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On Subsets Contained in a Family of Non-commensurable Subsets of a Finite Set

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In a previous note [1] the author presented an argument leading to the following conclusion. Let X be a collection of subsets of an n-element set no two of which are ordered by inclusion; let $Y(X)$ be the collection of subsets each member of which is contained in some member of X. Then with $k \leqslant n/2$ if X has not less than $\binom{n}{k}$ members, $Y(X)$ must have at least $\sum_{j=0}^{k} \binom{n}{j}$ members.

In fact the argument previously presented fails to provide justification for a particular statement from which the conclusion is derived.

The present note contains an alternate argument leading to the same conclusion which makes use of the same ideas. The proof is based upon the following remark. Let H_k be a collection of k-element subsets and let T_l be a set of l-element subsets including all of those containing a member of H_k. Then

$$\frac{|T_l|}{\binom{n}{l}} \geqslant \frac{|H_k|}{\binom{n}{k}}$$

or, in other words, the proportion of l-element subsets in T_l is at least as great as the proportion of k-element subsets in H_k. This fact can be proved by counting the number of containment pairs between members of T_l and H_k. This is at least $\binom{n-k}{n-l} | H_k |$ and at most $\binom{k}{l} | T_l |$; thus

$$\binom{n-k}{n-l} | H_k | \leqslant \binom{l}{k} | T_l |,$$

which can be rewritten in the form given above. The same results hold if $l < k$ and T_l includes all l-element subsets contained in members of H_k.

Consider a collection X containing $\binom{n}{k}$ members. We show that it can be transformed into the collection S_k consisting of all subsets having k elements by a certain sequence of operations. We then show that if these

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operations take X into X' and $|X| = |X'|$ then $|Y(X)| \geqslant |Y(X')|$, from which the desired conclusion follows.

Let X be a collection of subsets of the type under consideration (an "antichain") and let the subcollection of X consisting of its j element members be denoted by X_j. We assume that the size of X, $|X|$, is $\binom{n}{k}$ and that the largest members of X are of size l. Let m be the smallest integer for which $Y(X) \cap S_j \neq S_j$ or the smallest for which $X_j \neq \emptyset$, whichever is smaller.

We define the operation L as follows. Let Z_{l-1} be the maximal collection of $(l-1)$-element subsets each of which is contained in some member of X. If $|Z_{l-1}| > |X_l|$ then let $\bar{Z}_{l-1}$ be any $|X_l|$ sized subset of Z_{l-1}; otherwise let $\bar{Z}_{l-1} = Z_{l-1}$. We then let

$$L(X) = X - X_l + \bar{Z}_{l-1}.$$

We define the operation C by

$$C(X) = X + S_m - Y(X) \cap S_m$$

and finally R by

$$R(X) = X - X_m + (S_{m+1} - Y(X) \cap S_{m+1}).$$

It is obvious from the definition of these operations that

$$R^{k-m} C L^{l-k}(X) = S_k.$$

Let us notice the effect of these operations upon the size of a collection of subsets. If $|Z_{l-1}| \geqslant |X_l|$, then

$$|L(X)| = |X| \qquad \text{and} \qquad |Y(L(X))| < |Y(X)|,$$

so that the resulting collection $L(X)$ has the same size and smaller value of $|Y|$ than did X itself. If on the other hand $|Z_{l-1}| = |X_l| - \mu_l$ for $\mu_l > 0$, then

$$|L(X)| - |X| = -\mu_l$$

while

$$|Y(L(X)| - |Y(X)| = |X_l|.$$

According to our fundamental inequality we must have

$$\frac{|Z_{l-1}|}{\binom{n}{l-1}} \geqslant \frac{|X_l|}{\binom{n}{l}},$$

and hence we have

$$\mu_l \leqslant |X_l| \left(1 - \frac{\binom{n}{l-1}}{\binom{n}{l}}\right),$$

or

$$\mu_l \leqslant \frac{\binom{n}{l} - \binom{n}{l-1}}{\binom{n}{l}} \mid X_l \mid \leqslant \left(\frac{n-2l-1}{n-l-1}\right) \mid X_l \mid.$$

Thus the difference in the size of $Y(L)(X)$ and $Y(X)$ is at least $(n-l-1)/(n-2l-1)$ times the difference in the size of $L(X)$ and X. By similar argument, the increase in size of $Y(C(X))$ over $Y(X)$ is not more than the difference between $\mid C(X)\mid$ and $\mid X \mid$, while the difference in size of $Y(R(X))$ and $Y(X)$ is no greater than $(n-m-1)/(n-2m-1)$ times the difference between $\mid R(X)\mid$ and $\mid X \mid$. The latter statement follows from the fact that every $m+1$-element subset containing a member of X_m must be in $(S_{m+1} - Y(X) \cap S_{m+1})$.

Since for each application of R and each of L we have $m < l$, and since $(n-x-1)/(n-2x-1)$ is an increasing function of x, this ratio is always smaller in any R than it is in any L. Thus, the operations L lead to a decrease in the size of X but at a great cost in decreasing the size of Y. The operations C and R lead to increase in the size of X but these are accompanied by relatively smaller increases in the size of Y. The net total effect of these operations is to leave the size of X equal to the size of the final product. Hence $\mid Y(X)\mid$ must have been originally greater than the final value $\mid Y(S_k)\mid$. This is the desired conclusion.

Reference

1. D. Kleitman, On Subsets Containing a Family of Non-commensurable Subsets of a Finite Set, *J. Combinatorial Theory* **1** (1966), 297–299.