esposito and Lebowitz [7], Golse and Levermore [8], Golse and Saint-Raymond [9], Lachowicz [10], Lions and Masmouidi [11], Nishida [12], Ukai and Asano [13], Bellouquid [14] for the Boltzmann equation, Golse and Saint-Raymond [15], Poupaud and Soler [16], Saint-Raymond [17] for the Vlasov–Poisson system, Bellouquid [18–21] for discrete velocity models, and Graffi, Martinez and Pulvirenti [22] for the quantum Boltzmann equation.

Specifically, the asymptotic approach of the scaled Boltzmann equation to the incompressible nonlinear Navier–Stokes equations is proved rigorously in [3], when the Knudsen number \( \varepsilon \) tends to zero, for small, smooth initial data. The smallness of the initial data is a necessary item in the proof.

We should mention that the classical Boltzmann equation was considered over any periodic spatial domain of dimension two or more in [8]. It was shown that the scaled families of DiPerna–Lions renormalized solutions have fluctuations that, globally in time, converge weakly to a unique limit governed by a solution of the Stokes equation provided that the fluid moments of their initial fluctuations converge to appropriate \( L^2 \) initial data and the scaling of the fluctuations with respect to the Knudsen number is essentially optimal, and the limit becomes strong when the initial fluctuations converge entropically to appropriate \( L^2 \) initial data. The mathematical justification of the Stokes limit for all scalings for which its formal derivation holds is an open problem and is specifically dealt with in this work, in the case of regular solutions, where it will be shown that the solution of the Boltzmann equation converges strongly to a unique limit of a regular solution of the Stokes equation provided that the initial data is regular, but not necessarily small.

The analysis developed in this work applies to the Boltzmann equation linked to initial condition rescaled in such a way that

\[
\begin{cases}
\varepsilon \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f_\varepsilon) & \text{in } \mathbb{R}^{2n} \times (0, \infty), \\
 f_\varepsilon(t = 0) = f_0(x, v). 
\end{cases}
\]

The dimensionless parameter \( \varepsilon > 0 \) is the Knudsen number. Here \( f_\varepsilon(t, x, v) \) is a nonnegative function representing the density of particles with position \( x \) and velocity \( v \) in the single-particle phase space \( \mathbb{R}_x^n \times \mathbb{R}_v^n \) at time \( t \). The interaction of particles through collisions is modeled by the operator \( Q(f, f) \); this operator acts only on the variable \( v \) and is given by

\[
Q(f, f) = \int_{\mathbb{R}_v^n \times S^{n-1}} B(v - v_*, w) (f' f_* - f_* f) dv_* dw,
\]

where the notation \( f_* = f(t, x, v_*) \), \( f' = f(t, x, v') \), \( f'_* = f(t, x, v'_*) \) is used with

\[
v' = v - (v - v_*, w) w, \quad v'_* = v_* + (v - v_*, w) w,
\]

and where the kernel \( B \) is a nonnegative function which characterizes the details of the binary interactions. During the evolution process, mass, momentum and energy are conserved:

\[
\int_{\mathbb{R}_v^n} Q(f, f) \phi(v) dv = 0, \quad \phi(v) = 1, v, |v|^2.
\]
The equilibrium distribution function satisfies the equation \( Q(f, f) = 0 \), and has the form of a locally Maxwellian distribution:

\[
M(\rho, u, T) = \frac{\rho}{(2\pi T)^{\frac{n}{2}}} \exp \left( -\frac{|v - u|^2}{2T} \right),
\]

where \( \rho \) refers to the density, \( u \) to velocity and \( T \) to the temperature of the gas.

The contents of the work, following the above introduction, are organized in three parts: first the formal derivation of the macroscopic equations is given; then the existence and strong convergence theorems are stated; finally we provide the proofs of the above theorems.

2. Formal derivation of Stokes limits

The Stokes system is the linearization about the zero state of an incompressible Navier–Stokes system. It governs \((\rho, u, T)\), the fluctuations of mass density, bulk velocity, and temperature about their spatially homogeneous equilibrium values. After making a suitable choice of units, these fluctuations satisfy the incompressibility and Boussinesq relations:

\[
\nabla_x u = 0, \quad \nabla_x (\rho + T) = 0,
\]

while their evolution is given by the motion and heat equations:

\[
\partial_t u + \nabla_x p = \mu \Delta u, \quad \partial_t T = k \Delta T,
\]

where \( \mu > 0 \) is the kinematic viscosity and \( k > 0 \) is the thermal conductivity. This is one of the simplest systems of fluid dynamical equations. It may be derived directly from the kinetic model (1.1) by taking the formal limit of the moment equations.

The derivation of the Stokes system from the kinetic equation (1.1) assumes the sizes of the fluctuations of \( g_\varepsilon \) about \( m \) are of order \( \varphi(\varepsilon) \), where \( \varphi(\varepsilon) \longrightarrow 0 \) as \( \varepsilon \longrightarrow 0 \).

Accordingly, it is convenient to write a solution \( f_\varepsilon \) (resp. \( f_0 \)) as a perturbation of \( m, f = m + \varphi(\varepsilon)\sqrt{mg_0} \), (resp. \( f_0 = m + \varphi(\varepsilon)\sqrt{mg_0} \)). Therefore, the Cauchy problem (1.1) can be rewritten in the following way:

\[
\begin{cases}
\varepsilon \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon = \frac{1}{\varepsilon}L(g_\varepsilon) + \frac{\varphi(\varepsilon)}{\varepsilon} \Gamma(g_\varepsilon, \varepsilon), \\
g_\varepsilon(t = 0) = g_0(x, v),
\end{cases}
\]

where \( L \) is the linearized collision operator \( L \) near the global Maxwellian \( m(v) \), and \( \Gamma \) is the nonlinear quadratic term (see Refs. [14] and [23]). It is known (see for example [24]) that the null space of \( L \) is given by

\[
\text{Null}(L) = \left\{ \sqrt{m}, \ v_i \sqrt{m}, \ |v|^2 \sqrt{m}; \quad 1 \leq i \leq n \right\},
\]

and each of the components of the tensor \( B(v) = v \otimes v - \frac{|v|^2}{n}Id \) and the vector \( A(v) = \frac{|v|^2 - (n+2)}{2}v \) are orthogonal to \( N(L) \), which guarantees the existence and uniqueness of the vector \( A'(v) \) and the tensor \( B'(v) \) such that

\[
L(A') = A(v), \quad L(B') = B(v).
\]

The following theorem is due to [25]:
Theorem 2.1. Suppose that as $\varepsilon \to 0$,
\begin{align*}
g_\varepsilon &\to g, \quad Lg_\varepsilon \to Lg, \quad \langle \phi, g_\varepsilon \rangle \to \langle \phi, g \rangle, \quad \text{in } D'_t \times \Omega, \tag{2.7}
\end{align*}
for any $\phi \in L^2(\mathbb{R}^n_\varepsilon)$, with some limit $g$ and that every formally small term vanishes. Then $g$ must have the form
\begin{align*}
g = \left\{ \rho + v \cdot u + \frac{|v|^2 - n}{2} T \right\} m^\frac{1}{2}, \tag{2.8}
\end{align*}
where the coefficients $\rho, u = (u_1, \ldots, u_n)$ and $T$ are solutions of (2.1)–(2.3), where the kinematic viscosity $\mu_*$ and thermal conductivity $k_*$ are given by
\begin{align*}
\mu_* = \frac{1}{(n-1)(n+2)} \langle A' : LA' \rangle, \quad k_* = \frac{1}{n} \langle B' : LB' \rangle, \tag{2.9}
\end{align*}
where $\langle , \rangle$ is the integral with respect to the variable $v$.

The aim of this note is to prove the convergence (2.7) for the Cauchy problem (2.4). Throughout the work, Grad’s cutoff hard potential [23] is assumed.

3. Main results

The rigorous derivation of the linear incompressible Navier–Stokes form requires strong estimates of solution in order to pass to the limit. Therefore uniform estimates in $\varepsilon$ would likely be needed. The proof was successfully performed by Bardos and Ukai [3] in the case of the nonlinear incompressible limit. There is a class of functional spaces consistent with the problem that was introduced. Here we consider in what follows similar function spaces.

Let $C(\Omega, X)$ and $L^\infty(\Omega, X)$ denote the spaces of the continuous and bounded functions on $\Omega \subset \mathbb{R}^n$ with values in a Banach space $X$, respectively. Denote the norm of the Sobolev space $H^l = H^l(\mathbb{R}^n_\varepsilon)$ by $| \cdot |_{H^l}$ and define the spaces
\begin{align*}
H_{l, \beta} &= \{ f = f(x, v)/(1 + |v|)\beta f \in L^\infty(\mathbb{R}^n_\varepsilon, H^l(\mathbb{R}^n_\varepsilon)), \\
&\quad \text{sup} \ (1 + |v|)^\beta |f(\cdot, v)|_{H^l} \to 0 (R \to \infty)\}, \\
X_{l, \beta} &= C((0, \infty), H_{l, \beta}) \cap L^\infty((0, \infty), H_{l, \beta}).
\end{align*}
The norm of $H_{l, \beta}$ is defined by
\begin{align*}
\| f \|_{H_{l, \beta}} = \sup_v (1 + |v|)^\beta |f(\cdot, v)|_{H^l}.
\end{align*}

Theorem 3.1. Let $g_0 \in H_{l, \beta}$ ($l > \frac{n}{2}, \beta > \frac{n}{2} + 1$); then there exists $\varepsilon_0$ ($0 < \varepsilon_0 < 1$), $a_0$ such that $\forall \varepsilon < \varepsilon_0$, there exists a unique strong solution in time $g_\varepsilon$ of (2.4) satisfying
\begin{align*}
g_\varepsilon(t) \in X_{l, \beta}, \quad \| g_\varepsilon(t) \|_{H_{l, \beta}} \leq a_0 \| g_0 \|_{H_{l, \beta}}, \quad \forall t \geq 0. \tag{3.1}
\end{align*}

Theorem 3.2. $g_\varepsilon$ converges, as $\varepsilon$ tends to 0, to $g$ weakly *$L^\infty((0, \infty), H_{l, \beta})$ and strongly in $C([0, \infty) \times \mathbb{R}^n, *L^\infty_\beta)$, $\forall \delta > 0$ where $g$ is the form (2.8), and the functions $\rho(t, x), u(t, x)$ and $T(t, x)$ are solutions of the Stokes equations (2.1)–(2.3).
**Remark 3.1.** The derivation of the nonlinear incompressible nonlinear Navier–Stokes equation in the case of a regular solution was performed in [3] for initial conditions smooth and small. Here we get the Stokes limit, globally in time, provided that the fluctuation (with respect to an absolute Maxwellian) of the initial data is smooth, but not necessary small.

Letting now $\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon$ be given by

$$
\rho_\varepsilon = \langle m^\frac{1}{2}, g_\varepsilon \rangle, \quad u_\varepsilon = \langle m^\frac{1}{2}, v g_\varepsilon \rangle, \quad T_\varepsilon = \frac{1}{n}((|v|^2 - n)m^\frac{1}{2}, g_\varepsilon),
$$

(3.2)

the following result holds:

**Theorem 3.3.** $(\rho_\varepsilon, u_\varepsilon, T_\varepsilon)$, as $\varepsilon$ goes to 0, converges to $(\rho, u, T)$ weakly * in $L^\infty((0, \infty), H_{l,\beta})$, and strongly in $C((\delta, \infty) \times \mathbb{R}^n)$, $\forall \delta > 0$. Moreover:

- $\rho, u$ and $T$ belong to $C((0, \infty), H^l) \cap L^\infty((0, \infty), H^l)$.
- $\rho + T = 0$.
- $T$ is the unique global solution of
  
  $$
  \begin{aligned}
  &\partial_t T = k_\varepsilon \Delta T, \\
  &T(t = 0) = \frac{1}{2}(T_0 - \rho_0).
  \end{aligned}
  $$

(3.3)

- $u$ is the unique global solution of

$$
\begin{aligned}
&\partial_t u + \nabla_x p = \mu_\varepsilon \Delta u, \\
&u(t = 0) = P u_0.
\end{aligned}
$$

(3.4)

Here, $P$ is the projection on the divergence-free subspace, $\rho_0, u_0$ and $T_0$ being given by (3.2) in terms of $g_0$.

### 4. Proofs of theorems

First, we write (2.4) in the form of the evolution problem

$$
\begin{aligned}
&\frac{dg}{dt} = B_\varepsilon g + \frac{1}{\varepsilon} \Gamma(g, g), \\
g(t = 0) = g_0(x, v),
\end{aligned}
$$

(4.1)

where $B = \frac{1}{\varepsilon}(-\varepsilon v \cdot \nabla_x + L)$. Let now $S_\varepsilon(t)$ denote the semigroup generated by $B_\varepsilon : S_\varepsilon(t) = \exp(tB_\varepsilon)$. Then, Eq. (4.1) can be reduced to an integral $g_\varepsilon = M_\varepsilon(g_\varepsilon)$, where $M_\varepsilon$ is a map defined by

$$
M_\varepsilon(g)(t) = S_\varepsilon(t)g_0 + \frac{\varphi(\varepsilon)}{\varepsilon} \int_0^t S_\varepsilon(t - s) \Gamma(g, g)(s)ds.
$$

(4.2)

**Theorem 3.1** can be proved by showing that the map $M_\varepsilon$ is a contraction in a ball of $X_{l,\beta}$. This requires uniform estimates of $M_\varepsilon(g)(t)$. The following two lemmas give the desired estimates. The proof uses the technique of [3,22].

**Lemma 4.1.** Let $l \in \mathbb{R}, \beta > \frac{n}{2}$. $S_\varepsilon(t)$ is a strongly continuous semigroup on $H_{l,\beta}$, and

$$
\exists C_1 > 0, \quad \forall \varepsilon \in (0, 1) \quad \forall t \geq 0 \quad \|S_\varepsilon(t)g\|_{l,\beta} \leq C_1 \|g\|_{l,\beta}.
$$

(4.3)
In addition, let
\[G_\varepsilon(g)(t) = \frac{\varphi(\varepsilon)}{\varepsilon} \int_0^t S_\varepsilon(t-s) \Gamma(g,g)(s) ds.\] (4.4)

Lemma 4.2. Let \(g, h \in H_{l,\beta}\) (\(l > \frac{m}{2}, \beta > \frac{m}{2} + 1\)); then \(G_\varepsilon(g)(t) \in X_{l,\beta}\), and \(C_2 > 0, \forall \varepsilon \in (0, 1),\)

\[\|G_\varepsilon(g)\|_{l,\beta} \leq C_2 \varphi(\varepsilon) \|g\|_{l,\beta}^2,\] (4.5)

\[\|G_\varepsilon(g) - G_\varepsilon(h)\|_{l,\beta} \leq C_2 \varphi(\varepsilon) \|g - h\|_{l,\beta}(\|g\|_{l,\beta} + \|h\|_{l,\beta}),\] (4.6)

where \(\| \cdot \|_{l,\beta}\) denotes, here (and in what follows), the norm \(X_{l,\beta}\).

Proof of Theorem 3.1. Owing to Lemmas 4.1 and 4.2, \(M_\varepsilon\) maps \(X_{l,\beta}\) into itself, and \(\forall \varepsilon \in (0, 1)\) the following inequalities hold:

\[\|M_\varepsilon(g)\|_{l,\beta} \leq C_1 \|g_0\|_{l,\beta} + C_2 \varphi(\varepsilon) \|g\|_{l,\beta}^2,\]

\[\|M_\varepsilon(g) - M_\varepsilon(h)\|_{l,\beta} \leq C_2 \varphi(\varepsilon) \|g - h\|_{l,\beta}(\|g\|_{l,\beta} + \|h\|_{l,\beta}).\]

On the other hand, there exists \(\eta > 0\) such that

\[\varphi(\varepsilon) \leq \frac{1}{4C_1C_2\|g_0\|_{l,\beta}}, \quad \text{for } \varepsilon \leq \eta.\]

Then if \(d = 1 - 4C_1C_2\varphi(\varepsilon)\|g_0\|_{l,\beta}\), one has \(d \geq 0\) for \(\varepsilon \leq \eta\). Now let \(\varepsilon \leq \varepsilon_0 = \inf(1, \eta)\), and let

\[a_0 = \frac{1 - \sqrt{d}}{2C_2\varphi(\varepsilon_0)\|g_0\|_{l,\beta}};\]

then \(a_0\) is a solution of

\[C_1 \|g_0\|_{l,\beta} + C_2 \varphi(\varepsilon_0)a_0^2 \|g_0\|_{l,\beta}^2 = a_0\|g_0\|_{l,\beta}.\]

Moreover,

\[2C_2\varphi(\varepsilon_0)a_0\|g_0\|_{l,\beta} = 1 - \sqrt{d} < 1.\]

This implies that there exist a constant \(a_0\) determined only by \(C_1, C_2\) and \(\|g_0\|_{l,\beta}\) such that \(M_\varepsilon\) is a contraction on a ball in \(X_{l,\beta}\) of radius \(a_0\). This proves Theorem 3.1. \(\square\)

Proof of Theorem 3.2 and 3.3. Using the approach of Bardos and Ukai [3] the following lemma is immediately obtained:

Lemma 4.3. Let \(l > \frac{m}{2}, \beta > \frac{m}{2} + 1\); then there exists a linear operator \(S(t)\) having the following properties:

(i) \(\forall g_0 \in H_{l,\beta}, S(t)g_0 \in X_{l,\beta}\) and \(\|S(t)g_0\|_{l,\beta} \leq C_1 \|g_0\|_{l,\beta}, \text{with } S(0) = P^0.\)

(ii) \(S_\varepsilon(t)g_0 \to S(t)g_0\) \((\varepsilon \to 0)\) strongly in \(C([\delta, \infty] \times \mathbb{R}^n, L^\infty)\), for any \(\delta > 0\), and with \(\delta = 0\) if and only if \(P^0g_0 = g_0\).

The definition of \(P^0\) is given in [3]. Let now \(g_\varepsilon\) be the solution of Theorem 3.1. According to (3.1), \(g_\varepsilon\) is bounded in \(X_{l,\beta}\), which yields the subsequence

\[g_\varepsilon \to g \quad \text{weakly*} \quad L^\infty(0, T, H_{l,\beta}),\]
and the limit $g \in L^\infty(0, \infty, H_{l,\beta})$. By (4.5), it is easy to show that $G_\varepsilon \longrightarrow 0$ strongly in $C((0, \infty) \times \mathbb{R}^n, L^{\infty, \beta})$, as $\varepsilon \longrightarrow 0$. Then, using Lemma 4.3 yields

$$g_\varepsilon \longrightarrow g = S(t)g_0, \quad \text{strongly in} \quad C((\delta, \infty) \times \mathbb{R}^n, L^{\infty, \beta}), \forall \delta > 0.$$  

This completes the proof of Theorem 3.2.

The proof of Theorem 3.3 is easily derived from Theorem 3.2 simply by taking into account (3.2).

\[\square\]

References