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ENUMERATION OF RISES AND FALLS BY POSITION*

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Abstract. Let $\pi = (\pi_1, \pi_2, ..., \pi_n)$ denote a permutation of $Z_n = \{1, 2, ..., n\}$. The pair (π_i, π_{i+1}) is a *rise* if $\pi_i < \pi_{i+1}$ or a *fall* if $\pi_i > \pi_{i+1}$. Also a conventional rise is counted at the beginning of π and a conventional fall at the end. Let k be a fixed integer ≥ 1. The rise π_i, π_{i+1} is said to be in a in a j (mod k) position if $i \equiv j$ (mod k); similarly for a fall. The conventional rise at the beginning is in a 0 (mod k) position, while the conventional fall at the end is in an n (mod k) position.

Let $P_n \equiv P_n(r_0, ..., r_{k-1}, f_0, ..., f_{k-1})$ denote the number of permutations having r_i rises in i (mod k) positions and f_i falls in i (mod k) positions. A generating function for P_n is obtained. In particular, for $k = 2$ the generating function is quite explicit and also, for certain special cases when $k = 4$.

1. Introduction

We consider all permutations $\pi = (\pi_1, \pi_2, ..., \pi_n)$ of $Z_n = \{1, 2, ..., n\}$. The pair (π_i, π_{i+1}) from the permutation π is called a *rise* if $\pi_i < \pi_{i+1}$ or a *fall* if $\pi_i > \pi_{i+1}$. We also agree to count a conventional rise at the beginning of π and a conventional fall at the end. We will say that π contains a *maximum* whenever a rise immediately precedes a fall and that π contains a *minimum* whenever a fall immediately precedes a rise. We remark that the words *peak* and *trough* are sometimes used in place of maximum and minimum.

Let $A(r, s)$ be the number of permutations of Z_{r+s+1} with $r+1$ rises and $s+1$ falls. Then it is known [3; 6, Chapter 8] that

$$(1.1) \qquad \sum_{r,s=0}^{\infty} A(r,s) \frac{x^r y^s}{(r+s+1)!} = \frac{e^x - e^y}{xe^y - ye^x} .$$

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If we let $P(r, s, m)$ be the number of permutations of Z_n having r rises, s falls and m maxima, then we have [3]

$$(1.2)\qquad \sum_{r,s,m} P(r,s,m)\, x^r y^s u^m \frac{z^{r+s-1}}{(r+s-1)!} = x + xyu \frac{e^{Uz}-e^{Vz}}{Ue^{Vz}-Ve^{Uz}},$$

$$U = \tfrac{1}{2}\left(x+y+\sqrt{(x+y)^2-4xyu}\right), V = \tfrac{1}{2}\left(x+y-\sqrt{(x+y)^2-4xyu}\right).$$

For a further refinement see [2].

Let A_n be the number of up-down permutations of Z_n. An up-down permutation is one in which rises and falls occur alternately. It is well known [4, pp. 105–112] that

$$(1.3)\qquad \sum_{n=0}^{\infty} A_n \frac{x^n}{n!} = \tan x + \sec x .$$

(Note that in enumerating up-down permutations, the conventional rise and fall are not counted.) Along this line, one of the authors [1] has found the generating functions for the number of permutations having k initial rises followed by one fall, this pattern continuing for as long as possible.

In this paper, we will discuss another variation. Let k be a fixed integer ≥ 1. If $\pi = (\pi_1, \pi_2, ..., \pi_n)$ is a permutation of Z_n, we will say that the rise (or fall) (π_i, π_{i+1}) is in a j (mod k) *position* if $i \equiv j$ (mod k). We will say that the conventional rise at the beginning is in a 0 (mod k) position and that the conventional fall at the end is in an n (mod k) position. We may say more briefly that a rise in a j (mod k) position is a j (mod k) rise, and similarly for falls.

Let

$$P_n \equiv P_n(r_0, ..., r_{k-1}, f_0, ..., f_{k-1})$$

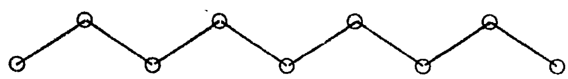
be the number of permutations having r_i rises in i (mod k) positions and f_i falls in i (mod k) positions, $i = 0, 1, ..., k-1$.

For fixed k, we obtain the generating function

$$\sum P_n\, x_0^{r_0} x_1^{r_1} \cdots x_{k-1}^{r_{k-1}} y_0^{f_0} y_1^{f_1} \cdots y_{k-1}^{f_{k-1}} \frac{t^n}{n!}$$

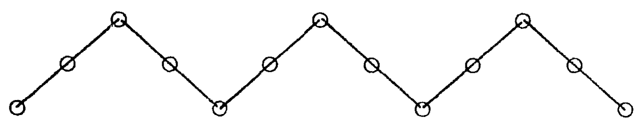
in terms of certain $k \times k$ matrices.

For $k = 2$, we give the generating function quite explicitly, verifying certain special cases obtained previously. For instance, the number A_{2n-1} of up-down permutations of Z_{2n-1} having the form



is obtained in agreement with (1.3).

For $k = 4$, we consider the special case of the number B_{4n-1} of permutations of Z_{4n-1} having the form



We show that

$$(1.4) \qquad \sum_{n=1}^{\infty} B_{4n-1} \frac{t^{4n-1}}{(4n-1)!} = \frac{\phi_1(t)\,\phi_2(t) - \phi_0(t)\,\phi_3(t)}{\phi_0^2(t) - \phi_1(t)\,\phi_3(t)},$$

where

$$(1.5) \qquad \phi_i(t) = \sum_{n=0}^{\infty} \frac{t^{4n+i}}{(4n+i)!} \qquad (i = 0, 1, 2, 3)$$

are the Olivier functions [5].

Finally we estimate B_{4n-1} for large n by finding all zeros of $\cos z \cosh z + 1$. The result is

$$(1.6) \qquad B_{4n-1} \sim 4(4n-1)!\,(2/\gamma)^{4n},$$

where $\gamma = 3.7502 \ldots$. This may be compared with the fact that

$$A_{4n-1} \sim 2(4n-1)!\,(2/\pi)^{4n}.$$

The actual values and the estimates are given in Table 1. We can offer no intuitively plausible argument implying that $A_{4n-1} \geq B_{4n-1}$.

Table 1.

n	A_n	$2(4n-1)!\,(2/\pi)^{4n}$	B_n	$4(4n-1)!\,(2/\gamma)^{4n}$
1	2	1.97 ...	2	1.94
2	272	271.76 ...	132	131.91
3	353792	353791 ...	84512	84460 ...

2. Calculation of the generating function

Suppose $\pi = (\pi_1, \pi_2, ..., \pi_n)$ is a permutation of Z_n. We will say that the rise (or fall) (π_i, π_{i+1}) is in a j (mod k) *position* whenever $i \equiv j$ (mod k). We define the position of the conventional rise at the beginning to be 0 (mod k) and the position of the conventional fall at the end to be n (mod k).

Let $k \geq 1$ be fixed. Let $P_n(r_0, ..., r_{k-1}, f_0, ..., f_{k-1})$ be the number of permutations of Z_n having r_i rises in i (mod k) positions and f_i falls in i (mod k) positions, $i = 0, 1, ..., k-1$.

Let $M(x_0, x_1, ..., x_{k-1})$ denote the $k \times k$ matrix

$$M(x_0, x_1, ..., x_{k-1}) = \begin{pmatrix} 0 & x_0 & 0 & \cdots & & 0 \\ 0 & 0 & x_1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & 0 & x_{k-2} \\ x_{k-1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} . \tag{2.1}$$

We set

$$\begin{aligned} R_0 &= M(x_0, x_1, ..., x_{k-1}) , \\ F_0 &= M(y_0, y_1, ..., y_{k-1}) . \end{aligned} \tag{2.2}$$

To each permutation $\pi = (\pi_1, \pi_2, ..., \pi_n)$ of Z_n we assign the matrix

$$\phi(\pi) = R_0 \cdot A_1 \ldots A_{n-1} \cdot F_0 , \tag{2.3}$$

where $A_i = R_0$ if (π_i, π_{i+1}) is a rise and $A_i = F_0$ if (π_i, π_{i+1}) is a fall.

Theorem 2.1. *The sum of the entries of the first row of $\phi(\pi)$ is*

$$x_0^{r_0} x_1^{r_1} \dots x_{k-1}^{r_{k-1}} \cdot y_0^{f_0} \dots y_{k-1}^{f_{k-1}} ,$$

where r_i (f_i) is the number of rises (falls) of π in position i (mod k).

Proof. It is clear that any product of m R_0's and F_0's is a matrix having only one non-zero entry in each row. Furthermore, the non-zero entry in the first row is in column m (mod k), assuming the columns are numbered $0, 1, 2, \dots, k-1$.

For n fixed, suppose $\phi(\pi)$ is defined by (2.3). Assume that the sum of the entries of the first row of $R_0 A_1, \dots, A_j$ is

$$x_0^{r'_0} x_1^{r'_1} \dots y_{k-1}^{f'_{k-1}},$$

where r'_i (f'_i) is the number of rises (falls) in i (mod k) positions up to the j^{th} interval. This sum lies in the column numbered $j+1$ (mod k). Hence the only non-zero entry in the first row of $R_0 A_1 \dots A_j A_{j+1}$ is

$$x_0^{r'_0} \dots y_k^{f'_{k-1}} \cdot a_{j+1} \quad ,$$

where $a_{j+1} = x_{k+1}$ if $A_{j+1} = R_0$ and $j \equiv h$ (mod k) or $a_{j+1} = y_{h+1}$ if $A_{j+1} = F_0$ and $j \equiv h$ (mod k). Since the assumption is true for $j = 0$, we see by induction that the theorem is true.

Now suppose we have two $k \times k$ matrix-valued functions $R(t)$ and $F(t)$ satisfying

$$(2.4) \qquad \begin{aligned} &R'(t) = F'(t) = R(t)\,F(t) \\ &R(0) = R_0, \quad F(0) = F_0. \end{aligned}$$

Theorem 2.2. *Let $P_n(r_0, r_1, \dots, r_{k-1}, f_0, \dots, f_{k-1})$ be the number of permutations of Z_n having r_i rises and f_i falls in positions i* (mod k) ($i = 0, \dots, k-1$). *Then*

$$\sum_{n=1}^{\infty} \sum_{r_i, f_i} P_n(r_0, r_1, \dots, f_{k-1})\, x_0^{r_0} x_1^{r_1} \dots y_0^{f_0} \dots y_{k-1}^{f_{k-1}}\, t^n/(n!)$$

is the sum of the entries of the first row of the matrix $R(t) - R_0$.

Proof. From Taylor's Theorem and the previous theorem it is enough to show that

$$R^{(n)}(0) = \sum \phi(\pi) ,$$

where the sum is extended over all permutations of Z_n. But this is clear by induction since $R'(t) = F'(t) = R(t)F(t)$: the $(n+1)!$ permutations of Z_{n+1} can be obtained by inserting $n+1$ in any interval of any permutation π of Z_n.

We now find a solution of the equations (2.4). Since $R' = F'$, we have $F = R + U$ with $U = F_0 - R_0$. We get

(2.5) $$R' = R(R+U).$$

Assume a solution of the form

(2.6) $$R = -Q^{-1} Q'.$$

Since $Q \cdot Q^{-1} = I$, we have

$$Q' \cdot Q^{-1} + Q \cdot [Q^{-1}]' = 0,$$

so that

$$[Q^{-1}]' = -Q^{-1} Q' Q^{-1} .$$

Hence (2.5) becomes

$$Q^{-1} Q' Q^{-1} Q' - Q^{-1} Q'' = (Q^{-1} Q')(Q^{-1} Q') - Q^{-1} Q' U,$$

that is,

(2.7) $$Q'' = Q' U .$$

Hence we may take $Q' = e^{tU}$ and

(2.8) $$Q = U^{-1} e^{tU} + K .$$

Hence we have

(2.9) $$R = -(U^{-1} e^{tU} + K)^{-1} e^{tU} ,$$

where K is given by

(2.10) $R_0 = -(U^{-1} + K)^{-1}$.

From (2.9), we see that

$$-UR^{-1} = e^{-tU}(e^{tU} + UK) = 1 + e^{-tU}UK.$$

Since $R_0^{-1} = -(U^{-1} + K)$ and $F_0 - R_0 = U$, we get

$$UR_0^{-1} = F_0 R_0^{-1} - 1 = -(1 + UK),$$

so that

(2.11) $UK = -F_0 R_0^{-1}$.

Hence we have

$$R_0^{-1} R = -(R_0 - e^{-tU} F_0)^{-1} U,$$

so that

$$R_0^{-1}(R - R_0) = (e^{-tU} F_0 - R_0)^{-1}(U - (e^{-tU} F_0 - R_0)),$$

that is,

(2.12) $R_0^{-1}(R - R_0)F_0^{-1} = -(R_0 - e^{-tU} F_0)^{-1}(1 - e^{-tU})$,

or

(2.13) $R - R_0 = R_0 (e^{-tU} F_0 - R_0)^{-1}(1 - e^{-tU}) F_0$.

3. Special cases

If the matrices are one-dimensional, (2.13) becomes, setting $F_0 = y$ and $R_0 = x$,

(3.1) $$\frac{r - x}{xy} = \frac{e^{ty} - e^{tx}}{ye^{tx} - xe^{ty}},$$

the symmetric generating function [3] for the Eulerian polynomials

$$A_n(x,y) = \sum_{r+s=n} A(r,s)\, x^r\, y^s\ .$$

If $k = 2$, formula (2.13) can be given more explicitly if we invert the matrix $e^{-tU} F_0 - R_0$. We have

$$R_0 = \begin{pmatrix} 0 & x_0 \\ x_1 & 0 \end{pmatrix},\quad F_0 = \begin{pmatrix} 0 & y_0 \\ y_1 & 0 \end{pmatrix},\quad U = F_0 - R_0\ .$$

Since

$$e^{-tU} = \cosh = \cosh\tau - (U/\sqrt{\alpha})\sinh\tau\ ,$$

where we have put

$$(3.2)\qquad \alpha = (y_0 - x_0)(y_1 - x_1),\quad \tau = t\sqrt{\alpha}\ ,$$

we have

$$e^{-tU} F_0 - R_0 = \begin{pmatrix} y_1(x_0 - y_0)\alpha^{-\frac{1}{2}}\sinh\tau & y_0\cosh\tau - x_0 \\ y_1\cosh\tau - x_1 & y_0(x_1 - y_1)\alpha^{-\frac{1}{2}}\sinh\tau \end{pmatrix},$$

$$D = \det(e^{-tU}F_0 - R_0) = -x_0x_1 + x_0y_1\cosh\tau + x_1y_0\cosh\tau - y_0y_1$$

and

$$R_0(e^{-tU}F_0 - R_0)^{-1} = \frac{1}{D}\begin{pmatrix} x_0(x_1 - y_1\cosh\tau) & x_0y_1(x_0 - y_0)\alpha^{-\frac{1}{2}}\sinh\tau \\ x_1y_0(x_1 - y_1)\alpha^{-\frac{1}{2}}\sinh\tau & x_1(x_0 - y_0\cosh\tau) \end{pmatrix}.$$

Also

$$(e^{-tU} - 1)F_0 = \begin{pmatrix} y_1(x_0 - y_0)\alpha^{-\frac{1}{2}}\sinh\tau & y_0\cosh\tau - y_0 \\ y_1\cosh\tau - y_1 & y_0(x_1 - y_1)\alpha^{-\frac{1}{2}}\sinh\tau \end{pmatrix}.$$

Hence for the sum of the entries of the first row of $R-R_0$ we get

$$(y_1-x_1)x_0y_1(x_0-y_0)\alpha^{-\frac{1}{2}}\sinh\tau - x_0y_0(x_1+y_1)(\cosh\tau-1).$$

We state the result as a theorem.

Theorem 3.1. *The generating function for the number $P_n(r_0, r_1, f_0, f_1)$ of permutations of Z_n having r_0 even rises, r_1 odd rises, f_0 even falls and f_1 odd falls is given by*

$$(3.3)\qquad \sum_{n\geq 1} P_n(r_0, r_1, f_0, f_1)\, x_0^{r_0} x_1^{r_1} y_0^{f_0} y_1^{f_1}\, t^n/n!$$

$$= x_0y_1 S/E + x_0y_0(x_1+y_1)C/E,$$

where $r_0+r_1+f_0+f_1 = n+1$ and

$$(3.4)\qquad \begin{aligned} S &= \frac{t}{1!}+\frac{\alpha t^3}{3!}+\frac{\alpha^2 t^5}{5!}+\dots, \qquad C=\frac{t^2}{2!}+\frac{\alpha t^4}{4!}+\frac{\alpha^2 t^6}{6!}+\dots,\\ E &= 1-(x_0y_1+x_1y_0)C, \qquad \alpha=(y_0-x_0)(y_1-x_1). \end{aligned}$$

The first few terms are

$$(3.5)\qquad \frac{t}{1!}(x_0y_1)+\frac{t^2}{2!}(x_0x_0y_0+x_0y_1y_0)$$

$$+\frac{t^3}{3!}(x_0y_1)(x_0x_1+2x_0y_1+2x_1y_0+y_0y_1)$$

$$+\frac{t^4}{4!}(x_0y_0)(x_0x_1^2+6x_0x_1y_1+5x_1^2y_0+5x_0y_1^2+6x_1y_0y_1+y_0y_1^2)$$

$$+\frac{t^5}{5!}(x_0y_1)(x_0^2x_1^2+13x_0^2x_1+13x_0x_1^2y_0+16x_0^2y_1^2+34x_0x_1y_0y_1$$

$$+16x_1^2y_0^2+13x_0y_0y_1^2+13x_1y_0^2y_1+y_0^2y_1^2)+\dots .$$

It is evident from (3.3) and (3.5) that the numbers $P_n(r_0, r_1, f_0, f_1)$ furnish a refinement of the Eulerian numbers $A(r, s)$. This is somewhat clearer in the following tabular form:

1				
1	1			
1	2 + 2	1		
1	6 + 5	5 + 6	1	
1	13 + 13	16 + 34 + 16	13 + 13	1

We observe that if we interchange the x_i and y_i, (3.3) becomes

$$\sum_{n\geq 1} P_n(f_0, f_1, r_0, r_1)x_0^{r_0} x_1^{r_1} y_0^{f_0} y_1^{f_1} t^n/n!$$
$$= x_1 y_0 S/E + x_0 y_0 (x_1 + y_1) C/E .$$

It follows that

(3.6) $$P_n(f_0, f_1, r_0, r_1) = P_n(r_0, r_1, f_0, f_1) \qquad (n \text{ even}),$$

while

(3.7) $$P_n(r_0+1, r_1, f_0, f_1+1) = P_n(f_0+1, f_1, r_0, r_1+1) \quad (n \text{ odd}) .$$

If we put

$$P_n(r_0, r_1 | f) = \sum_{f_0+f_1=f} P_n(r_0, r_1 | f_0, f_1) ,$$

$$P_n(r | f_0, f_1) = \sum_{r_0+r_1=r} P_n(r_0, r_1 | f_0, f_1) ,$$

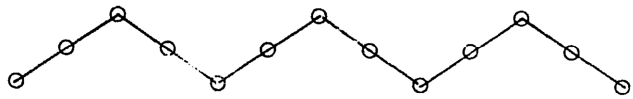
we get

(3.8) $$P_n(r_0, r_1 | f) = P_n(f | r_0, r_1) \qquad (n \text{ even}) ,$$
$$P_n(r_0 + 1, r_1 | f) = P_n(f | r_0, r_1 + 1) \quad (n \text{ odd}) .$$

The formulas (3.6), (3.7) can also be proved by a simple combinatorial argument, namely with each permutation $\pi = (a_1 a_2 \dots a_n)$ associate the complementary permutation $\pi' = (b_1 b_2 \dots b_n)$, where

$$b_i = n - a_i + 1 \qquad (i = 1, \ldots, n) .$$

As another special case we enumerate those permutations of Z_{4n-1} of the form

(3.9)  .

We have

(3.10)

$$R_0 = x\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F_0 = x\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$-U = xP, \qquad P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

We get

(3.11) $$e^{-U} = \sum_{i=0}^{3} \sum_{n=0}^{\infty} \frac{(xP)^{4n+i}}{(4n+i)!} = \sum_{i=0}^{3} P^i \phi_i(x) ,$$

where the ϕ_i are the Olivier functions defined by

(3.12) $$\phi_i(x) = \sum_{n=0}^{\infty} \frac{x^{4n+i}}{(4n+i)!} \qquad (i = 0, 1, 2, 3).$$

Then we get

(3.13) $$e^{-U}F_0 - R_0 = x\begin{pmatrix} -\phi_3 & -1 & 0 & \phi_2 \\ -\phi_2 & 0 & 0 & \phi_1 \\ -\phi_1 & 0 & 0 & \phi_0 \\ \phi_0 & 0 & 0 & -\phi_3 \end{pmatrix},$$

(3.14) $$D = \det(e^{-U}F_0 - R_0) = x^4(\phi_1\phi_3 - \phi_0^2) .$$

Now we can find (M_{ij}) so that

$$(3.15)\qquad (\mathrm{e}^{-U}F_0 - R_0)^{-1} = \frac{x^3}{x^4(\phi_1\phi_3 - \phi_0^2)}(M_{ij}) .$$

Note that

$$(3.16)\qquad M_{21}(-\phi_3)+M_{22}(-\phi_2)+M_{23}(-\phi_1)+M_{24}(\phi_0) = 0.$$

Furthermore, we have

$$(3.17)\qquad (\mathrm{e}^{-U}-1)F_0 = x\begin{pmatrix} -\phi_3 & 0 & 0 & \phi_2 \\ -\phi_2 & 0 & 0 & \phi_1 \\ -\phi_1 & 0 & 0 & \phi_0 - 1 \\ \phi_0 - 1 & 0 & 0 & -\phi_3 \end{pmatrix}.$$

Hence from (2.13), (3.15) and (3.17) we get

$$(3.18)\qquad R-R_0 = \frac{x^3}{D}\begin{pmatrix} M_{21} & M_{22} & M_{23} & M_{24} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} -\phi_3 & \cdot & \cdot & \cdot \\ -\phi_2 & \cdot & \cdot & \cdot \\ -\phi_1 & \cdot & \cdot & \cdot \\ -\phi_0 - 1 & \cdot & \cdot & \cdot \end{pmatrix}.$$

so by (3.16) the entry in the first row and first column of $R-R_0$ is

$$-M_{24} = \det\begin{pmatrix} -\phi_3 & 0 & \phi_2 \\ -\phi_2 & -1 & \phi_1 \\ -\phi_1 & 0 & \phi_0 \end{pmatrix} = \phi_1\phi_2 - \phi_0\phi_3 .$$

Altogether, then, we get

Theorem 3.2. *The number B_{4n-1} of up-up-down-down permutations of Z_{4n-1} satisfies*

$$\sum_{n=1}^{\infty} B_{4n-1}\frac{x^{4n-1}}{(4n-1)!} = \frac{\phi_1(x)\phi_2(x) - \phi_0(x)\phi_3(x)}{\phi_0^2(x) - \phi_1(x)\phi_2(x)}$$

$$= \frac{2}{3!}x^3 + \frac{132}{7!}x^7 + \frac{84512}{11!}x^{11} + \dots .$$

Next we estimate B_{4n-1} for large n. We note that ϕ_0, ϕ_1, ϕ_2 and ϕ_3 are four linearly independent solutions of $f^{(4)} = f$. Let $w = (1+i)/\sqrt{2}$. Then $\phi_i(w\sqrt{2}\,x)$ $(i = 0, 1, 2, 3)$ are four linearly independent solutions of

$$\text{(3.19)} \qquad f^{(4)} = -4f\,.$$

Since $\phi_0\phi_3 - \phi_1\phi_2 = (\phi_0^2 - \phi_1\phi_3)'$ is also a solution of (3.19), and since $\phi_0(0) = 1$, we see that

$$\text{(3.20)} \qquad \phi_0^2(z) - \phi_1(z)\phi_3(z) = \tfrac{1}{2}[\phi_0(w\sqrt{2}z) + 1]\,.$$

Let

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is easily verified that

$$\text{(3.21)} \qquad e^{aQ} = \phi_0(a) + \phi_1(a)Q + \phi_2(a)Q^2 + \phi_3(a)Q^3\,,$$

so from $e^{(a+b)Q} = e^{aQ}e^{bQ}$ we get

$$\text{(3.22)} \qquad \phi_0(a+b) = \phi_0(a)\phi_0(b) + \phi_1(a)\phi_3(b) + \phi_2(a)\phi_2(b) + \phi_3(a)\phi_1(b).$$

Setting $a + b = 0$ we have

$$\text{(3.23)} \qquad \phi_0^2 - 2\phi_1\phi_3 + \phi_2^2 = 1.$$

Hence from (3.20) we get

$$\text{(3.24)} \qquad \cosh z \cos z = \phi_0^2(z) - \phi_2^2(z) = \phi_0(w\sqrt{2}\,z)\,.$$

Theorem 3.3. *Let* $0 < x_1 < x_2 < x_3 < \ldots$ *denote the positive solutions of* $\cos x \cosh x + 1 = 0$. *Then any solution of*

$$\text{(3.25)} \qquad \cos z \cosh z + 1 = 0$$

is of one of the forms $x_n, -x_n, \mathrm{i}x_n, -\mathrm{i}x_n$. *Furthermore, if*

(3.26) $$f(x) = \frac{\phi_1(x)\phi_2(x) - \phi_0(x)\phi_3(x)}{\phi_0^2(x) - \phi_1(x)\phi_2(x)} = \sum_{n=1} B_{4n-1} \frac{x^{4n-1}}{(4n-1)!},$$

then

(3.27) $$B_{4n-1} = 4(4n-1)! \sum_{j=1}^{\infty} x_j^{-4n} .$$

Proof. First it is clear that x_n, $-x_n$, ix_n, $-ix_n$ are solutions of (3.22) and that $x_n - \frac{1}{2}(2n+1)\pi \to 0$. On the square determined by the lines $x = \pm 2n\pi$, $y = \pm 2n\pi$, we see that

$$|\cos z|^2 = \cosh^2 y - \sin^2 x \geq 1 .$$

(As a matter of fact, $|\cos z| \geq 1$ for $|z| = 2n\pi$, but this is not needed here.) Thus we have

$$1 = |\cos z \cosh z + 1 - \cos z \cosh z| \geq |\cos z \cosh z| .$$

By Rouché's theorem [7, p. 116], $\cos z \cosh z + 1$ has the same number of zeros in the square as $\cos z \cosh z$, namely $8n$. Hence we have accounted for all of them.

As for the second part of the theorem, we note that the denominator of f is entire of order 1 and that $-f$ is a logarithmic derivative. Hence from Hadamard's factorization theorem [7, p. 250] we obtain the result.

If we take only the first term in the sum (3.27), we get

(3.28) $$B_{4n-1} \sim 4(4n-1)!\, x_1^{-4n} .$$

As pointed out by the referee, it follows from (3.26) that B_{4n-1} satisfies the following recurrence:

(3.29) $$\beta_{n+1} + \sum_{j=0}^{n} \binom{4n+3}{4j} \beta_j B_{4n-4j+3} = 0,$$

with

$$\beta_n = (-1)^n\, 2^{2n-1} + \tfrac{1}{2}\,\delta_{n0}\,,$$

where δ_{n0} is the Kronecker delta.

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