The Operator Factorization Problems

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ABSTRACT

We survey various results concerning operator factorization problems. More precisely, we consider the following setting. Let H be a complex Hilbert space, and let $\mathscr{B}(H)$ be the algebra of all bounded linear operators on H. For a given subset $\mathscr{C}$ of $\mathscr{B}(H)$, we are interested in the characterization of operators in $\mathscr{B}(H)$ which are expressible as a product of finitely many operators in $\mathscr{C}$ and, for each such operator, the minimal number of factors in a factorization. The classes of operators we consider include normal operators, involutions, partial isometries together with their various subclasses, and other miscellaneous classes of operators. Most of the known results are for operators on finite-dimensional spaces or finite matrices. The paper concludes with some applications, due to Hochwald, concerning the uniqueness of the adjoint operation on operators.

1. INTRODUCTION

Which bounded linear operator on a Hilbert space can be factored as the product of finitely many normal operators? What is the answer if "normal operators" is replaced by "involutions," "partial isometries," or other classes of familiar operators? Just as in the case of the factorization of integers, polynomials, or other objects in mathematics, such operator factorization problems seem to arise very naturally in the course of study of operators. In the cases when they are solved, the solutions are usually neat and elegant; otherwise, they pose interesting and challenging questions whose solutions may lead us to a deeper understanding of the nature of the operators under consideration. However, in the past such problems seem to have attracted

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little attention among operator and matrix theorists and found very few applications even within operator or matrix theory itself. The purpose of this paper is to give a survey of problems of this nature, collect together the relevant references, and briefly indicate their proofs and interrelationships. Hopefully, this paper can serve as a convenient reference so that to enhance people's interest in this area of research.

Although some of the problems discussed below can be considered for operators on Hilbert spaces over more general fields or even for linear transformations on vector spaces over division rings, for ease of exposition we will restrict ourselves to bounded linear operators on complex, separable Hilbert spaces whose dimension can be either finite or infinite. Formally, let H be such a space, and let $\mathscr{B}(H)$ be the algebra of all bounded linear operators on H. For a given subset $\mathscr{C}$ of $\mathscr{B}(H)$, we would like to characterize the class $\mathscr{C}^{\infty}$ of operators in $\mathscr{B}(H)$ which are expressible as a product of finitely many operators in $\mathscr{C}$. For each operator T in $\mathscr{C}^{\infty}$, let $l(T;\mathscr{C})$ denote the minimal number of factors in such a factorization of T, called the *length* of T with respect to $\mathscr{C}$. The determination of $l(T;\mathscr{C})$ is, in general, pretty difficult. In such cases, we content ourselves with the quantity $L(\mathscr{C})$, the *length* of $\mathscr{C}$, defined as

$$L(\mathscr{C}) = \sup\{\, l(T;\mathscr{C}) : T \in \mathscr{C}^{\infty} \}.$$

As it turns out, for most factorization problems $L(\mathscr{C})$ is finite, and it may or may not depend on the dimension of H. (The one exception occurs for the problem of factorization into orthogonal projections; see Theorem 4.6.) For each $k < L(\mathscr{C})$, the characterization of operators expressible as the product of k operators in $\mathscr{C}$ is also of interest. Such problems can usually be solved completely for operators on finite-dimensional spaces or finite matrices. Much less is known for operators on infinite-dimensional spaces.

In Section 2 below, we first consider the factorization problem for symmetric, skew-symmetric, Hermitian, positive definite, positive semidefinite, accretive, positive stable, and normal operators. The factorizations into various kinds of involutions will be taken up in Section 3. They include involutions, pseudoinvolutions, orthogonal involutions, symmetries, reflections, dilatations, orthogonal reflections, and unitary reflections. (The precise definitions will be given in the discussions below.) Section 4 characterizes the finite products of partial isometries, unilateral shifts, orthogonal projections, and idempotent operators. Section 5 concerns products of operators in other miscellaneous classes such as nilpotent, quasinilpotent, and unipotent operators, EP matrices, and commutators (both additive and multiplicative). In the last section, results of Hochwald's concerning the uniqueness of the adjoint operation on operators will be discussed. Their proofs utilize various factorization results considered in this paper.

Due to the expository nature of the paper, we will omit most of the proofs in the discussions below, referring instead to the original papers. However, from time to time, we try to give the main idea behind and the flavor of the exact proofs, especially of those results which appeared recently. Along the way, we will also indicate the open problems and unexplored areas.

In the following, we will let I_n denote the $n \times n$ identity matrix and $\sigma(T)$ denote the spectrum of an operator T.

2. NORMAL OPERATOR

In this section, we consider the factorization problems when the factors are normal operators, its subclasses, and other related operators. We start with the symmetric ones. An operator T is *symmetric* if there exists an orthonormal basis $\{e_i\}$ for the underlying space such that $(Te_i, e_j) = (Te_j, e_i)$ for all i, j. In particular, a matrix representation of T, $T = [a_{ij}]$, is *symmetric* if $a_{ij} = a_{ji}$ or, equivalently, $T = T^t$, where T^t denotes the transpose of T. In 1910, Frobenius [32] proved probably the first result in operator factorizations.

THEOREM 2.1. *Any finite matrix is the product of two symmetric ones, one of which may be taken to be nonsingular.*

Actually, this result holds not only for complex matrices but for matrices over any field. Not well known among mathematicians, it has since appeared in the literature quite a few times (cf. [51] and the references in [14] or [71]). Its proof can be based on factoring the Jordan canonical form (for the complex field) or rational form (for arbitrary fields), since the property of being the product of two symmetric matrices is preserved under similarity. For operators on infinite-dimensional spaces, this problem seems not to have been explored.

Closely related to symmetric matrices are skew-symmetric ones. A matrix T is *skew-symmetric* if $T^t = -T$. Products of two skew-symmetric matrices have been completely characterized by Gow and Laffey [33].

THEOREM 2.2. *A finite matrix is the product of two skew-symmetric matrices if and only if*

(1) *its elementary divisors corresponding to nonzero eigenvalues occur with even multiplicity, and*

(2) *its elementary divisors corresponding to the zero eigenvalue are of the form*

$$x^{k_1}, x^{k_1'}, \ldots, x^{k_s}, x^{k_s'} \quad or \quad x^{k_1}, x^{k_1'}, \ldots, x^{k_s}, x^{k_s'}, x,$$

where $k_j' = k_j$ *or* $k_j + 1$, $1 \leqslant j \leqslant s$.

In particular, the following corollary is the closest to Theorem 2.1 for skew-symmetric products.

COROLLARY 2.3. *A matrix of even size is the product of two skew-symmetric matrices, one of which is nonsingular, if and only if each of its elementary divisors occurs with even multiplicity.*

Gow and Laffey [33] proved the preceding two results, basing their study on alternating forms and alternate transformations. More recently, in work to appear, they obtained the following result concerning factorizations with skew-symmetric factors (cf. also [46] and [47]).

THEOREM 2.4. *If* $n > 2$ *is even, then every* $n \times n$ *nonsingular matrix is the product of five skew-symmetric matrices and five is the smallest such number.*

The conjugate version of the symmetric matrix is the Hermitian one. Next we consider factorizations of this type. For finite matrices, this is solved by Radjavi [51].

THEOREM 2.5. *A finite matrix is the product of finitely many Hermitian matrices if and only if its determinant is real. The length of Hermitian matrices is* 4.

The proof of the sufficiency part proceeds by first factoring the matrix or rather, its rational form, into the product of two matrices which are either a real matrix or similar to a real one, and then invoking Frobenius's result, Theorem 2.1. The minimality of 4 can be easily seen from the following result, also due to Radjavi [52], which holds even for operators on infinite-dimensional spaces.

THEOREM 2.6. *Let* T *be an operator on* H. *If the closure of the numerical range of* T *lies entirely within one of the open quadrants, then* T *is not the product of three Hermitian operators.*

This is an easy consequence of the Toeplitz-Hausdorff theorem on the convexity of the numerical range (cf. [38, Problem 210]).

Products of Hermitian operators on infinite-dimensional spaces will be considered later on, together with those of normal, positive definite, and positive semidefinite operators.

For the products of two Hermitian operators on finite-dimensional spaces, we have the following characterization (cf. [18], [55, Theorem 1] and [71]).

THEOREM 2.7. *The following statements are equivalent for a finite matrix T:*

(1) *T is the product of two Hermitian matrices;*

(2) *T is the product of two Hermitian matrices, one of which is nonsingular;*

(3) *T is similar to a matrix with real entries;*

(4) *T is similar to T^*.*

The proof is an easy exercise in linear algebra. In the context of infinite-dimensional spaces, the implication $(1) \Rightarrow (4)$ is no longer true. Examples have been furnished in [55] and [34] (cf. Example 2.11 below). As a consolation, it was conjectured in [34] that T is similar to T^* if T is the product of two Hermitian Fredholm operators. This seems still to be open. As for the implication $(4) \Rightarrow (1)$, despite the abundance of supporting special cases as given in [55], its validity is still unconfirmed even under the extra assumption that T is invertible. The problem of characterizing the products of three Hermitian operators was proposed in [51]. It is unsolved even in the finite-dimensional case.

Next we consider the products of positive definite and positive semidefinite operators. Recall that an operator T on the space H is *positive definite*, denoted $T > 0$ [*positive semidefinite*, denoted $T \geqslant 0$] if $(Tx, x) > 0$ [$(Tx, x) \geqslant 0$] for any $x \neq 0$ in H. For convenience, we will abbreviate them as *positive* (*nonnegative*) operators. For operators on finite-dimensional spaces, the factorization problem concerning the former has been tackled by Ballantine in a series of papers [3], [4], and [5], which culminate in its complete characterization in [6]. For the latter, it was (partially) solved only recently in [68]. In the following, we will describe such results, starting with the products of two. Note that, in the next two theorems, the spaces on which the operators act may be infinite-dimensional.

THEOREM 2.8. *An operator is the product of two positive operators one of which is invertible if and only if it is similar to a positive one.*

The first appearance of results of this sort (for finite matrices) is probably in [61]. Here is a simple argument for its proof. If $T = AB$, where $A, B > 0$ and A is invertible, then T is similar to $A^{-1/2}TA^{1/2} = A^{1/2}BA^{1/2}$, which is positive. Conversely, if $T = X^{-1}CX$, where X is invertible and $C > 0$, then $T = (X^{-1}X^{-1*})(X^*CX)$ is the product of two positive operators. A slight modification of the above argument yields the following result for the product of two nonnegative operators.

THEOREM 2.9. *An operator is the product of two nonnegative operators one of which is invertible if and only if it is similar to a nonnegative one.*

In the case of finite dimensions, Theorem 2.8 settles completely the factorization problem with two positive factors, since positive operators are automatically invertible. For infinite-dimensional spaces, the situation is not clear. We next strengthen Theorem 2.9 for operators on finite-dimensional spaces.

THEOREM 2.10. *The following statements are equivalent for a finite matrix T:*

(1) *T is the product of two nonnegative matrices;*

(2) *T is the product of two nonnegative matrices one of which is nonsingular;*

(3) *T is similar to a nonnegative matrix.*

This result is proved in [68]. The proof for the implication (1) $\Rightarrow$ (3) can be reduced to the corresponding one for Theorem 2.8.

If the space is infinite-dimensional, then the product of two nonnegative operators may not be similar to a nonnegative one. In fact, this is the case even when one factor is positive but noninvertible. Here we re-cite the example given in [55], since it contains the genesis of the idea by which we obtain the results characterizing products of normal operators.

EXAMPLE 2.11. Let A be a one-to-one positive operator on H such that $AH \neq H$. Let $x \in H \setminus AH$, and let B be the orthogonal projection onto $\langle x \rangle^{\perp}$, the orthogonal complement of the subspace generated by x. Then $Bx = 0$ implies that AB is not one-to-one. On the other hand, if $BAy = 0$, then $Ay \in \ker B \cap \operatorname{ran} A = \{0\}$, which shows that BA is one-to-one. Thus if AB is similar to a nonnegative operator, then AB must be similar to $(AB)^* = BA$, which is absurd.

In the infinite-dimensional case, characterizing products of two positive or two nonnegative operators would make interesting and challenging problems.

For the products of three positive operators on a finite-dimensional space, Ballantine obtained a characterization following a long proof [6, Theorem 3] (cf. also Taussky's survey [62]). We will only give the statement of this result and refer the reader to [6] for the details. For an operator T on H, let $\Gamma(T) = \{(Tx, x) : x \in H\}$. Note that $\Gamma(T)$ is the cone generated by the numerical range of T.

THEOREM 2.12. *A finite matrix T is the product of three positive matrices if and only if one of the following holds:*

(1) $\det T > 0$ *and $\Gamma(T)$ is the whole plane, or*
(2) $1 \in \Gamma(T)$, $-1 \notin \Gamma(T)$, *and $\sum_j \theta_j = 0$, where the θ_j's are the arguments between $-\pi$ and π of the eigenvalues of T.*

For the products of three nonnegative matrices, only some partial results are known (cf. [68]). The next theorem gives a sufficient condition in terms of the "invertible part" of an operator on a finite-dimensional space. Note that any such operator can be uniquely expressed as the direct sum (not necessarily orthogonal) of an invertible operator and a nilpotent one (cf. [36, p. 113]).

THEOREM 2.13. *Let T be an operator on a finite-dimensional space and $T = T_1 + T_2$, where T_1 is invertible and T_2 is nilpotent. If T_1 is the product of three positive operators, then T is the product of three nonnegative operators.*

In particular, we have the following

COROLLARY 2.14. *Any nilpotent matrix is the product of three nonnegative matrices.*

It seems plausible that the converse of Theorem 2.13 is also true. This we don't know at present. However, the following supporting special case seems worth mentioning, whose proof makes use of Theorem 2.10 on the product of two nonnegative operators.

THEOREM 2.15. *Let $T = T_1 \oplus 0$ on a finite-dimensional space. Then T is the product of three nonnegative operators if and only if T_1 is.*

Finally, we come to the products of four and five positive or nonnegative matrices. The results for positive matrices are in [6]; those for nonnegative ones are in [68].

THEOREM 2.16. *A finite matrix T is the product of four positive matrices if and only if* $\det T > 0$ *and T is not a scalar matrix cI with c in* $\mathbb{C}\setminus\{z: z \geqslant 0\}$.

THEOREM 2.17. *A finite matrix T is the product of finitely many positive matrices if and only if* $\det T > 0$. *The length of positive matrices is* 5.

These two theorems can be proved with a factorization theorem for finite matrices (cf. [60] and [47]). Since this theorem is also useful for other factorization problems discussed later on, we state it next.

THEOREM 2.18. *Let T be an $n \times n$ nonscalar nonsingular matrix. For any complex numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ with $\prod_{j=1}^{n} a_j b_j = \det T$, there exist matrices A and B such that $\sigma(A) = \{a_1, \ldots, a_n\}$, $\sigma(B) = \{b_1, \ldots, b_n\}$, and $T = AB$.*

Its proof through induction and some clever manipulation is elementary. For nonnegative matrices, we start with the following theorem in [68].

THEOREM 2.19. *Any singular matrix is the product of four nonnegative matrices, and four is the smallest such number.*

The proof of the factorization assertion is based on Theorem 2.12, Ballantine's characterization of products of three positive matrices, through the Jordan canonical form of the given matrix; the optimality of four follows from Theorem 2.15.

Combining Theorem 2.19 with Theorems 2.16 and 2.17 yields the following characterizations of products of four or more nonnegative matrices.

THEOREM 2.20. *A finite matrix T is the product of four nonnegative matrices if and only if* $\det T \geqslant 0$ *and T is not a scalar matrix cI with c in* $\mathbb{C}\setminus\{z: z \geqslant 0\}$.

THEOREM 2.21. *A finite matrix T is the product of finitely many nonnegative matrices if and only if* $\det T \geqslant 0$. *The length of nonnegative matrices is* 5.

Further generalizations of the above results concern the products of accretive and positive stable operators. Recall that an operator T is *accretive* if its real part $\operatorname{Re} T = \frac{1}{2}(T + T^*)$ is positive, and *positive stable* if $\operatorname{Re} \sigma(T) > 0$. Note that, on finite-dimensional spaces, positive operators are accretive, accretive operators are positive stable, and positive stable operators are invertible. Products of these operators have been completely characterized in [12]. The following are some sample results from there.

THEOREM 2.22. *Let T be a finite matrix.*

(1) *T is the product of two accretive matrices if and only if all the real eigenvalues of T are positive.*

(2) *T is the product of finitely many accretive matrices if and only if T is nonsingular. The length of accretive matrices is* 3.

THEOREM 2.23. *Let T be a finite matrix.*

(1) *T is the product of two positive stable matrices if and only if T is nonsingular and is not a negative scalar matrix.*

(2) *T is the product of finitely many positive stable matrices if and only if T is nonsingular. The length of positive stable matrices is* 3.

The proofs of these theorems need a well-known result of Lyapunov's [49] that a finite matrix is the product of a positive one and an accretive one, in any order, if and only if it is positive stable.

We conclude this section with results on the factorizations with normal, Hermitian, positive, and nonnegative factors on infinite-dimensional spaces. Note that the normal factorization in finite dimensions is trivial, since, according to the polar decomposition, every finite matrix is the product of a unitary matrix and a nonnegative one, both of which are normal. In infinite-dimensional case, things become more interesting. There are operators, namely, the unilateral shift, which cannot be factored as the product of finitely many normal operators (cf. [38, Solution 144(a)]). The characterization of operators expressible as such has been obtained only recently in [69]. As it turns out, the classes of operators expressible as products of normal, Hermitian, or nonnegative operators are identical. More precisely, we have the following

THEOREM 2.24. *The following statements are equivalent for an operator T on an infinite-dimensional space:*

(1) *T is the product of finitely many normal operators;*
(2) *T is the product of finitely many Hermitian operators;*
(3) *T is the product of finitely many nonnegative operators;*
(4) $\dim\ker T = \dim\ker T^*$ *or* $\operatorname{ran} T$ *is not closed.*

Moreover, in this case, the number of normal factors may be limited to 3, *that of Hermitian factors to* 6, *and that of nonnegative factors to* 18.

An immediate corollary is the following

COROLLARY 2.25. *Any compact operator on an infinite-dimensional space is the product of finitely many nonnegative operators.*

For positive products, we have

THEOREM 2.26. *An operator on an infinite-dimensional space is the product of finitely many positive operators if and only if it is one-to-one with dense range. In this case,* 17 *factors suffice.*

It is not known whether the numbers 6, 18, and 17 here are best possible in each case. As for the proof, the necessity of condition (4) in Theorem 2.24 follows by an easy argument based on the theory of Fredholm operators (cf. [69, Proposition 2.4]). For the sufficiency, if $\dim\ker T = \dim\ker T^*$, then T is the product of two normal operators by the polar decomposition; otherwise (say, $\dim\ker T = \dim\ker T^* + 1$), T may be factored, as in Example 2.11, as the product of an operator A with $\dim\ker A = \dim\ker A^*$ and an orthogonal projection B, whence it is a product of three normal operators. For the general case $\dim\ker T \geqslant \dim\ker T^* + 2$, the above construction may be carried through analogously via the result that for nonclosed $\operatorname{ran} T$, there always exists a closed, infinite-dimensional subspace K such that $K \cap \operatorname{ran} T = \{0\}$ (cf. [69, Lemma 2.5]). The implication (1) $\Rightarrow$ (2) in Theorem 2.24 follows by a result of Radjavi's [52] that any normal operator is the product of four Hermitian operators, which is proved by an elaborate argument based on the spectral theorem for normal operators. For the proof of (2) $\Rightarrow$ (3), we need only consider factoring unitary operators into positive ones due to the polar decomposition. Since every unitary operator is the product of four symmetries (cf. Theorem 3.12 below), this is further reduced to the factorization of symmetries. Using Ballantine's characterizations of products of positive matrices (Theorems 2.12, 2.16, and 2.17), it can be shown that every symmetry is the product of six positive operators [69, Lemma 2.7] thus completing the proof of (2) $\Rightarrow$ (3).

3. INVOLUTION

An operator T is an *involution* if $T^2 = I$. In this section, we consider products of involutions and operators from its various subclasses. We start with the product of two involutions on a finite-dimensional space. The following characterization has been obtained by various mathematicians (cf. [66], [24], [44], and [8]):

THEOREM 3.1. *Let T be a finite matrix, and let $J=\Sigma_i \oplus J_{k_i}(a_i)$ be its Jordan canonical form, where each*

$$J_{k_i}(a_i)=\begin{bmatrix} a_i & & & & \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & 1 & a_i \end{bmatrix}$$

is a Jordan block with eigenvalue a_i and size k_i. Then the following statements are equivalent:

(1) *T is the product of two involutions;*
(2) *T is nonsingular and is similar to T^{-1};*
(3) *except those $J_{k_i}(a_i)$ with $a_i=\pm 1$, all the rest are in pairs $J_{k_j}(a_j)$ and $J_{k_l}(a_l)$ such that $k_j=k_l$ and $a_j a_l=1$.*

(1) $\Rightarrow$ (2) is true for operators even on infinite-dimensional spaces, and the proof of the rest is an easy exercise in linear algebra. It is unknown whether (2) $\Rightarrow$ (1) in general. That is true if T is normal (cf. [54, p. 12]).

Which operators can be factored as the product of three involutions? The complete characterization is not known at present even on finite-dimensional spaces. Here are some fragmentary results obtained recently by K.-M. Liu [48].

THEOREM 3.2. *An $n\times n$ matrix T with $\det T=\pm 1$ and satisfying $\dim\ker(T-zI)\leqslant[n/2]$ for any complex number z is the product of three involutions.*

Its proof is rather long. It is based on Theorem 3.1, the following lemma, and the idea of representing T with respect to a new basis as constructed in [13].

LEMMA 3.3. *Let T be an $n\times n$ nonsingular cyclic matrix. If $a_1, a_2,\dots,a_n$ are complex numbers satisfying $\prod_{j=1}^n a_j=-\det T$, then there exist an involution A and a cyclic matrix B with $\sigma(B)=\{a_1,\dots,a_n\}$ such that $T=AB$.*

From Theorem 3.2, we can easily derive the following corollary, which has also been noted in [35] and [9].

COROLLARY 3.4. *If the rational form of a matrix with determinant* ± 1 *has at most two blocks, then it is the product of three involutions.*

The next result, from [48], gives a necessary condition for the product of three involutions. Note that the gap of the conditions in Theorems 3.2 and 3.5 is between $n/2$ and $3n/4$.

THEOREM 3.5. *If the* $n \times n$ *matrix* T *is the product of three involutions, then* $\dim\ker(T - zI) \leqslant [3n/4]$ *for any complex number* z, $z^4 \neq 1$.

In [9], $n \times n$ matrices expressible as the product of three involutions are completely characterized for n up to 4. This is further pushed to five in [48].

So much for the product of three involutions. A result of Gustafson, Halmos, and Radjavi [35] takes care of the products of any finite number of involutions.

THEOREM 3.6. *A finite matrix is the product of a finite number of involutions if and only if its determinant is* ± 1. *The length of involutions is* 4.

The proof in [35] exploits the similarity between certain $n \times n$ weighted permutation matrices and the permutations of the index set $\{1, 2, \ldots, n\}$, and essentially reduces the problem to that of expressing a permutation as the composite of two involutory permutations. Sourour [60] gives a simple proof of this based on his factorization theorem, Theorem 2.18 (cf. also [9] and [47]). That the length is four is seen by the matrix ωI_n, where $\omega = \frac{1}{2}(-1 + \sqrt{3}\, i)$ (cf. [38, Solution 143]).

For products of involutions on infinite-dimensional spaces, much less is known. The following is one positive result (cf. [54]).

THEOREM 3.7. *An operator on an infinite-dimensional space is the product of finitely many involutions if and only if it is invertible. In this case, seven involutions suffice.*

It is not known whether seven is the smallest such number, although it was shown in [54] that four won't be enough.

A class of operators analogous to involutions is obtained by replacing the requirement ± 1 with modulus equal to 1. Formally, a finite matrix T is a *pseudoinvolution* if $T\overline{T} = I$, where $\overline{T}$ denotes the entrywise conjugate of T. This notion was first introduced by Ballantine [11]. The factorizations into pseudoinvolutions are very much like those into involutions. Indeed, the following are proved in [11].

THEOREM 3.8. *A finite matrix T is the product of two pseudoinvolutions if and only if T is nonsingular and is similar to T^{*-1}.*

THEOREM 3.9. *A finite matrix T is the product of finitely many pseudoinvolutions if and only if $|\det T| = 1$. The length of pseudoinvolutions is* 4.

Recall that a matrix T is *orthogonal* if $T^tT = TT^t = I$. If we require the involution to be orthogonal, then we have the orthogonal involutions. In other words, T is an *orthogonal involution* if $T = T^{-1} = T^t$. Wonenburger [65, 66] proved the following result.

THEOREM 3.10. *Any orthogonal finite matrix is the product of two orthogonal involutions.*

In particular, the length of orthogonal involutions is 2. This result is easiest to perceive for real orthogonal matrices, since in this case it is orthogonally similar to a matrix of the form

$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}$$

$$\oplus \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix},$$

and the rotation in the plane corresponding to each

$$\begin{bmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{bmatrix}$$

is, geometrically, the product of two reflections.

An operator T is a *symmetry* if it is a unitary involution, that is, T satisfies $T = T^{-1} = T^*$. On finite-dimensional spaces, products of symmetries are characterized by Radjavi [51].

THEOREM 3.11. *A finite matrix T is the product of finitely many symmetries if and only if T is unitary and* $\det T = \pm 1$. *The length of symmetries is* 4.

Note the similarity of this theorem to Theorem 3.6 for products of involutions. On infinite-dimensional spaces, the product of symmetries was considered much earlier by Halmos and Kakutani [39] (cf. also [38, Problem 143]).

THEOREM 3.12. *An operator on an infinite-dimensional space is the product of finitely many symmetries if and only if it is unitary. The length of symmetries is* 4.

To achieve this factorization, first factor the unitary operator into the product of a right shift and a left shift, and then factor each of the latter as the product of two symmetries. The factorizations make use of the spectral theorem for unitary operators. The length assertion follows by Theorem 2.6.

For the products of two symmetries, the characterization is due to Davis [20], and holds on an arbitrary Hilbert space.

THEOREM 3.13. *An operator T is the product of two symmetries if and only if T is unitary and T is similar to T^{-1}.*

In [55], this is derived as a corollary of a more general result concerning the product of a symmetry and a Hermitian operator whose proof, again, uses the spectral theorem for unitary operators.

It is not known which operators can be expressed as the product of three symmetries on either finite or infinite-dimensional spaces.

Next we consider factorizations into another kind of involutions, the so-called reflections. An operator T is a *reflection* if $T^2 = I$ and $\operatorname{rank}(T - I) = 1$. Geometrically, such a transformation is an "oblique" symmetry with respect to a hyperplane. Note that a finite matrix is a reflection if and only if it is similar to the diagonal matrix

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{bmatrix}.$$

Products of reflections were first considered by Radjavi [53]. (There reflections are called simple involutions.) He showed that any $n \times n$ ($n \geqslant 2$) matrix with determinant ± 1 is the product of at most $2n - 1$ reflections. The number of factors is reduced to the optimal $n+2$ for $n \geqslant 3$ by Cater [19]. More precisely, he proved

THEOREM 3.14. *An $n \times n$ matrix T is the product of finitely many reflections if and only if* $\det T = \pm 1$. *More precisely, for $n \geqslant 3$,*

(1) *if T is nonscalar and* $\det T = (-1)^n$, *then at most n factors are needed;*

(2) *if* $\det T = (-1)^{n+1}$, *then at most $n+1$ factors are needed;*

(3) *if T is scalar and* $\det T = (-1)^n$, *then at most $n+2$ factors are needed.*

If $n = 2$, then at most 3 factors are needed. In each case, the respective number is the smallest possible. In particular, the length of reflections is 3 if $n = 2$ and $n+2$ if $n \geqslant 3$.

The proof of this theorem depends on a factorization theorem developed by Cater [19, Lemma 8]. The assertions above can be further improved by precisely determining the length, with respect to the reflections, of each matrix with determinant ± 1. This is done independently by Ellers [28], Djoković and Malzan [27], and Yuan and Zou [70].

THEOREM 3.15. *Let T be an $n \times n$ $(n \geqslant 2)$ matrix with* $\det T = \pm 1$, *and let*

$$\varepsilon(T) = \begin{cases} 0 & \text{if } \det T = (-1)^{\operatorname{rank}(T-I)}, \\ 1 & \text{otherwise.} \end{cases}$$

Then the length of T with respect to the reflections equals $\operatorname{rank}(T-I) + 2 - \varepsilon(T)$ *if either T is similar to $I_m \oplus aI_{n-m}$, where $n - m \geqslant 2$ and $a \neq -1$, or $(T-I)^2 = 0$ and* $\operatorname{rank}(T-I) \geqslant 2$; *otherwise, it equals* $\operatorname{rank}(T-I) + \varepsilon(T)$.

Slightly more general than reflections is the class of dilatations. On a finite-dimensional space, a *dilatation* is one which is similar to a matrix of the form

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & a \end{bmatrix},$$

where $a \neq 0, 1$. Products of dilatations were considered by Djoković [25].

THEOREM 3.16. *An $n \times n$ matrix is the product of finitely many dilatations if and only if it is nonsingular. The length of dilatations is n.*

It can be easily seen that the matrix aI_n, where $a \neq 0$, 1, needs n dilatations in such a factorization.

In the literature, there are factorizations of matrices into special kinds of reflections. One such factorization has orthogonal reflections as factors. An *orthogonal reflection* is an operator T such that $T = T^{-1} = T^t$ and $\operatorname{rank}(T - I) = 1$. The next theorem is a classical result of Cartan and Dieudonné's (cf. [23, p. 20]).

THEOREM 3.17. *An $n \times n$ matrix is the product of finitely many orthogonal reflections if and only if it is orthogonal. The length of orthogonal reflections is n.*

The exact length for such decompositions was obtained by Scherk [57].

THEOREM 3.18. *If T is a finite orthogonal matrix, then the length of T with respect to orthogonal reflections equals* $\operatorname{rank}(T - I) + 2$ *or* $\operatorname{rank}(T - I)$ *depending on whether $T - I$ is skew-symmetric or not.*

We conclude this section by considering the factorization into unitary reflections. As one might expect, a *unitary reflection* is a reflection which is also unitary. Factorizations of this type were first considered by Radjavi [53]. (Note that, in [53], unitary reflections are called simple symmetries.)

THEOREM 3.19. *A finite matrix T is the product of finitely many unitary reflections if and only if T is unitary and* $\det T = \pm 1$.

The length of T in the factorization above was obtained later by Djoković and Malzan [26]. To state the result, we need one more bit of notation. For an $n \times n$ unitary matrix T, let $\theta_1, \ldots, \theta_n$ $(0 \leqslant \theta_j < 2\pi)$ be the arguments of its eigenvalues, and let $k(T) = (1/\pi)(\theta_1 + \cdots + \theta_n)$. Note that $k(T)$ is a nonnegative integer if and only if $\det T = \pm 1$.

THEOREM 3.20. *If T is an $n \times n$ unitary matrix, $T \neq I_n$, and* $\det T = \pm 1$, *then its length with respect to the unitary reflections is* $\max\{k(T), k(\overline{T})\}$. *In particular, the length of unitary reflections is* $2n - 1$.

This latter length assertion was first conjectured by Radjavi [53] and can be seen by considering the matrix

$$T=\begin{bmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{bmatrix},$$

where $0<\theta_j<\pi$ for all j and $\theta_1+\cdots+\theta_n=\pi$, in which case $k(T)=1$ and $k(\overline{T})=2n-1$, whence the length of T equals $2n-1$.

4. PARTIAL ISOMETRY

We start this section by considering the factorization into partial isometries. Recall that an operator is a *partial isometry* if T is isometric on $(\ker T)^{\perp}$. As is well known, this is equivalent to requiring that $T=TT^*T$ (cf. [38, Corollary 3 to Problem 127]). On finite-dimensional spaces, the factorization problem with partial isometry factors is solved completely in [45].

THEOREM 4.1. *Let T be a finite matrix and $k\geqslant 1$. Then T is the product of k partial isometries if and only if T is a contraction* ($\|T\|\leqslant 1$) *and* $\operatorname{rank}(I-T^*T)\leqslant k\dim\ker T$.

Since the product of a unitary operator and a partial isometry is again a partial isometry, in the above factorization we need only consider, via the singular-value decomposition, a nonnegative diagonal matrix. The details are in [45]. As an immediate corollary, we obtain the following

COROLLARY 4.2. *An $n\times n$ matrix is the product of finitely many partial isometries if and only if it is either unitary or a singular contraction. The length of partial isometries is n.*

The optimality of n here is observed through matrices of the form

$$\begin{bmatrix} a_1 & & & \\ & \ddots & & \\ & & a_{n-1} & \\ & & & 0 \end{bmatrix},$$

where $0<a_j<1$ for all j.

Since the supply of partial isometries is more abundant on infinite-dimensional spaces, we would expect a simpler factorization in this case. This is indeed true, as proved by Brown [17]. Note that the product of partial isometries must be a contraction, since all the factors have norm 0 or 1. The next theorem says that the converse is also true.

THEOREM 4.3. *If T is a contraction on an infinite-dimensional space, then there exist two unilateral shifts S_1 and S_2 with* $\operatorname{corank} S_1 = \operatorname{corank} S_2 = \infty$ *such that $T = S_1^* S_2$.*

If we only require that S_1 and S_2 be isometries, then an easier proof, based on the notion of unitary dilations of a contraction, has been given by Arveson (cf. [41, Lemma 2.1]).

Historically, the result above was preceded by the factorizations into unilateral shifts or unitary operators (cf. [37]).

THEOREM 4.4. *On an infinite-dimensional space, an operator T is the product of a unitary operator and a unilateral shift with* corank n *if and only if T is an isometry and* $\operatorname{corank} T = n$.

THEOREM 4.5. *On an infinite-dimensional space, an operator T is the product of two unilateral shifts if and only if T is an isometry and* $\operatorname{corank} T \geqslant 2$.

The proofs are through some clever constructions. Factorizations of the above type with more precise conditions on the multiplicities of the unilateral shift factors seem worth investigating.

A special kind of partial isometry is the orthogonal projection. Another result in [45] is concerned with the factorization into operators of this type. Here the situation is quite different from what we have discussed so far in that the number of factors can be arbitrarily large.

THEOREM 4.6. *An $n \times n$ matrix T is the product of finitely many orthogonal projections if and only if T is unitarily equivalent to a matrix of the form $\begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix}$, where S is singular with $\|S\| < 1$. The length of orthogonal projections is infinity for $n \geqslant 2$.*

The proof is based on some geometric considerations. The main step consists in transforming x to y through finitely many orthogonal projections, for any given vectors x and y satisfying $\|x\| > \|y\|$. As for the length, it was proved that, for any $m \geqslant 2$, the singular strict contraction

$$\begin{bmatrix} 0 & \left(\cos \dfrac{\pi}{2m}\right)^m \\ 0 & 0 \end{bmatrix}$$

can be factored as the product of no fewer than $m+1$ orthogonal projections (cf. [45, Lemma 3.6]).

For the products of two orthogonal projections, Crimmins found a characterization (cf. [55, Theorem 8]).

THEOREM 4.7. *An operator T is the product of two orthogonal projections if and only if $TT^*T = T^2$. In this case, $T = P_1P_2$, where P_1 and P_2 are the orthogonal projections onto $\overline{\operatorname{ran} T}$ and $(\ker T)^\perp$, respectively.*

Dropping the requirement of self-adjointness from the definition of orthogonal projections, we obtain the class of *idempotent operators*, that is, operators T satisfying $T^2 = T$. The products of idempotent operators on finite-dimensional spaces were first considered by Erdos [29]. He showed that every singular matrix can be written as a product of idempotent operators. This result was also obtained independently by Hawkins and Kammerer [40] and Dawlings [21]. [In the former, it was shown more generally that every finite-rank operator on a (possibly infinite-dimensional) Banach space can be expressed as such.] The final words on this matter are contained in [10].

THEOREM 4.8. *Let T be a finite matrix and $k \geqslant 1$. Then T is the product of k idempotent matrices if and only if* $\operatorname{rank}(T - I) \leqslant k \dim\ker T$.

The proof involves only elementary matrix operations and thus is constructive. Note the similarity of the conditions in Theorems 4.1 and 4.8, which links the factorization with partial isometry factors and that with idempotent factors. From Theorem 4.8, we can easily derive the following

COROLLARY 4.9. *An $n \times n$ matrix is the product of finitely many idempotent matrices if and only if it is either the identity matrix or singular. The length of idempotent matrices is n.*

The optimality of n is realized by the matrix

$$\begin{bmatrix} 0 & & & & \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & 1 & 0 \end{bmatrix}.$$

For infinite-dimensional spaces, a complete characterization of products of idempotent operators was obtained by Dawlings [22].

THEOREM 4.10. *On an infinite-dimensional space H, an operator T is the product of finitely many idempotent operators if and only if one of the following holds:*

(1) $T = I$;
(2) $\dim\ker T = \dim\ker T^* = \infty$;
(3) $0 < \dim\ker T = \dim\ker T^*$, $\dim F_T^{\perp} < \infty$,

where $F_T = \{x \in H : Tx = x\}$ is the subspace of fixed points of T. The length of idempotent operators is infinity.

Indeed, in [22], it was shown that the operator $T = I \oplus S$, where

$$S = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & & 1 & 0 \end{bmatrix}$$

is of size n, can be factored into no fewer than n idempotent operators.

The analogous factorization problem for linear transformations on an arbitrary vector space was considered in [56].

5. MISCELLANIES

In this section, we will consider factorizations into operators in other miscellaneous classes. These include nilpotent, quasinilpotent, and unipotent operators, EP matrices, and commutators (both additive and multiplicative).

For nilpotent operators on finite-dimensional spaces, the factorization problem was solved recently by Wu [67]. Recall that an operator T is *nilpotent* if $T^n = 0$ for some $n \geqslant 1$.

THEOREM 5.1. *A finite matrix is the product of finitely many nilpotent matrices if and only if it is singular. The length of nilpotent matrices is* 2.

In fact, more precise information concerning the factors is proved in [67]: any singular T can be expressed as the product of two nilpotent matrices A and B with $\operatorname{rank} A = \operatorname{rank} B = \operatorname{rank} T$ except when T is a 2×2 nilpotent matrix of rank one. The proof is via factoring the Jordan canonical form of T. It may also be proved using the rational form, thus extending the result to matrices over arbitrary fields. Whether a result of this nature holds for operators on an infinite-dimensional space is not known. As a matter of fact, this is unknown even if we enlarge our candidates for the factors from nilpotent to quasinilpotent operators. Recall that an operator T is *quasinilpotent* if its spectrum $\sigma(T)$ consists of zero only. Note that on finite-dimensional spaces, quasinilpotent operators coincide with nilpotent ones. The factorization with quasinilpotent factors was considered in [30]. Although no complete characterization is obtained, some partial results are quite interesting. The next two theorems give necessary or sufficient conditions for an operator to be the product of two quasinilpotent operators on infinite-dimensional spaces. Here $\sigma_e(A)$ denotes the essential spectrum of an operator A.

THEOREM 5.2. *If T is a product of two quasinilpotent operators, then* $0 \in \sigma_e(T^*T) \cap \sigma_e(TT^*)$.

THEOREM 5.3. *If* $0 \in \sigma_e(T^*T + TT^*)$, *then T is the product of two quasinilpotent operators.*

For special kinds of operators, characterizations are known.

THEOREM 5.4. *Every compact operator is the product of two compact quasinilpotent operators.*

THEOREM 5.5. *A normal operator T is the product of two quasinilpotent operators if and only if* $0 \in \sigma_e(T)$.

The proofs of the above results make extensive use of the infinite-dimensionality of the underlying space, compact operator theory, and the spectral theorem for normal operators.

An operator T is *unipotent* if $(T - I)^n = 0$ for some $n \geqslant 1$. Note that, on finite-dimensional spaces, T is unipotent if and only if $\sigma(T) = \{1\}$. The factorization with unipotent factors was recently considered by Fong and Sourour [31]. They obtained the following characterizations.

THEOREM 5.6. *An $n \times n$ matrix T is the product of two unipotent matrices if and only if $T = I_n$ or T is a nonscalar matrix with* $\det T = 1$.

THEOREM 5.7. *A finite matrix T is the product of finitely many unipotent matrices if and only if* $\det T = 1$. *The length of unipotent matrices is* 3.

Alternative proofs, only slightly different, based on the factorization theorem (Theorem 2.18), are given in [60].

For operators on infinite-dimensional spaces, Fong and Sourour [31] proved the following

THEOREM 5.8. *An operator on an infinite-dimensional space is the product of finitely many unipotent operators if and only if it is invertible. In this case, six unipotent factors suffice.*

Whether six is optimal in the preceding theorem is not known, although, as in the finite-dimensional case, two unipotent factors are not enough.

A finite matrix T is an *EP matrix* if $\ker T = \ker T^*$; it is an *EP_r matrix* if, in addition, $\operatorname{rank} T = r$. In [7], Ballantine characterized the products of EP_r matrices.

THEOREM 5.9. *Let $0 \leqslant r_1, r_2 \leqslant n$. Then an $n \times n$ matrix T is the product of an EP_{r_1} and an EP_{r_2} matrix if and only if* $\min\{r_1, r_2\} \geqslant \operatorname{rank} T \geqslant r_1 + r_2 - n$ *and* $\operatorname{rank} T^2 \geqslant 3 \operatorname{rank} T - r_1 - r_2$.

THEOREM 5.10. *Let $0 \leqslant r_1, \ldots, r_m \leqslant n$. Then an $n \times n$ matrix T is the product of matrices of classes $EP_{r_1}, \ldots, EP_{r_m}$ if and only if* $\min\{r_1, \ldots, r_m\} \geqslant \operatorname{rank} T \geqslant \sum_{j=1}^{m} r_j - (m-1)n$.

Finally, we consider commutators and their products. An operator is a *commutator* if it can be expressed as $AB - BA$ for some operators A and B. On finite-dimensional spaces, we have the following characterization of commutators (cf. [58] or [1]).

THEOREM 5.11. *A finite matrix is a commutator if and only if its trace is zero.*

What are the products of commutators? If the underlying space is 1-dimensional, then the answer is trivial. The next theorem settles the remaining finite-dimensional case.

THEOREM 5.12. *Any $n \times n$ $(n \geqslant 2)$ matrix is the product of two commutators.*

This follows by Theorem 5.11, the observation that commutators are preserved under the similarity of matrices, and the following decompositions:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -b \\ a & 0 \end{bmatrix}$$

and

$$C \equiv \begin{bmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \cdots & 0 & a_2 \\ 0 & 1 & \cdots & 0 & a_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots 1 & 0 & a_{n-1} \\ 0 & 0 & \cdots 0 & 1 & a_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & -a_1/(n-1) \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & a_2 \\ 0 & 1 & \cdots & 0 & a_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_n \\ 0 & 0 & \cdots & 0 & -(n-1) \end{bmatrix}$$

$$(n \geqslant 2)$$

together with a similar factorization for $\begin{bmatrix} C & 0 \\ 0 & a \end{bmatrix}$. In particular, this shows that Theorem 5.12 holds for matrices over any field with characteristic 0.

On infinite-dimensional spaces, commutators have also been characterized after many years' effort (cf. [15] and [2]).

THEOREM 5.13. *The following conditions are equivalent for an operator T on an infinite-dimensional space:*

(1) *T is a commutator;*

(2) *T is not the sum of a nonzero scalar and a compact operator;*

(3) *the essential numerical range of T cannot consist of a single nonzero scalar.*

In this context, the products of commutators seem not to have been considered. It would also be interesting to characterize products of a special kind of commutators: *self-commutators*, operators of the form $A^*A - AA^*$. (For their characterizations, see [63] and [50].)

Another type of commutator is the multiplicative one. Recall that an operator T is a *multiplicative commutator* if $T = ABA^{-1}B^{-1}$ for some invertible operators A and B. The next theorem characterizes such commutators on finite-dimensional spaces. It was first obtained by Shoda [59]. Thompson [64] extended it to all fields with more than three elements; Sourour [60] gave another proof (for complex matrices) based on his factorization theorem, Theorem 2.18.

THEOREM 5.14. *A finite matrix is a multiplicative commutator if and only if its determinant is* 1.

For infinite-dimensional spaces, it was conjectured in [16] that an invertible operator T is a multiplicative commutator if and only if either T is not the sum of a nonzero scalar and a compact operator or T is such a sum with the scalar having modulus 1. Although some partial results are known, the general case remains open. In particular, it is known that any unitary operator is a multiplicative commutator (cf. [16, Corollary 3.2] or [38, Problem 239]). However, for products of multiplicative commutators, we have the following characterization (cf. [16, Corollary 4.7]).

THEOREM 5.15. *An operator on an infinite-dimensional space is the product of finitely many multiplicative commutators if and only if it is invertible. The length of multiplicative commutators is* 2.

The proof is rather involved. For an easier proof of factoring an invertible operator into three multiplicative commutators, consult [38, Problem 240].

6. APPLICATION

In this final section, we discuss an application of the previous factorization results. The application is due to Hochwald and concerns the uniqueness of the adjoint operation on operators (cf. [41], [42], and [43]). He considered an operation h on $\mathscr{B}(H)$, the algebra of operators on the space H, or some subset of $\mathscr{B}(H)$, and asked whether the properties $h(ST) = h(T)h(S)$ and $h(T)T \geqslant 0$ for all S and T in the domain of h will characterize the adjoint operation $T \to T^*$. The answer turns out to depend on the domain of h. It is affirmative when h is nontrivial and defined on all of $\mathscr{B}(H)$ for infinite-dimensional H. If h is defined only on part of $\mathscr{B}(H)$, then other possibilities such as the inverse operation $T \to T^{-1}$ may arise. The following are the precise statements. We first consider the operation h defined on $I(H)$, the set of all isometries on H.

THEOREM 6.1. *For an infinite-dimensional space H, the following statements are equivalent for a function $h: I(H) \to \mathscr{B}(H)$:*

(1) *$h(ST) = h(T)h(S)$ and $h(T)T \geqslant 0$ for all S and T in $I(H)$;*

(2) *there is a homomorphism ϕ from the additive semigroup $\{0, 1, 2, \ldots, \dim H\}$ to the multiplicative semigroup $\mathbb{R}^+$ such that $h(T) = \phi(\dim \ker T^*)T^*$ for all T in $I(H)$.*

A special case of the preceding theorem is the following

THEOREM 6.2. *Let H be infinite-dimensional, and $h: I(H) \to \mathscr{B}(H)$ be a function such that $h(U) \neq 0$ for some unilateral shift U of infinite multiplicity. Then $h(ST) = h(T)h(S)$ and $h(T)T \geqslant 0$ for all S and T in $I(H)$ if and only if $h(T) = T^*$ for all T in $I(H)$.*

Next, for h defined on all of $\mathscr{B}(H)$, we have

THEOREM 6.3. *Let H be infinite-dimensional and h be a function on $\mathscr{B}(H)$ such that $h(I) \neq 0$. Then $h(ST) = h(T)h(S)$ and $h(T)T \geqslant 0$ for all S and T in $\mathscr{B}(H)$ if and only if $h(T) = T^*$ for all T in $\mathscr{B}(H)$.*

All the theorems above are in [41]. Their proofs exploit the various factorization results discussed in previous sections. For example, Theorem 6.2 is proved by first showing that $h(I) = I$ and, successively, that h behaves properly for symmetries, unitary operators (using Theorem 3.12), unilateral shifts, and finally isometries (using Theorem 4.4). As for Theorem 6.3, we proceed from isometries through coisometries, contractions (using Theorem 4.3), and finally arbitrary operators in $\mathscr{B}(H)$.

If the domain of h is restricted to $\mathrm{GL}(H)$, the set of all invertible operators on H, then h can be either the adjoint or the inverse operation (cf. [43]).

THEOREM 6.4. *Let H be infinite-dimensional. If h is a function on $\mathrm{GL}(H)$ such that $h(ST) = h(T)h(S)$ and $h(T)T \geqslant 0$ for all S and T in $\mathrm{GL}(H)$, then $h(T) = T^*$ for all $T \in \mathrm{GL}(H)$ or $h(T) = T^{-1}$ for all $T \in \mathrm{GL}(H)$.*

As before, the assertion in the preceding theorem is proved successively for symmetries, scalar operators, direct sums of a scalar operator and the identity operator, diagonal positive operators, positive operators (by Weyl's theorem on the perturbation of positive operators by diagonal ones), and finally, invertible operators (via the polar decomposition).

The finite-dimensional version of the preceding results are in [42].

THEOREM 6.5. *Let H be finite-dimensional, and let h be a function on* $\mathrm{GL}(H)$. *Then the following statements are equivalent*:

(1) $h(ST)=h(T)h(S)$ *and* $h(T)T\geqslant 0$ *for all S and T in* $\mathrm{GL}(H)$;

(2) *there is a multiplicative homomorphism* $\sigma:\mathbb{C}\setminus\{0\}\to\mathbb{R}^{+}$ *such that either* $h(T)=\sigma(\det T)T^{*}$ *for all* $T\in\mathrm{GL}(H)$ *or* $h(T)=\sigma(\det T)T^{-1}$ *for all* $T\in\mathrm{GL}(H)$.

In particular, the following holds:

THEOREM 6.6. *Let H be finite-dimensional, and let h be a function on* $\mathscr{B}(H)$ *such that* $h(I)\neq 0$. *Then* $h(ST)=h(T)h(S)$ *and* $h(T)T\geqslant 0$ *for all S and T in* $\mathscr{B}(H)$ *if and only if either* $h(T)=0$ *for all singular T and h is given as in Theorem* 6.5(2) *on* $\mathrm{GL}(H)$ *or* $h(T)=T^{*}$ *for all T on H.*

The idea of the proofs is the same as before: working through various classes of operators using the factorizations if possible.

This concludes the paper. We hope that it fulfills our stated purpose of directing people's attention to this area of research and arousing their interest so that they work on problems of this nature and find more applications.

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