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Global Methods for Solving Systems of Nonlinear Algebraic Equations

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Abstract—Systems of nonlinear algebraic equations (SNAE) are ubiquitous in the many applications requiring numerical simulation, and more robust and efficient methods for solving SNAE are continuously being sought. In this paper, we present an overview of existing algorithmic approaches for solving SNAE such as reduction to a Groebner basis, the multidimensional resultant method, and the spectral method. A major deficiency in all of these methods is the lack of a theoretical foundation that will allow *a priori* information about the number of solutions. In the present work, we recognize that SNAE are the principal object of an algebraic geometry and seek to derive qualitative criteria about the solution in an algebraic form. Desirable qualitative criteria include solvability and uniqueness. We show here that the problem of solving SNAE is equivalent to the problem of solving matrices of rank 1 in a given subspace of matrices. Recognizing such equivalencies is an important step to future success in developing improved methods for the solution of SNAE.

Keywords—Groebner bases, Multidimensional resultants, Singular pencils, Singular value decomposition, Jordan algebra.

1. INTRODUCTION

Nowadays numerical simulation is widely used in the practice of scientific research, design investigations, and industrial applications all over the world. The governments of the U.S.A., some countries of Europe, and Southeast Asia compel producers to create complete computer models of future technologies and installations before project development [1]. The great success in space technologies, design of the rational forms of flight vehicles, practice of optimum control, and development of different chemical technological processes are obliged to use so-called *computing experiments* and the information obtained in this way. Well-designed and well-simulated computing codes can provide information from calculations that very often is more comprehensive and less expensive than can be obtained by existing experimental methods.

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Modern computers with processors effectively realizing integer and floating point calculations carry out direct numerical simulation of various problems connected to the solution of nonlinear partial differential equations. The typical way for a numerical simulation is to transform from *a continuous model*, describing the initial system of the differential equations and boundary conditions, to *a discrete model*, obtained on the basis of various finite-difference schemes for the equations under investigation. This transformation results in a representation of the model in terms of a system of nonlinear algebraic equations (SNAE) of a large dimension. Such a transformation is necessary because in the modern methods of calculations, there is no possibility of direct implementation of many mathematical operators. For example, differential operators are approximated by difference operators, integral operators are approximated by summation operators, and so forth. Thus, it is necessary to take into account properties of an initial operator of the problem; otherwise, some characteristics can be distorted during numerical simulation.

This practice described above has generated intensive development of techniques for calculations and, in particular, methods for solution of SNAE that in the common form can be presented as

$$f_i(x_1, \dots, x_n) = 0, \qquad i = 1, \dots, m, \tag{1}$$

where f_i are polynomials from variables $x_1, \dots, x_n$. For example, most geometric objects (curves and surfaces) are formulated in terms of polynomial equations, thereby reducing many applications' problems to manipulating polynomials' systems. Therefore, solving SNAE is a fundamental problem in these geometric calculations. Applications that reduce to finding the roots of the nonlinear polynomial equations include

(i) the problems of curve intersection,
(ii) curve and surface ray-tracing,
(iii) collision detection,
(iv) computing the distance from a point to a curve, and
(v) finding a point on the bisector between two curves or a point equidistant from three curves.

Surface intersection algorithms use algebraic equations to find starting points on each component and to locate singularities. This field of research has received much attention through the academic and industrial communities for the last three decades.

One can classify current techniques for solving SNAE into two methods, *symbolic* and *numeric* ones. *Symbolic methods*, based on the theory of resultants and a Groebner basis, can eliminate variables, reducing the problem to finding the roots of univariate polynomials. However, the current algorithms are efficient only for low-degree polynomial systems (no more than three or four polynomials). The major problem arises from the possible ill-conditioned results of computing the roots of univariate polynomials having degree greater than 14 or 15. This possibility makes it difficult to implement these techniques using finite-precision arithmetic. Circumstances in many cases do not allow these methods to be used in the case of fixed accuracy arithmetic because of slow convergence of the appropriate algorithms.

Numeric techniques for solution of SNAE are based on either iterative or homotopy methods. As is known, the iterative methods, like the well-known Newton's method, are good only for local analysis and work well only with a good initial guess for each solution. Homotopy methods, based on continuation techniques, follow paths in complex space. In this theory, each path converges to a geometrically isolated solution. These methods have been implemented in a variety of applications, and, in practice, the different paths might not be geometrically isolated. This can cause problems with robustness. Moreover, continuation techniques are considered computationally demanding and are currently restricted to dense polynomial systems.

In the present paper, the features of implementation of various methods used in practice for solution of SNAE are considered, and a new approach to the problem of qualitative research in nonlinear systems of the algebraic equations is offered.

2. METHOD OF REDUCTION TO A GROEBNER BASIS

Recently, a method of solving SNAE (1) by reduction to a Groebner basis received widespread attention. The basis is constructed from the concept of ordering of monomials $x_1^{k_1} \cdots x_n^{k_n}$. For a solution to problems as defined in equation (1), the lexicographic ordering T, with which $x_1^{k_1} \cdots x_n^{k_n} <_T x_1^{l_1} \cdots x_n^{l_n}$, if $k_n < l_n$ or $k_n = l_n$, but $k_{n-1} < l_{n-1}$ or $k_n = l_n$, $k_{n-1} = l_{n-1}$, but $k_{n-2} < l_{n-2}$ and so on is the best ordering. Let us denote by F a system of polynomials $f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)$, standing on the left side of equation (1). For each polynomial, $g(x_1, \dots, x_n)$, we define the operation *reduction* $g \rightarrow_F h$ *(modulo* F*)* as follows: $h = buf$, where b is a number, u is some monomial, and f is some polynomial from F, such that the higher monomial (by the ordering T) of the polynomial buf is equal to any monomial of a polynomial g. Factor b thus is selected so that at a polynomial h, this monomial is absent. It is easy to show that any polynomial g by a finite sequence of reductions modulo F can be reduced to a normal form, that is to such polynomial h, which already cannot be reduced modulo F to another polynomial. Thus, the various sequences of reductions can reduce to various normal forms. The following concept is principal.

DEFINITION. *The set F of polynomials is called a Groebner basis, if for any polynomial $g(x_1, \dots, x_n)$ all its normal forms coincide.*

Buchberger in [2] has proved that for any system F of polynomials

$$f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n),$$

there is a Groebner basis G of polynomials $g_1(x_1, \dots, x_n), \dots, g_r(x_1, \dots, x_n)$, generating the same ideal, Ideal(G), in a ring $K(x_1, \dots, x_n)$ all polynomials from a variable $x_1, \dots, x_n$, as the system F. Therefore, the set of equations (1) is equivalent to a system

$$g_i(x_1, \dots, x_n) = 0, \qquad i = 1, \dots, r. \tag{2}$$

Thus, system (2) is easily solved in view of the following lemma [2].

LEMMA. *Let G be a Groebner basis relative to lexicographic ordering T of monomials, and let $x_1 <_T x_2 <_T \cdots <_T x_n$. Then, for any i ($i = 1, \dots, n$), the equality* $\mathrm{Ideal}(G) \cap K(x_1, \dots, x_n) = \mathrm{Ideal}(G \cap K(x_1, \dots, x_n))$ *holds.*

This lemma implies that system (2) with the set of polynomials

$$g_1(x_1, \dots, x_n), \dots, g_r(x_1, \dots, x_n),$$

a Groebner basis relative to a lexicographic ordering, has a triangular aspect. Thus, nonsolvability of system (1) is equivalent to the presence of a polynomial $g = 1$ in the corresponding system (2). If system (1) has a finite number of solutions, then its Groebner basis has exactly one polynomial $g_1(x_1)$. After solving the equations $g_1(x_1) = 0$ and substituting its roots in the remaining equations of system (2), we can apply this reasoning to the system so obtained for sequential determination of values of the remaining unknown variables $x_2, \dots, x_n$. The infinite number of solutions of the given set of equations (1) is equivalent to the absence of some variable x_i as argument of all polynomials which comprise the Groebner basis that allows this variable to be selected as a free unknown variable.

Buchberger [2] has constructed some algorithms for reducing a system of polynomials to a Groebner basis; but, nevertheless, until now it was impossible to consider the task of reduction to Groebner basis completely solved. Buchberger's algorithms are realized within the framework of systems of character evaluations MAPLE and Mathematica. The algorithm realization frequently takes a lot of computer time. Besides, as numerous experiments have shown, the complexity of the algorithm depends critically on the lexicographic ordering to the renaming of the unknowns [2].

Thus, there is no possibility of an *a priori* prediction of what renaming will give the least operating time.

The approach based on reduction to a Groebner basis can be generalized in the case that the evaluations are done in floating point arithmetic. However, as emphasized in [3], if the polynomials have a high degree or contain many terms, such an approach cannot be applied in practice. In fact, a Groebner basis in floating point arithmetic lexicographic ordering of variables cannot be calculated at this time for a polynomial of eighth degree with three variables when it contains only 21 terms.

3. MULTIDIMENSIONAL RESULTANT METHOD

Manocha [4] has offered an original algorithm for finding roots of polynomial equations. The algorithm is based on a combination of symbolic and numeric methods and uses multidimensional resultants to eliminate variables from polynomial equations. The introduction of resultants allows SNAE to be linearized with the resultant expressed in terms of matrices and determinants. In particular, Manocha has shown that the resultant of a polynomial equation system corresponds to the determinant of a matrix polynomial system's roots and reduces to an eigenvalue problem or a generalized eigenvalue of a matrix pencil. The algorithm developed, using LINPACK and EISPACK libraries of linear algebra, has three basic steps.

(1) Use a suitable resultant formulation to linearize the problem in terms of matrix polynomials.
(2) Reduce the problem to an eigenvalue problem.
(3) Compute the eigendecomposition of the given matrix and recover the common roots from the eigenvalues and eigenvectors.

All three parts of the algorithm are relatively simple to implement given the linear algebra routines. A major feature of the algorithm is that the numerical accuracy of the operations at each stage of the calculations is well understood. In fact, the higher multiplicity eigenvalues are determined from their condition numbers. For most cases, one can accurately compute the roots using the 64-bit IEEE floating-point arithmetic available on most modern workstations.

The algorithm described solves SNAE (1) assuming that they have a finite number of common solutions. The use of matrix polynomials reduces the problem of computing roots to one of finding the eigenvalues of a matrix polynomial and the corresponding vectors in its kernel.

For example, suppose the degrees of the equations in system (1) are $d_1, d_2, \dots, d_n$, respectively. By eliminating the variables $x_2, x_3, \dots, x_n$ from these equations, one obtains the resultant $R(x_1)$. The resultant is a polynomial in x_1 whose roots correspond to the x_1 coordinate of each solution of the given multivariate system. Different formulations of the resultant express it as the determinant of a matrix or the ratio of two determinants. In either case, entries of the resulting matrices are polynomial functions of x_1.

If a single matrix formulation is not possible for the given system, one can solve it using the u-resultant formulation [5]. Thus, it requires adding to the given system of equations a polynomial

$$f_{n+1}(x_1, x_2, \dots, x_n) = u_0 + u_1 x_1 + \dots + u_n x_n, \tag{3}$$

where u_1 is a symbolic variable. The resultant is obtained by eliminating the variables $x_1, x_2, x_3, \dots, x_n$ from the $n+1$ equations. The resultant, a polynomial in $u_0, u_1, \dots, u_n$, is called the u-resultant of the original SNAE. Moreover, the u-resultant is expressed as the ratio of two determinants, $\det(\mathbf{M})/\det(\mathbf{D})$. However, the entries of matrix $\mathbf{D}$ are independent of u. This independence follows from the theory of multidimensional resultants [6]. As a result, if matrix $\mathbf{D}$ is nonsingular, the resultant of $f_1, f_2, \dots, f_{n+1}$ corresponds exactly to the determinant $\mathbf{M}$. If $\mathbf{D}$ is singular, $\mathbf{M}$ is replaced by its largest nonsingular minor. Given $\mathbf{M}$, whose entries are polynomials u_l, one can factor the u-resultant corresponding to its determinant into linear factors of the form

$$\det(\mathbf{M}) = \prod_{i=1}^{k} (\alpha_{i0}\mathbf{u}_0 + \alpha_{i1}\mathbf{u}_1 + \cdots + \alpha_{in}\mathbf{u}_n), \tag{4}$$

where k is the total number of nontrivial solutions, and $(\alpha_{i0}, \alpha_{i1}, \ldots, \alpha_{in})$ are the projective coordinates of a solution of the given SNAE.

Let us choose a specialization of the variables $u_0 = x_1$, $u_1 = -1$, $u_2 = 0$, $\ldots$, $u_n = 0$. The determinant of $\mathbf{M}$ obtained after specialization is a polynomial in x_1, and its roots correspond exactly to the x_1-coordinate of each solution of the given multivariate system. Thus, the determinant corresponds exactly to $R(x_1)$, the resultant of $f_1, f_2, \ldots, f_n$, obtained after eliminating $x_2, x_3, \ldots, x_n$. As a result, given any system of n polynomial equations whose coefficients are numerical constants, one can eliminate $n-1$ variables and express the resultant as the determinant of $\mathbf{M}(x_1)$. So, multipolynomial resultants linearize SNAE. In other words, they reduce nonlinear algebraic equations, say, $f_1, f_2, \ldots, f_n$ to a linear system of algebraic equations

$$\mathbf{M}(x_1)\,(1 \quad x_2 \quad \cdots \quad x_n \quad \cdots \quad x_2^d \quad x_3^d \quad \cdots \quad x_n^d)^{\top} = (0 \quad 0 \quad \cdots \quad 0)^{\top}, \tag{5}$$

where $\mathbf{M}(x_1)$ is a square matrix and its entries are polynomials in x_1. The entries of the vector consist of power products of $x_1, x_2, \ldots, x_n$. This linearization has the property that for any given solution $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the given SNAE, $\mathbf{M}(\alpha_i)$ is a singular matrix. Thus, one obtains the vector in the kernel by substituting $x_1 = \alpha_1$, $x_2 = \alpha_2$, $\ldots$, $x_n = \alpha_n$ in the vector consisting of power products highlighted in the system of equation (1). This property is used along with those of matrix polynomials to compute the roots of SNAE.

The algorithm on the basis of multidimensional resultants offered by Manocha includes both numeric and symbolic calculations and is realized as a software package, written in a high level language for IBM RS/360 workstations.

4. SPECTRAL METHOD

In 1978, Kublanovskaya [7] offered a new approach to the solution of SNAE based on reduction of an initial task to spectral problems or systems of spectral problems for pencils of rectangular matrices. The construction of such pencils is stipulated on the algebraic structure of the null-space of a matrix of the SNAE which is presented in a special basis.

The Kronecker product of n vectors of an algebraic structure is used as such a basis for SNAE with n variables. The vector of an algebraic structure $\mathbf{x}_i$ for variable x_I is a column vector of all degrees of the given variable, except zero, that is

$$\mathbf{x}_i = [\,x_i^{s_i} \quad x_i^{s_i - 1} \quad \cdots \quad x_i \quad 1\,]^{\top}. \tag{6}$$

Its size is equal $l_i \times 1$, and $l_i = s_i + 1$. For convenience, the vector is noted as a line with the indication of a symbol of transpose $[\;]^{\top}$.

Then, the initial SNAE with p of the equations and n variables can be written as

$$\mathbf{A}_{p\times N}[\mathbf{x}_1 \otimes \mathbf{x}_2 \cdots \otimes \mathbf{x}_n] = \mathbf{0}. \tag{7}$$

This is a particular form of equation (1) in a special basis. The basis used for the algebraic structure, $[\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \cdots \otimes \mathbf{x}_n]$, can be compared to lexicographic ordering of variables in the Groebner basis method.

The approach offered by Kublanovskaya has been developed in papers [8,9], where the procedure of a systematic construction for spectral problems was updated and the practical algorithms of a solution were presented.

4.1. Construction of Systems of Spectral Problems

Here, we consider the construction of spectral problems and systems of spectral problems (SSP) for SNAE (7) as examples of solving equations with two and three variables.

Solving SNAE with two variables. We can describe SNAE with two variables using two equivalent forms

$$\mathbf{A}_x[\mathbf{x}\otimes\mathbf{y}] = \mathbf{0}, \tag{8a}$$

$$\mathbf{A}_y[\mathbf{y}\otimes\mathbf{x}] = \mathbf{0}, \tag{8b}$$

$\mathbf{x}, \mathbf{y}$ are vectors of algebraic structure, i.e.,

$$\mathbf{x} = \begin{bmatrix} x^{s_x} & \cdots & x & 1 \end{bmatrix}^\top, \quad \text{size of } l_x \times 1, \quad l_x = s_x + 1,$$

$$\mathbf{y} = \begin{bmatrix} y^{s_y} & \cdots & y & 1 \end{bmatrix}^\top, \quad \text{size of } l_y \times 1, \quad l_y = s_y + 1,$$

$\mathbf{A}_x, \mathbf{A}_y$ are constant matrices of size $p \times (l_x \times l_y)$; p is the number of equations.

SNAE with two variables may be reduced to the following system of spectral problems:

$$\begin{aligned} (\mathbf{a}_x - x\mathbf{b}_x)\,\mathbf{R}_x(x) &= 0,\\ (\mathbf{a}_y - y\mathbf{b}_y)\,\mathbf{R}_y(y) &= 0,\\ [\mathbf{R}_x(x) - \mathbf{R}_y(y)]\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix} &= 0. \end{aligned} \tag{9}$$

If the pencils have regular solutions, the eigenvectors' matrices are constant and do not depend on x and y, respectively. In this case, the solutions $\{x_i\}$ are the eigenvalues of the regular part $\mathbf{D}_f(x)$ of pencil $\mathbf{D}(x)$, and $\{y_i\}$ are the eigenvalues of the regular part $\mathbf{D}_f(y)$ of pencil $\mathbf{D}(y)$.

If the pencils have no regular solutions, then the solutions of SNAE may be generated by polynomial solutions of pencils $\mathbf{D}(x), \mathbf{D}(y)$. In this case, the eigenvectors' matrices are matrix polynomials and may be described as

$$\mathbf{R}_x(x) = \mathbf{S}_{\lambda_x}\lambda_x, \tag{10a}$$

$$\mathbf{R}_y(y) = \mathbf{S}_{\lambda_y}\lambda_y, \tag{10b}$$

where $\lambda_x = \begin{bmatrix} x^t & x^{t-1} & \cdots & x & 1\end{bmatrix}^\top$, of size $(t+1)\times 1$; $\lambda_y = \begin{bmatrix} y^v & y^{v-1} & \cdots & y & 1\end{bmatrix}^\top$, of size $(v+1)\times 1$. Substitute (10a,b) into (9) and make an auxiliary SNAE (ASNAE)

$$\begin{bmatrix}\mathbf{S}_{\lambda_x} & -\mathbf{S}_{\lambda_y}\end{bmatrix}\begin{bmatrix}\lambda_x\otimes\mathbf{X}\\ \lambda_y\otimes\mathbf{Y}\end{bmatrix} = 0, \tag{11}$$

where the vectors and matrices are the sizes given as follows:

$$\begin{aligned} &\text{size } \lambda_x = [t+1 1], && \text{size } \lambda_y = [v+1 1],\\ &\text{size } \mathbf{X} = [\mu_x 1], && \text{size } \mathbf{Y} = [\mu_y 1],\\ &\text{size } \mathbf{S}_{\lambda x} = [N-p \quad \mu_x\times(t+1)], && \text{size } \mathbf{S}_{\lambda y} = [N-p \quad \mu_y\times(v+1)],\\ &\text{size } [\mathbf{S}_{\lambda x} - \mathbf{S}_{\lambda y}] = [N-p \quad Nl], && Nl = \mu_x\times(t+1)+\mu_y\times(v+1). \end{aligned}$$

The matrices $\mathbf{S}_{\lambda x}$ and $\mathbf{S}_{\lambda y}$ can be calculated using the definition for polynomial solutions given in paper [10].

Let us construct SSP for auxiliary SNAE (11). Consider the following matrix:

$$Q_{0\lambda} = \ker[S_{\lambda x} - S_{\lambda y}] = \begin{bmatrix}Q_{0\lambda x}\\ Q_{0\lambda y}\end{bmatrix} = \begin{bmatrix}\lambda_x\otimes\mathbf{X}\\ \lambda_y\otimes\mathbf{Y}\end{bmatrix}, \tag{12}$$

with size of $Nl \times pl$, where pl is the number of equations in equation (11). Using the algebraic structure of vectors λ_x, λ_y, we construct the following SSP:

$$\begin{aligned} (\mathbf{a}_{\lambda x} - \lambda_x \mathbf{b}_{\lambda x})\,\mathbf{R}_{\lambda x} &= 0, \\ (\mathbf{a}_{\lambda y} - \lambda_y \mathbf{b}_{\lambda y})\,\mathbf{R}_{\lambda y} &= 0, \end{aligned} \tag{13}$$

where $\mathbf{D}_{\lambda x} = \mathbf{a}_{\lambda x} - \lambda_x \mathbf{b}_{\lambda x}$ is a matrix pencil of size $\mu_x t \times Nl - pl$; $\mathbf{D}_{\lambda y} = \mathbf{a}_{\lambda y} - \lambda_y \mathbf{b}_{\lambda y}$ is a matrix pencil of size $\mu_y v \times Nl - pl$. The admissible solutions of SNAE are generated by the regular parts, $\mathbf{D}_{f\lambda x}$ and $\mathbf{D}_{f\lambda y}$, of matrix pencils $\mathbf{D}_{\lambda x}$ and $\mathbf{D}_{\lambda y}$, respectively. If the regular blocks in both cases (regular and polynomial) generate no admissible eigenvalues, then the initial SNAE has an infinite set of solutions or is reducible. The latter means that initial SNAE should have been previously reduced to a Groebner basis.

Solving SNAE with three variables. The case of SNAE with three variables is described by the following equations:

$$\mathbf{A}_{xy}[\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}] = \mathbf{0}, \tag{14a}$$

$$\mathbf{A}_{xz}[\mathbf{x} \otimes \mathbf{z} \otimes \mathbf{y}] = \mathbf{0}, \tag{14b}$$

$$\mathbf{A}_{yz}[\mathbf{y} \otimes \mathbf{z} \otimes \mathbf{x}] = \mathbf{0}. \tag{14c}$$

By analogy with the case of two variables, we may generate the following systems of spectral problems:

$$\begin{aligned} (\mathbf{a}_{xy} - x\mathbf{b}_{xy})\,\mathbf{R}_{xy}(x) &= 0, \\ (\mathbf{a}_{yx} - y\mathbf{b}_{yx})\,\mathbf{R}_{yx}(y) &= 0, \\ [\mathbf{R}_{xy}(x) - \mathbf{R}_{yx}(y)] \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} &= 0, \end{aligned} \tag{15a}$$

$$\begin{aligned} (\mathbf{a}_{xz} - x\mathbf{b}_{xz})\mathbf{R}_{xz}(x) &= 0, \\ (\mathbf{a}_{zx} - z\mathbf{b}_{zx})\mathbf{R}_{zx}(z) &= 0, \\ [\mathbf{R}_{xz}(x) - \mathbf{R}_{zx}(z)] \begin{bmatrix} \mathbf{X} \\ \mathbf{Z} \end{bmatrix} &= 0, \end{aligned} \tag{15b}$$

$$\begin{aligned} (\mathbf{a}_{yz} - y\mathbf{b}_{yz})\,\mathbf{R}_{yz}(y) &= 0, \\ (\mathbf{a}_{zy} - z\mathbf{b}_{zy})\,\mathbf{R}_{zy}(z) &= 0, \\ [\mathbf{R}_{yz}(y) - \mathbf{R}_{zy}(z)] \begin{bmatrix} \mathbf{Y} \\ \mathbf{Z} \end{bmatrix} &= 0. \end{aligned} \tag{15c}$$

Now, we may solve these SSP with two variables and find all possible triplets $\{x, y, z\}$; these solutions should be put into initial the SNAE and the residuals calculated. Grouping pairs of variables, we may solve SNAE with any number of them, so for n variables it is necessary to solve $\binom{n}{2}$ SSP equations with respect to two variables.

Within the framework of a system for engineering and scientific calculations, the appropriate algorithms have been realized as the SNAE Toolbox of MATLAB [11]. The approach explained was applied to the calculation of the stationary solutions for partial differential equations describing the process of diffusion for multicomponent isotope mixtures in a gas centrifuge [12].

5. A QUALITATIVE APPROACH TO SOLVING SNAE

The history of numerical methods for solving SNAE leads one to believe that this problem probably will be not solved completely in the near future. However, rapid development of computer capabilities during the last decade and, in particular, the emergence of parallel calculations opens a path to solving SNAE directly without a preliminary linearization (which can lead to the loss of some important physical effects).

The methods described above for solving SNAE based on a Groebner basis, multidimensional resultants, and spectral decomposition of singular pencils are algorithmic ones. They do not allow for the theoretical investigation of solvability of the equations, i.e., to know *a priori* the number of solutions before the end of an algorithm process.

SNAE are the principal object of an algebraic geometry, but in this theory, as a rule, the primary interest is in the properties of appropriate algebraic varieties. On the other hand, in a numerical solution, more practical problems such as criteria of solvability, uniqueness of a solution, etc., are important. The latter criteria may not be obtained by means of a pure algorithmic approach. Therefore, one reason to research the problem of solving SNAE is to derive qualitative criteria in an algebraic form.

In traditional techniques of solving SNAE, the process is reduced to a sequence of linear tasks and equations for one variable. In the present work, a contrasting approach is offered which consists of searching matrices of rank 1 in the given linear subspaces of a space of matrices.

Let us consider systems of nonlinear algebraic equations, each degree of which does not exceed two. Such systems arise frequently in cases of grid approximations of large numbers of equations from systems of mathematical physics.

In the present work, the problem of solving SNAE is shown to be equivalent to the problem of solving matrices of rank 1 in a given subspace of a space of matrices. Let

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)} x_i x_j + 2\sum_{i=1}^{n} a_i^{(k)} x_i + a^{(k)} = 0, \qquad 1 \le k \le p, \tag{16}$$

be the given system of p equations ($p \ge 1$) with real or complex entries, in which $A^{(k)} = (a_{ij}^{(k)})$ are symmetric matrices of a size $n \times n$. Let us denote through $\hat{A}^{(k)}$ an accompanying symmetric matrix of a size $(n+1) \times (n+1)$ of the polynomials in the left part of (16), that is, the matrix obtained by bordering matrix $A^{(k)}$ on the right and from below by a column and a row of numbers $a_1^{(k)}, \dots, a_n^{(k)}, a^{(k)}$. The null space of system (16) is a set L_0 of all symmetrical matrices B of size $(n+1) \times (n+1)$ satisfying a system of equalities for traces

$$\operatorname{tr}\left(\hat{A}^{(k)} B\right) = 0, \qquad 1 \le k \le p. \tag{17}$$

Let us note that the problem of determination of the space L_0 or, what is the same, of a basis $B^{(1)}, \dots, B^{(q)}$ of the space L_0, is the standard problem of linear algebra for determining a solution of a system of linear homogeneous equations.

Let $\mathbf{C}$ be some symmetric matrix of a size $(n+1) \times (n+1)$ and rank 1. Then, the matrix $\mathbf{C}$ can be written as

$$\mathbf{C} = (c_1, \dots, c_n, c_{n+1})^{\top} (c_1, \dots, c_n, c_{n+1}),$$

where $c_1, \dots, c_n, c_{n+1}$ are some ordered set of numbers. We will call such matrix $\mathbf{C}$ an affine one (rank 1), if $c_{n+1} = 1$. We use this term because the condition $c_{n+1} = 1$ selects some affine space in the projective space. If $\mathbf{C} = (c_1, \dots, c_n, 1)^{\top}(c_1, \dots, c_n, 1)$ is an affine matrix of rank 1, then it is easy to show, by immediate evaluations, the accuracy of the following equalities for any k, $1 \le k \le p$, for the trace:

$$\operatorname{tr}\left(\hat{A}^{(k)} C\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)} c_i c_j + 2\sum_{i=1}^{n} a_i^{(k)} c_i + a^{(k)}. \tag{18}$$

Therefore, if $\mathbf{C} \in L_0$, then the ordered set of numbers $c_1, \dots, c_n$ is a solution of system (16). Inversely, because of (18), for any solution $c_1, \dots, c_n$ of system (16), there is an affine (rank 1) matrix $\mathbf{C} = (c_1, \dots, c_n, c_{n+1})^{\top}(c_1, \dots, c_n, c_{n+1}) \in L_0$. Thus, the following theorem is proved.

THEOREM 1. *For each ordered set of numbers $c_1, \dots, c_n$, the matrix* $\mathbf{C}$ *may be formed as* $\mathbf{C} = (c_1, \dots, c_n, 1)^\top (c_1, \dots, c_n, 1)$ *that provides the one-to-one correspondence between all solutions of system (16) and the set of all affine matrices of rank 1 of zero-space of system (16).*

For practical use of Theorem 1, it is enough to determine a basis $\mathbf{B}^{(1)}, \dots, \mathbf{B}^{(q)}$ of the null space L_0 of system (16). Then, the problem of solving system (16) is equivalent to the problem of searching all such numbers $\alpha_1, \dots, \alpha_q$ for which matrices $\alpha_1\mathbf{B}^{(1)} + \cdots + \alpha_q\mathbf{B}^{(q)}$ are affine matrices of rank 1.

The theorem allows for simplification in the solving of some concrete systems of the type in equation (16). For example, all matrices of a basis $\mathbf{B}^{(1)}, \dots, \mathbf{B}^{(q)}$ in the null space L_0 of system (16) are real and pair-wise permutable matrices. One may show the existence of such a nondegenerate real (even orthogonal) matrix $\mathbf{T}$ that all matrices $\mathbf{T}^1\mathbf{B}^{(1)}\mathbf{T}, \dots, \mathbf{T}^1\mathbf{B}^{(q)}\mathbf{T}$ are diagonal ones. Let $\mathbf{T}^{-1}\mathbf{B}^{(l)}\mathbf{T} = \operatorname{diag}(\lambda_1^{(l)}, \dots, \lambda_{n+1}^{(l)})$, $1 \le l \le q$. In this case, $\alpha_1\mathbf{B}^{(1)}, \dots, \alpha_q\mathbf{B}^{(q)}$ is a matrix of rank 1 if and only if the diagonal matrix $\operatorname{diag}(\sum_{l=1}^q \alpha_l\lambda_1^{(l)}, \dots, \sum_{l=1}^q \alpha_l\lambda_{n+1}^{(l)})$ is a matrix of rank 1. That is, exactly one of its coefficients is not equal to 0. The determination of such numbers $\alpha_1, \dots, \alpha_q$ is a linear problem. Thus, solving system (16) in this case is reduced to the standard problem of simultaneous reduction to a diagonal form of several pair-wise permutable symmetric matrices and a sequence of linear problems.

We can extend all our previous considerations to arbitrary complex systems of the type

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{a}_{ij}^{(k)} x_i \bar{x}_j + \sum \left(\mathbf{a}_i^{(k)} x_i + \bar{\mathbf{a}}_i^{(k)} \bar{x}_i\right) + \mathbf{a}^{(k)} = 0, \qquad 1 \le k \le p, \tag{19}$$

in which all matrices $\mathbf{A}^{(k)} = (\alpha_{ij}^{(k)})$ are Hermitian ($\bar{x}_i, \bar{x}_j, \bar{a}_j^k$, as usual, denote the complex conjugate of x_i, x_j, a_i^k). In this case, the system may be reduced to a system with real coefficients. The null space of system (19) consists of Hermitian matrices.

Let us consider the following type of systems of nonlinear algebraic equations with unknowns $x_1, \dots, x_m, y_1, \dots, y_n$:

$$\sum\sum \mathbf{a}_{ij}^{(k)} x_i y_j + \sum \mathbf{a}_i^{(k)} x_i + \sum \mathbf{a}_j^{(k)} y_j + \mathbf{a}^{(k)} = 0, \qquad 1 \le k \le p. \tag{20}$$

The following differential equations of parabolic type:

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{v}\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} - \lambda_1 \mathbf{u} + f_1(x,t), \qquad \frac{\partial \mathbf{v}}{\partial \mathbf{t}} = \mathbf{u}\frac{\partial^2 \mathbf{v}}{\partial \mathbf{x}^2} - \lambda_2 \mathbf{v} + f_2(x,t),$$

lead to SNAE of type (20). The system (20) is a special case of system (16) with unknowns $x_1, \dots, x_m, x_{m+1} = y_1, \dots, x_{m+n} = y_n$, but this point of view calls to consideration matrices of size $(m+n+1) \times (m+n+1)$. We reduce this problem for system (20) to a matrix problem for matrices of size $(m+1) \times (n+1)$.

Let us denote by $\hat{\mathbf{A}}^{(k)}$ a matrix of a size $(m+1) \times (n+1)$, obtained by bordering a matrix $\mathbf{A}^{(k)}$ on the right and from below by a column of numbers $(\mathbf{a}_1'^{(k)}, \dots, \mathbf{a}_m'^{(k)}, \mathbf{a}^{(k)})^\top$ and a row $(\mathbf{a}_1''^{(k)}, \dots, \mathbf{a}_n''^{(k)}, \mathbf{a}^{(k)})$. We call the null space of system (20) a set L_0 of all matrices $\mathbf{B}$ of size $(m+1) \times (n+1)$, satisfying the following system of traces:

$$\operatorname{tr}\left(\mathbf{A}^{(k)}\mathbf{B}^t\right) = 0, \qquad 1 \le k \le p. \tag{21}$$

The affine matrix (rank 1) is a matrix of a size $(m+1) \times (n+1)$ of the following kind:

$$\mathbf{C} = (c_1', \dots, c_m', 1)^\top (c_1'', \dots, c_n'', 1). \tag{22}$$

By implementing the discussion used above for system (16), it is possible to show that the matrix defined by expression (21) belongs to the null space L_0 of system (20) only in the case when the ordered set of numbers $x_1 = c_1', \dots, x_m = c_m', y_1 = c_1'', \dots, y_n = c_n''$ is a solution of system (20). Therefore, the following theorem holds.

THEOREM 2. *The comparison to each ordered set of numbers* $c'_1, \dots, c'_m, c''_1, \dots, c''_n$ *of the matrix* $C = (c'_1, \dots, c'_m, 1)^\top (c''_1, \dots, c''_n, 1)$ *realizes the one-to-one correspondence of the set of all solutions of system (20) on the set of all affine (rank 1) matrices belonging to the null space of system (20).*

The common algebraic problem on the existence of matrices of rank 1 in matrix subspaces deserves independent interest. The following result specifies connection of this problem to the presence of some algebraic structures in given subspaces of matrices.

THEOREM 3. *Let us assume that* L_0 *is a linear subspace of the space of all square matrices of a given size. Then,* L_0 *contains a matrix of a rank 1 if and only if* L_0 *contains some Jordan subalgebra* J *as its linear subspace, and* J *has a matrix having even one simple nonzero characteristic number.*

PROOF. If L_0 contains some matrix $\mathbf{B}$ of rank 1, then the one-dimensional subspace $J = \{\alpha B \mid \alpha \in F\}$ is the required Jordan subalgebra. Inversely, let a linear subspace of matrices J be closed concerning operation $\mathbf{B}_1 \circ \mathbf{B}_2 = \mathbf{B}_1\mathbf{B}_2 + \mathbf{B}_2\mathbf{B}_1$. Then J, obviously, is closed for any degree of this matrix. Let $\mathrm{B} \in J$ and $\lambda_1, \dots, \lambda_k$ be the spectrum of the matrix $\mathbf{B}$, and λ_1 be a simple nonzero characteristic number. Then, if l is the nilpotency index of the matrix $\mathbf{B}$ (that is, the nilpotency index of the nilpotent part of a matrix $\mathbf{B}$ in its Jordan form), then the matrix $\mathbf{B}_1$ can be constructed

$$\mathbf{B}_1 = (\mathbf{B}(\mathbf{B} - \lambda_2\mathbf{E})(\mathbf{B} - \lambda_3\mathbf{E}) \cdots (\mathbf{B} - \lambda_k\mathbf{E}))^l,$$

where $\mathbf{E}$ is the unit matrix having rank 1 and one nonzero characteristic number which is equal to $\lambda_1^l(\lambda_1 - \lambda_2)^l \cdots (\lambda_1 - \lambda_k)^l$. Thus, Theorem 3 is proved.

In particular, if a Jordan algebra J of the vector space of matrices L_0 contains a nondecomposable matrix $\mathbf{B}$ with nonnegative entries, then J also contains a matrix of a rank 1, because the Perron root of the matrix $\mathbf{B}$ is its simple nonzero characteristic number. A similar approach allows matrices of rank 1 to be found in the null space L_0 of system (16), when L_0 contains a compact matrix group G.

6. CONCLUSION

Several approaches to the SNAE solution, based on methods of Groebner basis, multidimensional resultants, and spectral decomposition of singular pencils have been presented.

The algorithmic character of these methods does not provide answers to a line of algebraic questions connected with the solution and its structure for different classes of SNAE. However, it offers a new approach for establishing equivalence of the problem of solving SNAE to a problem of the presence of rank 1 matrices in a subspace of matrices that was constructed by a special way for given SNAE. The correlation of the last problem with the presence of some algebraic structures in given subspaces of matrices has been demonstrated. In particular, if a Jordan subalgebra J of a linear subspace of matrices L_0 contains a nondecomposable matrix $\mathbf{B}$ with nonnegative factors, J contains a matrix of a rank 1.

Based on mathematical models developed in floating point arithmetic, the numerical modelling requires adequate software with algorithms suitable for solution of partial differential equations.

The authors have much experience in the implementation of the systems of numerical computation MATLAB and MathCAD as well as the systems for symbolic calculations REDUCE, MAPLE, and Mathematica. This experience has shown that a combination of a system of numerical computation with MATLAB and a system of symbolic calculations with MAPLE provides an excellent way to find the solution of modelling problems arising from nonlinear partial differential equations. Moreover, the fundamental problem of the analysis of discrete models for partial differential equations connected to the SNAE solution by direct methods using a Groebner basis leads to the necessity of application of systems of symbolic calculations, the most mobile of which is the MAPLE system. This is due to the circumstance that the core of the MAPLE system is integrated in the MATLAB system as a package of the applied programs Extended Symbolic

Mathematics Toolbox. Thus, the necessary interface of data transfer between both systems is achieved in a standard way.

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