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# Bockstein basis and resolution theorems in extension theory

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ABSTRACT

We prove a generalization of the Edwards–Walsh Resolution Theorem:

**Theorem.** *Let $G$ be an abelian group with $P_G = \mathbb{P}$, where $P_G = \{p \in \mathbb{P}\colon \mathbb{Z}_{(p)} \in$ Bockstein basis $\sigma(G)\}$. Let $n \in \mathbb{N}$ and let $K$ be a connected CW-complex with $\pi_n(K) \cong G$, $\pi_k(K) \cong 0$ for $0 \leqslant k < n$. Then for every compact metrizable space $X$ with $X\tau K$ (i.e., with $K$ an absolute extensor for $X$), there exists a compact metrizable space $Z$ and a surjective map $\pi : Z \to X$ such that*

(a) *$\pi$ is cell-like,*
(b) *$\dim Z \leqslant n$, and*
(c) *$Z\tau K$.*



## 1. Introduction

The objective of this paper will be to prove the following resolution theorem:

**Theorem 1.1.** *Let $G$ be an abelian group with $P_G = \mathbb{P}$, where $P_G = \{p \in \mathbb{P}\colon \mathbb{Z}_{(p)} \in$ Bockstein basis $\sigma(G)\}$. Let $n \in \mathbb{N}$ and let $K$ be a connected CW-complex with $\pi_n(K) \cong G$, $\pi_k(K) \cong 0$ for $0 \leqslant k < n$. Then for every compact metrizable space $X$ with $X\tau K$ (i.e., with $K$ an absolute extensor for $X$), there exists a compact metrizable space $Z$ and a surjective map $\pi : Z \to X$ such that*

(a) *$\pi$ is cell-like,*
(b) *$\dim Z \leqslant n$, and*
(c) *$Z\tau K$.*

The word resolution refers to a map between topological spaces where the domain is in some way better than the range, and the fibers (point preimages) meet certain requirements.

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Let us look at some examples of resolution theorems. Here is the cell-like resolution theorem, first stated by R. Edwards [8], and later proven by J. Walsh in [22]:

**Theorem 1.2.** *(R. Edwards (1978) [8]; J. Walsh (1981) [22]) For every compact metrizable space $X$ with $\dim_{\mathbb{Z}} X \leqslant n$, there exists a compact metrizable space $Z$ and a surjective map $\pi : Z \to X$ such that $\pi$ is cell-like, and $\dim Z \leqslant n$.*

If $n \in \mathbb{N}$, then a *subset* $Y \subset \mathbb{R}^n$ is called *cellular* if $Y$ can be written as the intersection of a nested collection of $n$-cells in $\mathbb{R}^n$. A *space* $Y$ is called *cell-like* if for some $n \in \mathbb{N}$, there is an embedding $F : Y \to \mathbb{R}^n$ so that $F(Y)$ is cellular. A *map* $\pi : Z \to X$ is called *cell-like* if for each $x \in X$, $\pi^{-1}(x)$ is cell-like. Whenever $X$ is a finite-dimensional compact metrizable space, then $X$ is cell-like if and only if $X$ has the shape of a point. To detect that a compact metrizable space has the shape of a point, it is sufficient to prove that there is an inverse sequence $(Z_i, p_i^{i+1})$ of compact metrizable spaces $Z_i$ whose limit is homeomorphic to $X$ and such that for each $i \in \mathbb{N}$, $p_i^{i+1} : Z_{i+1} \to Z_i$ is null-homotopic. It is also sufficient to show that every map of $X$ to a CW-complex is null-homotopic.

The Edwards–Walsh Theorem has been generalized to the class of arbitrary metrizable spaces by L. Rubin and P. Schapiro [20], and to the class of arbitrary compact Hausdorff spaces by S. Mardešić and L. Rubin [17].

A similar statement to the Edwards–Walsh Theorem was proven by A. Dranishnikov, for the group $\mathbb{Z}/p$, where $p$ is an arbitrary prime number:

**Theorem 1.3.** *(A. Dranishnikov (1988) [2]) For every compact metrizable space $X$ with $\dim_{\mathbb{Z}/p} X \leqslant n$, there exists a compact metrizable space $Z$ and a surjective map $\pi : Z \to X$ such that $\pi$ is $\mathbb{Z}/p$-acyclic, and $\dim Z \leqslant n$.*

A map $\pi : Z \to X$ between topological spaces is called *$G$-acyclic* if all its fibers $\pi^{-1}(x)$ have trivial reduced Čech cohomology with respect to the group $G$, or, equivalently, every map $f : \pi^{-1}(x) \to K(G, n)$ is nullhomotopic. Note that a map $\pi : Z \to X$ being cell-like implies that $\pi$ is also $G$-acyclic.

Akira Koyama and Katsuya Yokoi [13] were able to obtain this $\mathbb{Z}/p$-resolution theorem of Dranishnikov both for the class of metrizable spaces and for the class of compact Hausdorff spaces. Dranishnikov proved a statement similar to Theorem 1.3 for the group $\mathbb{Q}$ [4], but he could only obtain $\dim Z \leqslant n+1$, and if $n \geqslant 2$, then additionally $\dim_{\mathbb{Q}} Z \leqslant n$. This result was later improved by M. Levin:

**Theorem 1.4.** *(M. Levin (2005) [16]) Let $n \in \mathbb{N}_{\geqslant 2}$. Then for every compact metrizable space $X$ with $\dim_{\mathbb{Q}} X \leqslant n$, there exists a compact metrizable space $Z$ and a surjective map $\pi : Z \to X$ such that $\pi$ is $\mathbb{Q}$-acyclic, and $\dim Z \leqslant n$.*

The obvious question was whether a theorem similar to Theorem 1.3 could be stated for compact metrizable spaces and arbitrary abelian groups. In their work [14], Koyama and Yokoi made a substantial amount of progress in answering this question. Their method relied heavily on the existence of Edwards–Walsh complexes, which have been studied by J. Dydak and J. Walsh in [6], and which had been applied originally, in a rudimentary form, in [22]. However, using a different approach from the one in [14], M. Levin has proved a very strong generalization for Theorems 1.2 and 1.3, concerning compact metrizable spaces and arbitrary abelian groups:

**Theorem 1.5.** *(M. Levin (2003) [15]) Let $G$ be an abelian group and let $n \in \mathbb{N}_{\geqslant 2}$. Then for every compact metrizable space $X$ with $\dim_G X \leqslant n$, there exists a compact metrizable space $Z$ and a surjective map $\pi : Z \to X$ such that*:

(a) *$\pi$ is $G$-acyclic,*
(b) *$\dim Z \leqslant n+1$, and*
(c) *$\dim_G Z \leqslant n$.*

The requirement of $n \in \mathbb{N}_{\geqslant 2}$ in Levin's Theorem cannot be improved because there is a counterexample for $n = 1$ ($G = \mathbb{Q}$ [15]). The requirement that $\dim Z \leqslant n+1$ cannot be improved either – there is a counterexample for $\dim Z \leqslant n$ ($G = \mathbb{Z}/p^\infty$ [14]). The part that may be improved is $\dim_G X \leqslant n$, using the characterization of cohomological dimension by extension of maps. Namely, for any paracompact Hausdorff space $X$, any abelian group $G$ and $n \in \mathbb{N}$, $\dim_G X \leqslant n$ if and only if every map of a closed subspace of $X$ to $K(G, n)$ can be extended to a map of $X$ to $K(G, n)$. By $K(G, n)$ we will always mean an Eilenberg–MacLane CW-complex of type $(G, n)$, and such is characterized (up to homotopy equivalence) by having $\pi_n \cong G$ and $\pi_k$ trivial for all other $k$.

This fact about extending maps from any closed subspace of $X$ to a $K(G, n)$ can be written as $K(G, n) \in AE(X)$ ($K(G, n)$ is an *absolute extensor* for $X$). Another notation, and the one we will be using, is $X \tau K(G, n)$. In fact, for any two topological spaces $X$ and $Y$, $X \tau Y$ will mean that every map from a closed subspace of $X$ to $Y$ can be extended continuously over $X$.

So, in order to generalize the requirement $\dim_G X \leqslant n$ from Theorem 1.5, note that $\dim_G X \leqslant n \Leftrightarrow X \tau K(G, n)$, and replace a $K(G, n)$ with a CW-complex upon which the demands will be less strict. Here is a theorem generalizing Theorem 1.5 for some abelian groups.

**Theorem 1.6.** *(L. Rubin and P. Schapiro (2005) [21]) Let $G$ be an abelian group with $P_G \neq \mathbb{P}$, where $P_G = \{p \in \mathbb{P}\colon\ \mathbb{Z}_{(p)} \in$ Bockstein basis $\sigma(G)\}$. Let $n \in \mathbb{N}_{\geqslant 2}$, and let $K$ be a connected* CW*-complex with $\pi_n(K) \cong G$, $\pi_k(K) \cong 0$ for $0 \leqslant k < n$. Then for every compact metrizable space $X$ with $X\tau K$, there exists a compact metrizable space $Z$ and a surjective map $\pi : Z \to X$ such that*:

(a) *$\pi$ is $G$-acyclic,*
(b) $\dim Z \leqslant n+1$, *and*
(c) $Z\tau K$.

Note that the statement of Theorem 1.6 does not cover the case when $P_G = \mathbb{P}$. In fact, the statement of this theorem will be true when $P_G = \mathbb{P}$, but in this case the statement can be improved, as shown in Theorem 1.1.

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Before we proceed, let us review some basic facts from Bockstein theory.

## 2. Bockstein theory

The cohomological dimension of a given compact metrizable space depends on the coefficient group, which can be any abelian group and there are uncountably many of them. It turns out that in the case of compact metrizable spaces, it suffices to consider only countably many groups. M.F. Bockstein found an algorithm for computation of the cohomological dimension with respect to a given abelian group $G$ by means of cohomological dimensions with coefficients taken from a countable family of abelian groups $\sigma(G)$. His definition of $\sigma(G)$ was also used by V.I. Kuz'minov [12], and later adapted by E. Dyer [7], and then by A. Dranishnikov [3].

Thus there are three different definitions of a Bockstein basis $\sigma(G)$, which are not equivalent in general, but which are equivalent from the point of view of cohomological dimension. This can be shown using the Bockstein Theorem and Bockstein Inequalities, which will be stated in this section.

**Notation.**

(1) $\mathbb{P}$ stands for the set of all prime numbers,
(2) $\mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q}\colon\ n$ is not divisible by $p\}$ is called the *$p$-localization of the integers*, and
(3) $\mathbb{Z}/p^\infty = \{\frac{m}{n} \in \mathbb{Q}/\mathbb{Z}\colon\ n = p^k$ for some $k \geqslant 0\}$ is called *the quasi-cyclic $p$-group*.

For an abelian group $G$, we say that *an element $g \in G$ is divisible by* $n \in \mathbb{Z} \setminus \{0\}$ if the equation $nx = g$ has a solution in $G$, *$G$ is divisible by $n$* if all of its elements are divisible by $n$, and $G$ is a *divisible group* if $G$ is divisible by all $n \in \mathbb{Z} \setminus \{0\}$.

For an abelian group $G$, $\operatorname{Tor} G$ is the subgroup of all elements of $G$ of finite order, and $p$–$\operatorname{Tor} G$ is the subgroup of all elements whose order is a power of $p$, that is, $p\text{–}\operatorname{Tor} G = \{g \in G\colon\ p^k g = 0$ for some $k \geqslant 1\}$.

Here is the definition of a Bockstein basis $\sigma(G)$ that we will use, adapted from the original one by E. Dyer [7].

**Definition 2.1.** Let $G$ be an abelian group, $G \neq 0$. Then $\sigma(G)$ is the subset of $\{\mathbb{Q}\} \cup \{\mathbb{Z}/p, \mathbb{Z}/p^\infty, \mathbb{Z}_{(p)}\colon\ p \in \mathbb{P}\}$ defined by:

| | | | |
|---|---|---|---|
| (I) | $\mathbb{Q} \in \sigma(G)$ | $\Leftrightarrow$ | $G$ contains an element of infinite order |
| | | $\Leftrightarrow$ | $G/\operatorname{Tor} G \neq 0$, |
| (II) | $\mathbb{Z}_{(p)} \in \sigma(G)$ | $\Leftrightarrow$ | $G$ satisfies the following: $\exists g \in G$ such that $\forall k \in \mathbb{Z}_{\geqslant 0}$, $p^k g$ is not divisible by $p^{k+1}$ |
| | | $\Leftrightarrow$ | $G/\operatorname{Tor} G$ is not divisible by $p$, |
| (III) | $\mathbb{Z}/p \in \sigma(G)$ | $\Leftrightarrow$ | $G$ contains an element of order $p^k$, for some $k \in \mathbb{N}$, which is not divisible by $p$ |
| | | $\Leftrightarrow$ | $p$–$\operatorname{Tor} G$ is not divisible by $p$, |
| (IV) | $\mathbb{Z}/p^\infty \in \sigma(G)$ | $\Leftrightarrow$ | $p$–$\operatorname{Tor} G \neq 0$ and $p$–$\operatorname{Tor} G$ is divisible by $p$. |

**Theorem 2.2** *(Bockstein Inequalities). ([3]) For any compact metrizable space $X$ the following inequalities hold*:

(BI1) $\dim_{\mathbb{Z}/p^\infty} X \leqslant \dim_{\mathbb{Z}/p} X$,
(BI2) $\dim_{\mathbb{Z}/p} X \leqslant \dim_{\mathbb{Z}/p^\infty} X + 1$,
(BI3) $\dim_{\mathbb{Z}/p} X \leqslant \dim_{\mathbb{Z}_{(p)}} X$,
(BI4) $\dim_{\mathbb{Q}} X \leqslant \dim_{\mathbb{Z}_{(p)}} X$,
(BI5) $\dim_{\mathbb{Z}_{(p)}} X \leqslant \max\{\dim_{\mathbb{Q}} X, \dim_{\mathbb{Z}/p^\infty} X + 1\}$,
(BI6) $\dim_{\mathbb{Z}/p^\infty} X \leqslant \max\{\dim_{\mathbb{Q}} X, \dim_{\mathbb{Z}_{(p)}} X - 1\}$.

**Theorem 2.3** *(Bockstein Theorem). ([7]) If $G$ is an abelian group and $X$ is a locally compact space, then*

$$\dim_G X = \sup_{H\in\sigma(G)} \dim_H X.$$

Now let $P_G := \{p \in \mathbb{P}\colon\ \mathbb{Z}_{(p)} \in \sigma(G)\}$.

**Lemma 2.4.** *If $G$ is an abelian group such that $P_G = \mathbb{P}$, then for any compact metrizable space $X$, $\dim_G X = \dim_{\mathbb{Z}} X$.*

**Proof.** $P_G = \mathbb{P}$ means that for each $p \in \mathbb{P}$, $\mathbb{Z}_{(p)} \in \sigma(G)$. By the Bockstein Inequalities (BI4), (BI3) and (BI1), the supremum $\sup_{H\in\sigma(G)} \dim_H X$ has to be achieved at $\sup_{p\in\mathbb{P}} \dim_{\mathbb{Z}_{(p)}} X$. Since $\sigma(\mathbb{Z}) = \{\mathbb{Q}\} \cup \{\mathbb{Z}_{(p)}\colon\ p \in \mathbb{P}\}$, $\sup_{H\in\sigma(G)} \dim_H X = \sup_{H\in\sigma(\mathbb{Z})} \dim_H X$. $\square$

## 3. Walsh technical lemma and Edwards type theorem

This will be a statement needed to produce a resolution $\pi : Z \twoheadrightarrow X$, based on [22].

**Notation.** $B_r(x)$ stands for the *closed* ball with radius $r$, centered at $x$.

**Lemma 3.1** *(Generalized Walsh Lemma). Let $\mathbf{X} = (P_i, f_i^{i+1})$ be an inverse sequence of compact metric spaces $(P_i, d_i)$ of diameter less than 1, $\mathbf{Z} = (M_i, g_i^{i+1})$ an inverse sequence of Hausdorff compacta, $X = \lim \mathbf{X}$ and $Z = \lim \mathbf{Z}$. Assume also that we have maps $\phi_i : M_i \to P_i$, and, for each $i \in \mathbb{N}$ we have numbers $0 < \varepsilon(i) < \frac{\delta(i)}{3} < 1$, satisfying:*

(I) *for $i \geqslant 2$, $\phi_{i-1} \circ g_{i-1}^i$ and $f_{i-1}^i \circ \phi_i$ are $\frac{\varepsilon(i-1)}{3}$-close,*
(II) *for $i \geqslant 2$ and for any $y \in P_i$, $\operatorname{diam}(f_{i-1}^i(B_{\delta(i)}(y))) < \frac{\varepsilon(i-1)}{3}$, and*
(III) *for $i > j$ and for any $y \in P_i$, $\operatorname{diam}(f_j^i(B_{\varepsilon(i)}(y))) < \frac{\varepsilon(j)}{2^i}$.*

*Then there is a map $\pi : Z \to X$ such that for all $x = (x_i) \in X$:*

(IV) $\pi^{-1}(x) = \lim(\phi_i^{-1}(B_{\delta(i)}(x_i)), g_i^{i+1}) = \lim(\phi_i^{-1}(B_{\varepsilon(i)}(x_i)), g_i^{i+1})$

*(here $g_i^{i+1}$ stands for the appropriate restriction).*
*If, in addition, we have that:*

(V) *for all $x = (x_i) \in X$ and for all $i$, $\phi_i^{-1}(B_{\varepsilon(i)}(x_i)) \neq \emptyset$,*

*then $\pi^{-1}(x) \neq \emptyset$, so the map $\pi$ will be surjective.*

**Proof.** The following diagram will help in visualizing the steps of this proof.

$$\begin{array}{ccccccccc}
\cdots \longleftarrow & M_i & \xleftarrow{g_i^{i+1}} & M_{i+1} & \longleftarrow \cdots & & Z \\
& \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i+1}} & & & \downarrow{\scriptstyle \pi} \\
\cdots \longleftarrow & P_i & \xleftarrow[f_i^{i+1}]{} & P_{i+1} & \longleftarrow \cdots & & X
\end{array}$$

Let $z = (z_i)$ be an element of $Z \subset \prod_{i=1}^{\infty} M_i$; so $g_i^{i+1}(z_{i+1}) = z_i$ and $\phi_i(z_i) \in P_i$, for all $i \in \mathbb{N}$. Define a sequence in $\prod_{i=1}^{\infty} P_i$ as follows:

$$x^1 = \big(\phi_1(z_1), \phi_2(z_2), \phi_3(z_3), \phi_4(z_4), \ldots\big)$$
$$x^2 = \big(f_1^2\big(\phi_2(z_2)\big), \phi_2(z_2), \phi_3(z_3), \phi_4(z_4), \ldots\big)$$
$$x^3 = \big(f_1^3\big(\phi_3(z_3)\big), f_2^3\big(\phi_3(z_3)\big), \phi_3(z_3), \phi_4(z_4), \ldots\big)$$
$$\vdots$$

$$x^j = \left(f_1^j\left(\phi_j(z_j)\right), f_2^j\left(\phi_j(z_j)\right), \ldots, f_{j-1}^j\left(\phi_j(z_j)\right), \phi_j(z_j), \phi_{j+1}(z_{j+1}), \ldots\right)$$
$$x^{j+1} = \left(f_1^{j+1}\left(\phi_{j+1}(z_{j+1})\right), f_2^{j+1}\left(\phi_{j+1}(z_{j+1})\right), \ldots, f_j^{j+1}\left(\phi_{j+1}(z_{j+1})\right), \phi_{j+1}(z_{j+1}), \phi_{j+2}(z_{j+2}), \ldots\right)$$
$$\vdots$$

Let $\pi_j : Z \to \prod_{i=1}^{\infty} P_i$ be defined by $\pi_j(z) := x^j$. Note that $\pi_j$ are continuous because coordinate maps $x^j$ are continuous. We shall employ the metric $d$ on $\prod_{i=1}^{\infty} P_i$ given by

$$d\left((s_i), (r_i)\right) := \sum_{i=1}^{\infty} \frac{d_i(s_i, r_i)}{2^i}.$$

We would like to show that $(\pi_j(z))_{j\in\mathbb{N}}$ is a Cauchy sequence in $\prod_{i=1}^{\infty} P_i$. Properties we will need are:

(1) for $j \geqslant 2$, $f_{j-1}^j(\phi_j(z_j))$ and $\phi_{j-1}(z_{j-1}) = \phi_{j-1}(g_{j-1}^j(z_j))$ are $\varepsilon(j-1)$-close, and
(2) for $i > j$, $f_j^{i+1}(\phi_{i+1}(z_{i+1}))$ and $f_j^i(\phi_i(z_i))$ are $\frac{\varepsilon(j)}{2^i}$-close.

Property (1) follows from (I). Property (2) is true because: by $(1)_{i+1}$, $f_i^{i+1}(\phi_{i+1}(z_{i+1}))$ and $\phi_i(z_i)$ are $\varepsilon(i)$-close, so $f_i^{i+1}(\phi_{i+1}(z_{i+1})) \in B_{\varepsilon(i)}(\phi_i(z_i))$. Therefore

$$f_j^{i+1}\left(\phi_{i+1}(z_{i+1})\right) = f_j^i\left(f_i^{i+1}(\phi_{i+1}(z_{i+1}))\right) \in f_j^i\left(B_{\varepsilon(i)}\left(\phi_i(z_i)\right)\right),$$

and $\operatorname{diam} f_j^i(B_{\varepsilon(i)}(\phi_i(z_i))) < \frac{\varepsilon(j)}{2^i}$, by (III). So $f_j^{i+1}(\phi_{i+1}(z_{i+1}))$ and $f_j^i(\phi_i(z_i))$ are $\frac{\varepsilon(j)}{2^i}$-close.

Note that by $(2)_{j>q}$ and $(1)_{j+1}$,

$$\begin{aligned} d\left(\pi_j(z), \pi_{j+1}(z)\right) &= \left(\sum_{q=1}^{j-1} \frac{d_q(f_q^j(\phi_j(z_j)), f_q^{j+1}(\phi_{j+1}(z_{j+1})))}{2^q}\right) + \frac{d_j(\phi_j(z_j), f_j^{j+1}(\phi_{j+1}(z_{j+1})))}{2^j} \\ &< \left(\sum_{q=1}^{j-1} \frac{\varepsilon(q)}{2^j}\frac{1}{2^q}\right) + \frac{\varepsilon(j)}{2^j} < \frac{1}{2^j}\left(\sum_{q=1}^{j-1}\frac{1}{2^q}\right) + \frac{1}{2^j} < \frac{1}{2^j}\left(\left(\sum_{q=1}^{\infty}\frac{1}{2^q}\right) + 1\right) \\ &= \frac{1}{2^{j-1}}. \end{aligned}$$

Therefore, for the indexes $j$ and $j+k$ we get:

$$\begin{aligned} d\left(\pi_j(z), \pi_{j+k}(z)\right) &\leqslant d\left(\pi_j(z), \pi_{j+1}(z)\right) + d\left(\pi_{j+1}(z), \pi_{j+2}(z)\right) + \cdots + d\left(\pi_{j+k-1}(z), \pi_{j+k}(z)\right) \\ &< \frac{1}{2^{j-1}} + \frac{1}{2^j} + \cdots + \frac{1}{2^{j+k-2}} < \frac{1}{2^{j-2}} \cdot \sum_{i=1}^{\infty}\frac{1}{2^i} = \frac{1}{2^{j-2}}. \end{aligned}$$

Thus $(\pi_j(z))_{j\in\mathbb{N}}$ is a Cauchy sequence in the compact metric space $\prod_{i=1}^{\infty} P_i$, and therefore it is convergent. Define $\pi(z) := \lim_{j\to\infty}\pi_j(z)$.

Notice that for any $k \in \mathbb{N}$, and for any $z \in Z$,

$$d\left(\pi_k(z), \pi(z)\right) \leqslant \sum_{j=k}^{\infty} d\left(\pi_j(z), \pi_{j+1}(z)\right) < \sum_{j=k}^{\infty}\frac{1}{2^{j-1}} = \frac{1}{2^{k-2}}.$$

So the sequence $(\pi_j)_{j\in\mathbb{N}}$ converges uniformly to $\pi$. Therefore $\pi : Z \to \prod_{i=1}^{\infty} P_i$ is a continuous function.

We would like to see that $\pi(Z) \subset X$. If $y_j$ is $j$-th coordinate of $\pi(z)$ for some $z \in Z$, then $y_j = \lim_{i>j} f_j^i(\phi_i(z_i))$. Therefore if $j > 1$,

$$f_{j-1}^j(y_j) = f_{j-1}^j\left(\lim_{i>j} f_j^i\left(\phi_i(z_i)\right)\right) = \lim_{i>j}\left(f_{j-1}^j\left(f_j^i\left(\phi_i(z_i)\right)\right)\right) = \lim_{i>j}\left(f_{j-1}^i\left(\phi_i(z_i)\right)\right) = \lim_{i>j-1}\left(f_{j-1}^i\left(\phi_i(z_i)\right)\right) = y_{j-1}.$$

So $\pi(z) \in X$, i.e., $\pi(Z) \subset X$.

Now that we have a map $\pi : Z \to X$, we need to see what its fibers are. Take any $x = (x_i) \in X$. From $(\text{II})_i$ and $(\text{I})_i$, we will get that

(3) $g_{i-1}^i(\phi_i^{-1}(B_{\delta(i)}(x_i))) \subset \phi_{i-1}^{-1}(B_{\varepsilon(i-1)}(x_{i-1}))$.

Here is why: take any $y \in \phi_i^{-1}(B_{\delta(i)}(x_i))$, i.e., $\phi_i(y) \in B_{\delta(i)}(x_i)$. Note that (II)$_i$: $\operatorname{diam}(f_{i-1}^i(B_{\delta(i)}(x_i))) < \frac{\varepsilon(i-1)}{3}$. Hence $d_{i-1}(f_{i-1}^i(\phi_i(y)), f_{i-1}^i(x_i)) < \frac{\varepsilon(i-1)}{3}$, i.e., $d_{i-1}(f_{i-1}^i(\phi_i(y)), x_{i-1}) < \frac{\varepsilon(i-1)}{3}$. By (I)$_i$: $d_{i-1}(\phi_{i-1}(g_{i-1}^i(y)), f_{i-1}^i(\phi_i(y))) < \frac{\varepsilon(i-1)}{3}$, and therefore

$$d_{i-1}\big(x_{i-1}, \phi_{i-1}\big(g_{i-1}^i(y)\big)\big) \leqslant d_{i-1}\big(x_{i-1}, f_{i-1}^i\big(\phi_i(y)\big)\big) + d_{i-1}\big(f_{i-1}^i\big(\phi_i(y)\big), \phi_{i-1}\big(g_{i-1}^i(y)\big)\big) < \frac{2\varepsilon(i-1)}{3} < \varepsilon(i-1).$$

So $\phi_{i-1}(g_{i-1}^i(y)) \in B_{\varepsilon(i-1)}(x_{i-1})$, and therefore $g_{i-1}^i(y) \in \phi_{i-1}^{-1}(B_{\varepsilon(i-1)}(x_{i-1}))$, so (3) is true.

As a consequence of (3) and the fact that $\varepsilon(i) < \delta(i)$, both $(\phi_i^{-1}(B_{\delta(i)}(x_i)), g_{i-1}^i|_{\phi_i^{-1}(B_{\delta(i)}(x_i))})$ and $(\phi_i^{-1}(B_{\varepsilon(i)}(x_i)), g_{i-1}^i|_{\phi_i^{-1}(B_{\varepsilon(i)}(x_i))})$ are inverse sequences with the same limit. Now we would like to show that this limit is $\pi^{-1}(x)$.

Let us show that $\lim(\phi_i^{-1}(B_{\varepsilon(i)}(x_i)), g_{i-1}^i) \subset \pi^{-1}(x)$, where $g_{i-1}^i$ stands for the appropriate restriction. Take any $z = (z_i) \in \lim(\phi_i^{-1}(B_{\varepsilon(i)}(x_i)), g_{i-1}^i)$. Note that

(4) the $j$-th coordinate of $\pi(z)$ is $\lim_{i>j} f_j^i(\phi_i(z_i))$.

Since $z_i \in \phi_i^{-1}(B_{\varepsilon(i)}(x_i))$, we have that $\phi_i(z_i) \in B_{\varepsilon(i)}(x_i)$. Condition (III)$_i$, which says that $\operatorname{diam}(f_j^i(B_{\varepsilon(i)}(x_i))) < \frac{\varepsilon(j)}{2^i}$, implies that $f_j^i(\phi_i(z_i))$ and $x_j = f_j^i(x_i)$ are $\frac{\varepsilon(j)}{2^i}$-close. Therefore $\lim_{i>j} f_j^i(\phi_i(z_i)) = x_j$, so $\pi(z) = x$, i.e., $z \in \pi^{-1}(x)$.

Let us demonstrate that $\pi^{-1}(x) \subset \lim(\phi_i^{-1}(B_{\delta(i)}(x_i)), g_{i-1}^i)$. Suppose that $z = (z_i) \in Z$, and $z \notin \lim(\phi_i^{-1}(B_{\delta(i)}(x_i)), g_{i-1}^i)$. We will show that $\pi(z) \neq x$.

Now $z \notin \lim(\phi_i^{-1}(B_{\delta(i)}(x_i)), g_{i-1}^i)$ means that there is an index $j \in \mathbb{N}$ such that $z_j \notin \phi_j^{-1}(B_{\delta(j)}(x_j))$. So $d_j(\phi_j(z_j), x_j) > \delta(j)$. The inequality $\varepsilon(j) < \frac{\delta(j)}{3}$ assures that $B_{2\varepsilon(j)}(\phi_j(z_j)) \cap B_{\varepsilon(j)}(x_j) = \emptyset$. If we look at the distance between $\phi_j(z_j)$ and the $j$-th coordinate of $\pi(z)$ (see (4)), from $(1)_{j+1}$ and $(2)_{k>j}$ we get:

$$d_j\Big(\phi_j(z_j), \lim_{i>j} f_j^i\big(\phi_i(z_i)\big)\Big) \leqslant d_j\big(\phi_j(z_j), f_j^{j+1}\big(\phi_{j+1}(z_{j+1})\big)\big) + \sum_{k=j+1}^{\infty} d_j\big(f_j^k\big(\phi_k(z_k)\big), f_j^{k+1}\big(\phi_{k+1}(z_{k+1})\big)\big)$$
$$< \varepsilon(j) + \sum_{k=j+1}^{\infty} \frac{\varepsilon(j)}{2^k} = \varepsilon(j) + \frac{\varepsilon(j)}{2^j} \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} < 2\varepsilon(j).$$

That is, the $j$-th coordinate of $\pi(z)$ is contained in $B_{2\varepsilon(j)}(\phi_j(z_j))$, implying $\pi(z) \neq x$, i.e., $z \notin \pi^{-1}(x)$.

So we get that

$$\lim\big(\phi_i^{-1}\big(B_{\varepsilon(i)}(x_i)\big), g_{i-1}^i\big) \subset \pi^{-1}(x) \subset \lim\big(\phi_i^{-1}\big(B_{\delta(i)}(x_i)\big), g_{i-1}^i\big),$$

and since the left and right side of this statement are equal, then (IV) is true.

If (V) is also true, i.e., $\pi^{-1}(x)$ is the inverse limit of an inverse sequence of compact nonempty spaces, then, according to Theorem 2.4 from Appendix II of [5], $\pi^{-1}(x) \neq \emptyset$. Thus, the map $\pi : Z \to X$ is surjective. $\square$

**Remark 3.2.** In some of the proofs that follow we will use stability theory, about which more details can be found in §VI.1 of [10]. Namely, we will use the consequences of the Theorem VI.1. from [10]: if $X$ is a separable metrizable space with $\dim X \leqslant n$, then for any map $f : X \to I^{n+1}$ all values of $f$ are unstable. A point $y \in f(X)$ is called an *unstable value* of $f$ if for every $\delta > 0$ there exists a map $g : X \to I^{n+1}$ such that:

(1) $d(f(x), g(x)) < \delta$ for every $x \in X$, and
(2) $g(X) \subset I^{n+1} \setminus \{y\}$.

Moreover, this map $g$ can be chosen so that $g = f$ on the complement of an arbitrary open neighborhood of $y$, and so that $g$ is homotopic to $f$ (see Corollary I.3.2.1 of [18]).

**Lemma 3.3** *(Special version of Walsh Lemma). Let $\mathbf{X} = (P_i, f_i^{i+1})$ be an inverse sequence of compact metric polyhedra $(P_i, d_i)$ with diameter less than 1, and let $L_i$ be triangulations of $P_i$. Suppose that we have maps $g_i^{i+1} : |L_{i+1}^{(n+1)}| \to |L_i^{(n+1)}|$ such that $g_i^{i+1}(|L_{i+1}^{(n)}|) \subset |L_i^{(n)}|$, and let $\mathbf{Z} = (|L_i^{(n)}|, g_i^{i+1})$ be the inverse sequence of subpolyhedra $|L_i^{(n)}| \subset P_i$, where each $g_i^{i+1}$ stands for the appropriate restriction. Let $X = \lim \mathbf{X}$, $Z = \lim \mathbf{Z}$. Assume that for each $i \in \mathbb{N}$ we have numbers $0 < \varepsilon(i) < \frac{\delta(i)}{3} < 1$, satisfying*:

(I) *for* $i \geqslant 2$, $g_{i-1}^{i}|_{|L_i^{(n)}|}$ *and* $f_{i-1}^{i}|_{|L_i^{(n)}|}$ *are* $\frac{\varepsilon(i-1)}{3}$*-close,*

*and conditions* (II) *and* (III) *from Lemma* 3.1.

*Then there is a map* $\pi : Z \to X$ *such that for all* $x = (x_i) \in X$:

$$\pi^{-1}(x) = \lim\bigl(B_{\delta(i)}(x_i) \cap |L_i^{(n)}|, g_i^{i+1}\bigr) = \lim\bigl(B_{\varepsilon(i)}(x_i) \cap \bigl|L_i^{(n)}\bigr|, g_i^{i+1}\bigr)$$

(*here* $g_i^{i+1}$ *stands for the appropriate restriction*).

*If, in addition, we have that*:

(IV) $\operatorname{mesh} L_i < \varepsilon(i)$, *for all* $i$,

*then for all* $x \in X$ *we have* $\pi^{-1}(x) \neq \emptyset$, *so the map* $\pi$ *will be surjective.*

*If we also have*

(V) *for* $i \geqslant 1$ *and for any* $y \in P_i$, $B_{\varepsilon(i)}(y) \subset P_{y,i} \subset B_{\delta(i)}(y)$, *where* $P_{y,i}$ *is a contractible subpolyhedron of* $|L_i|$, *and*
(VI) *for* $i \geqslant 2$, $g_{i-1}^{i}(|L_i^{(n+1)}|) \subset |L_{i-1}^{(n)}|$,

*then the map* $\pi$ *is cell-like.*

**Proof.** The following diagram will be useful.

$$\begin{array}{ccccccccc}
\cdots & \longleftarrow & |L_i^{(n)}| & \xleftarrow{\;g_i^{i+1}|_{|L_{i+1}^{(n)}|}\;} & |L_{i+1}^{(n)}| & \longleftarrow & \cdots & & Z \\
 & & \downarrow & & \downarrow & & & & \downarrow \pi \\
\cdots & \longleftarrow & P_i = |L_i| & \xleftarrow[\;f_i^{i+1}\;]{} & P_{i+1} = |L_{i+1}| & \longleftarrow & \cdots & & X
\end{array}$$

The existence of $\pi : Z \to X$ with the required properties of fibers follows from Lemma 3.1, when $P_i = |L_i|$, $M_i = |L_i^{(n)}|$ and $\phi_i$ is the inclusion $i : |L_i^{(n)}| \hookrightarrow |L_i|$.

Note that $\phi_i^{-1}(B_{\delta(i)}(x_i)) = B_{\delta(i)}(x_i) \cap |L_i^{(n)}|$, so (IV) of Lemma 3.1 becomes:

(IV*) $\pi^{-1}(x) = \pi^{-1}((x_i)) = \lim(B_{\delta(i)}(x_i) \cap |L_i^{(n)}|, g_i^{i+1}) = \lim(B_{\varepsilon(i)}(x_i) \cap |L_i^{(n)}|, g_i^{i+1})$.

Property (IV) will guarantee that, for any $x \in X$, $\pi^{-1}(x) \neq \emptyset$. This is true because, if we take any $x = (x_i) \in X$, $x_i \in P_i = |L_i|$ implies that there is a simplex $\sigma \in L_i$ such that $x_i \in \sigma$. Since $\operatorname{mesh} L_i < \varepsilon(i)$, we get that $\operatorname{diam} \sigma < \varepsilon(i)$, so $\sigma \subset B_{\varepsilon(i)}(x_i)$. Therefore $\sigma^{(n)} \subset B_{\varepsilon(i)}(x_i) \cap |L_i^{(n)}|$, so

$$\emptyset \neq B_{\varepsilon(i)}(x_i) \cap \bigl|L_i^{(n)}\bigr| \subset B_{\delta(i)}(x_i) \cap \bigl|L_i^{(n)}\bigr| = \phi_i^{-1}\bigl(B_{\delta(i)}(x_i)\bigr).$$

By (V) of Lemma 3.1, $\pi : Z \to X$ is surjective.

It remains to show that properties (V) and (VI) imply that $\pi$ is cell-like. Note that from (V) and (IV*) we get that $\pi^{-1}(x) = \lim(P_{x_i,i} \cap |L_i^{(n)}|, g_i^{i+1})$, where $g_i^{i+1}$ stands for the appropriate restriction. It will be sufficient to show that the maps $g_i^{i+1} : P_{x_{i+1},i+1} \cap |L_{i+1}^{(n)}| \to P_{x_i,i} \cap |L_i^{(n)}|$ are null-homotopic.

First note that $P_{x_{i+1},i+1}$ being contractible implies that the inclusion map $i : P_{x_{i+1},i+1} \cap |L_{i+1}^{(n)}| \hookrightarrow P_{x_{i+1},i+1}$ is null-homotopic. Since $\dim P_{x_{i+1},i+1} \cap |L_{i+1}^{(n)}| \leqslant n$, $i$ is null-homotopic as a map into $P_{x_{i+1},i+1} \cap |L_{i+1}^{(n+1)}|$, that is, this homotopy happens within the $(n+1)$-skeleton of $L_{i+1}$. This is because $\dim((P_{x_{i+1},i+1} \cap |L_{i+1}^{(n)}|) \times I) \leqslant n+1$, so if $H : (P_{x_{i+1},i+1} \cap |L_{i+1}^{(n)}|) \times I \to P_{x_{i+1},i+1}$ is our homotopy, then, by Remark 3.2, in each cell of $P_{x_{i+1},i+1} = |L_{i+1}|$ with dimension $\geqslant n+2$, the map $H$ will have unstable values.

Using the last part of Remark 3.2, as well as properties of deformation retracts, we can find a map $\widetilde{H} : (P_{x_{i+1},i+1} \cap |L_{i+1}^{(n)}|) \times I \to P_{x_{i+1},i+1}$ such that $\widetilde{H}|_{H^{-1}(|L_{i+1}^{(n+1)}|)} = H|_{H^{-1}(|L_{i+1}^{(n+1)}|)}$, $\widetilde{H}((P_{x_{i+1},i+1} \cap |L_{i+1}^{(n)}|) \times I) \subset |L_{i+1}^{(n+1)}|$, and so that $\widetilde{H}$ is a homotopy between $i : P_{x_{i+1},i+1} \cap |L_{i+1}^{(n)}| \hookrightarrow P_{x_{i+1},i+1} \cap |L_{i+1}^{(n+1)}|$ and a constant map.

Composing such a homotopy with $g_i^{i+1}|_{|L_{i+1}^{(n+1)}|} : |L_{i+1}^{(n+1)}| \to |L_i^{(n)}|$ yields the sought after null-homotopy for the restriction $g_i^{i+1}|_{P_{x_{i+1},i+1} \cap |L_{i+1}^{(n)}|}$. $\square$

We will now prepare for Lemma 3.7, which will be useful in the proof of the new version of Edwards' Theorem. First note the following:

**Remark 3.4.** *Each k-dimensional simplex is homeomorphic to $I^k$, so it is an absolute extensor for normal spaces, hence also for CW-complexes. In particular, for a simplex $\sigma$ we have $|\sigma|\tau|\sigma|$.*

**Lemma 3.5.** *Let $\sigma$ be a k-dimensional simplex. Then there exists an open neighborhood N of $|\partial\sigma|$ in $|\sigma|$, and a surjective map $s : |\sigma| \to |\sigma|$ such that $s(N) \subset |\partial\sigma|$, and $s|_{|\partial\sigma|} = id$.*

**Proof.** It suffices to prove the lemma in the case when $\sigma = \Delta \subset \mathbb{R}^{k+1}$ is the standard $k$-dimensional simplex. Consider the homothety $h_{B,\frac{1}{2}} : \Delta \to \Delta$, centered at the barycenter $B$ of $\Delta$ with scale $\frac{1}{2}$, that is, every point $P \in \Delta$ is mapped to $h_{B,\frac{1}{2}}(P)$ so that $B - h_{B,\frac{1}{2}}(P) = \frac{1}{2}(B - P)$. Since $h_{B,\frac{1}{2}}(\Delta)$ is contained in the interior of $\Delta$, we see that $N := \Delta \setminus h_{B,\frac{1}{2}}(\Delta)$ is an open neighborhood of $\partial\Delta$. Let $s : \Delta \to \Delta$ be the map which on $h_{B,\frac{1}{2}}(\Delta)$ coincides with $(h_{B,\frac{1}{2}})^{-1}$, and on $N$ coincides with the restriction to $N$ of the central projection $\Delta \setminus B \to \partial\Delta$. $\square$

Using the previous lemma we get the following technical result helpful in the proof of Lemma 3.7:

**Lemma 3.6.** *Let C be a finite simplicial complex with $\dim C = q$. Then for each $0 \leqslant k \leqslant q$ there is an open neighborhood U of $|C^{(k)}|$ in $|C|$, and a surjective map $r : |C| \to |C|$ so that*

(1) $r|_{|C^{(k)}|} = id_{|C^{(k)}|}$,
(2) *r preserves simplexes, i.e., for any $\tau \in C$, $r(\tau) \subset \tau$, and*
(3) $r(U) \subset |C^{(k)}|$.

**Proof.** The statement of this lemma is true when $q = 0$. If $q \geqslant 1$ and $k = q - 1$, the statement can be easily proven using Lemma 3.5.

Assume that $q > 1$, and assume inductively that the statement of this lemma is true when $q$ is replaced by $n$, and $0 \leqslant n < q$.

Choose an open neighborhood $M$ of $|C^{(q-1)}|$ in $|C|$, and a surjective map $p : |C| \to |C|$ so that

$(1)_{q-1}$ $p|_{|C^{(q-1)}|} = id_{|C^{(q-1)}|}$,
$(2)_{q-1}$ $p(\tau) \subset \tau$ for any $\tau \in C$, and
$(3)_{q-1}$ $p(M) \subset |C^{(q-1)}|$.

If $k = q - 1$, put $U := M$ and $r := p$ and we are done. If $k < q - 1$, proceed as follows. By the inductive assumption, we may select an open neighborhood $N$ of $|C^{(k)}|$ in $|C^{(q-1)}|$, and a surjective map $s : |C^{(q-1)}| \to |C^{(q-1)}|$ such that

(a) $s|_{|C^{(k)}|} = id_{|C^{(k)}|}$,
(b) $s(\tau) \subset \tau$ for any $\tau \in C^{(q-1)}$, and
(c) $s(N) \subset |C^{(k)}|$.

For each $q$-simplex $\sigma$ of $C$, $s(|\partial\sigma|) = |\partial\sigma|$ and $s|_{|\partial\sigma|} : |\partial\sigma| \to |\partial\sigma|$ is homotopic to identity. Hence there is a map $s_\sigma : |\sigma| \to |\sigma|$ such that $s_\sigma|_{|\partial\sigma|} = s|_{|\partial\sigma|}$, and $s_\sigma$ must be surjective. Put $\tilde{s} := s \cup (\bigcup\{s_\sigma | \sigma \text{is a } q\text{−simplex of } C\})$. Then $\tilde{s} : |C| \to |C|$ is surjective, $\tilde{s}(\tau) \subset \tau$ for any $\tau \in C$, and $\tilde{s}|_{|C^{(q-1)}|} = s$.

Note that $p|_M : M \to |C^{(q-1)}|$ is continuous, and $N$ is open in $|C^{(q-1)}|$, so $(p|_M)^{-1}(N)$ is open in $M$ and therefore open in $|C|$.

Define $U := (p|_M)^{-1}(N) = M \cap p^{-1}(N)$ and $r := \tilde{s} \circ p : |C| \to |C|$. Observe that $U$ is a neighborhood of $|C^{(k)}|$ in $|C|$ and that $r$ is surjective. It is routine to check that (1)–(3) are true. $\square$

**Lemma 3.7.** *For any finite simplicial complex C, there is a map $r : |C| \to |C|$ and an open cover $\mathcal{V} = \{V_\sigma :\ \sigma \in C\}$ of $|C|$ such that for all $\sigma, \tau \in C$:*

(i) $\mathring{\sigma} \subset V_\sigma$,
(ii) *if $\sigma \neq \tau$ and $\dim\sigma = \dim\tau$, $V_\sigma$ and $V_\tau$ are disjoint,*
(iii) *if $y \in \mathring{\tau}$, $\dim\sigma \geqslant \dim\tau$ and $\sigma \neq \tau$, then $y \notin V_\sigma$,*
(iv) *if $y \in \mathring{\tau} \cap V_\sigma$, where $\dim\sigma < \dim\tau$, then $\sigma$ is a face of $\tau$, and*
(v) $r(V_\sigma) \subset \sigma$.

**Proof.** Since $C$ is finite, let us suppose that $\dim C = q$. For $k = 0, \dots, q-1$, let $U_k$ correspond to $U$ and $r_k$ correspond to $r$ from Lemma 3.6. Note that for vertices $v \in C^{(0)}$ we have that $\mathring{v} = v$.

Here is how we will define the open cover $\mathcal{V} = \{V_\sigma \colon \sigma \in C\}$ for $|C|$:

(a) for each $k$-simplex $\sigma$ of $C$, where $k = 0, \dots, q-1$, put $V_\sigma := (r_k \circ r_{k+1} \circ \dots \circ r_{q-1})^{-1}(\mathring{\sigma})$ into $\mathcal{V}$, and
(b) for each $q$-simplex $\sigma$ of $C$, put $V_\sigma := \mathring{\sigma}$ into $\mathcal{V}$.

Note that all elements of $\mathcal{V}$ are open sets: in (b) that is clear, and in (a):

$$(r_k \circ r_{k+1} \circ \dots \circ r_{q-1})^{-1}(\mathring{\sigma}) = r_{q-1}^{-1}\big(\dots\big(r_{k+1}^{-1}\big(r_k^{-1}(\mathring{\sigma})\big)\big)\big),$$

and $r_k^{-1}(\mathring{\sigma})$ is open because $r_k|_{U_k} : U_k \to |C^{(k)}|$ is continuous, and $\mathring{\sigma}$ is open in $|C^{(k)}|$.

Let us check that (i) is true: $\mathring{\sigma} \subset V_\sigma$ is clear for case (b), and, for case (a), since $r_k, r_{k+1}, \dots, r_{q-1}$ are all the identity on $|C^{(k)}|$ and $\mathring{\sigma} \subset |C^{(k)}|$, then $\mathring{\sigma} \subset V_\sigma$. Hence $\mathcal{V}$ is a cover for $|C|$ because of (i).

If $\sigma$ and $\tau$ are two different simplexes of the same dimension, then $\mathring{\sigma}$ and $\mathring{\tau}$ are disjoint. If $\dim\sigma = \dim\tau = q$, (ii) is clear. If $\dim\sigma = \dim\tau < q$, then (a) implies that $V_\sigma$ and $V_\tau$ are disjoint, i.e., (ii) is true.

Let us prove property (iii). We know that $y \in \mathring{\tau} \subset V_\tau$. If $\tau$ and $\sigma$ are of the same dimension, then (ii) implies $y \notin V_\sigma$. If $\dim\tau < \dim\sigma \leqslant q-1$, then $V_\sigma := (r_{\dim\sigma} \circ \dots \circ r_{q-1})^{-1}(\mathring{\sigma})$, so if $y$ would be in $V_\sigma$, then $r_{\dim\sigma} \circ \dots \circ r_{q-1}(y) \in \mathring{\sigma}$. But $r_{\dim\sigma}, \dots, r_{q-1}$ are the identity on $|C^{(\dim\tau)}| \supset \tau$, so $r_{\dim\sigma} \circ \dots \circ r_{q-1}(y) = y \in \mathring{\sigma}$, which is in contradiction with $y \in \mathring{\tau}$. Thus $y \notin V_\sigma$. If $\dim\tau < \dim\sigma = q$, then $V_\sigma = \mathring{\sigma}$, so $y \in \mathring{\tau}$ and $\tau \neq \sigma$ imply that $y \notin V_\sigma$.

To prove (iv), suppose that $y \in V_\sigma$ for some $\sigma \in C$ with $\dim\sigma < \dim\tau$. Then $V_\sigma := (r_{\dim\sigma} \circ \dots \circ r_{q-1})^{-1}(\mathring{\sigma})$, so $r_{\dim\sigma} \circ \dots \circ r_{q-1}(y) \in \mathring{\sigma}$. Notice that $r_{\dim\tau}, r_{\dim\tau+1}, \dots, r_{q-1}$ are the identity on $\tau$, so $r_{\dim\sigma} \circ \dots \circ r_{q-1}(y) = r_{\dim\sigma} \circ \dots \circ r_{\dim\tau-1}(y) \in \mathring{\sigma}$. The maps $r_{\dim\sigma}, \dots, r_{\dim\tau-1}$ preserve simplexes, by (2) of Lemma 3.6, so $y \in \mathring{\tau}$ implies that $r_{\dim\sigma} \circ \dots \circ r_{\dim\tau-1}(y) \in \tau$. Thus $\tau \cap \mathring{\sigma} \neq \emptyset$, so $\sigma$ must be a face of $\tau$.

It remains to define the map $r$ and prove the property (v). Define $r := r_0 \circ r_1 \circ \dots \circ r_{q-1} : |C| \to |C|$. For any $k$-simplex $\sigma$ of $C$ where $k = 1, \dots, q-1$, by (a) we get that

$$r(V_\sigma) = r_0 \circ r_1 \circ \dots \circ r_{q-1}\big((r_k \circ r_{k+1} \circ \dots \circ r_{q-1})^{-1}(\mathring{\sigma})\big) = r_0 \circ r_1 \circ \dots \circ r_{k-1}(\mathring{\sigma}),$$

since all $r_i$ are surjective. Also, by (2) of Lemma 3.6, $r(V_\sigma) = r_0 \circ r_1 \circ \dots \circ r_{k-1}(\mathring{\sigma}) \subset \sigma$.

Likewise, for any $q$-simplex $\sigma$ of $C$, we get $r(V_\sigma) = r(\mathring{\sigma}) \subset \sigma$ for the same reason. For vertices $v \in C^{(0)}$, $r(V_v) = r \circ r^{-1}(v) = v$. So we conclude that (v) is true. □

Next we will see a version of Theorem 4.2 from [22], adapted for our situation. In order to proceed, however, we will need to be reminded of two definitions.

Let $K$ be a simplicial complex, $X$ a space, and $f : X \to |K|$ a map. Recall that a map $g : X \to |K|$ is called a *K-modification* of $f$ if whenever $x \in X$ and $f(x) \in \sigma$, for some $\sigma \in K$, then $g(x) \in \sigma$. This is equivalent to the following: whenever $x \in X$ and $f(x) \in \mathring{\sigma}$, for some $\sigma \in K$, then $g(x) \in \sigma$.

In the course of the proof of the following theorem, we will need the notion of *resolution in the sense of inverse sequences*. This usage of the word resolution is completely different from the notion from the title of this paper. The definition can be found in [18] for the more general case of inverse systems. Here, however, we will give the definition for inverse sequences.

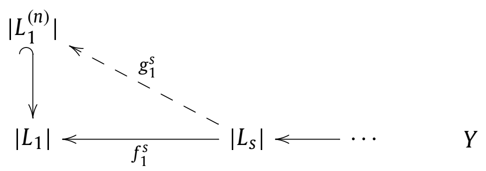
**Definition 3.8.** Let $X$ be a topological space. A *resolution* of $X$ *in the sense of inverse sequences* consists of an inverse sequence of topological spaces $\mathbf{X} = (X_i, p_i^{i+1})$ and a family of maps $(p_i : X \to X_i)$ with the following two properties:

(R1) Let $P$ be an ANR, $\mathcal{V}$ an open cover of $P$ and $h : X \to P$ a map. Then there is an index $s \in \mathbb{N}$ and a map $f : X_s \to P$ such that the maps $f \circ p_s$ and $h$ are $\mathcal{V}$-close.
(R2) Let $P$ be an ANR and $\mathcal{V}$ an open cover of $P$. There exists an open cover $\mathcal{V}'$ of $P$ with the following property: if $s \in \mathbb{N}$ and $f, f' : X_s \to P$ are maps such that the maps $f \circ p_s$ and $f' \circ p_s$ are $\mathcal{V}'$-close, then there exists an $s' \geqslant s$ such that the maps $f \circ p_s^{s'}$ and $f' \circ p_s^{s'}$ are $\mathcal{V}$-close.

By Theorem I.6.1.1 from [18], if all $X_i$ in $\mathbf{X}$ are compact Hausdorff spaces, then $\mathbf{X} = (X_i, p_i^{i+1})$ with its usual projection maps $(p_i : \lim X \to X_i)$ is a resolution of $\lim X$ in the sense of inverse sequences.

Moreover, since every compact metrizable space $X$ is the inverse limit of an inverse sequence of compact polyhedra $\mathbf{X} = (P_i, p_i^{i+1})$ (see Corollary I.5.2.4 of [18]), this inverse sequence $\mathbf{X}$ will have the property (R1) mentioned above, and we will refer to this property as the *resolution property* (R1) *in the sense of inverse sequences*.

**Theorem 3.9** *(New statement of Edwards Theorem). Let $n \in \mathbb{N}$ and let $Y$ be a compact metrizable space such that $Y = \lim(|L_i|, f_i^{i+1})$, where $|L_i|$ are compact polyhedra with $\dim L_i \leqslant n+1$, and $f_i^{i+1}$ are surjections. Then $\dim_{\mathbb{Z}} Y \leqslant n$ implies that there exists an $s \in \mathbb{N}$, $s > 1$, and there exists a map $g_1^s : |L_s| \to |L_1^{(n)}|$ which is an $L_1$-modification of $f_1^s$.*




**Proof.** There will be two separate parts of this proof, for $n \geqslant 2$ and for $n = 1$.

Let us start with $n \geqslant 2$. We will build an Edwards–Walsh complex $\widehat{L}_1$ above $L_1^{(n)}$. Since $\dim L_1 \leqslant n+1$ and $L_1$ is finite, $L_1$ has to have finitely many $(n+1)$-simplexes, say, $\sigma_1, \ldots, \sigma_m$. Focus on $L_1^{(n)}$, and above each of $\sigma_i^{(n)} = \partial\sigma_i \approx S^n$, build a $K(\mathbb{Z}, n)$ by attaching cells of dimension $(n+2)$ and higher. Name the CW-complex that we get in this fashion $\widehat{L}_1$. Notice that we can write $\widehat{L}_1 = L_1^{(n)} \cup K(\sigma_1) \cup K(\sigma_2) \cup \cdots \cup K(\sigma_m)$, where each $K(\sigma_i)$ is a $K(\mathbb{Z}, n)$ attached to $\partial\sigma_i$. Also notice that we can make the attaching maps piecewise linear, so that we will be able to triangulate $\widehat{L}_1$ keeping $L_1^{(n)}$ as a subcomplex.

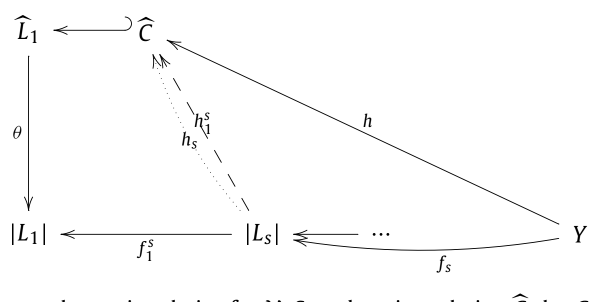
Let $\theta : \widehat{L}_1 \to |L_1|$ be a map such that $\theta|_{|L_1^{(n)}|} = id_{|L_1^{(n)}|}$ and $\theta(K(\sigma_i)) \subset \sigma_i$. This $\theta$ can be constructed as follows: first, define $\theta|_{|L_1^{(n)}|} := id_{|L_1^{(n)}|}$. By Remark 3.4, each $\sigma_i$ is an absolute extensor for CW-complexes, so the inclusion map $j : \sigma_i^{(n)} \to \sigma_i$ can be extended over $K(\sigma_i)$. Call this extension $\theta|_{K(\sigma_i)}$. Gluing together all of the extensions $\theta|_{K(\sigma_i)}$ for $i = 1, \ldots, m$ with $\theta|_{|L_1^{(n)}|}$ will produce the map $\theta$.

Let $f_1 : Y \to |L_1|$ be the projection map from the inverse sequence. The map $f_1$ is surjective since all $f_i^{i+1}$ are surjective. Extend $f_1|_{f_1^{-1}(|L_1^{(n)}|)} : f_1^{-1}(|L_1^{(n)}|) \to |L_1^{(n)}|$ to a map $h : Y \to \widehat{L}_1$ such that

(a) $h(f_1^{-1}(\sigma_i)) \subset \theta^{-1}(\sigma_i) = K(\sigma_i)$, for $i = 1, \ldots, m$.

This can be done using $\dim_{\mathbb{Z}} Y \leqslant n \;\Leftrightarrow\; Y \tau K(\mathbb{Z}, n)$: for any $(n+1)$-dimensional $\sigma_i$, take $f_1|_{f_1^{-1}(\sigma_i^{(n)})} : f_1^{-1}(\sigma_i^{(n)}) \to \sigma_i^{(n)}$ and compose it with the inclusion $i : \sigma_i^{(n)} \hookrightarrow K(\sigma_i) = K(\mathbb{Z}, n)$. Now $Y \tau K(\mathbb{Z}, n)$ implies $f_1^{-1}(\sigma_i) \tau K(\mathbb{Z}, n)$, so the map $i \circ f_1|_{f_1^{-1}(\sigma_i^{(n)})} : f_1^{-1}(\sigma_i^{(n)}) \to K(\sigma_i)$ can be extended over $f_1^{-1}(\sigma_i)$. Call this extension $h|_{f_1^{-1}(\sigma_i)}$. So we get the map $h$ that we need by gluing together all of the extensions $h|_{f_1^{-1}(\sigma_i)}$, for $i = 1, \ldots, m$, with $h|_{f_1^{-1}(|L_1^{(n)}|)} = f_1|_{f_1^{-1}(|L_1^{(n)}|)}$.

Note that our inverse sequence $(|L_i|, f_i^{i+1})$ is a compact resolution for $Y$ in the sense of inverse sequences (see Definition 3.8), so, in particular, it has the resolution property (R1) (in the sense of inverse sequences): if we choose an open cover $\mathcal{V}$ for the minimal and hence finite subcomplex $\widehat{C}$ in $\widehat{L}_1$ such that $h(Y) \subset \widehat{C}$, then we can find an $s > 1$ and a map $h_1^s : |L_s| \to \widehat{C}$ such that $h$ and $h_1^s \circ f_s$ are $\mathcal{V}$-close.




Let us make a wise choice for $\mathcal{V}$. Start by triangulating $\widehat{C}$: let $C$ denote a finite simplicial complex which is a triangulation of $\widehat{C}$ whose restriction to $|L_1^{(n)}|$ is a subcomplex. So $|C| = \widehat{C}$. Since $C$ is finite, let us suppose that $\dim C = q$.

Define an open cover $\mathcal{V}$ for $|C|$, and a map $r : |C| \to |C|$ as in Lemma 3.7. For this cover $\mathcal{V}$ for $|C|$, we may apply resolution property (R1) (in the sense of inverse sequences): we can find an $s > 1$ and a map $h_1^s : |L_s| \to |C|$ such that $h$ and $h_1^s \circ f_s$ are $\mathcal{V}$-close. Define $h_s := r \circ h_1^s : |L_s| \to |C|$. Because of our choices, we get that

(b) whenever $h(y) \in \mathring{\tau}$ for some $\tau \in C$, then $(h_s \circ f_s)(y) \in \tau$.

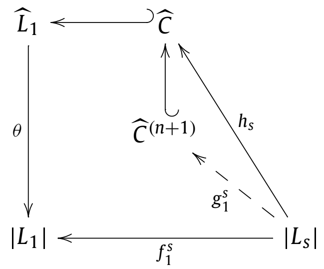
This is true because, by (i)–(iv) of Lemma 3.7, $h(y) \in \mathring{\tau}$ implies that $h(y) \in V_\tau$, and possibly also $h(y) \in V_\sigma$ for some $\sigma$ which is a face of $\tau$, but $h(y)$ is in no other elements of $\mathcal{V}$. Since $h_1^s \circ f_s$ is $\mathcal{V}$-close to $h$, we have that either $h_1^s \circ f_s(y) \in V_\tau$, or $h_1^s \circ f_s(y) \in V_\sigma$, for some face $\sigma$ of $\tau$. But by (v) of Lemma 3.7, $r(V_\tau) \subset \tau$ and $r(V_\sigma) \subset \sigma \subset \tau$. Thus $h_s \circ f_s(y) = r \circ h_1^s \circ f_s(y) \in \tau$.

If $f_1(y) \in \sigma_i$ for some $(n+1)$-simplex $\sigma_i$ of $L_1$, then, by (a), $h(y) \in K(\sigma_i)$, so $h(y) \in \overset{\circ}{\tau}$ for some $\tau \in C$ and $\tau \subset K(\sigma_i)$. By (b), $h_s(f_s(y)) \in \tau$. So we can conclude that

(c) if $f_1(y) \in \sigma_i$, for some $(n+1)$-simplex $\sigma_i$ of $L_1$, then both $h(y)$ and $h_s \circ f_s(y)$ land in $K(\sigma_i)$.

Now we will construct a map $g_1^s : |L_s| \to |L_1^{(n)}|$ such that:

(d) $g_1^s|_{h_s^{-1}(|L_1^{(n)}|)} = h_s|_{h_s^{-1}(|L_1^{(n)}|)}$, and
(e) whenever $h_s(z) \in K(\sigma_i)$ for some $(n+1)$-simplex $\sigma_i$ of $L_1$, then $g_1^s(z) \in \sigma_i$.




We know that $h_s : |L_s| \to |C| = \widehat{C}$, where $C$ is a triangulation of the finite CW-subcomplex $\widehat{C}$ of $\widehat{L}_1$. Since $\widehat{C}$ is finite, we can pick a cell $\gamma$ of maximal possible dimension $\dim \gamma = q$ (we have assumed that $\dim C = q$, so $\dim \widehat{C} = q$). It is safe to assume that $q \geqslant n+2$.

Pick a point $w$ in $\overset{\circ}{\gamma}$ with an open neighborhood $W \subset \overset{\circ}{\gamma}$. Since $\dim |L_s| \leqslant n+1$ and $\dim \gamma > n+1$, the point $w$ we picked is an unstable value for $h_s$, so we can construct a new map $g_{1,\gamma}^s : |L_s| \to \widehat{C} \setminus \{w\}$ that agrees with $h_s$ on $h_s^{-1}(\widehat{C} \setminus W)$, and $g_{1,\gamma}^s(h_s^{-1}(\gamma)) \subset \gamma \setminus \{w\}$ (see Remark 3.2). Retract $\gamma \setminus \{w\}$ to $\partial\gamma$ by a retraction $\tilde{r} : \widehat{C} \setminus \{w\} \to \widehat{C} \setminus \overset{\circ}{\gamma}$, such that $\tilde{r}|_{\widehat{C} \setminus \overset{\circ}{\gamma}} = id$. Replace $h_s$ with $\tilde{r} \circ g_{1,\gamma}^s : |L_s| \to \widehat{C} \setminus \overset{\circ}{\gamma}$.

We will repeat this process, starting with $\widehat{C} \setminus \overset{\circ}{\gamma}$ and the map $\tilde{r} \circ g_{1,\gamma}^s$ instead of $\widehat{C}$ and $h_s$: pick a cell of maximal dimension in $\widehat{C} \setminus \overset{\circ}{\gamma}$, etc. This is done one cell at a time, until we get rid of all cells in $\widehat{C}$ with dimension $\geqslant n+2$. The map we end up with will be $g_1^s : |L_s| \to \widehat{C}^{(n+1)}$, where $\widehat{C}^{(n+1)}$ stands for the CW-skeleton of dimension $n+1$ for $\widehat{C}$. Notice that $\widehat{C}^{(n+1)} \subset \widehat{L}_1^{(n+1)}$, but the CW-skeleton of dimension $n+1$ for $\widehat{L}_1$ is equal to the CW-skeleton of dimension $n$ for $\widehat{L}_1$, since we have built $\widehat{L}_1$ by attaching cells of dimension $n+2$ and higher to $L_1^{(n)}$. Thus $\widehat{L}_1^{(n+1)} = \widehat{L}_1^{(n)} = |L_1^{(n)}|$, where $L_1^{(n)}$ is the simplicial $n$-skeleton of $L_1$. So in fact, $g_1^s : |L_s| \to |L_1^{(n)}|$.

By our construction, $g_1^s$ agrees with $h_s$ on $h_s^{-1}(|L_1^{(n)}|)$, so (d) is true. To prove property (e), let $h_s(z) \in K(\sigma_i)$. Then $h_s(z) \in \gamma$, for some cell $\gamma$ of $K(\sigma_i)$. So $\tilde{r} \circ g_{1,\gamma}^s(z) \in \partial\gamma \subset K(\sigma_i)$. As we go on with our construction, we get $g_1^s(z) \in (K(\sigma_i))^{(n+1)} = \partial\sigma_i \subset \sigma_i$.

Finally, for any $z \in |L_s|$ we have that either $f_1^s(z) \in \overset{\circ}{\tau}$, for some $\tau \in L_1^{(n)}$, or $f_1^s(z) \in \overset{\circ}{\sigma}_i$, for some $(n+1)$-simplex $\sigma_i$ of $L_1$. Since $f_s$ is surjective, there is a $y \in Y$ such that $f_s(y) = z$.

So, if $f_1^s(z) \in \overset{\circ}{\tau}$ for some $\tau \in L_1^{(n)}$, then $f_1(y) = f_1^s(f_s(y)) = f_1^s(z) \in \overset{\circ}{\tau} \subset |L_1^{(n)}|$. Recall that on $f_1^{-1}(|L_1^{(n)}|)$, $f_1$ and $h$ coincide. Thus $f_1(y) = h(y) \in \overset{\circ}{\tau}$. There is a simplex $\tau' \in C \cap |L_1^{(n)}|$ such that $\tau' \subset \tau$, and $f_1(y) = h(y) \in \overset{\circ}{\tau}{}'$. By (b) we get that $h_s \circ f_s(y) \in \tau' \subset \tau$, i.e., $h_s(z) \in \tau \in L_1^{(n)}$, so by (d), $g_1^s(z) = h_s(z) \in \tau$.

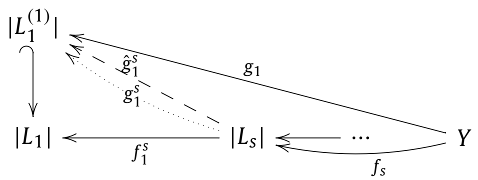
On the other hand, if $f_1^s(z) \in \overset{\circ}{\sigma}_i$, for some $(n+1)$-simplex $\sigma_i$ of $L_1$, then $f_1(y) = f_1^s(f_s(y)) = f_1^s(z) \in \overset{\circ}{\sigma}_i$. By (c), $h_s \circ f_s(y) \in K(\sigma_i)$, i.e., $h_s(z) \in K(\sigma_i)$. Property (e) implies that $g_1^s(z) \in \sigma_i$.

So $g_1^s$ is an $L_1$-modification of $f_1^s$.

It remains to prove this theorem for $n = 1$. First note that $\dim_{\mathbb{Z}} Y \leqslant 1$ implies that $\dim Y \leqslant 1$, because $S^1$ is a $K(\mathbb{Z}, 1)$-complex. We will not need to construct an Edwards–Walsh complex $\widehat{L}_1$ here. Instead, look at the map $f_1 : Y \to |L_1|$. Let $g_1 : Y \to |L_1^{(1)}|$ be a stability theory version of $f_1$. We construct $g_1$ as before: since we know that $\dim L_1 \leqslant 2$, pick any 2-simplex $\sigma$ of $L_1$. We can pick a point $w \in \overset{\circ}{\sigma}$ with an open neighborhood $W \subset \overset{\circ}{\sigma}$, and since $\dim \sigma = 2$, the point $w$ is an unstable value for $f_1$. So there exists a map $g_{1,\sigma} : Y \to |L_1| \setminus \{w\}$ which agrees with $f_1$ on $f_1^{-1}(|L_1| \setminus W)$, and such that $g_{1,\sigma}(f_1^{-1}(\sigma)) \subset \sigma \setminus \{w\}$. Now retract $\sigma \setminus \{w\}$ to $\partial\sigma$ by a retraction $\tilde{r}$ which is the identity on $|L_1| \setminus \overset{\circ}{\sigma}$. Finally, replace $f_1$ by

$\tilde{r} \circ g_{1,\sigma} : Y \to |L_1| \setminus \overset{\circ}{\sigma}$. Continue the process with one 2-simplex at a time. Since $L_1$ is finite, in finitely many steps we will reach the needed map $g_1 : Y \to |L_1^{(1)}|$. Note that from the construction of $g_1$, we get

(f) $g_1|_{f_1^{-1}(|L_1^{(1)}|)} = f_1|_{f_1^{-1}(|L_1^{(1)}|)}$, and for every 2-simplex $\sigma$ of $L_1$, $g_1(f_1^{-1}(\sigma)) \subset \partial\sigma$.




Let us choose an open cover $\mathcal{V}$ of $L_1^{(1)}$ as before: apply Lemma 3.7 to $C = L_1^{(1)}$. Note that $q = 1$, so the map $r = r_0 : |L_1^{(1)}| \to |L_1^{(1)}|$.

Now we can use resolution property (R1) (in the sense of inverse sequences): there is an index $s > 1$ and a map $\hat{g}_1^s : |L_s| \to |L_1^{(1)}|$ such that $\hat{g}_1^s \circ f_s$ and $g_1$ are $\mathcal{V}$-close. Define $g_1^s := r_0 \circ \hat{g}_1^s : |L_s| \to |L_1^{(1)}|$.

Notice that for any $y \in Y$, if $g_1(y) \in \overset{\circ}{\tau}$ for some $\tau \in L_1^{(1)}$ (vertices included), then $g_1(y) \in V_\tau$, and possibly also $g_1(y) \in V_v$, where $v$ is a vertex of $\tau$. Then either $\hat{g}_1^s \circ f_s(y) \in V_\tau$, or $\hat{g}_1^s \circ f_s(y) \in V_v$. In any case, $r_0 \circ \hat{g}_1^s \circ f_s(y) \in \tau$. Hence,

(g) for any $y \in Y$, $g_1(y) \in \overset{\circ}{\tau}$ for some $\tau \in L_1^{(1)}$, implies that $g_1^s(f_s(y)) \in \tau$.

Finally, for any $z \in |L_s|$, $f_s$ is surjective implies that there is a $y \in Y$ such that $f_s(y) = z$. Then $f_1^s(z) = f_1^s(f_s(y)) = f_1(y)$. Now $f_1^s(z)$ is either in $\overset{\circ}{\sigma}$ for some 2-simplex $\sigma$ in $L_1$, or in $\overset{\circ}{\tau}$ for some $\tau \in L_1^{(1)}$.

If $f_1^s(z) \in \overset{\circ}{\sigma}$, that is $f_1(y) \in \overset{\circ}{\sigma}$ for some 2-simplex $\sigma$, by (f) we get that $g_1(y) \in \partial\sigma$. Then by (g), $g_1^s(f_s(y)) \in \partial\sigma$, i.e., $g_1^s(z) \in \sigma$.

If $f_1^s(z) = f_1(y) \in \overset{\circ}{\tau}$ for some $\tau \in L_1^{(1)}$, then (f) implies that $g_1(y) = f_1(y) \in \overset{\circ}{\tau}$, so by (g), $g_1^s(f_s(y)) \in \tau$, i.e., $g_1^s(z) \in \tau$.

Therefore, $g_1^s$ is indeed an $L_1$-modification of $f_1^s$. $\square$

**Lemma 3.10.** *Let $n \in \mathbb{N}$, $G$ be an abelian group and $K$ be a connected* CW*-complex with $\pi_n(K) \cong G$, $\pi_k(K) \cong 0$ for $0 \leqslant k < n$. If $Y$ is a compact metrizable space with* $\dim Y \leqslant n+1$, *then $Y \tau K \Leftrightarrow \dim_G Y \leqslant n$.*

**Proof.** Build a $K(G,n)$ by attaching cells of dimension $n+2$ and higher to our CW-complex $K$.

First assume that $Y \tau K$, and let us show that $\dim_G Y \leqslant n$. If we look at any closed set $A \subset Y$ and any map $f : A \to K(G,n)$, we have that $\dim A \leqslant \dim Y \leqslant n+1$, so we can homotope $f$ into $K(G,n)^{(n+1)} = K^{(n+1)} \subset K$, i.e., there is a map $\overline{f} : A \to K$ which is homotopic to $f$. Now $Y \tau K$ implies the existence of a map $g : Y \to K$ which extends $\overline{f}$. Therefore, by the homotopy extension theorem, $f$ can be extended continuously over $Y$, so we get that $Y \tau K \Rightarrow Y \tau K(G,n) \Rightarrow \dim_G Y \leqslant n$.

Second, assume that $\dim_G Y \leqslant n$, and let us show $Y \tau K$. Look at any closed set $A \subset Y$ and any map $f : A \to K$. Let $i : K \hookrightarrow K(G,n)$ be the inclusion map. Then $Y \tau K(G,n)$ implies that there is a map $\tilde{f} : Y \to K(G,n)$ extending $i \circ f : A \to K(G,n)$, i.e., $\tilde{f}|_A = i \circ f$.

Since $Y$ is compact, $\tilde{f}(Y)$ is contained in a finite subcomplex $\widehat{C}$ of $K(G,n)$. There are finitely many cells in $\widehat{C} \setminus K$, and all of them have dimension $\geqslant n+2$. Pick a cell of maximal dimension $\gamma \in \widehat{C} \setminus K$, and a point $w \in \overset{\circ}{\gamma}$ with an open neighborhood $W \subset \overset{\circ}{\gamma}$. Since $\dim Y \leqslant n+1$ and $\dim \gamma \geqslant n+2$, the point $w$ is an unstable value of the map $\tilde{f}$, so there is a map $g_\gamma : Y \to \widehat{C} \setminus \{w\}$ which agrees with $\tilde{f}$ on $\tilde{f}^{-1}(\widehat{C} \setminus W)$, and such that $g_\gamma(\tilde{f}^{-1}(\gamma)) \subset \gamma \setminus \{w\}$. Retract $\gamma \setminus \{w\}$ to $\partial\gamma$ by a retraction $\tilde{r} : \widehat{C} \setminus \{w\} \to \widehat{C} \setminus \overset{\circ}{\gamma}$, such that $\tilde{r}|_{\widehat{C} \setminus \overset{\circ}{\gamma}} = id$. Replace $\tilde{f}$ with $\tilde{r} \circ g_\gamma : Y \to \widehat{C} \setminus \overset{\circ}{\gamma}$. Repeat this process one cell at a time until all cells of $\widehat{C} \setminus K$ are exhausted. The map we end up with will be $g : Y \to K$ such that $g|_{\tilde{f}^{-1}(K)} = \tilde{f}|_{\tilde{f}^{-1}(K)}$. Since $\tilde{f}(A) = f(A) \subset K$, that is, $A \subset \tilde{f}^{-1}(K)$, we get $g|_A = \tilde{f}|_A$. So $g : Y \to K$ is an extension of $f : A \to K$. Therefore $Y \tau K$. $\square$

## 4. Lemmas for inverse sequences

The proof of the main result will require certain manipulations of inverse sequences of metric compacta. This section will contain the needed results, mostly taken from Section 3 of [21]. The next lemma follows from Corollary 1 of [19], or from [1].

**Lemma 4.1.** *Let $\mathbf{X} = (X_i, p_i^{i+1})$ be an inverse sequence of metric compacta $(X_i, d_i)$. Then there exists a sequence $(\gamma_i)$ of positive numbers such that if $\mathbf{Y} = (X_i, q_i^{i+1})$ is an inverse sequence and $d_i(p_i^{i+1}, q_i^{i+1}) < \gamma_i$ for each $i$, then $\lim \mathbf{Y} = \lim \mathbf{X}$.*

We shall call such $(\gamma_i)$ a *sequence of stability* for **X**.

Let $K$ be a simplicial complex, $X$ a space, and $f : X \to |K|$ a map. One calls $f$ a *$K$-irreducible* map if each $K$-modification $g$ of $f$ is surjective. Note that $f$ being $K$-irreducible implies that $f$ is surjective, and for any subdivision $M$ of $K$, $f$ is $M$-irreducible.

**Lemma 4.2.** *If $f : X \to |K|$ is a $K$-irreducible map, and $g : X \to |K|$ is a $K$-modification of $f$, then $g$ is $K$-irreducible.*

The following fact may be deduced from Theorem 3.11 of [11], or found in [9] (Hauptsatz I, p. 191).

**Lemma 4.3.** *Let $X$ be a compact metrizable space. Then we may write $X$ as the inverse limit of an inverse sequence $\mathbf{Q} = (|Q_i|, q_i^{i+1})$ of compact metric polyhedra, where each bonding map $q_i^{i+1}$ is $Q_i$-irreducible.*

**Lemma 4.4.** *Let $X$ be a compact metrizable space. Then there exists an inverse sequence $\mathbf{K} = (|K_i|, p_i^{i+1})$ of compact metric polyhedra $(|K_i|, d_i)$ along with a sequence of stability $(\gamma_i)$ for $\mathbf{K}$ such that $\lim \mathbf{K} = X$, and for each $i \in \mathbb{N}$, $\operatorname{mesh} K_i < \gamma_i$. We may also specify that for some $m \in \mathbb{N}$, whenever $i \geqslant m$, then $p_i^{i+1} : |K_{i+1}| \to |K_i|$ is a $K_i$-irreducible simplicial map.*

**Proof.** Write $X = \lim \mathbf{Q}$, where $\mathbf{Q} = (|Q_i|, q_i^{i+1})$ is an inverse sequence of compact metric polyhedra $(|Q_i|, d_i)$ as in Lemma 4.3. By Lemma 4.1, we know that there is a sequence of stability $(\rho_i)$ for $\mathbf{Q}$. For each $i$, put $\gamma_i = \rho_i/2$. Note that $(\gamma_i)$ is also a sequence of stability for $\mathbf{Q}$.

Let $K_1$ be a subdivision of $Q_1$ with $\operatorname{mesh} K_1 < \gamma_1$. Suppose that $i \in \mathbb{N}$ and for each $1 \leqslant j \leqslant i$, we have chosen a subdivision $K_j$ of $Q_j$ with $\operatorname{mesh} K_j < \gamma_j$ and, when $1 < j$, a map $p_{j-1}^j : |K_j| \to |K_{j-1}|$ which is a simplicial approximation to $q_{j-1}^j$. Then select a subdivision $K_{i+1}$ of $Q_{i+1}$ with $\operatorname{mesh} K_{i+1} < \gamma_{i+1}$, and which supports a simplicial approximation $p_i^{i+1} : |K_{i+1}| \to |K_i|$ of $q_i^{i+1}$. Note that $d_i(q_i^{i+1}, p_i^{i+1}) < \gamma_i$.

Let us check that $\mathbf{K} := (|K_i|, p_i^{i+1})$ and $m = 1$ satisfy all of the requirements. Clearly $X = \lim \mathbf{K}$, since $(\gamma_i)$ is a sequence of stability for $\mathbf{Q}$. It remains to show that the new bonding maps $p_i^{i+1}$ are $K_i$-irreducible. First, note that $q_i^{i+1}$ being $Q_i$-irreducible implies that $q_i^{i+1}$ is also $K_i$-irreducible. Since $p_i^{i+1}$ is a simplicial approximation of $q_i^{i+1}$, $p_i^{i+1}$ is a $K_i$-modification of $q_i^{i+1}$. By Lemma 4.2, $p_i^{i+1}$ is $K_i$-irreducible too. □

**Definition 4.5.** Whenever $X$ is a compact metrizable space, then we shall refer to an inverse sequence **K** of metric polyhedra $(|K_i|, d_i)$ which admits a sequence $(\gamma_i)$ of positive numbers and $m \in \mathbb{N}$ so that the properties of Lemma 4.4 are satisfied as a *representation* of $X$ which is *stable and simplicially irreducible from index $m$* with *associated sequence of stability* $(\gamma_i)$.

Of course, Lemma 4.4 and its proof show that every compact metrizable space $X$ has a representation **K** which is stable and simplicially irreducible from index $m = 1$.

Next, we want to define a certain procedure which when applied to such $\mathbf{K} = \mathbf{K}_0$ as in Definition 4.5 results in a $\mathbf{K}_1$ which is also a stable and simplicially irreducible (from some index $m$) representation of $X$. We will then show that if this procedure is repeated recursively in a controlled manner, resulting in a sequence $\mathbf{K}_1, \mathbf{K}_2, \ldots$, then there will be a limit $\mathbf{K}_\infty = \lim_{j\to\infty}(\mathbf{K}_j)$ which also will be a representation of $X$.

**Lemma 4.6.** *Let $(\varepsilon_i)$ be a sequence of positive numbers. Let $X$ be a compact metrizable space, let $\mathbf{K} = (|K_i|, p_i^{i+1})$ be a representation of $X$ which is stable and simplicially irreducible from index $m_1$ with an associated sequence of stability $(\gamma_i)$, and let $m \in \mathbb{N}_{\geqslant m_1}$. Define $\gamma_i' = \gamma_i$ if $1 \leqslant i < m$, $\gamma_m' = \frac{1}{2}[\gamma_m - \operatorname{mesh} K_m]$, and $\gamma_i' = \gamma_i/2$ if $i > m$. Let $\Sigma$ be a subdivision of $K_m$ with $\operatorname{mesh} \Sigma < \min\{\varepsilon_m, \gamma_m'\}$. Then there exists an inverse sequence $\mathbf{L} = (|L_i|, l_i^{i+1})$ as follows*:

(a) *in case $1 \leqslant i < m$, then $L_i = K_i$ and $l_i^{i+1} = p_i^{i+1}$,*
(b) *$L_m = \Sigma$,*
(c) *for each $i \geqslant m+1$, $L_i$ is a subdivision of $K_i$ with $\operatorname{mesh} L_i < \min\{\varepsilon_i, \gamma_i'\}$, and*
(d) *if $i \geqslant m+1$, $l_{i-1}^i : |L_i| \to |L_{i-1}|$ is a simplicial approximation to the map $p_{i-1}^i$.*

**Definition 4.7.** We shall call a pair $(\mathbf{L}, (\gamma_i'))$ as in Lemma 4.6 an *m-shift* of $(\mathbf{K}, (\gamma_i))$ from $\Sigma$.

Observe that $d_m(p_m^{m+1}, l_m^{m+1}) \leqslant \operatorname{mesh} \Sigma < \frac{1}{2}[\gamma_m - \operatorname{mesh} K_m] = \gamma'_m$. Hence if $g : |L_{m+1}| \to |L_m|$ is a map and $d_m(g, l_m^{m+1}) < \gamma'_m$, we may conclude that $d_m(g, p_m^{m+1}) < \gamma_m$. Indeed, the following is true:

(e) for each $i$, if $g : |L_{i+1}| \to |L_i|$ is a map and $d_i(g, l_i^{i+1}) < \gamma'_i$, then $d_i(g, p_i^{i+1}) < \gamma_i$.

Therefore we conclude:

**Lemma 4.8.** *Whenever* $(\mathbf{L}, (\gamma'_i))$ *is an m*-shift *of* $(\mathbf{K}, (\gamma_i))$ *from* $\Sigma$, *then* $\mathbf{L}$ *is a stable and simplicially irreducible representation of X from index m with associated sequence of stability* $(\gamma'_i)$.

By exercising some additional care in the construction of **L**, we may guarantee that for all $i$, $d_i(p_i^{i+1}, l_i^{i+1}) < \varepsilon_i$ (of course, $p_i^{i+1} = l_i^{i+1}$ if $i < m$).

It is routine to check that the next lemma holds true.

**Lemma 4.9.** *Let X be a compact metrizable space, and let* $\mathbf{K}_0$ *be a representation of X which is stable and simplicially irreducible from index* $m_1$, *with* $(\gamma_{(0),i})$ *a sequence of stability. For every* $m_1$*-shift* $(\mathbf{K}_1, (\gamma_{(1),i}))$ *of* $(\mathbf{K}_0, (\gamma_{(0),i}))$ *from* $\Sigma_1$ *(an appropriate subdivision of the triangulation of the* $m_1$*-term of* $\mathbf{K}_0$*),* $\mathbf{K}_1$ *is a representation of X which is stable and simplicially irreducible from index* $m_1$, *with* $(\gamma_{(1),i})$ *an associated sequence of stability. It satisfies property* (e) *with* $(\gamma'_i) = (\gamma_{(1),i})$ *and* $(\gamma_i) = (\gamma_{(0),i})$. *The terms (as metric spaces) in* $\mathbf{K}_0$ *and* $\mathbf{K}_1$ *are equal. For* $i < m_1$, $\gamma_{(1),i} = \gamma_{(0),i}$, *the terms with index i have the same triangulations in* $\mathbf{K}_0$ *and* $\mathbf{K}_1$, *and the bonding maps in* $\mathbf{K}_0$ *and* $\mathbf{K}_1$ *with subscript i are equal. For* $i \geqslant m_1$, $\gamma_{(1),i}$ *need not equal* $\gamma_{(0),i}$, *the triangulation of the term in* $\mathbf{K}_1$ *with index i is a subdivision of that in* $\mathbf{K}_0$ *with the same index, and the bonding map with subscript i in* $\mathbf{K}_1$ *may differ from that in* $\mathbf{K}_0$ *with subscript i.*

*If* $i_0 \in \mathbb{N}$, $m_1 < \cdots < m_{i_0}$ *is a finite sequence in* $\mathbb{N}$, *and successively we have chosen* $(\mathbf{K}_j, (\gamma_{(j),i}))$ *an* $m_j$*-shift of* $(\mathbf{K}_{j-1}, (\gamma_{(j-1),i}))$ *from* $\Sigma_j$ *(an appropriate subdivision of the* $m_j$*-term of* $\mathbf{K}_{j-1}$*),* $1 \leqslant j \leqslant i_0$, *then we may conclude that* $\mathbf{K}_{i_0}$ *is a representation of X which is stable and simplicially irreducible from index* $m_{i_0}$, *with* $(\gamma_{(i_0),i})$ *an associated sequence of stability; it satisfies property* (e) *with* $(\gamma'_i) = (\gamma_{(i_0),i})$ *and* $(\gamma_i) = (\gamma_{(i_0-1),i})$. *The terms (as metric spaces) in* $\mathbf{K}_0$ *and* $\mathbf{K}_{i_0}$ *are equal. For* $i < m_{i_0}$, $\gamma_{(i_0),i} = \gamma_{(i_0-1),i}$, *the terms with index i have the same triangulations in* $\mathbf{K}_{i_0-1}$ *and* $\mathbf{K}_{i_0}$, *and the bonding maps in* $\mathbf{K}_{i_0-1}$ *and* $\mathbf{K}_{i_0}$ *with subscript i are equal. For* $i \geqslant m_{i_0}$, $\gamma_{(i_0),i}$ *need not equal* $\gamma_{(i_0-1),i}$, *the triangulation of the term in* $\mathbf{K}_{i_0}$ *with index i is a subdivision of that in* $\mathbf{K}_{i_0-1}$ *with the same index, and the bonding map with subscript i in* $\mathbf{K}_{i_0}$ *may differ from that in* $\mathbf{K}_{i_0-1}$ *with subscript i.*

Henceforth we typically shall write $(|K_{(j),i}|, p_{(j),i}^{i+1})$ to denote such a representation $\mathbf{K}_j$, $0 \leqslant j \leqslant i_0$. One should note that, whenever $i_0 \geqslant j_0 \geqslant j \geqslant 1$, then $K_{(j),m_j} = K_{(j_0),m_j} = \Sigma_j$ when this occurs from the procedure in Lemma 4.9.

**Definition 4.10.** Let $X$ be a compact metrizable space and let $r : \mathbb{N} \to \mathbb{N}$ be an increasing function. Let $\mathbf{K}_0$ be a representation of $X$ which is stable and simplicially irreducible from index $r(1)$, with $(\gamma_{(0),i})$ a sequence of stability. Suppose that $(\mathbf{K}_j, (\gamma_{(j),i}))$, $j \in \mathbb{N}$, is a sequence such that for each $j$, $(\mathbf{K}_j, (\gamma_{(j),i}))$ is an $r(j)$-shift of $(\mathbf{K}_{j-1}, (\gamma_{(j-1),i}))$ from $\Sigma_j$.

Then for each $k \in \mathbb{N}$, if $m$, $l$, and $i$ are chosen so that $m \geqslant l \geqslant r(k) > i$, one sees that $p_{(l),i}^{i+1} = p_{(m),i}^{i+1}$ and $\gamma_{(l),i} = \gamma_{(m),i}$. So for each $i$, the sequences $(\gamma_{(j),i})_{j\in\mathbb{N}}$ and $(p_{(j),i}^{i+1})_{j\in\mathbb{N}}$ are eventually constant. Hence we may define an inverse sequence $\mathbf{K}_\infty = (|K_{(\infty),i}|, p_{(\infty),i}^{i+1}) = \lim_{j\to\infty}\mathbf{K}_j$ and a sequence $(\gamma_{(\infty),i}) = \lim_{j\to\infty}(\gamma_{(j),i})$ of positive numbers by putting $K_{(\infty),i} = \lim_{j\to\infty} K_{(j),i}$ and $p_{(\infty),i}^{i+1} = \lim_{j\to\infty} p_{(j),i}^{i+1}$.

From our construction and this definition, we can deduce the following:

**Lemma 4.11.** *Assume the notation of Definition* 4.10. *Then* $\mathbf{K}_\infty$ *is a representation of X. If* $i \in \mathbb{N}$, $g : |K_{(\infty),i+1}| \to |K_{(\infty),i}|$ *is a map, and* $d_i(g, p_{(\infty),i}^{i+1}) < \gamma_{(\infty),i}$, *then* $d_i(g, p_{(0),i}^{i+1}) < \gamma_{(0),i}$ *and hence* $(\gamma_{(\infty),i})$ *is a sequence of stability for* $\mathbf{K}_\infty$.

**Proof.** To show that $\mathbf{K}_\infty$ is a representation of $X$, it is enough to check that for all $i \in \mathbb{N}$, $d_i(p_{(\infty),i}^{i+1}, p_{(0),i}^{i+1}) < \gamma_{(0),i}$.

Take an $i \in \mathbb{N}$. If $i < r(1)$, then $p_{(\infty),i}^{i+1} = p_{(0),i}^{i+1}$ and $\gamma_{(\infty),i} = \gamma_{(0),i}$. Hence the statement $d_i(g, p_{(\infty),i}^{i+1}) < \gamma_{(\infty),i}$ implies that $d_i(g, p_{(0),i}^{i+1}) < \gamma_{(0),i}$.

If $i \geqslant r(1)$, then we know that $r(k-1) \leqslant i < r(k)$ for some $k \in \mathbb{N}_{\geqslant 2}$. The fact that $i < r(k)$ implies that $p_{(\infty),i}^{i+1} = p_{(k-1),i}^{i+1}$. On the other hand, $r(k-1) \leqslant i$ implies that $\gamma_{(j),i}$ has changed in every step of the construction from step 0 to $(k-1)$. That is, $\gamma_{(j),i} \leqslant \frac{1}{2}\gamma_{(j-1),i}$ for all $1 \leqslant j \leqslant k-1$, so $\gamma_{(j),i} \leqslant \frac{1}{2^j}\gamma_{(0),i}$. Therefore

$$d_i\big(p^{i+1}_{(\infty),i}, p^{i+1}_{(0),i}\big) = d_i\big(p^{i+1}_{(k-1),i}, p^{i+1}_{(0),i}\big) \leqslant d_i\big(p^{i+1}_{(k-1),i}, p^{i+1}_{(k-2),i}\big) + \cdots + d_i\big(p^{i+1}_{(1),i}, p^{i+1}_{(0),i}\big)$$
$$< \gamma_{(k-1),i} + \cdots + \gamma_{(1),i} \leqslant \frac{\gamma_{(0),i}}{2^{k-1}} + \cdots + \frac{\gamma_{(0),i}}{2} < \gamma_{(0),i} \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} = \gamma_{(0),i}.$$

By Lemma 4.1, $\lim \mathbf{K}_\infty = X$.

It remains to show that $d_i(g, p^{i+1}_{(\infty),i}) < \gamma_{(\infty),i}$ implies $d_i(g, p^{i+1}_{(0),i}) < \gamma_{(0),i}$. The fact that $i < r(k)$ implies that $\gamma_{(\infty),i} = \gamma_{(k-1),i}$. So $d_i(g, p^{i+1}_{(\infty),i}) = d_i(g, p^{i+1}_{(k-1),i}) < \gamma_{(k-1),i}$. Therefore

$$\begin{aligned} d_i\big(p^{i+1}_{(0),i}, g\big) &\leqslant d_i\big(p^{i+1}_{(0),i}, p^{i+1}_{(1),i}\big) + d_i\big(p^{i+1}_{(1),i}, p^{i+1}_{(2),i}\big) + \cdots + d_i\big(p^{i+1}_{(k-2),i}, p^{i+1}_{(k-1),i}\big) + d_i\big(p^{i+1}_{(k-1),i}, g\big) \\ &< (\gamma_{(1),i} + \gamma_{(2),i} + \cdots + \gamma_{(k-1),i}) + \gamma_{(k-1),i} \\ &\leqslant \gamma_{(0),i} \cdot \left(\left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}}\right) + \frac{1}{2^{k-1}}\right) \\ &= \gamma_{(0),i}. \qquad \square \end{aligned}$$

## 5. Proof of the main theorem

Let us now prove Theorem 1.1.

**Proof.** We will construct, using induction:

- ⋄ an increasing function $r : \mathbb{N} \to \mathbb{N}$,
- ⋄ sequences of numbers $(\delta(i))_{i\in\mathbb{N}}$ and $(\varepsilon(i))_{i\in\mathbb{N}}$ such that $0 < \varepsilon(i) < \frac{\delta(i)}{3} < 1$, for all $i$,
- ⋄ a sequence of inverse sequences $\mathbf{K}_j = (|K_{(j),i}|, p^{i+1}_{(j),i})$, for $j \in \mathbb{Z}_{\geqslant 0}$, as described in Lemma 4.9, with terms that are compact polyhedra and with surjective bonding maps, and with $\lim \mathbf{K}_j = X$ (in fact, these sequences are representations for $X$ that are stable and simplicially irreducible from index $r(j)$, with stability sequences $(\gamma_{(j),i})$, and $|K_{(j),i}| = |K_{(0),i}|$, for all $i$ and $j$ in $\mathbb{N}$),
- ⋄ a sequence of subdivisions $\Sigma_i$ of $K_{(i-1),r(i)}$, for $i \in \mathbb{N}$, and
- ⋄ a sequence of maps $g^{r(i)}_{r(i-1)} : |\Sigma^{(n+1)}_i| \to |\Sigma^{(n)}_{i-1}|$, for $i \geqslant 2$,

such that for each $i$ for which the statement makes sense, we have:

(I)$_i$ $g^{r(i)}_{r(i-1)}$ and $p^{r(i)}_{(i-1),r(i-1)}|_{|\Sigma^{(n+1)}_i|}$ are $\frac{\varepsilon(i-1)}{3}$-close,

(II)$_i$ for any $y \in |K_{(i-1),r(i)}| = |\Sigma_i|$, $\operatorname{diam}(p^{r(i)}_{(i-1),r(i-1)}(B_{\delta(i)}(y))) < \frac{\varepsilon(i-1)}{3}$,

(III)$_i$ for $i > j$ and for any $y \in |K_{(i-1),r(i)}| = |\Sigma_i|$, $\operatorname{diam}(p^{r(i)}_{(j),r(j)}(B_{\varepsilon(i)}(y))) < \frac{\varepsilon(j)}{2^i}$,

(IV)$_i$ mesh $\Sigma_i < \min\{\frac{\varepsilon(i)}{3}, \gamma_{(i-1),r(i)}\}$, so mesh $\Sigma_i < \varepsilon(i)$, and

(V)$_i$ for any $y \in |K_{(i-1),r(i)}| = |\Sigma_i|$, $B_{\varepsilon(i)}(y) \subset P_{y,i} \subset B_{\delta(i)}(y)$, where $P_{y,i}$ is a contractible subpolyhedron of $|\Sigma_i|$.

In fact, this will prepare us to use Walsh's Lemma 3.3 with

$$\mathbf{X} = \big(|K_{(0),r(i)}|, p^{r(i+1)}_{(i),r(i)}\big), \qquad \mathbf{Z} = \big(\big|\Sigma^{(n)}_i\big|, g^{r(i+1)}_{r(i)}\big|_{|\Sigma^{(n)}_{i+1}|}\big).$$

Let us start the construction by taking a representation for $X$ which is stable and simplicially irreducible from index 1: $\mathbf{K}_0 = (|K_{(0),i}|, p^{i+1}_{(0),i})$, $\lim \mathbf{K}_0 = X$, with stability sequence $(\gamma_{(0),i})$.

Define $r(1) := 1$.

We will choose $0 < \delta(1) < 1$ any way we want. Next, we pick an intermediate subdivision $\widetilde{\Sigma}_1$ of $K_{(0),1}$ so that for any $y \in |K_{(0),1}|$, any closed $\widetilde{\Sigma}_1$-vertex star containing $y$ is contained in the closed $\delta(1)$-ball $B_{\delta(1)}(y)$. (A closed $\widetilde{\Sigma}_1$-vertex star is a closed star $\overline{\mathrm{st}}(w, \widetilde{\Sigma}_1)$ in the complex $\widetilde{\Sigma}_1$ whose center $w$ is a vertex of $\widetilde{\Sigma}_1$.) It is enough to make mesh $\widetilde{\Sigma}_1 < \frac{\delta(1)}{2}$, so $\operatorname{diam}(\overline{\mathrm{st}}(w, \widetilde{\Sigma}_1)) \leqslant 2\,\text{mesh}\,\widetilde{\Sigma}_1 < \delta(1)$).

Now choose an $\varepsilon(1)$ so that $0 < \varepsilon(1) < \frac{\delta(1)}{3}$, and for any $y \in |K_{(0),1}|$, the closed $\varepsilon(1)$-ball $B_{\varepsilon(1)}(y)$ sits inside an open vertex star with respect to $\widetilde{\Sigma}_1$. (This can be done as follows: form the open cover for $|K_{(0),1}|$ consisting of the open stars $\mathrm{st}(w, \widetilde{\Sigma}_1)$. There is a Lebesgue number $\lambda$ for this cover, so make your $\varepsilon(1) < \frac{\lambda}{2}$. Then for any $y \in |K_{(0),1}|$, $\operatorname{diam} B_{\varepsilon(1)}(y) < \lambda \Rightarrow B_{\varepsilon(1)}(y) \subset \mathrm{st}(w_0, \widetilde{\Sigma}_1)$, for some $w_0 \in \widetilde{\Sigma}^{(0)}_1$. Fix such $w_0$ for each $y$.)

Note that for any $y \in |K_{(0),1}|$, $B_{\varepsilon(1)}(y) \subset |\overline{\mathrm{st}}(w_0, \widetilde{\Sigma}_1)| \subset B_{\delta(1)}(y)$. Define $P_{y,1} := |\overline{\mathrm{st}}(w_0, \widetilde{\Sigma}_1)|$, which is a contractible subpolyhedron of $|K_{(0),1}|$, so (V)$_1$ is satisfied.

Choose a subdivision $\Sigma_1$ of $\widetilde{\Sigma}_1$ with mesh $\Sigma_1 < \min\{\frac{\varepsilon(1)}{3}, \gamma_{(0),1}\}$, which implies (IV)$_1$.

Let $(\mathbf{K}_1, (\gamma_{(1),i}))$ be a 1-shift of $(\mathbf{K}_0, (\gamma_{(0),i}))$ from $\Sigma_1$, i.e., $\mathbf{K}_1 = (|K_{(1),i}|, p^{i+1}_{(1),i})$ is an inverse sequence with $K_{(1),1} = \Sigma_1$, limit equal $X$, and stability sequence $(\gamma_{(1),i})$. Note that at this point, all bonding maps in $\mathbf{K}_1$ are simplicial because $\mathbf{K}_1$ is simplicially irreducible from index 1. This concludes the basis of induction.

*Step of induction.* Let $k \in \mathbb{N}_{\geqslant 2}$. Suppose that we have chosen, as required above,

- ⋄ for $j = 1, \dots, k-1$, the numbers $r(j)$, $\delta(j)$, $\varepsilon(j)$,
- ⋄ for $j = 0, \dots, k-1$, the inverse sequences $\mathbf{K}_j = (|K_{(j),i}|, p^{i+1}_{(j),i})$, which are stable and simplicially irreducible from index $r(j)$, with stability sequences $(\gamma_{(j),i})$,
- ⋄ for $j = 1, \dots, k-1$, subdivisions $\Sigma_j$ of $K_{(j-1),r(j)}$, and
- ⋄ for $j = 2, \dots, k-1$, maps $g^{r(j)}_{r(j-1)} : |\Sigma_j^{(n+1)}| \to |\Sigma_{j-1}^{(n)}|$,

so that the properties $(\mathrm{I})_j$–$(\mathrm{V})_j$ are satisfied for each $j = 1, \dots, k-1$ for which they make sense.

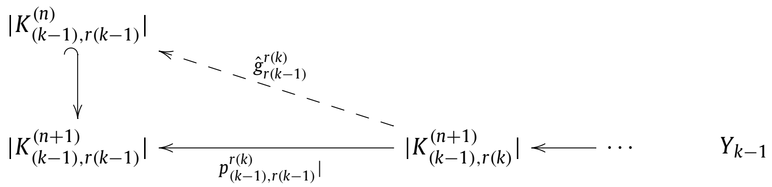
Focus on the inverse sequence $\mathbf{K}_{k-1} = (|K_{(k-1),i}|, p^{i+1}_{(k-1),i})$. For $i \geqslant r(k-1)$, the bonding maps $p^{i+1}_{(k-1),i}$ are simplicial. Recall that $\lim \mathbf{K}_{k-1} = X$, and notice that $K_{(k-1),r(k-1)} = \Sigma_{k-1}$. Let

$$\mathbf{Y}_{k-1} := \left(\left|K^{(n+1)}_{(k-1),i}\right|, p^{i+1}_{(k-1),i}\big|_{|K^{(n+1)}_{(k-1),i+1}|}\right)_{i \geqslant r(k-1)}$$

be the inverse sequence of the $(n+1)$-skeleta of the polyhedra in $\mathbf{K}_{k-1}$, starting with the $(r(k-1))$-th polyhedron onward, where the bonding maps are the restrictions of the original bonding maps. Notice that every $p^{i+1}_{(k-1),i}|_{|K^{(n+1)}_{(k-1),i+1}|} : |K^{(n+1)}_{(k-1),i+1}| \to |K^{(n+1)}_{(k-1),i}|$ is still simplicial and surjective: since $p^{i+1}_{(k-1),i}$ is simplicial and surjective, for every simplex $\sigma \in K^{(n+1)}_{(k-1),i}$ with $\dim \sigma = k$, there exists a simplex $\tau \in K_{(k-1),i+1}$ such that $\dim \tau \geqslant k$ and $p^{i+1}_{(k-1),i}(\tau) = \sigma$. So there must be a $k$-face of $\tau$ which is mapped by $p^{i+1}_{(k-1),i}$ onto $\sigma$. In particular, for every $(n+1)$-dimensional $\sigma \in K^{(n+1)}_{(k-1),i}$, there exists an $(n+1)$-simplex in $K_{(k-1),i+1}$ that is mapped onto $\sigma$ by $p^{i+1}_{(k-1),i}$.

Now let $Y_{k-1} = \lim \mathbf{Y}_{k-1}$. Then $\dim Y_{k-1} \leqslant n+1$, because $\dim |K^{(n+1)}_{(k-1),i}| \leqslant n+1$, and $X \tau K$ implies $Y_{k-1} \tau K$, because $Y_{k-1} \subset X$. So by Lemma 3.10, we get $\dim_G Y_{k-1} \leqslant n$. Since $P_G = \mathbb{P}$, Lemma 2.4 implies $\dim_{\mathbb{Z}} Y_{k-1} = \dim_G Y_{k-1} \leqslant n$, so we can apply Edwards' Theorem 3.9 to $\mathbf{Y}_{k-1}$, noticing that the first entry in $\mathbf{Y}_{k-1}$ has index $r(k-1)$.

So there exists an $s \in \mathbb{N}$, $s > r(k-1)$ and a map $\hat{g}^s_{r(k-1)} : |K^{(n+1)}_{(k-1),s}| \to |K^{(n)}_{(k-1),r(k-1)}|$ so that if $z \in |K^{(n+1)}_{(k-1),s}|$, and $p^s_{(k-1),r(k-1)}(z)$ lands in the combinatorial interior $\mathring{\sigma}$ of a simplex $\sigma$ of $K^{(n+1)}_{(k-1),r(k-1)}$, then $\hat{g}^s_{r(k-1)}(z)$ lands in $\sigma$. This will help us get the property $(\mathrm{I})_k$.




Define $r(k) := s$. Using the uniform continuity of the map $p^{r(k)}_{(k-1),r(k-1)}$, choose $0 < \delta(k) < 1$ so that $(\mathrm{II})_k$ is true:

$$\forall y \in |K_{(k-1),r(k)}|, \quad \operatorname{diam}\left(p^{r(k)}_{(k-1),r(k-1)}\left(B_{\delta(k)}(y)\right)\right) < \frac{\varepsilon(k-1)}{3}.$$

Pick an intermediate subdivision $\widetilde{\Sigma}_k$ of $K_{(k-1),r(k)}$ so that for any $y \in |K_{(k-1),r(k)}|$, any closed $\widetilde{\Sigma}_k$-vertex star containing $y$ is contained in $B_{\delta(k)}(y)$.

Now choose an $\varepsilon(k)$ so that $0 < \varepsilon(k) < \frac{\delta(k)}{3}$, and so that $(\mathrm{III})_k$ and $(\mathrm{V})_k$ will hold, namely: first make sure that for all $y \in |K_{(k-1),r(k)}|$, the closed $\varepsilon(k)$-ball centered at $y$ sits inside an open $\widetilde{\Sigma}_k$-vertex star, i.e., $B_{\varepsilon(k)}(y) \subset \mathrm{st}(w_0, \widetilde{\Sigma}_k)$, for some $w_0 \in \widetilde{\Sigma}_k^{(0)}$. Therefore $B_{\varepsilon(k)}(y) \subset |\overline{\mathrm{st}}(w_0, \widetilde{\Sigma}_k)| \subset B_{\delta(k)}(y)$. Define $P_{y,k} := |\overline{\mathrm{st}}(w_0, \widetilde{\Sigma}_k)|$, which is a contractible subpolyhedron of $|K_{(k-1),r(k)}|$, so $(\mathrm{V})_k$ is satisfied. Next, we know that for all $j < k$, the maps $p^{r(k)}_{(j),r(j)}$ are uniformly continuous. We also know that, in our notation, $j < k$ implies that $p^{r(k)}_{(j),r(j)} = p^{r(k)}_{(k-1),r(j)}$. So we can make a choice of $\varepsilon(k)$ so that we have: for any $y \in |K_{(k-1),r(k)}|$,

$$\operatorname{diam}\left(p^{r(k)}_{(1),r(1)}\left(B_{\varepsilon(k)}(y)\right)\right) < \frac{\varepsilon(1)}{2^k},$$

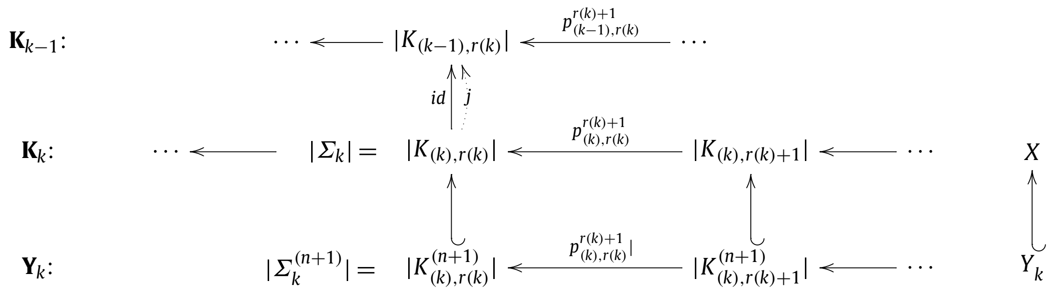
$$\operatorname{diam}\left(p^{r(k)}_{(2),r(2)}\left(B_{\varepsilon(k)}(y)\right)\right) < \frac{\varepsilon(2)}{2^k},$$

$$\vdots$$

$$\operatorname{diam}\big(p^{r(k)}_{(k-1),r(k-1)}\big(B_{\varepsilon(k)}(y)\big)\big) < \frac{\varepsilon(k-1)}{2^k}.$$

So $(\mathrm{III})_k$ is true.

Choose a subdivision $\Sigma_k$ of $\widetilde{\Sigma}_k$ with mesh $\Sigma_k < \gamma_{(k-1),r(k)}$, where $\gamma_{(k-1),r(k)}$ is from the stability sequence $(\gamma_{(k-1),i})$ for $\mathbf{K}_{k-1}$. Also make sure that mesh $\Sigma_k < \frac{\varepsilon(k)}{3}$, which implies $(\mathrm{IV})_k$. Note that $\Sigma_k$ is a subdivision of $K_{(k-1),r(k)}$.



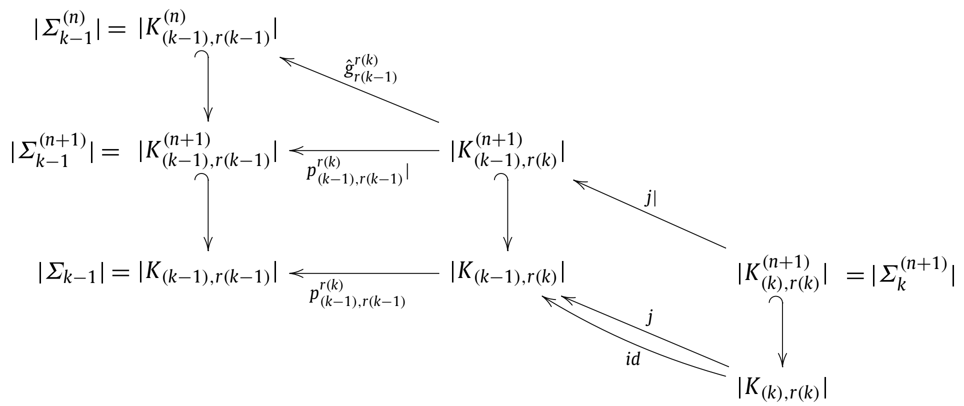

Now we can build $\mathbf{K}_k = (|K_{(k),i}|, p^{i+1}_{(k),i})$ as an $r(k)$-shift of $(\mathbf{K}_{k-1}, (\gamma_{(k-1),i}))$ from $\Sigma_k$, i.e., $\mathbf{K}_k = (|K_{(k),i}|, p^{i+1}_{(k),i})$ is an inverse sequence with $K_{(k),r(k)} = \Sigma_k$ and limit $X$, and stability sequence $(\gamma_{(k),i})$. For index $i \geqslant r(k)$, the bonding maps $p^{i+1}_{(k),i}$ are simplicial.

Let $j : |\Sigma_k| \to |K_{(k-1),r(k)}|$ be a simplicial approximation to the identity map. Since $j$ is simplicial, $j(|\Sigma_k^{(n+1)}|) \subset |K^{(n+1)}_{(k-1),r(k)}|$, so treat $j|_{|\Sigma_k^{(n+1)}|} : |\Sigma_k^{(n+1)}| \to |K^{(n+1)}_{(k-1),r(k)}|$.

Define $g^{r(k)}_{r(k-1)} := \hat{g}^{r(k)}_{r(k-1)} \circ j|_{|\Sigma_k^{(n+1)}|} : |\Sigma_k^{(n+1)}| \to |K^{(n)}_{(k-1),r(k-1)}| = |\Sigma^{(n)}_{k-1}|$. For any $y \in |\Sigma_k^{(n+1)}|$, $y$ and $j(y)$ have to be contained in the same simplex of $K_{(k-1),r(k)}$. Since $p^{r(k)}_{(k-1),r(k-1)} : |K_{(k-1),r(k)}| \to |K_{(k-1),r(k-1)}|$ is simplicial, $p^{r(k)}_{(k-1),r(k-1)}(y)$ and $p^{r(k)}_{(k-1),r(k-1)}(j(y))$ land in the same simplex $\tau$ of $K_{(k-1),r(k-1)} = \Sigma_{k-1}$. On the other hand, because of our choice of $\hat{g}^{r(k)}_{r(k-1)}$, if $p^{r(k)}_{(k-1),r(k-1)}(j(y))$ lands in $\mathring{\sigma}$, for some simplex $\sigma$ of $K^{(n+1)}_{(k-1),r(k-1)}$ which is a face of $\tau$, then $\hat{g}^{r(k)}_{r(k-1)}(j(y))$ lands in $\sigma$, too. Therefore

$$d_{k-1}\big(p^{r(k)}_{(k-1),r(k-1)}(y), \hat{g}^{r(k)}_{r(k-1)}\big(j(y)\big)\big) \leqslant \operatorname{mesh} K_{(k-1),r(k-1)} = \operatorname{mesh} \Sigma_{k-1} < \frac{\varepsilon(k-1)}{3}.$$

Hence $g^{r(k)}_{r(k-1)}$ and $p^{r(k)}_{(k-1),r(k-1)}|_{|\Sigma_k^{(n+1)}|}$ are $\frac{\varepsilon(k-1)}{3}$-close, so $(\mathrm{I})_k$ is true. This concludes the inductive step. The following diagram summarizes the preceding construction.




Notice that the inverse sequence

$$\mathbf{X} := \big(|K_{(0),r(i)}|, p^{r(i+1)}_{(i),r(i)}\big) = \big(|K_{(i),r(i)}|, p^{r(i+1)}_{(i),r(i)}\big) = \big(|\Sigma_i|, p^{r(i+1)}_{(i),r(i)}\big)$$

is a subsequence of $\mathbf{K}_\infty = (|K_{(\infty),i}|, p^{i+1}_{(\infty),i}) = (|K_{(0),i}|, p^{i+1}_{(\infty),i})$. By Lemma 4.11, $\lim \mathbf{K}_\infty = X$, so $\lim \mathbf{X}$ is homeomorphic to $X$. Without loss of generality, assume that $\lim \mathbf{X} = X$.

Let $\mathbf{Z} := (|\Sigma_i^{(n)}|, g^{r(i+1)}_{r(i)}|_{|\Sigma^{(n)}_{i+1}|})$. Since $|\Sigma_i^{(n)}|$ are metrizable, compact and nonempty, $\lim \mathbf{Z} = Z$ is a nonempty compact metrizable space. Clearly, $\dim Z \leqslant n$, which also implies that $\dim_G Z \leqslant n$. Now $Z \tau K$ follows from Lemma 3.10.

Apply Walsh's Lemma 3.3 to these **X** and **Z**: since the requirements (I)–(VI) of Lemma 3.3 are satisfied, there is a cell-like surjective map $\pi : Z \to X$. □

**Corollary 5.1.** *Let $G$ be an abelian group with $P_G = \mathbb{P}$. Let $K$ be a connected* CW*-complex with $\pi_1(K) \cong G$. Then every compact metrizable space $X$ with $X\tau K$ has to have* $\dim X \leqslant 1$.

**Proof.** Theorem 1.1 is true for $n = 1$, so for any compact metrizable space $X$ with $X\tau K$, we can find a compact metrizable space $Z$ with $\dim Z \leqslant 1$, $Z\tau K$ and a cell-like map $\pi : Z \to X$. Note that cell-like maps are always surjective. Also, cell-like maps are $G$-acyclic, so in particular, $\pi$ is a $\mathbb{Z}$-acyclic map.

The Vietoris–Begle Theorem implies that a $G$-acyclic map cannot raise $\dim_G$-dimension. Since $\dim Z \leqslant 1$ implies that $\dim_{\mathbb{Z}} Z \leqslant 1$, and since $\pi$ is a $\mathbb{Z}$-acyclic map, we have that $\dim_{\mathbb{Z}} X \leqslant 1$, too. Recall that $\dim_{\mathbb{Z}} X \leqslant 1 \Leftrightarrow \dim X \leqslant 1$. □

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