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The q-WZ method for infinite series

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ABSTRACT

Motivated by the telescoping proofs of two identities of Andrews and Warnaar, we find that infinite q-shifted factorials can be incorporated into the implementation of the q-Zeilberger algorithm in the approach of Chen, Hou and Mu to prove nonterminating basic hypergeometric series identities. This observation enables us to extend the q-WZ method to identities on infinite series. We give the q-WZ pairs for some classical identities such as the q-Gauss sum, the ${}_6\phi_5$ sum, the Ramanujan's ${}_1\psi_1$ sum and Bailey's ${}_6\psi_6$ sum.

1. Introduction

The objective of this paper is to give an extension of the q-WZ method (Amdeberhan and Zeilberger, 1998; Mohammed and Zeilberger, 2004; Wilf and Zeilberger, 1990a; Zeilberger, 1990, 1994) to nonterminating basic hypergeometric series identities. We will follow the standard notation on q-series (Gasper and Rahman, 2004) and always assume $|q| < 1$. The q-shifted factorials $(a;q)_n$ and $(a;q)_\infty$ are defined by

$$(a;q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n \geq 1, \end{cases}$$

$$(a;q)_{-n} = \frac{1}{(aq^{-n};q)_n},$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots,$$

$$(a_1,a_2,\ldots,a_k;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n.$$

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An ${}_r\phi_s$ *basic hypergeometric series* is defined by

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] := \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{1.1} $$

where $q \neq 0$ when $r > s+1$. An ${}_r\psi_s$ *bilateral basic hypergeometric series* is defined by

$$ {}_r\psi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] := \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} z^n. \tag{1.2} $$

It is assumed that q, z and the parameters are such that each term of the series is well-defined. We say that an ${}_r\phi_s$ series terminates if only a finite number of terms contribute. Otherwise, we say that the series ${}_r\phi_s$ is nonterminating.

For the ordinary nonterminating hypergeometric identities, Gessel (1995) and Koornwinder (1998) provided computer proofs of Gauss' summation formula and Saalschütz's summation formula by means of a combination of Zeilberger's algorithm and asymptotic estimates. Vidunas (2002) (see also Koepf (2003) and Koornwinder (2005)) presented a method to evaluate ${}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix} \middle| -1\right)$ when $c-a+b$ is an integer. Recently, Chen et al. (2008) developed an approach to proving nonterminating basic hypergeometric identities based on the q-Zeilberger algorithm (Koornwinder, 1993; Wilf and Zeilberger, 1992). In this paper we will show how to apply the q-WZ method to prove nonterminating basic hypergeometric summation formulas by finding the q-WZ pairs. We will give some examples including the q-Gauss sum, the very-well-poised ${}_6\phi_5$ sum, the Ramanujan's ${}_1\psi_1$ sum and Bailey's very-well-poised series ${}_6\psi_6$ sum (Gasper and Rahman, 2004).

2. The Andrews–Warnaar identities

In this paper, we give telescoping proofs of the following two identities on partial theta functions (Andrews, 1981):

$$ \left(\sum_{n=0}^{\infty}(-1)^n a^n q^{\binom{n}{2}}\right)\left(\sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}}\right) = (q, a, b; q)_\infty \sum_{n=0}^{\infty} \frac{(abq^{n-1}; q)_n}{(q, a, b; q)_n} q^n, \tag{2.1} $$

$$ 1 + \sum_{n=1}^{\infty}(-1)^n q^{\binom{n}{2}}(a^n + b^n) = (a, b, q; q)_\infty \sum_{n=0}^{\infty} \frac{(ab/q; q)_{2n}}{(q, a, b, ab; q)_n} q^n. \tag{2.2} $$

The identity (2.2) was first proved by Warnaar (2003). Andrews and Warnaar (2007) derived the identity (2.1) and used it to prove (2.2).

As will be seen, the telescoping proofs suggest that the approach developed by Chen et al. (2008) for proving nonterminating basic hypergeometric identities can be extended so that infinite q-shifted factorials can be allowed in a q-hypergeometric term. This idea immediately leads to an extension of the q-WZ method to identities on infinite series.

Note that formula (2.2) is a generalization of the well-known Jacobi's triple product identity. When $b = q/a$, we get Jacobi's triple product identity

$$ \sum_{n=-\infty}^{\infty} (-1)^n a^n q^{\binom{n}{2}} = (a, q/a, q; q)_\infty, \tag{2.3} $$

where $|q| < 1$ and $a \neq 0$.

We now describe how to prove the identities (2.1) and (2.2) by the telescoping method. Let us consider (2.1) first. Put

$$ f(a) = \left(\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}} a^n\right)\left(\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}} b^n\right). $$

Note that the second factor does not contain the parameter a. It is easily verified that

$$f(a) = (1-a)f(aq) + aqf(aq^2). \tag{2.4}$$

We proceed to show that the right hand side of (2.1) satisfies the same recurrence relation. Of course, we still need to verify the boundary condition. Let

$$g(a) = \sum_{n=0}^{\infty} D_n(a), \quad \text{where } D_n(a) = (q, a, b; q)_\infty \frac{(abq^{n-1}; q)_n q^n}{(q, a, b; q)_n}.$$

Then it is necessary to show that

$$g(a) - (1-a)g(aq) - aqg(aq^2) = 0. \tag{2.5}$$

Here comes the key step of finding a telescoping relation for $D_n(a)$. Note that, for any $n \geq 0$, we have

$$\begin{aligned}
&D_n(a) - (1-a)D_n(aq) - aqD_n(aq^2) \\
&= \frac{(abq^n; q)_n (q, a, b; q)_\infty q^n}{(q, a, b; q)_n} \left(\frac{1-abq^{n-1}}{1-abq^{2n-1}} - \frac{1-a}{1-aq^n} - \frac{aq(1-abq^{2n})}{(1-aq^{n+1})(1-aq^n)(1-abq^n)} \right) \\
&= \frac{(abq^n; q)_n (q, a, b; q)_\infty q^n}{(q, a, b; q)_n} \left(\frac{a(1-q^n)(1-bq^{n-1})}{(1-aq^n)(1-abq^{2n-1})} - \frac{aq(1-abq^{2n})}{(1-aq^{n+1})(1-aq^n)(1-abq^n)} \right) \\
&= z_{n+1} - z_n,
\end{aligned} \tag{2.6}$$

where

$$z_n = -\frac{(1-q^n)(1-bq^{n-1})(abq^n; q)_n (q, a, b; q)_\infty aq^n}{(1-aq^n)(1-abq^{2n-1})(q, a, b; q)_n}.$$

The above relation reveals that the infinite q-shifted factorial $(q, a, b; q)_\infty$ can be incorporated into the telescoping relation and this step can be automated by the q-Gosper algorithm (Gosper, 1978; Koornwinder, 1993). Moreover, one sees that infinite q-shifted factorials can be incorporated into the q-Zeilberger algorithm so that the approach of Chen et al. (2008) can be extended to terms containing infinite q-shifted factorials. In particular, one can make the q-WZ method work for nonterminating hypergeometric series.

Now, let us return to the proof of (2.1). Clearly, $z_0 = 0$. It is also easily seen that $\lim_{n\to+\infty} z_n = 0$. Summing (2.6) over the nonnegative integers, we obtain the recurrence relation (2.5). In order to show that $f(a) = g(a)$, we will use the recurrence relation of $f(a) - g(a)$ to reach this goal.

Let $H(a) = f(a) - g(a)$. From the recurrence relations for $f(a)$ and $g(a)$, it follows that $H(a)$ satisfies the recurrence relation

$$H(a) = (1-a)H(aq) + aqH(aq^2). \tag{2.7}$$

Iterating the above relation yields that

$$H(a) = A_n H(aq^{n+1}) + B_n H(aq^{n+2}), \tag{2.8}$$

where A_n and B_n are given by

$$A_0 = (1-a), \qquad B_0 = aq, \qquad A_1 = (1-a)(1-aq) + aq, \qquad B_1 = (1-a)aq^2,$$

and

$$A_{n+1} = (1-aq^{n+1})A_n + aq^{n+1}A_{n-1}, \qquad B_{n+1} = aq^{n+2}A_n, \quad n \geq 1.$$

Hence we have

$$A_{n+1} - A_n = -aq^{n+1}(A_n - A_{n-1}),$$

which implies that

$$\begin{aligned}
|A_{n+1} - A_n| &= |(-1)^n a^n q^{\binom{n+2}{2}-1}||(A_1 - A_0)| \\
&\leq |a^n q^{\binom{n+2}{2}-1}| \left(|A_1| + |A_0|\right).
\end{aligned}$$

So, for fixed a and $|q| < 1$, $\lim_{n\to+\infty} A_n$ exists. Since $B_{n+1} = aq^{n+2}A_n$, $\lim_{n\to+\infty} B_n$ also exists. Again, by relation (2.8), we find

$$H(a) = H(0)\left(\lim_{n\to+\infty} A_n + \lim_{n\to+\infty} B_n\right).$$

It remains to show that $H(0) = 0$, that is,

$$\sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}} = (q, b; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q, b; q)_n}. \tag{2.9}$$

We can use the telescoping method to prove (2.9). Let

$$G(b) = \sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}} - (q, b; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q, b; q)_n}.$$

It is easy to check that

$$G(b) = (1 - b)G(bq) + bqG(bq^2).$$

We aim to show that $G(b) = 0$. Since $G(b)$ satisfies the same recurrence relation as $H(a)$, it suffices to confirm $G(0) = 0$, that is,

$$(q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} = 1,$$

which is a special case of Euler's identity (Gasper and Rahman, 2004, P. 354)

$$\sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_\infty}, \quad |z| < 1.$$

Indeed, relation (2.9) is a limiting case of Heine's transformation of ${}_2\phi_1$. For completeness, we give a proof based on Euler's identities:

$$\begin{aligned}
(q, b; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q, b; q)_n} &= (q; q)_\infty \sum_{m=0}^{\infty} \frac{q^m}{(q; q)_m} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-bq^m)^n}{(q; q)_n} \\
&= (q; q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n b^n q^{\binom{n}{2}}}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(q^{n+1})^m}{(q; q)_m} \\
&= \sum_{n=0}^{\infty}(-1)^n b^n q^{\binom{n}{2}}.
\end{aligned} \tag{2.10}$$

Thus, we have verified that $H(a) = 0$. This completes the proof. ■

We remark that once the recurrence relation (2.7) is derived, one can also use the theorem of Chen et al. (2008, Theorem 3.1) to prove the existence of the limits of A_n and B_n.

We next present a telescoping proof of (2.2). Let

$$f(a) = 1 + \sum_{n=1}^{\infty}(-1)^n q^{\binom{n}{2}}(a^n + b^n).$$

It is easily seen that

$$(1 + aq)f(a) - (1 - a^2q)f(aq) - (aq + a^2q)f(aq^2) = (q - 1)a. \tag{2.11}$$

Let

$$g(a) = \sum_{n=0}^{\infty} D_n(a), \quad \text{where } D_n(a) = (q, a, b; q)_\infty \frac{(ab/q; q)_{2n} q^n}{(q, a, b, ab; q)_n}.$$

It will be shown that

$$(1+aq)g(a)-(1-a^2q)g(aq)-(aq+a^2q)g(aq^2)=(q-1)a. \tag{2.12}$$

Since

$$\begin{aligned}
&\frac{q^n-abq^{n-1}}{1-abq^{2n-1}}-\frac{(1-a^2q)(1-ab)q^n}{(1+aq)(1-aq^n)(1-abq^n)}\\
&\quad-\frac{(a^2q+aq)(1-abq^{2n})(1-abq)q^n}{(1+aq)(1-aq^n)(1-aq^{n+1})(1-abq^n)(1-abq^{n+1})}\\
&=\frac{(1-abq^{2n})(-1+q+abq^{n+1}+a^2bq^{n+2}-aq^{n+2}-q^{n+1}-a^2bq^{2n+2}+a^2bq^{2n+3})a}{(1-aq^{n+1})(1-aq^n)(1-abq^n)(1+aq)(1-abq^{n+1})}\\
&\quad-\frac{(-1+q+abq^n+a^2bq^{n+1}-aq^{n+1}-q^n-a^2bq^{2n}+a^2bq^{2n+1})a(1-bq^{n-1})(1-q^n)}{(1-aq^n)(1-abq^{2n-1})(1+aq)(1-abq^n)},
\end{aligned}$$

multiplying both sides by

$$\frac{(ab;q)_{2n}(a,b,q;q)_\infty}{(q,a,b,ab;q)_n},$$

we deduce that

$$D_n(a)-\tfrac{(1-a^2q)}{1+aq}D_n(aq)-\tfrac{(a^2q+aq)}{1+aq}D_n(aq^2)=z_{n+1}-z_n, \tag{2.13}$$

where

$$\begin{aligned}
z_n&=\frac{(-1+q-aq^{n+1}-q^n+a^2bq^{n+1}+abq^n-a^2bq^{2n}+a^2bq^{2n+1})a}{(1-aq^n)(1-abq^{2n-1})}\\
&\quad\times\frac{(1-bq^{n-1})(1-q^n)(ab;q)_{2n}(q,a,b;q)_\infty}{(1+aq)(1-abq^n)(ab;q)_n(q,a,b;q)_n}.
\end{aligned}$$

Clearly, $z_0=0$ and $\lim_{n\to+\infty}z_n=\frac{(q-1)a}{1+aq}$. Summing (2.13) over nonnegative integers, we obtain the recurrence relation (2.12).

Let $H(a)=f(a)-g(a)$. Then $H(a)$ satisfies the following recurrence relation

$$H(a)=\frac{1-a^2q}{1+aq}H(aq)+\frac{aq+a^2q}{1+aq}H(aq^2). \tag{2.14}$$

By iteration, we obtain

$$H(a)=A_nH(aq^{n+1})+B_nH(aq^{n+2}), \tag{2.15}$$

where A_n and B_n are given by

$$\begin{aligned}
A_0&=\frac{1-a^2q}{1+aq}, & A_1&=\frac{1+a^3q^3}{1+aq^2},\\
B_0&=\frac{aq+a^2q}{1+aq}, & B_1&=\frac{aq^2(1-a^2q)}{(1+aq^2)},
\end{aligned}$$

and for $n\geq 1$,

$$A_{n+1}=\frac{1-a^2q^{2n+3}}{1+aq^{n+2}}A_n+\frac{aq^{n+1}+a^2q^{2n+1}}{1+aq^{n+1}}A_{n-1}, \tag{2.16}$$

$$B_{n+1}=\frac{aq^{n+2}+a^2q^{2n+3}}{1+aq^{n+2}}A_n. \tag{2.17}$$

Based on the above recurrence relations, one can deduce that both $\lim_{n\to+\infty} A_n$ and $\lim_{n\to+\infty} B_n$ exist. We note that Zeilberger (2008) has shown that

$$A_n = \frac{1+(-1)^{n+1}a^{n+2}q^{\binom{n+2}{2}}}{1+aq^{n+1}}$$

and

$$B_n = \frac{aq^{n+1}\left(1+(-1)^n a^{n+1}q^{\binom{n+1}{2}}\right)}{1+aq^{n+1}}.$$

Now we see that the limits $\lim_{n\to+\infty} A_n$ and $\lim_{n\to+\infty} B_n$ exist. By relation (2.15), we deduce that

$$H(a) = H(0)\left(\lim_{n\to+\infty} A_n + \lim_{n\to+\infty} B_n\right).$$

The identity (2.10) implies that $f(0) = g(0)$. So we have $H(a) = 0$. This completes the proof. ■

We also note that once the recurrence relation (2.14) is established, one may assume that $|a| < 1$ and may use the theorem in Chen et al. (2008, Theorem 3.1) to prove the existence of the limits of A_n and B_n. Moreover, we may drop the assumption $|a| < 1$ by analytic continuation.

3. The *q*-WZ pairs for infinite series

Our approach to the q-WZ method for infinite series can be described as follows. The key step is to construct q-WZ pairs for infinite sums. Suppose that we aim to prove an identity of the form:

$$\sum_{k=N_0}^{\infty} F_k(a_1, a_2, \ldots, a_t) = R(a_1, a_2, \ldots, a_t), \tag{3.1}$$

where t is a positive integer, and the sum is either a unilateral or bilateral basic hypergeometric series, namely, $N_0 = 0$ or $N_0 = -\infty$, $R(a_1, a_2, \ldots, a_t)$ is either zero or a quotient of two products of infinite q-shifted factorials.

First, we set some parameters, say, $a_1, \ldots, a_p$, $(1 \le p \le t)$ to $a_1q^n, \ldots, a_pq^n$, so that we get

$$\sum_{k=N_0}^{\infty} F_k(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t) = R(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t). \tag{3.2}$$

If $R(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t) \neq 0$, set

$$F(n,k) = \frac{F_k(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t)}{R(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t)}.$$

Otherwise, set

$$F(n,k) = F_k(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t).$$

Our goal is to show that

$$\sum_{k=N_0}^{\infty} F(n,k) = \text{constant}, \quad n = 0, 1, 2, \ldots. \tag{3.3}$$

The constant can be determined by setting $n = 0$ and setting $a_1, a_2, \ldots, a_t$ to special values. We claim that the above goal can be achieved by adopting the q-WZ method for finite sums.

Let us recall the boundary and limit conditions for the q-WZ-method. Let $f(n)$ denote the left hand side of (3.3), i.e.,

$$f(n) = \sum_{k=N_0}^{\infty} F(n,k)$$

and we aim to show that

$$f(n) = \text{constant}$$

for every nonnegative integer n. To this end, it suffices to show that $f(n+1) - f(n) = 0$ for every nonnegative integer n. This can be done by finding $G(n, k)$ such that

$$F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k). \tag{3.4}$$

A pair of functions $(F(n, k), G(n, k))$ satisfying (3.4) is called a *q-WZ pair*. Once a q-WZ pair is found, one can check the boundary and limit conditions to ensure that $f(n)$ equals the claimed constant. Here are the conditions:

(C1) For each integer $n \geq 0$, $\lim_{k\to\pm\infty} G(n, k) = 0$.
(C2) For each integer k, the limit

$$f_k = \lim_{n\to\infty} F(n, k) \tag{3.5}$$

exists and is finite.
(C3) We have $\lim_{L\to\infty} \sum_{n\geq 0} G(n, -L) = 0$.

The WZ method can be formally stated as follows.

Theorem 3.1 (Wilf and Zeilberger (1990b)). *Assume that $(F(n, k), G(n, k))$ is a WZ pair. If* (C1) *holds, then we have*

$$\sum_k F(n, k) = \text{constant}, \quad n = 0, 1, 2, \ldots. \tag{3.6}$$

If (C2) *and* (C3) *hold, then we have the companion identity*

$$\sum_{n=0}^{\infty} G(n, k) = \sum_{j\leq k-1} (f_j - F(0, j)), \tag{3.7}$$

where f_j is defined by (3.5).

We now explain how to compute the desired q-WZ pair for the identity (3.1). In fact, it can be obtained by applying the q-Gosper algorithm to $F(n+1, k) - F(n, k)$. It should be noted that $F(n+1, k) - F(n, k)$ is a q-hypergeometric term with respect to q^k, even if $F(n, k)$ contains infinite q-shifted factorials such as $(aq^n; q)_\infty$. Obviously, $F(n+1, k) - F(n, k)$ is a q-hypergeometric term when $R(a_1, \ldots, a_t) = 0$. Assume that $R(a_1, \ldots, a_t) \neq 0$. Let

$$M_1 = \frac{R(a_1q^{n+1}, \ldots, a_pq^{n+1}, a_{p+1}, \ldots, a_t)}{R(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t)},$$
$$M_2 = \frac{F_{k+1}(a_1q^{n+1}, \ldots, a_pq^{n+1}, a_{p+1}, \ldots, a_t)}{F_k(a_1q^{n+1}, \ldots, a_pq^{n+1}, a_{p+1}, \ldots, a_t)},$$
$$M_3 = \frac{F_{k+1}(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t)}{F_k(a_1q^{n+1}, \ldots, a_pq^{n+1}, a_{p+1}, \ldots, a_t)},$$
$$M_4 = \frac{F_k(a_1q^n, \ldots, a_pq^n, a_{p+1}, \ldots, a_t)}{F_k(a_1q^{n+1}, \ldots, a_pq^{n+1}, a_{p+1}, \ldots, a_t)}.$$

Since M_1 is a rational function in q^n and is independent of k, M_2, M_3, M_4 are all rational functions in q^k. Observe that

$$\frac{F(n+1, k+1) - F(n, k+1)}{F(n+1, k) - F(n, k)} = \frac{M_2 - M_1M_3}{1 - M_1M_4} \tag{3.8}$$

is a rational function in q^k, i.e., $F(n+1, k) - F(n, k)$ is a q-hypergeometric term with respect to q^k. It is necessary to mention that even if $F(n, k)$ contains infinite q-shifted factorials of the form $(aq^n; q)_\infty$, the quotient (3.8) no longer contains the q-shifted factorial $(aq^n;q)_\infty$ and it is still a rational function in q^k. Consequently, we can employ the q-Gosper algorithm to determine whether $G(n,k)$ exists. Nevertheless, it is also necessary to note that $G(n,k)$ contains infinite q-shifted factorials if $F(n,k)$ does.

There is another way to look at the above procedure. Suppose that $F(n,k)$ contains an infinite q-shifted factorial $(a;q)_\infty$, where a is a chosen parameter for the substitution $a \to aq^n$. If we set $G'(n,k) = R(aq^n)G(n,k)$, then Eq. (3.4) becomes

$$F(n+1,k)R(aq^n) - F(n,k)R(aq^n) = G'(n,k+1) - G'(n,k).$$

It is evident that the infinite q-shifted factorial $(aq^n;q)_\infty$ will disappear in the above equation, and one can use the q-Gosper algorithm to find a q-WZ pair if it exists.

We now take the q-binomial theorem (Gasper and Rahman, 2004, P. 354) as an example to explain the above steps:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1. \tag{3.9}$$

In this case, we have

$$F_k(a) = \frac{(a;q)_k}{(q;q)_k} z^k, \qquad R(a) = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

We choose the parameter a, and substitute a with aq^n. Then we set

$$F(n,k) = \frac{F_k(aq^n)}{R(aq^n)} = \frac{(aq^n;q)_k (z;q)_\infty}{(q;q)_k (azq^n;q)_\infty} z^k.$$

In order to find $G(n,k)$ such that (3.4) holds, it is easily checked that $F(n+1,k) - F(n,k)$ is a q-hypergeometric term. By examining the q-Gosper algorithm, one sees that it is capable of dealing with the input $F(n+1,k) - F(n,k)$, or we can set

$$G'(n,k) = R(aq^n)G(n,k)$$

and find a solution of the equation

$$(1-azq^n)\frac{(aq^{n+1};q)_k}{(q;q)_k} z^k - \frac{(aq^n;q)_k}{(q;q)_k} z^k = G'(n,k+1) - G'(n,k). \tag{3.10}$$

Finally, we obtain the q-WZ pair

$$F(n,k) = \frac{(aq^n;q)_k (z;q)_\infty}{(q;q)_k (azq^n;q)_\infty} z^k,$$

$$G(n,k) = -\frac{(aq^n;q)_k (z;q)_\infty (a-aq^k)}{(q;q)_k (azq^n;q)_\infty (1-aq^n)} q^n z^k.$$

If $|z| < 1$, it is easy to see that $F(n,k)$ and $G(n,k)$ satisfy conditions (C1), (C2) and (C3). By (3.6),

$$\sum_{k=-\infty}^{\infty} F(n,k) = \sum_{k=0}^{\infty} F(n,k) = \text{constant}, \quad n = 0, 1, 2, \ldots.$$

Setting $z = 0$ yields that the constant equals 1. Setting $n = 0$, we have

$$\sum_{k=0}^{\infty} F(0,k) = \text{constant} = 1.$$

By (3.7), we get the companion identity of (3.9)

$$\sum_{j=0}^{k} \frac{(a;q)_j}{(q;q)_j} z^j = (az;q)_\infty \sum_{j=0}^{k} \frac{z^j}{(q;q)_j} + \frac{az^{k+1}(a;q)_{k+1}}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(az;q)_n (aq^{k+1};q)_n}{(a;q)_{n+1}} q^n.$$

We remark that our algorithm depends on the choice of parameters. For a given choice of parameters, it is not guaranteed that one can find a q-WZ pair. Nevertheless, this approach applies to many classical identities.

We now give a few more examples.

Example 3.1. The q-Gauss sum (Gasper and Rahman, 2004, P. 354):

$$\sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(q, c; q)_k} \left(\frac{c}{ab}\right)^k = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \quad |c/ab| < 1. \tag{3.11}$$

By computation we get the q-WZ pair

$$F(n, k) = \frac{(b, aq^n; q)_k (c/ab, cq^n; q)_\infty}{(q, cq^n; q)_k (c/a, cq^n/b; q)_\infty} \left(\frac{c}{ab}\right)^k,$$

$$G(n, k) = -\frac{(a - aq^k)(b, aq^n; q)_k (c/ab, cq^n; q)_\infty}{(1 - aq^n)(q, cq^n; q)_k (c/a, cq^n/b; q)_\infty} \left(\frac{c}{ab}\right)^k q^n.$$

If $|c/ab| < 1$, it is easy to verify that the two functions $(F(n, k), G(n, k))$ satisfy relation (3.4) and conditions (C1), (C2) and (C3). By (3.6), we have

$$\sum_{k=-\infty}^{\infty} F(n, k) = \sum_{k=0}^{\infty} F(n, k) = \text{constant}, \quad n = 0, 1, 2, \ldots.$$

Setting $c = 0$ and $n = 0$, we find that the constant equals 1, and we have

$$\sum_{k=0}^{\infty} F(0, k) = \text{constant} = 1.$$

After simplification, we obtain the identity (3.11).

By (3.7), we obtain the companion identity of (3.11)

$$-\sum_{n=0}^{\infty} \frac{(a - aq^k)(b, aq^n; q)_k (c/ab, cq^n; q)_\infty}{(1 - aq^n)(q, cq^n; q)_k (c/a, cq^n/b; q)_\infty} \left(\frac{c}{ab}\right)^k q^n = \sum_{j \le k-1} (f_j - F(0, j)), \tag{3.12}$$

where

$$f_j = \lim_{n \to \infty} F(n, j) = \frac{(b; q)_j (c/ab; q)_\infty}{(q; q)_j (c/a; q)_\infty} \left(\frac{c}{ab}\right)^j,$$

which can be restated as

$$\sum_{j=0}^{k} \frac{(a, b; q)_j}{(q, c; q)_j} \left(\frac{c}{ab}\right)^j = \frac{(c/b; q)_\infty}{(c; q)_\infty} \sum_{j=0}^{k} \frac{(b; q)_j}{(q; q)_j} \left(\frac{c}{ab}\right)^j + \frac{(a, b; q)_{k+1} c^{k+1}}{(q; q)_k (c; q)_{k+1} a^k b^{k+1}} \sum_{n=0}^{\infty} \frac{(aq^{k+1}, c/b; q)_n}{(a; q)_{n+1} (cq^{k+1}; q)_n} q^n.$$

Example 3.2. The sum of a very-well-poised ${}_6\phi_5$ series (Gasper and Rahman, 2004, P. 356):

$$\sum_{k=0}^{\infty} \frac{(1 - aq^{2k})(a, b, c, d; q)_k}{(1 - a)(q, aq/b, aq/c, aq/d; q)_k} \left(\frac{aq}{bcd}\right)^k = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}, \quad |aq/bcd| < 1. \tag{3.13}$$

By computation we get the following q-WZ pair:

$$\begin{aligned} F(n,k) &= \frac{(1-aq^{n+2k})(c,d,aq^n,bq^n;q)_k}{(1-aq^n)(q,aq/b,aq^{n+1}/c,aq^{n+1}/d;q)_k} \\ &\quad\times \frac{(aq/b,aq/bcd,aq^{n+1}/c,aq^{n+1}/d;q)_\infty}{(aq/bc,aq/bd,aq^{n+1},aq^{n+1}/cd;q)_\infty}\left(\frac{aq}{bcd}\right)^k, \\ G(n,k) &= \frac{(c,d;q)_k(a/b,a/bcd;q)_\infty}{(q,a/b;q)_k(aq^n,aq^n/cd;q)_\infty}\frac{(aq^n,bq^n;q)_k(aq^n/c,aq^n/d;q)_\infty}{(aq^n/c,aq^n/d;q)_k(a/bd,a/bc;q)_\infty} \\ &\quad\times \frac{(a-bc)(a-bd)(aq^n-cd)(1-q^k)}{(a-bcd)(bq^n-1)(aq^{n+k}-c)(aq^{n+k}-d)}\left(\frac{aq}{bcd}\right)^k q^n. \end{aligned}$$

It is easily seen that $F(n,k)$ and $G(n,k)$ satisfy conditions (C1), (C2) and (C3). Therefore, by (3.6), we have the result that $\sum_{k=0}^{\infty} F(n,k)$ is a constant. Setting $n=0$ and $a=0$, we find that the constant equals 1. Thus we have

$$\sum_{k=0}^{\infty} F(0,k) = \text{constant} = 1,$$

which is nothing but (3.13). Since

$$f_k = \frac{(c,d;q)_k}{(q,aq/b;q)_k}\frac{(aq/b,aq/bcd;q)_\infty}{(aq/bc,aq/bd;q)_\infty}\left(\frac{aq}{bcd}\right)^k$$

and

$$F(0,j) = \frac{(1-aq^{2j})(a,b,c,d;q)_j(aq/b,aq/c,aq/d,aq/bcd;q)_\infty}{(1-a)(q,aq/b,aq/c,aq/d;q)_j(aq,aq/bc,aq/bd,aq/cd;q)_\infty}\left(\frac{aq}{bcd}\right)^j,$$

by (3.7), we obtain the companion identity

$$\begin{aligned} \sum_{j=0}^{k}\frac{(1-aq^{2j})(a,b,c,d;q)_j}{(1-a)(q,aq/b,aq/c,aq/d;q)_j}\left(\frac{aq}{bcd}\right)^k &= \frac{(aq,aq/cd;q)_\infty}{(aq/c,aq/d;q)_\infty}\sum_{j=0}^{k}\frac{(c,d;q)_j}{(q,aq/b;q)_j}\left(\frac{aq}{bcd}\right)^j \\ &+ \frac{b(aq;q)_k(b,c,d;q)_{k+1}}{(q,aq/b;q)_k(aq/c,aq/d;q)_{k+1}}\left(\frac{aq}{bcd}\right)^{k+1}\sum_{n=0}^{\infty}\frac{(aq/cd;q)_n(aq^{k+1},bq^{k+1};q)_n}{(b;q)_{n+1}(aq^{k+2}/c,aq^{k+2}/d;q)_n}q^n. \end{aligned}$$

Example 3.3. The Ramanujan's ${}_1\psi_1$ sum (Gasper and Rahman, 2004, P. 357)

$${}_1\psi_1(a;b;q,z) = \frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}, \quad |b/a|<|z|<1. \tag{3.14}$$

In this case, we find that

$$\begin{aligned} F(n,k) &= \frac{(aq^n;q)_k(z,b/az,bq^n,q^{1-n}/a;q)_\infty}{(bq^n;q)_k(q,b/a,azq^n,q^{1-n}/az;q)_\infty}z^k, \\ G(n,k) &= \frac{(z,b/az,bq^n,q^{-n}/a;q)_\infty(aq^n;q)_k(1-azq^n)}{(q,b/a,azq^n,q^{-n}/az;q)_\infty(bq^n;q)_k(z-azq^n)}z^k. \end{aligned}$$

If $|b/a|<|z|<1$, utilizing the following relation

$$(a;q)_{-n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}, \quad n=0,1,2,\ldots, \tag{3.15}$$

we can verify that $G(n,k)$ satisfies condition (C1). It follows that,

$$\sum_{k=-\infty}^{\infty} F(n,k) = \text{constant}, \quad n=0,1,2,\ldots. \tag{3.16}$$

Setting $n = 0$, $b = q$ and utilizing the q-binomial theorem (3.9), we see that the constant equals 1. Setting $n = 0$, we obtain the identity (3.14). However, we note that the conditions for the companion identity do not hold in this case.

Example 3.4. The sum of a very-well-poised ${}_6\psi_6$ series (Gasper and Rahman, 2004, P. 357):

$$\sum_{k=-\infty}^{\infty} \frac{(1-aq^{2k})(b,c,d,e;q)_k}{(1-a)(aq/b,aq/c,aq/d,aq/e;q)_k}\left(\frac{a^2q}{bcde}\right)^k$$
$$= \frac{(aq,aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de,q,q/a;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e,a^2q/bcde;q)_\infty}. \tag{3.17}$$

We obtain the following q-WZ pair:

$$\begin{aligned}
F(n,k) &= \frac{(1-aq^{n+2k})(d,e,bq^n,cq^n;q)_k(aq/b,aq/c;q)_\infty}{(1-aq^n)(aq/b,aq/c,aq^{n+1}/d,aq^{n+1}/e;q)_k(aq/bd,aq/be;q)_\infty}\\
&\quad\times\frac{(q/d,q/e,a^2q/bcde,aq^{n+1}/d,aq^{n+1}/e,q^{1-n}/b,q^{1-n}/c;q)_\infty}{(q,aq/cd,aq/ce,aq^{n+1},aq^{n+1}/de,q^{1-n}/a,aq^{1-n}/bc;q)_\infty}\left(\frac{a^2q}{bcde}\right)^k,\\
G(n,k) &= \frac{(d,e,bq^n,cq^n;q)_k(a/b,a/c,1/e,a^2/bcde,1/d;q)_\infty}{(a/b,a/c,aq^n/d,aq^n/e;q)_k(q,a/bd,a/be,a/cd,a/ce;q)_\infty}\\
&\quad\times\frac{(aq^n/d,aq^n/e,q^{-n}/b,q^{-n}/c;q)_\infty(-1+aq^n)}{(aq^n,aq^n/de,aq^{-n}/bc,q^{-n}/a;q)_\infty(1-bq^n)(1-cq^n)}\\
&\quad\times\frac{(a-bd)(a-be)(a-cd)(a-ce)(aq^n-de)q^n}{(aq^{n+k}-d)(aq^{n+k}-e)(a-ad)(1-e)(a^2-bcde)}\left(\frac{a^2q}{bcde}\right)^k.
\end{aligned}$$

Since $|a^2q/bcde| < 1$, from the identity (3.15) it follows that $G(n,k)$ satisfies condition (C1). By (3.6), we find

$$\sum_{k=-\infty}^{\infty} F(n,k) = \text{constant}, \quad n = 0,1,2,\ldots. \tag{3.18}$$

In order to determine the constant, we set $n = 0$ and $b = a$. From the ${}_6\phi_5$ summation formula (3.13), we see that the constant equals

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} F(0,k) &= \sum_{k=0}^{\infty}\frac{(1-aq^{2k})(a,c,d,e;q)_k}{(1-a)(aq/c,aq/d,aq/e;q)_k}\\
&\quad\times\frac{(aq,aq/cd,aq/ce,aq/de;q)_\infty}{(aq/c,aq/d,aq/e,aq/cde;q)_\infty}\left(\frac{aq}{cde}\right)^k = 1,
\end{aligned}$$

which can be restated as (3.17). Nevertheless, we note that the conditions for the companion identity do not hold in this case.

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