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(19, 9, 4) Hadamard Designs and Their Residual Designs

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In this paper we study (19, 9, 4) Hadamard designs and their residual designs. We prove that there are precisely six non-isomorphic solutions of (19, 9, 4) designs and that these six designs give rise to in all twenty-one mutually non-isomorphic residual designs.

1. INTRODUCTION

A *balanced incomplete block design* (BIBD) is an arrangement of v symbols called treatments in b subsets called blocks of size $k < v$ such that any two distinct treatments occur together in λ blocks. It then follows that each treatment occurs in r blocks and the following relations are satisfied:

$$vr = bk,$$

$$\lambda(v - 1) = r(k - 1).$$

Besides these necessary conditions we also have the inequality

$$b \geqslant v,$$

which is due to Fisher. We shall use the term "design" generally to indicate a BIBD. By a (v, b, r, k, λ) design we will mean a BIBD with these parameters. By a *symmetric* BIBD (SBIBD) we mean a BIBD with $b = v$ and hence $r = k$. We shall call such a design a (v, k, λ) design.

Two BIBD's D_1 and D_2 are said to be *isomorphic* if there exists a bijection of the set of treatments of D_1 onto the set of treatments of D_2 such that under this bijection the set of blocks of D_1 is mapped onto the set of blocks of D_2. Otherwise, they are said to be *non-isomorphic*. If D_1 and D_2 are isomorphic, we will write $D_1 \simeq D_2$.

It is well known that the existence of a (v, k, λ) design implies the existence of its residual designs which are $(v - k, v - 1, k, k - \lambda, \lambda)$

designs and its derived designs which are $(k, v-1, k-1, \lambda, \lambda-1)$ designs. They are obtained, respectively, by omitting a block of the (v, k, λ) design called the *initial block*, and retaining in the remaining blocks only those treatments which do not (do) occur in the initial block.

If treatments of a (v, b, r, k, λ) design D are $a_1, a_2, ..., a_v$ and blocks are $B_1, B_2, ..., B_b$, we define the usual incidence matric $N = (n_{ij})$ of D, by

$$n_{ij} = \begin{cases} 1, & \text{if} \quad a_j \in B_i, \\ 0, & \text{if} \quad a_j \notin B_i. \end{cases}$$

Obviously N is a (0, 1) matrix of order $b \times v$ and, if N' is the transpose of N, then $N'N = (r - \lambda)I + \lambda J$, where I is the identity matrix of order v and J is the square matrix of order v with all elements 1.

Two BIBD's D_1 and D_2 will then be isomorphic if and only if the corresponding incidence matrices N_1 and N_2 are such that each can be obtained from the other by a suitable permutation of its rows and columns.

Corresponding to any design D with its incidence matrix N, there exists the *complementary design* $\bar{D}$ with incidence matrix $\bar{N}$, which is obtained by interchanging 0 and 1 in N. If D is a (v, b, r, k, λ) design, then obviously $\bar{D}$ is a $(v, b, b-r, v-k, b-2r+\lambda)$ design.

For a (v, k, λ) design D with incidence matrix N it is known that the *dual* configuration D' of v treatments and v blocks with incidence matrix N' is again a (v, k, λ) design. These two designs in general are not isomorphic. We will call a (v, k, λ) design *self-dual* if it is isomorphic to its dual.

A *Hadamard matrix* H of order m is a square matrix of order m with elements ± 1 such that

$$HH' = mI.$$

Clearly, permuting rows and permuting columns of H, or multiplying rows or columns of H by -1, leaves this property unchanged. We consider such matrices as *equivalent*. Since this relation between two Hadamard matrices is an equivalence relation, we call two such matrices as equivalent. Any Hadamard matrix of a given order then belongs to precisely one of the set of mutually exclusive equivalence classes. Given a Hadamard matrix we can obtain another Hadamard matrix (obtained by multiplying suitable rows and columns by -1) which is equivalent to it and whose i-th row and j-th column consist entirely of $+1$'s. We will call such a Hadamard matrix *normalized*. It is known that order m of a Hadamard matrix is 1, 2 or necessarily a multiple of 4. It is well known that construction of a normalized Hadamard matrix H of order $4t + 4$ is equivalent to construction of a $(4t + 3, 2t + 1, t)$ design. In fact the matrix obtained from H, by deleting the normalized row and column of H and replacing -1 by 0, will be the incidence matrix of such a design. But it is to be noted that a given Hadamard matrix of order $4t + 4$ may give rise to several non-isomorphic designs with parameters $(4t + 3, 2t + 1, t)$ as a Hadamard matrix can be normalized in many ways. A $(4t + 3, 2t + 1, t)$ design is called a *Hadamard design.*

M. Hall [3] has proved that there are exactly three distinct classes of Hadamard matrices of order 20, if equivalent matrices are considered to be in the same *class*. He labeled them as class Q, class P, and class N. Class Q contains the matrix derivable from the quadratic residues modulo 19. Class P contains the matrix that Paley constructed from GF(9). The class N is a new class due to M. Hall. Using Hadamard matrices which are representatives of these classes V. N. Bhat [1] has constructed six non-isomorphic (19, 9, 4) designs.

In this paper we will prove that these are the only non-isomorphic solutions of (19, 9, 4) designs. We will also prove that all these six designs are self-dual and that they give rise to in all twenty-one mutually non-isomorphic residual designs.

2. Hadamard Designs

In this section we will prove some results on the Hadamard designs obtained from Hadamard matrices belonging to the same class.

Let H be a Hadamard matrix of order $4t + 4$. We will denote by $D_{i,j}(H)$ the $(4t + 3, 2t + 1, t)$ design obtained from H by normalizing the i-th row and j-th column. We will denote the block (treatment) of $D_{i,j}(H)$ corresponding to row (column) k of H by $B^k_{i,j}(b^k_{i,j})$, $k \neq i$ $(k \neq j)$. Suppose $D = D_{i,j}(H)$, then by $R_k(D)$ we will denote the residual design of D obtained from D with $B^k_{i,j}$ as the initial block. We now prove some results which will be used in the next two sections:

LEMMA 1. *If H is a Hadamard matrix of order $4t + 4$ and H_1 is the matrix obtained from H by multiplying i-th row and j-th column of H by -1 then $D_{l,k}(H) = D_{l,k}(H_1)$ for every $1 \leqslant l, k \leqslant 4t + 4$.*

We omit the proof as it is a trivial consequence of definitions.

LEMMA 2. *If H is a Hadamard matrix of order $4t + 4$ and H_1 is the matrix obtained from H by applying a permutation σ to rows of H and permutation τ to columns of H, then $D_{i,j}(H) \simeq D_{\sigma(i),\tau(j)}(H_1)$.*

Proof. The map f which takes the treatment $b^k_{i,j}$ of $D_{i,j}(H)$ to the treatment $b^{\tau(k)}_{\sigma(i),\tau(j)}$ of $D_{\sigma(i),\tau(j)}(H_1)$ is an isomorphism of $D_{i,j}(H)$ onto $D_{\sigma(i),\tau(j)}(H_1)$. In fact under this isomorphism the block $B^k_{i,j}$ of $D_{i,j}(H)$ is mapped onto the block $B^{\sigma(k)}_{\sigma(i),\tau(j)}$ of $D_{\sigma(i),\tau(j)}(H_1)$.

Let H_1 and H_2 be two Hadamard matrices belonging to the same class. By an *equivalence*, of H_1 to H_2 we mean a permutation and possibly sign changes of rows of H_1 followed by a permutation and possibly sign changes of columns, which take H_1 to H_2. In particular if $H_1 = H_2$, we will call an equivalence of H_1 to itself an *automorphism*, as in [3]. An easy consequence of Lemma 1 and Lemma 2 is the following:

LEMMA 3. *If f is an equivalence of Hadamard matrices H_1 and H_2 of order n which takes row i_1 of H_1 to row i_2 of H_2 and column j_1 of H_1 to column j_2 of H_2, then f defines an isomorphism of $D_{i_1,j_1}(H_1)$ to $D_{i_2,j_2}(H_2)$.*

Remark 1. If H is a Hadamard matrix of order $4t+4$ and $D = D_{i,j}(H)$ then the map taking treatment $B^k_{i,j}$ of D' to treatment $b^k_{j,i}$ of $D_{j,i}(H')$ is an isomorphism of D' onto $D_{j,i}(H')$. Thus $D'_{i,j}(H) \simeq D_{j,i}(H')$.

LEMMA 4. *If f is an automorphism of a Hadamard matrix H of order n which takes row i and column j onto themselves, row i_1 to row i_2 and column j_1 to column j_2, $i \neq i_1$, $j \neq j_1$, then, if $D = D_{i,j}(H)$,*

(i) $R_{i_1}(D) \simeq R_{i_2}(D)$,

(ii) $R_{j_1}(D') \simeq R_{j_2}(D')$.

Proof. Using Lemma 3, f defines an automorphism of D. Further under this automorphism the image of the block $B^{i_1}_{i,j}$ will be the block $B^{i_2}_{i,j}$, implying that $R_{i_1}(D) \simeq R_{i_2}(D)$. This proves the first part of the lemma. The second part follows from Remark 1 and the first part.

In [1] it has been shown that if H is a Hadamard matrix of order $4t+4$ then we can obtain $D_{i',j}(H)$ from $D_{i,j}(H)$ by the process of "natural embedding of the complement of a residual design," which is as follows: Let

$$\begin{pmatrix} \underline{1} & \underline{0} \\ P & Q \end{pmatrix}$$

be the incidence matrix of $D_{i,j}(H)$ where the first row corresponds to the block $B^{i'}_{i,j}$ and $\underline{1}$, $\underline{0}$ are row vectors consisting of all 1's and 0's, respectively. P and Q are then the incidence matrices of the derived and the residual design of $D_{i,j}(H)$ with $B^{i'}_{i,j}$ as the initial block. It now follows that

$$\begin{pmatrix} \underline{1} & \underline{0} \\ P & \bar{Q} \end{pmatrix}$$

is the incidence matrix of $D_{i',j}(H)$ where the first row corresponds to the block $B^{i}_{i',j}$. Moreover, we can obtain $D_{i',j'}(H)$ from $D_{i,j}(H)$ in the following manner:

(a) Obtain $D_{i',j}(H)$ from $D_{i,j}(H)$ by the natural embedding of the complement of the residual design of $D_{i,j}(H)$ with $B^{i'}_{i,j}$ as the initial block.

(b) Take the dual $D'_{i',j}(H)$ of $D_{i',j}(H)$.

(c) Obtain $D'_{i',j'}(H)$ from $D'_{i',j}(H)$ by the natural embedding of the complement of the residual design of $D'_{i',j}(H)$ with $b^{j'}_{i',j}$ as the initial block.

(d) Take the dual of $D'_{i',j'}(H)$ to get $D_{i',j'}(H)$.

We now prove the final result of this section.

PROPOSITION 1. *If H is a Hadamard matrix of order n such that, for some fixed i_0, $1 \leqslant i_0 \leqslant n$, the designs $D_{i_0,j}(H)$, $1 \leqslant j \leqslant n$ are all self-dual, then given i', j', $1 \leqslant i', j' \leqslant n$ there exists j, $1 \leqslant j \leqslant n$ such that $D_{i',j'}(H)$ and its dual both are isomorphic to $D_{i_0,j}(H)$.*

Similarly, if $D_{i,j_0}(H)$, $1 \leqslant i \leqslant n$ are all self-dual for some fixed j_0, $1 \leqslant j_0 \leqslant n$, then given i', j', $1 \leqslant i', j' \leqslant n$, there exists i, $1 \leqslant i \leqslant n$, such that $D_{i',j'}(H)$ and its dual both are isomorphic to $D_{i,j_0}(H)$.

Proof. If $i' = i_0$ there is nothing to prove. Suppose $i' \neq i_0$. Then we can obtain $D_{i',j'}(H)$ from $D_{i_0,j'}(H)$ by the natural embedding of the complement of the residual design of $D_{i_0,j'}(H)$ with $B^{i'}_{i_0,j'}$ as initial block. Now $D_{i_0,j'}(H)$ is self-dual. Suppose f is an isomorphism of $D_{i_0,j'}(H)$ to $D'_{i_0,j'}(H)$. Let the image of $B^{i'}_{i_0,j'}$ under f be $b^{k}_{i_0,j'}$. Then clearly the designs obtained from $D_{i_0,j'}(H)$ and $D'_{i_0,j'}(H)$ by the natural embedding of complement of residual designs, respectively, with $B^{i'}_{i_0,j'}$ and $b^{k}_{i_0,j'}$ as initial blocks will be isomorphic. Hence we have $D_{i',j'}(H) \simeq D'_{i_0,k}(H)$. But $D'_{i_0,k}(H) \simeq D_{i\cdot,k}(H)$. Therefore

$$D_{i',j'}(H) \simeq D_{i_0,k}(H) \qquad \text{and} \qquad D'_{i',j'}(H) \simeq D_{i_0,k}(H).$$

This proves the first part of the proposition.

Proof of the second part is similar.

3. (19, 9, 4) DESIGNS

M. Hall [3] has constructed the automorphism groups of the three classes Q, P, and N of Hadamard matrices of order 20. In this section we will use these automorphisms and Lemma 3 to get isomorphisms of (19, 9, 4) designs. We will use the terminology and notation of [3]. We will assume throughout that the class Q is represented by (C.2), the class P is represented by (A.3) and the class N is represented by (A.1) as given in [3]. We will denote these matrices themselves by Q, P, and N, respectively.

PROPOSITION 2. *Each of the matrices Q, P, and N give at most two non-isomorphic* (19, 9, 4) *designs. Moreover all these designs are self-dual.*

Proof. We first prove that Q gives at most two non-isomorphic (19, 9, 4) designs.

The automorphism σ [3, page 28] takes row 1 of Q onto itself and permutes columns in the following cycles:

$$(1, 17, 6, 7, 10, 5, 11, 20, 3, 12, 18, 9, 4, 14, 15, 13, 16, 2, 19)(8).$$

Hence using Lemma 3 it follows that the designs $D_{1,j}(Q)$, $1 \leqslant j \leqslant 20$, $j \neq 8$, are all mutually isomorphic.

In [3, page 36] it is proved that the matrix Q_1 obtained from Q by applying the permutation α_1 to rows of Q is symmetric where

$$\alpha_1 = \begin{pmatrix} 1, 2, 3, \quad 4, 5, 6, 7, 8, \quad 9, \quad 10, \quad 11, \quad 12, 13, 14, \\ \qquad\qquad\qquad\qquad\qquad 15, 16, 17, 18, 19, 20 \\ -16, 1, 6, -11, 4, 3, 2, 5, -8, -9, -7, -10, 14, 15, \\ \qquad\qquad\qquad\qquad\qquad 13, 12, 20, 18, 17, 19 \end{pmatrix}$$

(and negative sign indicates sign change). Now the automorphism $\sigma^{-7}\tau^{-1}\sigma^{-1}\rho^{-1}$ of Q takes row 1 of Q to row 9 and column 8 to column 16. Hence, using Lemma 3, $D_{1,8}(Q) \simeq D_{9,16}(Q)$. Also, using α_1, it follows that $D_{9,16}(Q) \simeq D_{8,16}(Q_1)$. Again, $Q_1{}'$ can be obtained from Q' by applying α_1 to columns of Q'. Hence, using Lemma 3, we have

$$D_{8,1}(Q') \simeq D_{8,16}(Q_1{}') = D_{8,16}(Q_1),$$

since Q_1 is symmetric. Therefore by Remark 1, $D_{1,8}(Q) \simeq D'_{1,8}(Q)$. Thus the design $D_{1,8}(Q)$ is self-dual. By similar argument, since using α_1 we have $D_{1,16}(Q) \simeq D_{16,16}(Q_1)$ and $D_{16,1}(Q') \simeq D_{16,16}(Q_1{}') = D_{16,16}(Q_1)$, the design $D_{1,16}(Q)$ is self-dual. Hence, by Proposition 1, it follows that $D_{i,j}(Q)$, $1 \leqslant i, j \leqslant 20$, is isomorphic to either $D_{1,16}(Q)$ or $D_{1,8}(Q)$. Consequently we have established that Q gives atmost two non-isomorphic (19, 9, 4) designs.

Proofs for matrices P and N are similar. For example in case of P, using automorphisms β_p and γ_p [3, Page 33], we can prove that the designs $D_{1,i}(P)$, $1 \leqslant i \leqslant 20$, $i \neq 2, 16$, are all mutually isomorphic while $D_{1,2}(P) \simeq D_{1,16}(P)$. Also, using automorphism $\beta_p\alpha_p\gamma_p{}^5\delta_p$ and the symmetric matrix P_1, we can show that the designs $D_{1,2}(P)$ and $D_{1,9}(P)$ are self-dual where P_1 is the matrix obtained from P by applying the following permutation α_2 to rows of P [3, page 37]:

$$\alpha_2 = \begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\ 2, & 3, & 4, & -5, & 16, & -11, & -6, & -1, & -19, & -20, & 18, & 17, & -13, & -12, & 14, & 15, & -8, & -7, & 9, & 10 \end{pmatrix}$$

In case of N automorphisms γ_N and α_N [3, page 35] show that the designs $D_{1,i}(N)$, $1 \leqslant i \leqslant 20$, $i \neq 1, 6, 11, 16$, are all mutually isomorphic and $D_{1,1}(N)$, $D_{1,6}(N)$, $D_{1,11}(N)$, and $D_{1,16}(N)$ are also mutually isomorphic. Again using the automorphism $\gamma_N{}^2\beta_N\gamma_N^{-1}\alpha_N^{-1}\gamma_N^{-1}$ and the symmetric matrix N_1 we can show that designs $D_{1,2}(N)$ and $D_{1,11}(N)$ are self-dual, where N_1 is obtained from N by applying the following permutation α_3 to rows of N [3, page 37].

$$\alpha_3 = \begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16, & 17, & 18, & 19, & 20 \\ 2, & 3, & 4, & -5, & 16, & -11, & -6, & -1, & -19, & 20, & 18, & 17, & -12, & -13, & 15, & 14, & -8, & -7, & 9, & 10 \end{pmatrix}.$$

This completes the proof of Proposition 2.

In [1] six non-isomorphic (19, 9, 4) designs, two from each class, have been constructed. Proposition 2 therefore gives the following:

THEOREM 1. *There are exactly six mutually non-isomorphic* (19, 9, 4) *Hadamard designs, two from each class. Moreover all these designs are self-dual.*

4. ISOMORPHISMS OF SOME RESIDUAL DESIGNS OF (19, 9, 4) DESIGNS

We have proved that the designs $D_{1,8}(Q)$ and $D_{1,1}(Q)$ are non-isomorphic. We will denote these designs, respectively, by D_1 and D_2. Likewise we will denote by D_3, D_4 the designs $D_{1,2}(P)$ and $D_{1,1}(P)$ and by D_5, D_6 the designs $D_{1,1}(N)$ and $D_{1,1}(N)$. We will denote by $n(D)$ the number of residual designs of a SBIBD D.

Since $D_i \simeq D_i'$, $1 \leqslant i \leqslant 6$, we have $n(D_i) = n(D_i')$.

Making use of automorphisms of Q and Lemma 4 we will show that among the residual designs of D_1 (of D_2) some are mutually isomorphic. Similar is the case with D_3, D_4 and D_5, D_6. To be precise we prove the following:

LEMMA 5. $n(D_1) = 1$, $n(D_2) \leqslant 3$, $n(D_3) \leqslant 3$, $n(D_4) \leqslant 7$, $n(D_5) \leqslant 3$, $n(D_6) \leqslant 5$.

Proof. Automorphism σ [3, page 28] of Q takes row 1 and column 8 of Q onto themselves and permutes the remaining columns in the cycle

$$(1, 17, 6, 7, 10, 5, 11, 20, 3, 12, 18, 9, 4, 14, 15, 13, 16, 2, 19).$$

Since $D_1' = D'_{1,8}$ we have by Lemma 4 $R_i(D_1')$, $1 \leqslant i \leqslant 20$, $i \neq 8$, are all mutually isomorphic. Therefore $n(D_1) = n(D_1') = 1$. Again $D_2' = D'_{1,1}$. Automorphisms $\sigma^{-1}\tau^{-1}\sigma^{-1}\tau^{-1}\sigma^{-1}$, $\sigma^{-2}\tau^{-1}$, $\tau^{-1}\sigma^{-1}\tau^{-2}$, $\tau^{-1}\sigma^{-2}\tau^{-3}\sigma^{-1}$, $\tau^{-2}\sigma^{-1}\tau^{-3}\sigma^{-4}$, $\tau^{-4}\sigma^{-4}\tau^{-4}$, $\tau^{-3}\sigma^{-5}\tau^{-4}$, $\tau^{-1}\sigma^{-1}\tau^{-1}\sigma^{-1}\tau^{-1}\sigma^{-1}\tau^{-3}$ of the matrix Q keep row 1 and column 1 fixed and take column 2, respectively, to columns 20, 12, 16, 10, 17, 9, 11, 5 and column 3, respectively, to columns 18, 6, 4, 7, 13, 14, 15, 19. Hence by Lemma 4, $R_2(D_2')$, $R_{20}(D_2')$, $R_{12}(D_2')$, $R_{16}(D_2')$, $R_{10}(D_2')$, $R_{11}(D_2')$, $R_9(D_2')$, $R_{17}(D_2')$, $R_5(D_2')$ are all mutually isomorphic and $R_3(D_2')$, $R_{18}(D_2')$, $R_6(D_2')$, $R_4(D_2')$, $R_7(D_2')$, $R_{13}(D_2')$, $R_{14}(D_2')$, $R_{15}(D_2')$, $R_{19}(D_2')$ are all mutually isomorphic.

All these automorphism leave column 8 also fixed. Therefore $n(D_2) = n(D_2') \leqslant 3$.

Similarly from automorphisms of P and N we can get isomorphisms of:

(a) $R_3(D_3')$ with $R_4(D_3')$, $R_5(D_3')$, $R_9(D_3')$, $R_{10}(D_3')$, $R_{14}(D_3')$, $R_{15}(D_3')$, $R_{19}(D_3')$ and $R_{20}(D_3')$;

(b) $R_1(D_3')$ with $R_6(D_3')$, $R_7(D_3')$, $R_8(D_3')$, $R_{11}(D_3')$, $R_{12}(D_3')$, $R_{13}(D_3')$, $R_{17}(D_3')$ and $R_{18}(D_3')$;

(c) $R_3(D_4')$ with $R_4(D_4')$, $R_{19}(D_4')$ and $R_{20}(D_4')$;

(d) $R_6(D_4')$ with $R_{11}(D_4')$, $R_{17}(D_4')$ and $R_{18}(D_4')$;

(e) $R_8(D_4')$ with $R_7(D_4')$, $R_{12}(D_4')$ and $R_{13}(D_4')$;

(f) $R_9(D_4')$ with $R_{10}(D_4')$, $R_{14}(D_4')$ and $R_{15}(D_4')$;

(g) $R_2(D_5')$ with $R_3(D_5')$, $R_4(D_5')$, $R_5(D_5')$, $R_7(D_5')$, $R_8(D_5')$, $R_9(D_5')$, $R_{10}(D_5')$, $R_{12}(D_5')$, $R_{13}(D_5')$, $R_{14}(D_5')$, $R_{15}(D_5')$;

(h) $R_6(D_5')$ with $R_{11}(D_5')$ and $R_{16}(D_5')$;

(i) $R_{17}(D_5')$ with $R_{18}(D_5')$, $R_{19}(D_5')$ and $R_{20}(D_5')$

(j) $R_2(D_6')$ with $R_3(D_6')$, $R_{13}(D_6')$, $R_{14}(D_6')$, $R_{17}(D_6')$, $R_{19}(D_6')$;

(k) $R_4(D_6')$ with $R_5(D_6')$, $R_8(D_6')$, $R_9(D_6')$, $R_{10}(D_6')$, $R_{12}(D_6')$, $R_{15}(D_6')$ and $R_{20}(D_6')$;

(l) $R_1(D_6')$ with $R_6(D_6')$ and $R_{16}(D_6')$.

Hence it follows that $n(D_3) = n(D_3') \leqslant 3$, $n(D_4) = n(D_4') \leqslant 7$, $n(D_5) = n(D_5') \leqslant 3$ and $n(D_6) = n(D_6') \leqslant 5$.

In the next section we will show that actually $n(D_2) = 3$, $n(D_3) = 3$, $n(D_4) = 7$, $n(D_5) = 3$, $n(D_6) = 5$. We will further show that one of the residual designs of D_4 is isomorphic to a residual design of D_6 and that the total number of mutually non-isomorphic (10, 18, 9, 5, 4) residual designs is twenty-one.

5. Non-isomorphisms of Some Residual Designs of (19, 9, 4) Designs

If R is a residual design of a (19, 9, 4) design D, then since, in D, 3 treatments cannot occur together in more than 3 blocks [1, Lemma 1], 3 treatments of R will also not occur together in more than 3 blocks of R. We will call a set of 3 treatments of R occurring in 3 blocks of R *special 3-tuple* of R and corresponding 3 blocks of R *B-triple*. We will denote by $\alpha(R)$ the number of special 3-tuples of R.

We note that the complementary design $\bar{R}$ is a derived design of the complementary design $\bar{D}$ [4]. Hence using Lemma 4 of [1] it can be easily seen that, in $\bar{R}$, 4 treatments cannot together in more than 2 blocks. We will call a set of 4 treatments of $\bar{R}$ occurring in a pair of blocks of $\bar{R}$ a *special 4-tuple* and corresponding pair of blocks a *B-pair* of $\bar{R}$. We will denote the number of special 4-tuple of $\bar{R}$ by $\beta(R)$.

It is obvious that $\alpha(R_a) \neq \alpha(R_b)$ (or $\beta(R_a) \neq \beta(R_b)$) will imply that the residual designs R_a and R_b are non-isomorphic whether they are obtained from the same or different SBIBD's. Even if the values of these numbers are the same analyzing special 3-tuples of R_a, R_b (or the special 4-tuples of $\bar{R}_a$, $\bar{R}_b$), it may still be possible to distinguish R_a and R_b for isomorphism.

Remark 2. It can be checked that our designs D_1, D_2, D_3, D_4, D_5, D_6 are respectively, isomorphic to the designs D_1, D_2, D_3, D_4, D_5, D_6 as given in [1]. Hereafter we will use the designs, the special-tuples and B-triples for them and for the complementary design, as given in [1].

Henceforward $R_j(D_i)$ will denote the residual design of D_i (as given in [1]) with the block numbered j as initial block. It should be noted that the special 3-tuples and the special 4-tuples of the residual designs can be immediately obtained from those of D_i and $\bar{D}_i$ given in [1]. The special 3-tuples of $R_j(D_i)$ will be precisely the special 3-tuples of D_i which do not contain any treatment of block numbered j of D_i, whereas, the special 4-tuples of $\overline{R_j(D_i)}$ will be precisely the special 4-tuples of $\bar{D}_i$ occurring in block numbered j of $\bar{D}_i$.

LEMMA 6. $n(D_2) = 3$, $n(D_3) = 3$, $n(D_4) = 7$, $n(D_5) = 3$, $n(D_6) = 5$.

Proof. We have already shown that $n(D_2) \leqslant 3$, $n(D_3) \leqslant 3$, $n(D_4) \leqslant 7$, $n(D_5) \leqslant 3$, $n(D_6) \leqslant 5$.

Consider the residual designs $R_1(D_2)$, $R_2(D_2)$ and $R_4(D_2)$. We show that they are mutually non-isomorphic. We have $\beta(R_1(D_2)) = 0$, $\beta(R_2(D_2)) = \beta(R_4(D_2)) = 2$. But then the treatment 18 is the only treatment of $\overline{R_2(D_2)}$, which occurs in both the special 4-tuple of $\overline{R_2(D_2)}$ while in the case of $\overline{R_2(D_4)}$ the treatment 5 and 17 are common to the two 4-tuples. Consequently $R_2(D_2)$ and $R_4(D_2)$ are non-isomorphic and $n(D_2) = 3$.

Consider $R_3(D_4)$ and $R_9(D_4)$. We have $\alpha(R_3(D_4)) = \alpha(R_9(D_4)) = 2$; $\beta(R_3(D_4) = \beta(R_9(D_4) = 1$. The special 3-tuples and the corresponding special B-triples are as shown below:

Design	Special 3-tuples	Special B-triples
$R_9(D_4)$	8, 10, 12	3, 14, 15
	8, 11, 13	1, 8, 15
$R_3(D_4)$	2, 11, 19	2, 12, 16
	2, 13, 17	2, 10, 18

Here the earlier argument does not distinguish $R_9(D_4)$ and $R_3(D_4)$. However, we observe that the blocks numbered 3 and 1 of $R_9(D_4)$ have three treatments in common but none of the four pair of blocks (12, 10), (12, 18), (16, 10), (16, 18) of $R_3(D_4)$ has three treatments in common. Hence $R_9(D_4)$ and $R_3(D_4)$ are non-isomorphic.

With such type of analysis it can be shown that

(a) $R_1(D_3)$, $R_2(D_3)$, $R_3(D_3)$ are mutually non-isomorphic and hence $n(D_3) = 3$.

(b) $R_1(D_4)$, $R_2(D_4)$, $R_3(D_4)$, $R_4(D_4)$, $R_9(D_4)$, $R_{10}(D_4)$, and $R_{13}(D_4)$ are mutually non-isomorphic and hence $n(D_4) = 7$.

(c) $R_1(D_5)$, $R_4(D_5)$, $R_8(D_5)$, are mutually non-isomorphic and hence $n(D_5) = 3$.

(d) $R_1(D_6)$, $R_3(D_6)$, $R_6(D_6)$, $R_{11}(D_6)$, and $R_4(D_6)$ are mutually non-isomorphic and hence $n(D_6) = 5$.

Using these techniques it can also be shown that the twenty-one designs

$$R_1(D_1),\ R_1(D_2),\ R_2(D_2),\ R_4(D_2),\ R_1(D_3),\ R_2(D_3),\ R_3(D_3),\ R_1(D_4)$$
$$R_2(D_4),\ R_3(D_4),\ R_4(D_4),\ R_9(D_4),\ R_{10}(D_4),\ R_{13}(D_4),\ R_1(D_5)$$
$$R_4(D_5),\ R_8(D_5),\ R_3(D_6),\ R_6(D_6),\ R_{11}(D_6) \text{ and } R_4(D_6)$$

are all mutually non-isomorphic. However $R_1(D_6)$ is isomorphic to $R_{13}(D_4)$.

In fact the following map of treatments of $R_{13}(D_4)$ onto those of $R_1(D_6)$ will give an isomorphism of these designs:

$3 \to 10$	$10 \to 19$	$14 \to 16$	$19 \to 12$
$4 \to 17$	$11 \to 11$	$15 \to 13$	
$5 \to 14$	$13 \to 15$	$18 \to 18$	

We have thus proved the following:

THEOREM 2. *The six mutually non-isomorphic designs D_i, $1 \leqslant i \leqslant 6$, give, in all, twenty-one mutually non-isomorphic residual designs.*

6. CONCLUDING REMARKS

It is known that, when $\lambda \geqslant 3$, a design with parameters of a residual design of a (v, k, λ) design may not be embeddable as a residual design in a (v, k, λ) design, and, even if it is embeddable, several non-isomorphic embeddings might be possible [2]. We do not know if there exists a (10, 18, 9, 5, 4) design which is not embeddable in a (19, 9, 4) design and hence naturally different from these twenty-one solutions of (10, 18, 9, 5, 4) design. We observe that among these twenty-one residual designs all except one are embeddable in a (up to isomorphism) unique (19, 9, 4) design. The design $R_{13}(D_4) \simeq R_1(D_6)$ is embeddable in precisely two (19, 9, 4) designs which are isomorphic to D_4 and D_6, respectively.

The methods used in this paper are quite general. Knowing the automorphisms of a family of Hadamard matrices and special-tuples and B-triples of corresponding Hadamard designs, it should be possible to enumerate Hadamard designs, their residual designs, and the derived designs.

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