ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Topology 43 (2004) 903–924
TOPOLOGY
www.elsevier.com/locate/top

Infra-solvmanifolds and rigidity of subgroups in solvable linear algebraic groups

Oliver Baues*

Departement Mathematik, ETH-Zentrum, Rämistrasse 101, Zürich CH-8092, Switzerland

Received 20 June 2003; accepted 18 November 2003

Abstract

We give a new proof that compact infra-solvmanifolds with isomorphic fundamental groups are smoothly diffeomorphic. More generally, we prove rigidity results for manifolds which are constructed using affine actions of virtually polycyclic groups on solvable Lie groups. Our results are derived from rigidity properties of subgroups in solvable linear algebraic groups.

MSC: 57S30; 57S25; 22E25; 22E40

Keywords: Aspherical manifold; Polycyclic group; Solvable Lie group; Rigidity of group actions; Solvmanifold

1. Introduction

A closed manifold M is called topologically rigid if every homotopy equivalence $h:N \to M$ from another manifold N is homotopic to a homeomorphism. The Borel conjecture expects every closed aspherical manifold to be topologically rigid. The manifold M is called smoothly rigid if every homotopy equivalence is homotopic to a diffeomorphism. Geometric methods are useful to prove smooth rigidity inside some classes of closed aspherical manifolds. Well-known cases are, for example, locally symmetric spaces of non-compact type [26], or flat Riemannian manifolds [6]. In this paper, we study the smooth rigidity problem for infra-solvmanifolds. These manifolds are constructed by considering isometric affine actions on solvable Lie groups.

The fundamental group of an infra-solvmanifold is a virtually polycyclic group. A result of Farrell and Jones [11] on aspherical manifolds with virtually polycyclic fundamental group shows that infra-solvmanifolds are topologically rigid. Yet, an argument due to Browder [9] implies that there

* Tel.: +41-1632-7024; fax: +41-1632-1570.
E-mail address: oliver@math.ethz.ch (O. Baues).

doi:10.1016/j.top.2003.11.002

exist smooth manifolds which are homeomorphic but not diffeomorphic to the n-torus, for $n \geqslant 5$. Farrell and Jones [12] proved that any two compact infrasolvmanifolds of dimension not equal to four, whose fundamental groups are isomorphic, are diffeomorphic. This generalizes previous results of Bieberbach [6] on compact flat Riemannian manifolds, Mostow [22] on compact solv-manifolds and of Lee and Raymond [19] on infra-nilmanifolds. The proof of Farrell and Jones requires smoothing theory and the topological rigidity result. Recent results of Wilking on rigidity properties of isometric actions on solvable Lie-groups [31] imply the smooth rigidity of infra-solvmanifolds in all dimensions, giving an essentially geometric proof.

In well-known cases, smooth rigidity properties of geometric manifolds are closely connected to rigidity properties of lattices in Lie groups. The aim of the present paper is to establish the smooth rigidity of infra-solvmanifolds from natural rigidity properties of virtually polycyclic groups in linear algebraic groups. More generally, we prove rigidity results for manifolds which are constructed using affine, not necessarily isometric, actions of virtually polycyclic groups on solvable Lie groups. This approach leads us to a new proof of the rigidity of infra-solvmanifolds, and also to a geometric characterization of infra-solvmanifolds in terms of polynomial actions on affine space $\mathbb{R}^n$. As an application of the latter point of view we compute the cohomology of an infra-solvmanifold using the finite-dimensional complex of polynomial differential forms. This generalizes a result of Goldman [14] on compact complete affine manifolds. As another application, we show that every infra-solvmanifold has maximal torus-rank. Our approach towards rigidity of infra-solvmanifolds also suggests to study the rigidity-problem for the potentially bigger class of manifolds which are constructed using affine actions of virtually polycyclic groups on solvable Lie groups. Our main result establishes smooth rigidity for virtually polycyclic affine actions if the holonomy of the action is contained in a reductive group, generalizing the particular case of isometric actions.

Infra-solvmanifolds: We come now to the definition of infra-solvmanifolds. Let G be a Lie-group and let $\mathrm{Aff}(G)$ denote the semi-direct product $G \rtimes \mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ is the group of automorphisms of G. We view $\mathrm{Aff}(G)$ as a group of transformations acting on G. If Δ is a subgroup of $\mathrm{Aff}(G)$ then let Δ_0 denote its connected component of identity, and $\mathrm{hol}(\Delta) \leqslant \mathrm{Aut}(G)$ its image under the natural homomorphism $\mathrm{Aff}(G) \to \mathrm{Aut}(G)$.

Definition 1.1. An *infra-solvmanifold* is a manifold of the form $\Delta \setminus G$, where G is a connected, simply connected solvable Lie group, and Δ is a torsion-free subgroup of $\mathrm{Aff}(G)$ which satisfies (1) the closure of $\mathrm{hol}(\Delta)$ in $\mathrm{Aut}(G)$ is compact.

The manifold $\Delta \setminus G$ is a smooth manifold with universal cover diffeomorphic to $\mathbb{R}^m$, $m = \dim G - \dim \Delta_0$, where Δ_0 is the connected component of identity in Δ. The fundamental group of $\Delta \setminus G$ is isomorphic to $\Gamma = \Delta/\Delta_0$. It is known that $\Delta \setminus G$ is finitely covered by a solv-manifold, i.e., a homogeneous space of a solvable Lie group. By a result of Mostow [23] the torsion-free group Γ is then a virtually polycyclic group. (Recall that a group Γ is called virtually polycyclic (or polycyclic by finite) if it contains a subgroup Γ_0 of finite index which is polycyclic, i.e., Γ_0 admits a finite normal series with cyclic quotients. The number of infinite cyclic factors in the series is an invariant of Γ called the rank of Γ [29].) If $\Delta \setminus G$ is compact then $\dim \Delta \setminus G$ equals the rank of Γ. Not every smooth manifold which is finitely covered by a compact solvmanifold is diffeomorphic to an infra-solvmanifold. By the work of Wall et al. (see [18]), there exist fake tori which are finitely covered by standard tori. Consequently, these smooth manifolds do not carry any infra-solv structure.

Main results: Let Γ be a torsion-free virtually polycyclic group. To Γ we associate in a functorial way a solvable by finite real linear algebraic group H_Γ which contains Γ as a discrete and Zariski-dense subgroup. The group H_Γ is called the real algebraic hull for Γ. The construction of the algebraic hull for Γ extends results of Malcev [21] on torsion-free nilpotent groups, and results of Mostow [24] on torsion-free polycyclic groups. The extended construction was first announced in [4]. The details are provided in Appendix A of this paper.

We explain now the role the real algebraic hull plays in the construction of infra-solvmanifolds. Let $T \leqslant H_\Gamma$ be a maximal reductive subgroup, and let U denote the unipotent radical of H_Γ. Then H_Γ decomposes as a semi-direct product $H_\Gamma = U \cdot T$. The splitting induces an injective homomorphism $\alpha_T : H_\Gamma \to \mathrm{Aff}(U)$ and a corresponding affine action of $\Gamma \leqslant H_\Gamma$ on U. The quotient space

$$M_\Gamma = {}_{\alpha_T(\Gamma)} \backslash U$$

is a compact aspherical manifold of dimension $n = \operatorname{rank} \Gamma$, and has universal cover $U = \mathbb{R}^n$. In fact, we show that M_Γ is an infra-solvmanifold. We call every manifold M_Γ which arises by this construction a standard Γ-manifold.

We prove:

Theorem 1.2. *Let Γ be a torsion-free virtually polycyclic group. Then M_Γ is a compact infra-solvmanifold and the fundamental group $\pi_1(M_\Gamma)$ is isomorphic to Γ. Every two standard Γ-manifolds are diffeomorphic and every given isomorphism of fundamental groups of standard Γ-manifolds is induced by a smooth diffeomorphism.*

Let G be a connected, simply connected Lie group, and let $\mathfrak{g}$ denote its Lie algebra. The group $\mathrm{Aut}(G)$ attains the structure of a real linear algebraic group since it has a natural identification with the group $\mathrm{Aut}(\mathfrak{g})$ of Lie algebra automorphisms of $\mathfrak{g}$. Our main result is:

Theorem 1.3. *Let G be a connected, simply connected solvable Lie group. Let $\Delta \leqslant \mathrm{Aff}(G)$ be a solvable by finite subgroup which acts freely and properly on G with compact quotient manifold $M = \Delta \backslash G$. Assume that one of the following two conditions is satisfied*:

(i) *G is nilpotent, or*
(ii) $\mathrm{hol}(\Delta) \leqslant \mathrm{Aut}(G)$ *is contained in a reductive subgroup of* $\mathrm{Aut}(G)$.

Then the group $\Gamma = \Delta/\Delta_0$ is virtually polycyclic, and M is diffeomorphic to a standard Γ-manifold.

We deduce:

Theorem 1.4. *Every compact infra-solvmanifold is smoothly diffeomorphic to a standard Γ-manifold.*

Corollary 1.5. *Compact infra-solvmanifolds are smoothly rigid. In particular, every two compact infra-solvmanifolds with isomorphic fundamental groups are smoothly diffeomorphic.*

Theorem 1.2 also implies the following result which was first proved by Auslander and Johnson [2]. Their construction is different from ours.

Corollary 1.6. *Every torsion-free virtually polycyclic group is the fundamental group of a compact infra-solvmanifold.*

The torus rank r of a manifold M is the maximum dimension of a torus which acts almost freely and smoothly on M. For a closed aspherical manifold M, r is bounded by the rank of the center of the fundamental group. If r equals the rank of the center then the torus rank of M is said to be maximal. It is known (see [20]) that the torus rank of a solvmanifold is maximal and the result is expected to hold for infra-solvmanifolds as well. It is straightforward to see that standard Γ-manifolds admit maximal torus actions. Therefore, we also have:

Corollary 1.7. *Every infra-solvmanifold has maximal torus rank.*

Let U be a connected, simply connected, nilpotent Lie group, and let $\mathfrak{u}$ denote its Lie-algebra. Nomizu [27] proved that the cohomology of a compact nilmanifold $M = U/\Gamma$, where $\Gamma \leqslant U$ is a lattice, is isomorphic to the cohomology of the complex of left invariant differential forms on U. This means that the cohomology of the nilmanifold M is computed by the Lie algebra cohomology $H^*(\mathfrak{u})$. Now let Γ be a torsion-free virtually polycyclic group and M_Γ a standard Γ-manifold. Let $H_\Gamma = U \cdot T$ be the real algebraic hull for Γ, where U is the unipotent radical and T is maximal reductive. Then T acts by automorphisms on U and on the cohomology ring $H^*(\mathfrak{u})$. Let $H^*(\mathfrak{u})^T$ denote the T-invariants in $H^*(\mathfrak{u})$. Let M be an infra-solvmanifold with fundamental group Γ. By Theorem 1.4, M is diffeomorphic to the standard Γ-manifold M_Γ. Hence, the following result computes the cohomology of M:

Theorem 1.8. *Let M_Γ be a standard Γ-manifold. Then the de Rham-cohomology ring $H^*(M_\Gamma)$ is isomorphic to $H^*(\mathfrak{u})^T$.*

We remark that the theorem implies that the discrete group cohomology of Γ, $H^*(\Gamma, \mathbb{R}) = H^*(M_\Gamma)$, is isomorphic to the rational cohomology (see [17, Theorem 5.2]) of the real linear algebraic group H_Γ.

Some historical remarks: We want to give a few more historical remarks about the context of our paper, and the techniques we use. As our main tool we employ the algebraic hull functor which naturally associates a linear algebraic group to a (torsion-free) virtually polycyclic group or to a solvable Lie group. This functor was considered by Mostow in his paper [24]. Auslander and Tolimieri solved the main open problems on solv-manifolds at their time using the technique of the nilpotent shadow and semi-simple splitting for solvable Lie groups (see [1,3]). Mostow remarked then in [25] that the nilpotent shadow and splitting construction may be derived naturally from the algebraic hull, and reproved the Auslander–Tolimieri results, as well as his older result on the rigidity of compact solv-manifolds. In our paper, we establish and use the properties of the algebraic hull functor for the class of virtually polycyclic groups not containing finite normal subgroups. We provide the necessary results and proofs about the hull functor in an appendix. Immediate applications are then our rigidity results and cohomology computations for infra-solvmanifolds. In [4] we give another application of the hull functor in the context of affine crystallographic groups and their deformation spaces.

Arrangement of the paper: We start in Section 2 with some preliminaries on real algebraic and syndetic hulls for virtually polycyclic groups, affine actions and splittings of real algebraic groups. The necessary results about the construction of algebraic hulls are provided in Appendix A. In Section 3 we prove Theorems 1.2 and 1.3. In Section 4 we provide some applications on the geometry of infrasolvmanifolds. In particular, we show that infra-solvmanifolds are distinguished in the class of aspherical compact differentiable manifolds with a virtually polycyclic fundamental group by the existence of a certain atlas whose coordinate changes are polynomial maps. As an application, we compute the cohomology of infra-solvmanifolds in terms of polynomial differential forms, and derive Theorem 1.8.

2. Hulls and splittings

We need some terminology concerning real algebraic groups. For terminology on algebraic groups see also Appendix A. Let $\mathbf{G}$ be a $\mathbb{R}$-defined linear algebraic group. The group of real points $G = \mathbf{G}_{\mathbb{R}} \leqslant \mathrm{GL}_n(\mathbb{R})$ will be called a real algebraic group. The group G has the natural Euclidean topology which turns it into a real Lie-group but it carries also the Zariski-topology induced from $\mathbf{G}$. Let $H = \mathbf{H}_{\mathbb{R}}$ be another real algebraic group. A group homomorphism $\phi: G \to H$ is called an algebraic homomorphism if it is the restriction of a $\mathbb{R}$-defined morphism $\mathbf{G} \to \mathbf{H}$ of linear algebraic groups. If ϕ is an isomorphism of groups which is algebraic with algebraic inverse, then ϕ is called an algebraic isomorphism. We let G^0 denote the Zariski-irreducible component of identity in G, and G_0 the connected component in the Euclidean topology. In particular, $G_0 \leqslant G^0$ is a subgroup of finite index in G. If g is an element of G then $g = g_u g_s$ denotes the Jordan-decomposition of g. Here $g_u \in G$ is unipotent, $g_s \in G$ is semisimple, and g_u, g_s commute. Let $M \subset G$ be a subset. Then $\overline{M}$ denotes the Zariski-closure of M in G. We put $M_u = \{g_u \mid g \in M\}$, $M_s = \{g_s \mid g \in M\}$. We let $\mathrm{u}(G)$ denote the unipotent radical of G, i.e., the maximal normal subgroup of G which consists of unipotent elements.

2.1. Solvable by finite real algebraic groups

A linear algebraic group $\mathbf{H}$ is called solvable by finite if $\mathbf{H}^0$ is solvable. Assume that $\mathbf{H}$ is solvable by finite. Then $\mathbf{H}_u = \mathrm{u}(\mathbf{H})$. In particular, for any subgroup G of $\mathbf{H}$, $\mathrm{u}(G) = G \cap G_u$. If G is a nilpotent subgroup then (cf. [7, Section 10]) G_u and G_s are subgroups of $\mathbf{H}$, and $G \leqslant G_u \times G_s$. A Zariski-closed subgroup $\mathbf{T} \leqslant \mathbf{H}$ which consists only of semi-simple elements is called a d-subgroup of $\mathbf{H}$. The group $H = \mathbf{H}_{\mathbb{R}}$ is called a solvable by finite real algebraic group. Every Zariski-closed subgroup $T \leqslant H$ consisting of semi-simple elements is called a d-subgroup of H. Any d-subgroup of H is an abelian by finite group, and its identity component T^0 is a real algebraic torus.

Proposition 2.1. *Let H be a solvable by finite real linear algebraic group. Let T be a maximal d-subgroup of H, and $U = \mathrm{u}(H)$ the unipotent radical of H. Then*

$$H = U \cdot T \quad (\textit{semi-direct product}).$$

Moreover, any two maximal d-subgroups T and T' of H are conjugate by an element of U.

Proof. Let us assume that $H \leqslant \mathbf{H}$ is a Zariski-dense subgroup. Let $\mathbf{T}$ be the Zariski-closure of T in $\mathbf{H}$. Then $\mathbf{T}$ is a $\mathbb{R}$-defined subgroup of $\mathbf{H}$, and a d-subgroup. Also $T \leqslant \mathbf{T}_{\mathbb{R}}$ and, by maximality of T, $T = \mathbf{T}_{\mathbb{R}}$. Moreover, $\mathbf{T}$ is a maximal reductive $\mathbb{R}$-defined subgroup of $\mathbf{H}$. Therefore, by a well-known result (see [8, Proposition 5.1]) $\mathbf{H} = \mathbf{U} \cdot \mathbf{T}$, where $\mathbf{U} = \mathrm{u}(\mathbf{H})$, and every two $\mathbb{R}$-defined maximal reductive subgroups $\mathbf{T}$ and $\mathbf{T}'$ are conjugate by an element of $\mathbf{U}_{\mathbb{R}} = U$. Then the decomposition of H follows. Since T and T' are the group of real-points in maximal d-subgroups $\mathbf{T}$ and $\mathbf{T}'$ they are conjugate by an element of U. □

2.2. Algebraic hulls

Let Γ be a torsion-free virtually polycyclic group. We introduce the concept of an algebraic hull for Γ. For more details and proofs see Appendix A. Let $\mathbf{G}$ be a linear algebraic group, and let $\mathbf{U}$ denote the unipotent radical of $\mathbf{G}$. We say that $\mathbf{G}$ has a *strong unipotent radical* if the centralizer $\mathrm{Z}_{\mathbf{G}}(\mathbf{U})$ is contained in $\mathbf{U}$.

Theorem 2.2. *There exists a $\mathbb{Q}$-defined linear algebraic group* $\mathbf{H}$ *and an injective homomorphism* $\psi : \Gamma \to \mathbf{H}_{\mathbb{Q}}$ *so that,*

(i) $\psi(\Gamma)$ *is Zariski-dense in* $\mathbf{H}$,
(ii) $\mathbf{H}$ *has a strong unipotent radical* $\mathbf{U}$,
(iii) $\dim \mathbf{U} = \operatorname{rank} \Gamma$.

We call the $\mathbb{Q}$-defined linear algebraic group $\mathbf{H}$ the *algebraic hull for* Γ. The homomorphism ψ may be chosen so that $\psi(\Gamma) \cap \mathbf{H}_{\mathbb{Z}}$ has finite index in $\psi(\Gamma)$. Let $k \leqslant \mathbb{C}$ be a subfield. The hull $\mathbf{H}$ together with a Zariski-dense embedding $\psi : \Gamma \to \mathbf{H}_k$ of Γ into the group of k-points of $\mathbf{H}$ satisfies the following rigidity property:

(∗) Let $\mathbf{H}'$ be another linear algebraic group and $\psi' : \Gamma \to \mathbf{H}'_k$ an injective homomorphism so that (i)–(iii) above are satisfied with respect to $\mathbf{H}'$. Then there exists a k-defined isomorphism $\Phi : \mathbf{H} \to \mathbf{H}'$ so that $\psi' = \Phi \circ \psi$.

In particular, the group $\mathbf{H}$ is determined by conditions (i)–(iii) up to $\mathbb{Q}$-defined isomorphism of linear algebraic groups.

The real algebraic hull for Γ: Let $\mathbf{H}$ be an algebraic hull for Γ, $H = \mathbf{H}_{\mathbb{R}}$ the group of real points. Put $U = \mathrm{u}(H)$. Then there exists an injective homomorphism $\psi : \Gamma \to H$ which satisfies: (i) $\psi(\Gamma) \leqslant H$ is a discrete, Zariski-dense subgroup, (ii) $\mathbf{H}$ has a strong unipotent radical, and (iii) $\dim U = \operatorname{rank} \Gamma$. Let $H' = \mathbf{H}'_{\mathbb{R}}$ be another real linear algebraic group, $\psi' : \Gamma \to H'$ an embedding of Γ into H' so that (i)–(iii) are satisfied with respect to $\mathbf{H}'$. Hence, as a consequence of the rigidity property (∗), there exists an algebraic isomorphism $\Phi : H \to H'$ so that $\psi' = \Phi \circ \psi$. We call the solvable by finite real linear algebraic group $H_\Gamma := H$ the real algebraic hull for Γ.

The real algebraic hull for G: Let G be a connected, simply connected solvable Lie-group. By Raghunathan [28, Proposition 4.40], there exists an algebraic hull for G. This means that there exists an $\mathbb{R}$-defined linear algebraic group $\mathbf{H}_G$, and an injective Lie-homomorphism $\psi : G \to (\mathbf{H}_G)_{\mathbb{R}}$ so that (i)′ $\psi(G) \leqslant \mathbf{H}_G$ is a Zariski-dense subgroup, (ii)′ $\mathbf{H}_G$ has a strong unipotent radical $\mathbf{U}$, and (iii)′ $\dim \mathbf{U} = \dim G$. Moreover, $\mathbf{H}_G$ satisfies rigidity properties analogous to the rigidity properties of $\mathbf{H}_\Gamma$. Let $H_G = (\mathbf{H}_G)_\mathbb{R}$. Then there exists a continuous injective homomorphism $\psi : G \to H_G$ which has Zariski-dense image in H_G. As for real hulls of discrete groups, these data are uniquely defined up to composition with an isomorphism of real algebraic groups, and we call H_G the real algebraic hull for G. We consider henceforth a fixed continuous Zariski-dense inclusion $G \leqslant H_G$.

Let N denote the nilpotent radical of G, i.e., the maximal, connected nilpotent normal subgroup of G, and let U_G denote the unipotent radical of H_G. Then (cf. the proof of Lemma A.7) $N \leqslant U_G = \mathrm{u}(H_G)$, so that N is the connected component of $\mathrm{u}(G) = G \cap \mathrm{u}(\mathbf{H}_G)$. We remark further that G is a normal subgroup of H_G. In fact, $N \leqslant U_G$ is Zariski-closed in H_G, and $[G,G] \leqslant N$ implies therefore that $[H_G, H_G] \leqslant N$. Let T be a maximal d-subgroup of H_G. We consider the decomposition $H_G = U_G \cdot T$. Since H_G is decomposed as a product of varieties, the projection map $\tau_T : H_G \to U_G$, $g = ut \mapsto u$, onto the first factor of the splitting is an algebraic morphism.

Proposition 2.3. *Let G be connected, simply connected solvable Lie group, and $G \leqslant H_G$ a continuous, Zariski-dense inclusion into its real algebraic hull. Then G is a closed normal subgroup of H_G. Moreover, if $T \leqslant H_G$ is a maximal d-subgroup then*

$$H_G = GT, \quad G \cap T = \{1\}.$$

Let $U_G = \mathrm{u}(H_G)$ denote the unipotent radical of H_G. Then the algebraic projection map $\tau_T : H_G \to U_G$ restricts to a diffeomorphism $\tau : G \to U_G$.

Proof. Let C be a Cartan subgroup of G. Then C is nilpotent, and $G = NC$, where $N \leqslant U_G$ is the nilradical of G. Let us put $S = C_s = \{g_s \mid g \in C\}$, so that $C \leqslant C_u \times S$. Note that C_u is a closed subgroup of U_G, and S is an abelian subgroup of H_G which is centralized by C. Let $T \leqslant H_G$ be a maximal d-subgroup which contains S. Since $H_G = \overline{G} \leqslant NC_uT$, we conclude that $U_G = NC_u$ and $H_G = GT$. It follows that the crossed homomorphism $\tau : G \to U_G$ is surjective, in fact, since $\dim U_G = \dim G$ it is a covering map. Since U_G is simply connected τ must be a diffeomorphism. Therefore $T \cap G = \{1\}$. From the above remarks, G is a normal subgroup of H_G. Let $\pi_T : H_G \to T$ denote the projection map onto the second factor of the splitting $H_G = U_G \cdot T$. Then $G = \{g = u\theta(u) \mid u \in U_G\}$, where $\theta = \pi_T \tau^{-1} : U \to T$ is a differentiable map. Therefore G is a closed subgroup. □

Let $\Gamma \leqslant G$ be a lattice. We call Γ a Zariski-dense lattice if Γ is Zariski-dense in H_G. We remark:

Proposition 2.4. *Let G be a connected, simply connected solvable Lie group, and $\Gamma \leqslant G$ a Zariski-dense lattice. Then the real algebraic hull H_G is a real algebraic hull for Γ.*

Proof. By the inclusion $\Gamma \leqslant G$ we have an inclusion $\Gamma \leqslant H_G$. Since Γ is cocompact, $\operatorname{rank} \Gamma = \dim G = \dim \mathrm{u}(\mathbf{H}_G)$. Therefore $\mathbf{H}_G$ is a $\mathbb{R}$-defined algebraic hull for Γ. By the rigidity property $(*)$, there exists an $\mathbb{R}$-defined isomorphism $\mathbf{H}_\Gamma \to \mathbf{H}_G$. In particular, there is an induced algebraic isomorphism of the groups of real points H_Γ and H_G. □

Identifying, $\mathrm{Aut}(G)$ with $\mathrm{Aut}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie-algebra of G, we obtain a natural structure of real linear algebraic group on $\mathrm{Aut}(G)$. Let $\mathbf{H}$ be a solvable by finite linear algebraic group, and let $\mathrm{Aut}_\mathrm{a}(\mathbf{H})$ denote its group of algebraic automorphisms. In [15], it is observed that the group $\mathrm{Aut}_\mathrm{a}(\mathbf{H})$ is itself a linear algebraic group if $\mathbf{H}$ has a strong unipotent radical. In particular, $\mathrm{Aut}_\mathrm{a}(H_G)$, the group of algebraic automorphisms of H_G, inherits a structure of a real linear algebraic group. The rigidity of the hull H_G induces an extension homomorphism

$$\mathscr{E}: \mathrm{Aut}(G) \hookrightarrow \mathrm{Aut}_{\mathrm{a}}(H_G), \quad \psi \mapsto \Psi.$$

Proposition 2.5. *The extension homomorphism* $\mathscr{E}: \mathrm{Aut}(G) \hookrightarrow \mathrm{Aut}_{\mathrm{a}}(H_G)$ *identifies the real linear algebraic group* $\mathrm{Aut}(G)$ *with a Zariski-closed subgroup of* $\mathrm{Aut}_{\mathrm{a}}(H_G)$.

Proof. Let $\mathfrak{h}_G$ denote the Lie-Algebra of H_G. From the inclusion $G \leqslant H_G$, we have that $\mathfrak{g} \subseteq \mathfrak{h}_G$. Since $H_G = H_G^0$, it follows from the discussion in [15, Section 3] that the Lie-functor identifies the group $\mathrm{Aut}_{\mathrm{a}}(H_G)$ with a Zariski-closed subgroup $\mathrm{Aut}_{\mathrm{a}}(\mathfrak{h}_G)$ of $\mathrm{Aut}(\mathfrak{h}_G)$. Consider

$$\mathrm{Aut}_{\mathrm{a}}(\mathfrak{h}_G, \mathfrak{g}) = \mathrm{Aut}_{\mathrm{a}}(\mathfrak{h}_G) \cap \{\varphi \mid \varphi(\mathfrak{g}) \subseteq \mathfrak{g}\}.$$

The rigidity property of the hull implies that the restriction map

$$\mathrm{Aut}_{\mathrm{a}}(\mathfrak{h}_G, \mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g})$$

is surjective. Since $G \leqslant H_G$ is Zariski-dense, the restriction map is injective as well. This implies that the restriction map induces an isomorphism of real linear algebraic groups. Since the image of $\mathrm{Aut}(G)$ in $\mathrm{Aut}_{\mathrm{a}}(H_G)$ corresponds to the Zariski-closed subgroup $\mathrm{Aut}_{\mathrm{a}}(\mathfrak{h}_G, \mathfrak{g}) \leqslant \mathrm{Aut}_{\mathrm{a}}(\mathfrak{h}_G)$, the proposition follows. □

2.3. *Affine actions by rational maps*

Let G be a group. We view the affine group $\mathrm{Aff}(G)$ as a group of transformations acting on G by declaring

$$(g, \phi) \cdot g' = g\phi(g'), \quad \text{where } (g, \phi) \in \mathrm{Aff}(G),\ g' \in G.$$

Let H be a solvable by finite real linear algebraic group with a strong unipotent radical. Let $\mathrm{Aut}_{\mathrm{a}}(H) \leqslant \mathrm{Aut}(H)$ denote its group of algebraic automorphisms. We remark that, since H has a strong unipotent radical, $\mathrm{Aut}_{\mathrm{a}}(H)$ *is a real linear algebraic group* (as follows from [15, Section 4]), and so is

$$\mathrm{Aff}_{\mathrm{a}}(H) = H \rtimes \mathrm{Aut}_{\mathrm{a}}(H).$$

Let T be a maximal d-subgroup of H, and $U = \mathrm{u}(H)$. For $h \in H$, let $c(h): H \to H$ denote the inner automorphism $l \mapsto hlh^{-1}$ of H. If $L \leqslant H$ is a normal subgroup $c_L(h)$ denotes the restriction of $c(h)$ on L. Let $h = ut$ be a decomposition of $h \in H$ with respect to the algebraic splitting $H = U \cdot T$. Then we have a homomorphism of real algebraic groups

$$\alpha_T: H \to \mathrm{Aff}_{\mathrm{a}}(U), \quad h = ut \mapsto (u, c_U(t)).$$

Since H has a strong unipotent radical, the homomorphism α_T is injective. Similarly, if $G \leqslant H$ is a normal subgroup of H, $H = GT$ and $G \cap T = \{1\}$, we define

$$\beta_T: H \to \mathrm{Aff}(G), \quad h = gt \mapsto (g, c_G(t)).$$

Lemma 2.6. *Let H be a solvable by finite real algebraic group with a strong unipotent radical U, and let $T \leqslant H$ be a maximal d-subgroup. Assume there exists a connected Lie subgroup G of H which is normal in H, so that $H = GT$, $H \cap T = \{1\}$. Then $\beta_T : H \to \mathrm{Aff}(G)$ is an injective continuous homomorphism. Moreover, the projection $\tau : G \to U$, induced by the splitting $H = U \cdot T$, is a diffeomorphism which is equivariant with respect to the affine actions β_T and α_T.*

Proof. We remark first that $H = GT$ implies that the Zariski-closure $\overline{G} \leqslant H$ contains the unipotent radical U of H. The argument given in the proof of Proposition 2.3 shows that τ is a diffeomorphism. Using the notation in the proof of Proposition 2.3, we can write

$$\beta_T(h) = (\tau_T(h)\theta(\tau_T(h)), c_G(\theta(\tau_T(h))^{-1}\pi_T(h))).$$

This shows that β_T is continuous. It is also injective: Assume that $\beta_T(gt) = 1$. Then, in particular, $c_G(t) = id_G$. Hence, t centralizes the unipotent radical U. Since t is semisimple and H has a strong unipotent radical, this implies $t = 1$. Therefore, $h \in G$. But β_T is clearly injective on G, proving that β_T is injective. Finally, let $h \in H$, $g \in G$. Then an elementary calculation shows that $\tau(\beta_T(h) \cdot g) = \alpha_T(h) \cdot \tau(g)$, proving that τ is equivariant. □

Finally, we briefly remark how the affine action α_T depends on the choice of maximal d-subgroup in H_Γ. Let $T' \leqslant H_\Gamma$ be another maximal d-subgroup. By Proposition 2.1, there exists $v \in U$ so that $T' = vTv^{-1}$. Let $h = ut$, where $u \in U$, $t \in T$. The decomposition $h = u't'$ of h relative to T' is given by $u' = uv^t v^{-1}$, $t' = t^v$. Hence,

Lemma 2.7. *Let $R_v : U \to U$ denote right-multiplication with v on U, and $T' = vTv^{-1}$. Then, for all $h \in H$, $\alpha_T(h) \circ R_v = R_v \circ \alpha_{T'}(h)$.*

2.4. Syndetic hulls

The notion of *syndetic hull* of a solvable subgroup of a linear group is due to Fried and Goldman, [13, Section 1.6]. Fried and Goldman introduced this notion in the context of affine crystallographic groups. We will employ the syndetic hull to prove that standard Γ-manifolds are infra-solvmanifolds. We use the slightly modified definition for the syndetic hull which is given in [16]. Let V be a finite-dimensional real vector space.

Definition 2.8. Let Γ be a polycyclic subgroup of $\mathrm{GL}(V)$, and G a closed, connected subgroup of $\mathrm{GL}(V)$ such that $\Gamma \leqslant G$. G is called a *syndetic hull* of Γ if Γ is a Zariski-dense (i.e., $G \leqslant \overline{\Gamma}$) uniform lattice in G, and $\dim G = \operatorname{rank} \Gamma$.

The syndetic hull for Γ is necessarily a connected, simply connected solvable Lie group. If $\Gamma \leqslant \mathrm{GL}(V)$ is discrete, $\Gamma \leqslant (\overline{\Gamma})_0$, and $\Gamma/\mathrm{u}(\Gamma)$ is torsion-free then it is proved in [16, Proposition 4.1, Lemma 4.2] that Γ has a syndetic hull. In particular, any virtually polycyclic linear group has a normal finite index subgroup which possesses a syndetic hull. We need the following slightly refined result:

Proposition 2.9. *Let $H \leqslant \mathrm{GL}(V)$ be a Zariski-closed subgroup. Let $\Delta \leqslant H$ be a virtually polycyclic discrete subgroup, Zariski-dense in H. Then there exists a finite index normal subgroup $\Gamma_0 \leqslant \Delta$, and a syndetic hull G for Γ_0, $\Gamma_0 \leqslant G \leqslant H$, so that G is normalized by Δ.*

A similar result is also stated in [13, Section 1.6]. However, the proof given in [13] is faulty. We refine the proof of [16, Proposition 4.1] a little to obtain Proposition 2.9. Also we warn the reader that a syndetic hull $\Gamma \leqslant G$ is (in general) not uniquely determined by Γ, neither a *good syndetic hull* (cf. [16]) is uniquely determined by Γ. (See [16, Section 9].)

Proof of Proposition 2.9. There exists a normal polycyclic subgroup $\Gamma \leqslant \Delta$ of finite index with the following properties: $\Gamma \leqslant H_0$, $[\Gamma,\Gamma] \leqslant \mathrm{u}(H)$, and $\Gamma/\mathrm{u}(\Gamma)$ is torsion-free. We consider the abelian by finite Lie group $T = H/N$, where $N = \overline{[\Gamma,\Gamma]}$. Let $p: H \to T$ denote the projection, and $\pi: \hat{E} \to T$ the universal cover. Then $\hat{E}$ is an extension of a vector space E by some finite group μ. Let $\hat{S} = \pi^{-1}(\Delta N/N)$, $S = \hat{S} \cap E$. Then $\mu = \hat{S}/S$. Now $K = \ker \pi \subset S$ is invariant by the induced action of μ on S. From Maschke's theorem we deduce that there exists a μ-invariant complement $P \subset S$ of K so that KP is of finite index in S.

Now define $\bar{P}$ to be the real vector space spanned by P, $G = p^{-1}(\pi(\bar{P}))$, and $\Gamma_0 = G \cap \Gamma$. Then Γ_0 is of finite index in Γ, and G is a syndetic hull for Γ_0 in the sense of Definition 2.8. Since $\bar{P}$ is invariant by μ the Lie group G is normalized by Δ. □

3. Standard Γ-manifolds

Let Γ be a torsion-free virtually polycyclic group. The purpose of this section is to explain the construction of standard Γ-manifolds and to prove Theorems 1.2 and 1.3.

3.1. Construction of standard Γ-manifolds

Let H_Γ be a real algebraic hull for Γ, and fix a Zariski-dense embedding $\Gamma \leqslant H_\Gamma$. Let T be a maximal d-subgroup of H_Γ, and put $U = \mathrm{u}(H_\Gamma)$ for the unipotent radical of H_Γ. We consider the affine action $\alpha_T : H_\Gamma \to \mathrm{Aff_a}(U)$ which is defined by the splitting $H_\Gamma = U \cdot T$. Since U is strong in H_Γ, the homomorphism α_T is injective. Let

$$M_{\Gamma,\alpha_T} = {}_{\alpha_T(\Gamma)}\backslash U$$

denote the quotient space of the affine action of Γ on U. We will show that M_{Γ,α_T} is a compact manifold with fundamental group isomorphic to Γ. In fact, the proof implies that M_{Γ,α_T} is an infra-solvmanifold. We also show that the diffeomorphism class of M_{Γ,α_T} depends only on Γ, not on the choice of maximal d-subgroup T in H_Γ, nor on the particular embedding of Γ into H_Γ. In fact, we show that the corresponding actions of Γ are affinely conjugate. We call $M_\Gamma = M_{\Gamma,\alpha_T}$ a *standard Γ-manifold.*

Proof of Theorem 1.2. We first show that M_Γ is an infra-solvmanifold. Let Γ_0 be a finite index normal subgroup of Γ so that there exists a syndectic hull $\Gamma_0 \leqslant G \leqslant H_\Gamma$ for Γ_0. By Proposition 2.9, we may also assume that G is normalized by Γ. By the defining properties of the hull, $(H_\Gamma)^0$ is a real algebraic hull for G. Since G is a normal subgroup in its hull, it follows that G is a normal subgroup of $H_\Gamma = \Gamma(H_\Gamma)^0$. Let T be a maximal d-subgroup of H_Γ. We infer from Proposition 2.3 that $H_\Gamma = GT$, $G \cap T = \{1\}$. Let $\beta_T : H_\Gamma \to \mathrm{Aff}(G)$ denote the affine action which is defined by this splitting. Lemma 2.6 implies that the affine action β_T is effective. Note that $\beta_T(\Gamma) \cap G$ contains Γ_0, hence $\beta_T(\Gamma)$ is discrete in $\mathrm{Aff}(G)$ and $\mathrm{hol}(\beta_T(\Gamma)) \leqslant \mathrm{Aut}(G)$ is finite. Therefore, the quotient space

$$M_{\beta_T} = _{\beta_T(\Gamma)} \backslash G$$

is an infra-solvmanifold. Since G is diffeomorphic to $\mathbb{R}^n$, the fundamental group $\pi_1(M_{\beta_T})$ is isomorphic to Γ. Since G is a syndetic hull, M_{β_T} is compact. Let $\tau : G \to U$ be the projection map which is induced by the splitting $H_\Gamma = U \cdot T$. By Lemma 2.6, τ induces a diffeomorphism $\bar{\tau} : \beta_T(\Gamma) \setminus G \to \alpha_T(\Gamma) \setminus U$. Hence $\alpha_T(\Gamma) \setminus U$ is diffeomorphic to a compact infra-solvmanifold.

Note that the diffeomorphism class of M_Γ does not depend on the choice of maximal d-subgroup in H_Γ. In fact, let $T' \leqslant H_\Gamma$ be another maximal d-subgroup. Then, by Proposition 2.1, there exists $v \in U$ so that $T' = vTv^{-1}$. By Lemma 2.7, $R_v : U \to U$ induces a smooth diffeomorphism $M_{\Gamma,\alpha_T} \to M_{\Gamma,\alpha_{T'}}$. The diffeomorphism class of M_Γ is also independent of the particular choice of Zariski-dense embedding of Γ into H_Γ. Let $\Gamma' \leqslant H_\Gamma$ be a Zariski dense subgroup isomorphic to Γ, and let $\phi : \Gamma \to \Gamma'$ be an isomorphism. By the rigidity of the real algebraic hull, there exists an algebraic automorphism $\bar{\phi} : H_\Gamma \to H_\Gamma$ extending ϕ. The restriction of $\bar{\phi}$ on the unipotent radical U of $\mathbf{H}_\Gamma$ projects to a diffeomorphism $M_{\Gamma,\alpha_T} \to M_{\Gamma',\alpha_{\bar{\phi}(T)}}$ which induces ϕ on the level of fundamental groups. Similarly, any given automorphism of $\pi_1(M_{\Gamma,\alpha_T})$ corresponds to an automorphism ϕ of Γ. The algebraic extension $\bar{\phi}$ projects to a diffeomorphism of M_{Γ,α_T} inducing ϕ on $\pi_1(M_{\Gamma,\alpha_T})$. □

We also remark:

Proposition 3.1. *Every standard Γ-manifold M_Γ admits a smooth effective action of an r-dimensional torus T^r, where $r = \mathrm{rank}\, \mathrm{Z}(\Gamma)$.*

Proof. Let $U_Z = \overline{\mathrm{Z}(\Gamma)}$ be the Zariski-closure of $\mathrm{Z}(\Gamma)$ in H_Γ. Since $U_Z \leqslant \mathrm{Z}(H_\Gamma)$, $U_Z \leqslant U$, and $\dim U_Z = r$. It follows that $\alpha_T(U_Z)$ induces a free maximal torus action on M_Γ. □

Theorem 1.4 and Proposition 3.1 imply Corollary 1.7.

3.2. Affine actions on unipotent groups

Here we show that every compact manifold which arises by (solvable by finite) affine actions on unipotent groups is diffeomorphic to a standard Γ-manifold.

Proposition 3.2. *Let U be a connected, simply connected nilpotent Lie group. Let $\Delta \leqslant \mathrm{Aff}(U)$ be a solvable by finite subgroup which acts freely and properly on U with compact quotient manifold $M = \Delta \backslash U$. Let $T \leqslant \bar{\Delta} \leqslant \mathrm{Aff}(U)$ be a maximal d-subgroup. Then M is diffeomorphic to $\alpha_T(\Delta) \backslash \mathrm{u}(\bar{\Delta})$.*

Proof. We decompose $\bar{\Delta} = \mathrm{u}(\bar{\Delta})T$. Since any maximal d-subgroup of $\mathrm{Aut}(U)$ is maximal in $\mathrm{Aff}(U)$, we may assume (after conjugation of Δ with a suitable element of $\mathrm{Aff}(U)$) that $T \leqslant \mathrm{Aut}(U)$. In particular, $\bar{\Delta} \cdot 1 = \mathrm{u}(\bar{\Delta}) \cdot 1$, and hence Δ acts on the orbit $O = \mathrm{u}(\bar{\Delta}) \cdot 1 \leqslant U$. Since O is the homogeneous space of a connected unipotent group acting on U, it is a submanifold diffeomorphic to $\mathbb{R}^k$, $k \leqslant \dim U$. The connected, simply connected solvable Lie group Δ_0 acts freely on O, and the quotient space $\Delta_0 \backslash O$ is a simply connected aspherical manifold. Hence, the quotient space $M_O = \Delta \backslash O$ is a manifold with fundamental group isomorphic to Γ, and homotopy equivalent to an Eilenberg–Mac Lane space $K(\Gamma, 1)$. Since M is an aspherical compact manifold with fundamental group Γ, its dimension equals the cohomological dimension of Γ. This implies that $\dim M \leqslant \dim M_O$, and consequently $O = U$. In particular, $\mathrm{u}(\bar{\Delta})$ acts transitively on U and the orbit map

$$o\colon \mathrm{u}(\bar{\Delta}) \to U, \quad \delta \mapsto \delta \cdot 1$$

in $1 \in U$ is a diffeomorphism. Using $T \leqslant \mathrm{Aut}(U)$, it is straightforward to verify that o is Δ-equivariant with respect to the affine action α_T of Δ on $\mathrm{u}(\bar{\Delta})$. Hence, M is diffeomorphic to $\alpha_T(\Delta) \setminus \mathrm{u}(\bar{\Delta})$. □

The next result shows how the algebraic hull enters the picture:

Proposition 3.3. *Let $\Gamma \leqslant \mathrm{Aff}(U)$ be virtually polycyclic, such that Γ acts freely and properly discontinuously on U, and with compact quotient $M = \Gamma \setminus U$. Then the Zariski-closure $\overline{\Gamma} \leqslant \mathrm{Aff}(U)$ is an algebraic hull for Γ. In particular, M is diffeomorphic to a standard Γ-manifold.*

Proof. Put $H = \overline{\Gamma} \leqslant \mathrm{Aff}(U)$. By the previous proposition, the orbit map $o\colon \mathrm{u}(H) \to U$ in $1 \in U$ is a diffeomorphism. In particular, $\dim \mathrm{u}(H) = \mathrm{rank}\, \Gamma$. Let $T \leqslant H$ be a maximal d-subgroup. We may assume that $T \leqslant \mathrm{Aut}(U)$. This shows that $T \cap \mathrm{Z}_H(U) = \{1\}$. Hence, H has a strong unipotent radical. It follows that H is an algebraic hull for Γ. Since o is equivariant with respect to the action α_T, M is diffeomorphic to a standard Γ-manifold. □

Next, we consider affine actions which arise from splittings of solvable by finite linear algebraic groups.

Proposition 3.4. *Let $H = UT$ be a solvable linear algebraic group. Let $\Theta \leqslant H$ be a Zariski-dense subgroup such that $\Delta = \alpha_T(\Theta) \leqslant \mathrm{Aff}(U)$ acts freely and properly on U with compact quotient manifold $M = \Delta \setminus U$. Then $\Gamma = \Delta/\Delta_0$ is virtually polycyclic and M is diffeomorphic to a standard Γ-manifold.*

Proof. Put $U_{\Delta_0} = \mathrm{u}(\overline{\Delta_0})$, and remark that $U_{\Delta_0} \leqslant U$ under the natural inclusion $U \leqslant \mathrm{Aff}(U)$. Since Δ_0 is normal in Δ, U_{Δ_0} is normal in $\bar{\Delta}$. From $\overline{\Delta_0} \cdot 1 = U_{\Delta_0} \cdot 1$ and $\dim \mathrm{u}(\overline{\Delta_0}) \leqslant \dim \Delta_0$, we deduce that $\Delta_0 \cdot 1 = U_{\Delta_0} \cdot 1$ and also that $\dim \Delta_0 = \dim U_{\Delta_0}$. Let $h \in \Theta$, such that $\alpha_T(h) \in \Delta_0$. Then $h = u_h t$, where $u_h \in U_{\Delta_0}$ and $t \in T$. Moreover,

$$\alpha_T(h) \cdot u = u_h u^t = u_h(tut^{-1}u^{-1})u.$$

Remark that $\overline{\Delta_0}$, and U_{Δ_0} are normal in $\alpha_T(H)$. This implies that $h^u = vt$, where $v \in U_{\Delta_0}$, $v = (uu_h u^{-1})(utu^{-1}t^{-1})$. Furthermore $uu_h u^{-1} \in U_{\Delta_0}$, and consequently $utu^{-1}t^{-1} \in U_{\Delta_0}$. Since Δ_0 acts freely, this implies that $\Delta_0 \cdot u = \mathrm{u}(\overline{\Delta_0}) \cdot u$. Hence, Δ_0 and U_{Δ_0} have the same orbits on U.

Put $L = H/U_{\Delta_0}$, and $U_L = U/U_{\Delta_0}$. Let $\pi\colon H \to L$ be the quotient homomorphism. Put $\Upsilon := \pi(\Theta)$. Then Υ is a Zariski-dense subgroup of L. Decompose $L = U_L \pi(T)$. Evidently, π induces a diffeomorphism of quotient spaces $\alpha_T(\Theta) \setminus U \to \alpha_{\pi(T)}(\Upsilon) \setminus U_L$. In particular, M is diffeomorphic to $\alpha_{\pi(T)}(\Upsilon) \setminus U_L$. Moreover, $\alpha_{\pi(T)}(\Upsilon)$ is a discrete solvable subgroup of $\mathrm{Aff}(U_L)$ and isomorphic to Δ/Δ_0. Since $\mathrm{Aff}(U_L)$ has only finitely many connected components, a theorem of Mostow [23] implies that $\Gamma = \Delta/\Delta_0$ is virtually polycyclic. By Proposition 3.3, M is diffeomorphic to a standard Γ-manifold. □

Putting the results together, we proved:

Theorem 3.5. *Let U be connected, simply connected nilpotent Lie group. Let $\Delta \leqslant \mathrm{Aff}(U)$ be a solvable by finite subgroup which acts freely and properly on U with compact quotient manifold $M = \Delta \backslash U$. Then $\Gamma = \Delta/\Delta_0$ is virtually polycyclic, and M is diffeomorphic to a standard Γ-manifold.*

3.3. Rigidity of reductive affine actions

Let G be a connected, simply connected solvable Lie group and $\Delta \leqslant \mathrm{Aff}(G)$ a solvable by finite subgroup which acts on G. Let H_G be an algebraic hull for G, and fix a Zariski-dense continuous inclusion $G \leqslant H_G$. By the rigidity of the hull, there are induced inclusions $\mathrm{hol}(\Delta) \leqslant \mathrm{Aut}_{\mathrm{a}}(H_G)$, and $\Delta \leqslant \mathrm{Aff}_{\mathrm{a}}(H_G)$. Let $T \leqslant H_G$ be a maximal d-subgroup, and U_G the unipotent radical of H_G. Then, G acts affinely on U_G via the action α_T, cf. Section 2.3. Note that the orbit map of this action in $1 \in U_G$, $o_T : G \to U_G$, coincides with the projection diffeomorphism $\tau : G \to U_G$. Via o_T, the affine action of Δ on G induces then a diffeomorphic action of Δ on U_G.

Lemma 3.6. *Suppose that $\mathrm{hol}(\Delta) \leqslant \mathrm{Aut}_{\mathrm{a}}(H_G)$ stabilizes T. Then the action of Δ on U_G induced by the orbit map $o_T : G \to U_G$ is affine.*

Proof. A straightforward computation shows that the lemma is true: Let $\delta = (h, \phi) \in \mathrm{Aff}(G)$, where $h \in G$, $\phi \in hol(\Delta)$. We consider ϕ henceforth as an element of $\mathrm{Aut}(H_G)$. Write $h = u_h t_h$, where $u_h \in U_G$, $t_h \in T$. Analogously, write $g = u_g t_g$, for $g \in G$. Now, $\delta \cdot g = h\phi(g) = u_h t_h \phi(u_g)\phi(t_g)$. By our assumption, $\phi(t_g) \in T$. Hence,

$$
\begin{aligned}
o_T(\delta \cdot g) &= \tau(u_h \phi(u_g)^{t_h} t_h \phi(t_g)) = u_h \phi(u_g)^{t_h} \\
&= \alpha_T(h) \cdot \phi(u_g) = (\alpha_T(h) \circ \phi) \cdot o_T(g).
\end{aligned}
$$

Therefore, the action of δ on G, corresponds to the action of $\alpha_T(h) \circ \phi$ on U_G. □

Lemma 3.7. *Let $L \leqslant \mathrm{Aut}_{\mathrm{a}}(H_G)$ be a reductive subgroup. Then L stabilizes a maximal torus $T \leqslant H_G$.*

Proof. Consider the semi-direct product $H_L = H_G \rtimes L$. Then $\mathrm{u}(H_L) = U_G$ is the unipotent radical of H_L. Let S be a maximal reductive subgroup in H_L which contains L such that $H_L = U_G S$. Then $T = S \cap H_G$ is a d-subgroup in H_G which is normalized by L, and $S = TL$. The latter equality shows that $H_G = U_G T$ and therefore T is a maximal torus of H_G. □

In the light of Theorem 3.5, Lemmas 3.6 and 3.7 prove the following:

Theorem 3.8. *Let $\Delta \leqslant \mathrm{Aff}(G)$ act freely on M with quotient space $\Delta \backslash G$ a compact manifold. Assume further that $\mathrm{hol}(\Delta) \leqslant \mathrm{Aut}_{\mathrm{a}}(H_G)$ is contained in a reductive subgroup of $\mathrm{Aut}_{\mathrm{a}}(H_G)$. Then $M = \Delta \backslash G$ is diffeomorphic to a standard Γ-manifold.*

Theorem 3.8 implies part (ii) of Theorem 1.3. In fact, by Proposition 2.5 assumption (ii) implies that $\mathrm{hol}(\Delta)$ is contained in a reductive subgroup of $\mathrm{Aut}_a(H_G)$. In particular, condition (ii) is satisfied if the Zariski-closure of $\mathrm{hol}(\Delta)$ in $\mathrm{Aut}(G)$ is compact. This proves Theorem 1.4.

4. Geometry of infra-solvmanifolds

We derive a few consequences of our proof which concern the existence and uniqueness of certain geometric structures on infra-solvmanifolds. As another application we construct a finite-dimensional complex which computes the cohomology of an infra-solvmanifold.

4.1. Infra-solv geometry

Let M be a compact infra-solvmanifold. A pair (G, Δ), $\Delta \subset \mathrm{Aff}(G)$, so that M is diffeomorphic to $\Delta \backslash G$ is called a presentation for M. By the proof of Theorem 1.2, every standard Γ-manifold admits a presentation (G, Γ) so that $\Gamma \leqslant \mathrm{Aff}(G)$ is discrete with finite holonomy group $\mathrm{hol}(\Gamma)$. Hence, by Theorem 1.4, every compact infra-solvmanifold has such a presentation. (The appendix of [12] is devoted to proving that every infra-solvmanifold has a presentation with finite holonomy.)

Corollary 4.1. *Every compact infra-solvmanifold M admits a discrete presentation with finite holonomy.*

Let (G, Γ) be a discrete presentation for M with finite holonomy. Then M is finitely covered by the homogeneous space $\Gamma \cap G \backslash G$ of the solvable Lie-group G. The group G carries a natural flat (but not necessarily torsion-free) left invariant connection which is preserved by $\mathrm{Aff}(G)$. Since the presentation is discrete, M has a flat connection inherited from G. The group of covering transformations of $\Gamma \cap G \backslash G \to M$ is acting by connection preserving diffeomorphisms. This geometric property distinguishes infra-solvmanifolds from the larger class of compact manifolds which admit a finite covering by a solv-manifold. (Compare also [30] for a similar discussion.) One should note however that the Lie group G, and discrete presentation (G, Γ) is not uniquely determined by M. However, Wilking [31] proved that every infra-solvmanifold is modeled in a canonical way on an affine isometric action on a super-solvable Lie-group.

Our approach implies that, dropping the condition of finite holonomy, there is a canonical choice of flat geometry on M which is modeled on a nilpotent Lie group. Let U be a unipotent real algebraic group, $\Gamma \leqslant \mathrm{Aff}(U)$ a discrete subgroup which acts properly discontinuously on U. (It is not required that the holonomy of Γ be finite.) Then Γ preserves the natural flat invariant connection on U, and there is an induced flat connection on the quotient manifold M. We say that M has an *affinely flat geometry modeled on U*. Let U_Γ denote the unipotent radical of the real algebraic hull of Γ. We call U_Γ the *unipotent shadow of Γ*.

Corollary 4.2. *Every compact infra-solvmanifold M admits an affinely flat geometry modeled on the simply connected nilpotent Lie group $U_{\pi_1(M)}$.*

Toral affine actions: A natural question is the following: *Given G a simply connected solvable Lie group. Which polycyclic groups act affinely on G with a compact quotient manifold*? In the particular case where G is abelian, this question asks for the classification of affine crystallographic groups. This is a well known and difficult geometric problem. (Compare [4], and also the references cited therein for some recent results).

Some answers to the above question can be given when putting restrictions on the holonomy. Let us call the holonomy $\mathrm{hol}(\Gamma) \leqslant \mathrm{Aut}(G)$ *toral* if the Zariski-closure of $\mathrm{hol}(\Gamma)$ is a reductive subgroup of $\mathrm{Aut}(G)$. In particular, $\mathrm{hol}(\Gamma)$ is toral if its closure is compact. Hence, infra-solvmanifolds come from toral actions. Note that also standard Γ-manifolds are constructed using toral affine actions. Now let U_G denote the unipotent radical of H_G, and U_Γ the radical of H_Γ. We remark:

Proposition 4.3. *Let Γ be torsion-free virtually polycyclic, acting on the connected, simply connected solvable Lie group G with compact quotient space and toral holonomy. Then $U_G = U_\Gamma$.*

Proof. Let T_1 denote the Zariski-closure of $\mathrm{hol}(\Delta)$ in $\mathrm{Aut_a}(H_G)$. By the assumption, $T_1 \leqslant \mathrm{Aut_a}(H_G)$ is a d-subgroup. By Proposition 2.5, T_1 stabilizes G, i.e., $T_1 \leqslant \mathrm{Aut}(G)$. Also T_1 stabilizes a maximal torus $T \leqslant H_G$. Then the corresponding projection map $\tau : G \to U_G$ induces an affine action of Γ on U_G. By the proof of Lemma 3.6, the image of Γ in $\mathrm{Aff}(U_G)$ is contained in $U_G \rtimes TT_1 \leqslant \mathrm{Aff}(U_G)$. By Proposition 3.3, the Zariski-closure of Γ in $U_G \rtimes TT_1$ is an algebraic hull for Γ. Hence, $U_G = U_\Gamma$. □

In particular, if Γ acts isometrically on G, then $U_G = U_\Gamma$.

4.2. Polynomial geometry

The construction of standard Γ-manifolds was carried out in the category of real algebraic groups. In fact, as noted above such a manifold is obtained as a quotient space $\Gamma \backslash U$, where U is a unipotent real algebraic group, and $\Gamma \leqslant \mathrm{Aff_a}(U)$ is a properly discontinuous subgroup. In particular, Γ acts algebraically on U. A differentiable map of $\mathbb{R}^n$ is called a polynomial map if its coordinate functions are polynomials. A polynomial diffeomorphism is a polynomial map which has a polynomial inverse. A group of polynomial diffeomorphisms of $\mathbb{R}^n$ is called bounded if there is a common bound for the degrees of the polynomials which describe its elements. It is known that any algebraic group action on $\mathbb{R}^n$ is bounded. Now, since U is diffeomorphic to $\mathbb{R}^n$, $n = \dim U$, and since the diffeomorphism is given by the exponential map $\exp : \mathfrak{u} = \mathbb{R}^n \to U$ which actually is an algebraic map, we obtain:

Corollary 4.4. *Every torsion-free virtually polycyclic group Γ acts faithfully as a discrete group of bounded polynomial diffeomorphisms on $\mathbb{R}^n$, $n = \mathrm{rank}\,\Gamma$. The quotient space $\Gamma \backslash \mathbb{R}^n$ is diffeomorphic to a standard Γ-manifold.*

Slightly more general, our proof works for all virtually polycyclic groups which do not have finite normal subgroups (see the appendix). The existence of such actions was shown previously in [10] by different methods. Recently, it was proved [5] that a bounded polynomial action of Γ on $\mathbb{R}^n$ is unique up to conjugation by a bounded polynomial diffeomorphism. Therefore Theorem 1.4 implies also the following characterization of infra-solvmanifolds.

Corollary 4.5. *Let M be a compact differentiable manifold, aspherical and with a virtually polycyclic fundamental group. Then M is diffeomorphic to an infra-solv manifold if and only if M is* *diffeomorphic to a quotient space of* $\mathbb{R}^n$ *by a properly discontinuous bounded group of polynomial diffeomorphisms.*

4.3. Polynomial cohomology

Let M be an infra-solvmanifold and fix a diffeomorphism of M to a quotient space $\mathbb{R}^n/\Gamma$, where Γ acts as a group of bounded polynomial diffeomorphisms. Recall that a differential form (more generally a tensor field) on $\mathbb{R}^n$ is called polynomial if its component functions relative to the standard coordinate system are polynomials. Since Γ acts by polynomial maps, the notion of polynomial differential form on M is well defined. This gives a subcomplex

$$\Omega^*_{\mathrm{poly}}(M) \subset \Omega^*(M)$$

of the C^∞-de Rham complex $\Omega^*(M)$. The following result generalizes a theorem of Goldman [14] on the cohomology of compact complete affine manifolds:

Theorem 4.6. *The induced map on cohomology* $H^*_{\mathrm{poly}}(M) \to H^*(M)$ *is an isomorphism.*

Proof. The idea of the proof which is given in [14] carries over to our situation. We pick up the notation of Section 3. Let $\Gamma \leqslant H_\Gamma$, where H_Γ is the algebraic hull of Γ. As explained in Section 3, H_Γ acts as a subgroup of $\mathrm{Aff}_{\mathrm{a}}(U)$ on U. Via $\exp : \mathfrak{u} \to U$, H_Γ acts by polynomial maps on $\mathbb{R}^n = U$. The cohomology of M is computed by the complex $\Omega^*(\mathbb{R}^n)^\Gamma$ of Γ-invariant differential forms on $\mathbb{R}^n$. Therefore, we have to show that the inclusion of complexes

$$\Omega^*_{\mathrm{poly}}(\mathbb{R}^n)^\Gamma \to \Omega^*(\mathbb{R}^n)^\Gamma$$

induces an isomorphism on cohomology. Let Γ_0 be a finite index normal subgroup of Γ with syndetic hull G, so that

$$\Gamma_0 \leqslant G \leqslant H = (H_\Gamma)_0.$$

We consider now the following inclusion maps of complexes

$$\Omega^*(\mathbb{R}^n)^H \to \Omega^*(\mathbb{R}^n)^G \to \Omega^*(\mathbb{R}^n)^{\Gamma_0}.$$

Decompose $H = US = GS$, where S is a maximal d-subgroup of H. By Lemma 2.6, G acts simply transitively on $\mathbb{R}^n$ via the affine action α_T of H_Γ on U. Hence, the complex $\Omega^*(\mathbb{R}^n)^G$ identifies with the complex $H^*(\mathfrak{g})$ of left invariant differential forms on G. The action of G on $H^*(\mathfrak{g})$ which is induced by conjugation is trivial. Since G is Zariski-dense in H, H acts trivially on $H^*(\mathfrak{g})$. The affine action of $S \leqslant H$ on U corresponds to conjugation on G. It follows that S acts trivially on the cohomology of the complex $\Omega^*(\mathbb{R}^n)^G$. Since S acts reductively on $\Omega^*(\mathbb{R}^n)^G$, this implies

$$H^*(\Omega^*(\mathbb{R}^n)^G) = H^*(\Omega^*(\mathbb{R}^n)^G)^S = H^*(\Omega^*(\mathbb{R}^n)^H).$$

In particular, $\Omega^*(\mathbb{R}^n)^H \to \Omega^*(\mathbb{R}^n)^G$ induces an isomorphism on cohomology. The map $\Omega^*(\mathbb{R}^n)^G \to \Omega^*(\mathbb{R}^n)^{\Gamma_0}$ is an isomorphism on the cohomology level by a theorem of Mostow [25] (see also [28, Corollary 7.29]). Hence, the induced map $H^*(\Omega^*(\mathbb{R}^n)^H) \to H^*(\Omega^*(\mathbb{R}^n)^{\Gamma_0})$ is an isomorphism.

Next remark that $\Omega^*(\mathbb{R}^n)^H = \Omega^*_{\mathrm{poly}}(\mathbb{R}^n)^H = \Omega^*_{\mathrm{poly}}(\mathbb{R}^n)^{\Gamma_0}$. The first equality follows since every left invariant differential form on U is polynomial relative to the coordinates given by the exponential map. The second equality follows since Γ_0 is Zariski-dense in H. We conclude that the natural map

$$H^*(\Omega^*_{\text{poly}}(\mathbb{R}^n)^{\Gamma_0}) \to H^*(\Omega^*(\mathbb{R}^n)^{\Gamma_0})$$

is an isomorphism. This proves that $H^*_{\text{poly}}(M_{\Gamma_0}) = H^*(M_{\Gamma_0})$.

Put $\mu = \Gamma/\Gamma_0$. Then μ acts on the cohomology of $\Omega^*(\mathbb{R}^n)^{\Gamma_0}$. The inclusion map $\Omega^*(\mathbb{R}^n)^{\Gamma} \to \Omega^*(\mathbb{R}^n)^{\Gamma_0}$ induces an isomorphism on cohomology

$$H^*(M_\Gamma) \to H^*(M_{\Gamma_0})^\mu.$$

Similarly, $H^*_{\text{poly}}(M_{\Gamma_0})^\mu$ is isomorphic to the cohomology of the μ-invariant forms in $\Omega^*_{\text{poly}}(\mathbb{R}^n)^{\Gamma_0}$, implying that $H^*_{\text{poly}}(M_\Gamma) = H^*_{\text{poly}}(M_{\Gamma_0})^\mu$. Hence,

$$H^*_{\text{poly}}(M_\Gamma) = H^*(M_\Gamma).$$

The theorem follows. □

Proof of Theorem 1.8. By the previous theorem, $H^*(M_\Gamma) = H^*_{\text{poly}}(M_\Gamma)$. Now

$$H^*_{\text{poly}}(M_\Gamma) = H^*(\Omega^*_{\text{poly}}(\mathbb{R}^n)^{H_\Gamma}).$$

Let T be a maximal d-subgroup of H_Γ. Since U acts simply transitively on $\mathbb{R}^n$, the complex $\Omega^*_{\text{poly}}(\mathbb{R}^n)^{H_\Gamma}$ is isomorphic to the left-invariant forms on U which are fixed by T. Let $\mathfrak{u}$ denote the Lie algebra of U. Since T acts reductively on the complex $\Omega^*_{\text{poly}}(\mathbb{R}^n)^U$, it follows that

$$H^*_{\text{poly}}(M_\Gamma) = H^*((\Omega^*_{\text{poly}}(\mathbb{R}^n)^U)^T) = H^*(\mathfrak{u})^T. \qquad \square$$

Acknowledgements

I thank Fritz Grunewald, Burkhard Wilking and Wilhem Singhof for helpful comments on an earlier draft of this article.

Appendix A. Algebraic hulls for virtually polycyclic groups

Let $k \leqslant \mathbb{C}$ be a subfield. A group $\mathbf{G}$ is called a k-defined linear algebraic group if it is a Zariski-closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ which is defined by polynomials with coefficients in k. A morphism of algebraic groups is a morphism of algebraic varieties which is also a group homomorphism. A morphism is defined over k if the polynomials which define it have coefficients in k. It is called a k-isomorphism if its inverse exists and is a morphism defined over k. Let $\mathbf{U} = \mathrm{u}(\mathbf{G})$ denote the unipotent radical of $\mathbf{G}$. We say that $\mathbf{G}$ has a *strong unipotent radical* if the centralizer $\mathrm{Z}_{\mathbf{G}}(\mathbf{U})$ is contained in $\mathbf{U}$.

A.1. The algebraic hull

Let Γ be a virtually polycyclic group. Its maximal nilpotent normal subgroup $\mathrm{Fitt}(\Gamma)$ is called the Fitting subgroup of Γ. Now assume that $\mathrm{Fitt}(\Gamma)$ is torsion-free and $\mathrm{Z}_\Gamma(\mathrm{Fitt}(\Gamma)) \leqslant \mathrm{Fitt}(\Gamma)$. In this case, we say that Γ has a *strong Fitting subgroup*. We remark (see also Corollary A.9) that this condition is equivalent to the requirement that Γ has no non-trivial finite normal subgroups. The following result was announced in [4].

Theorem A.1. *Let Γ be a virtually polycyclic group with a strong Fitting subgroup. Then there exists a $\mathbb{Q}$-defined linear algebraic group $\mathbf{H}$ and an injective group homomorphism $\psi:\Gamma \to \mathbf{H}_{\mathbb{Q}}$ so that,*

(i) $\psi(\Gamma)$ *is Zariski-dense in* $\mathbf{H}$,
(ii) $\mathbf{H}$ *has a strong unipotent radical* $\mathbf{U}$,
(iii) $\dim \mathbf{U} = \operatorname{rank} \Gamma$.

Moreover, $\psi(\Gamma) \cap \mathbf{H}_{\mathbb{Z}}$ is of finite index in $\psi(\Gamma)$.

We remark that the group $\mathbf{H}$ is determined by the conditions (i)–(iii) up to $\mathbb{Q}$-isomorphism of algebraic groups:

Proposition A.2. *Let $\mathbf{H}'$ be a $\mathbb{Q}$-defined linear algebraic group, $\psi':\Gamma \to \mathbf{H}'_{\mathbb{Q}}$ an injective homomorphism which satisfies* (i)–(iii) *above. Then there exists a $\mathbb{Q}$-defined isomorphism $\Phi:\mathbf{H} \to \mathbf{H}'$ so that $\psi' = \Phi \circ \psi$.*

Corollary A.3. *The algebraic hull $\mathbf{H}_\Gamma$ of Γ is unique up to $\mathbb{Q}$-isomorphism of algebraic groups. In particular, every automorphism ϕ of Γ extends uniquely to a $\mathbb{Q}$-defined automorphism Φ of $\mathbf{H}_\Gamma$.*

We call the $\mathbb{Q}$-defined linear algebraic group $\mathbf{H}$ the *algebraic hull for Γ*. If Γ is finitely generated torsion-free nilpotent then $\mathbf{H}$ is unipotent and Theorem A.1 and Proposition A.2 are essentially due to Malcev. If Γ is torsion-free polycyclic, Theorem A.1 is due to Mostow [24] (see also [28, Section IV, p. 74] for a different proof).

A.2. Construction of the algebraic hull

Let Δ be a virtually polycyclic group with $\mathrm{Fitt}(\Delta)$ torsion-free. Since Δ is virtually polycyclic it contains a torsion-free polycyclic subgroup Γ which is normal and of finite index. By Mostow's theorem there exists an algebraic hull $\psi_\Gamma:\Gamma \to \mathbf{H}_\Gamma$ for Γ. Hence, in particular Γ is realized as a subgroup of a linear algebraic group.

Embedding of finite extensions: We use a standard induction procedure to realize the finite extension group Δ of Γ in a linear algebraic group which finitely extends $\mathbf{H}_\Gamma$. The procedure is summarized in the next lemma.

Lemma A.4. *Let $\mathbf{G}$ be a linear algebraic group defined over $\mathbb{Q}$ and $\Gamma \leqslant \mathbf{G}_{\mathbb{Q}}$ a subgroup. Let Δ be a finite extension group of Γ so that Γ is normal in Δ. Let $\Delta = \Gamma r_1 \cup \cdots \cup \Gamma r_m$ be the decomposition of Δ into left cosets, and assume that there are $\mathbb{Q}$-defined algebraic group morphisms $f_1,\ldots,f_m:\mathbf{G} \to \mathbf{G}$ so that*

$$f_i(\gamma) = r_i \gamma r_i^{-1}, \quad i = 1,\ldots,m, \quad \textit{for all } \gamma \in \Gamma. \tag{A.1}$$

Then there exists a $\mathbb{Q}$*-defined linear algebraic group* $\mathbf{I}(\mathbf{G},\Gamma,\Delta)$, *an injective homomorphism* $\psi:\Delta\to\mathbf{I}(\mathbf{G},\Gamma,\Delta)$ *and a* $\mathbb{Q}$*-defined injective morphism of algebraic groups* $\Psi:\mathbf{G}\to\mathbf{I}(\mathbf{G},\Gamma,\Delta)$ *which extends* $\psi:\Gamma\to\mathbf{I}(\mathbf{G},\Gamma,\Delta)$ *so that* $\psi(\Delta)\leqslant\mathbf{I}(\mathbf{G},\Gamma,\Delta)_{\mathbb{Q}}$, $\mathbf{I}(\mathbf{G},\Gamma,\Delta)=\Psi(\mathbf{G})\psi(\Delta)$ *and* $\Psi(\mathbf{G})\cap\psi(\Delta)=\psi(\Gamma)$.

For more comments and details of the proof see [15, Proposition 2.2].

The algebraic hull for Δ: We continue with our standing assumption that Δ is a virtually polycyclic group with $\mathrm{Fitt}(\Delta)$ torsion-free and $\mathrm{Z}_\Delta(\mathrm{Fitt}(\Delta))\leqslant\mathrm{Fitt}(\Delta)$. Let $\Gamma\leqslant\Delta$ be a torsion-free polycyclic normal subgroup of finite index. By Mostow's result there exists an algebraic hull $\mathbf{H}_\Gamma$ for Γ, and we may assume that $\Gamma\leqslant(\mathbf{H}_\Gamma)_{\mathbb{Q}}$ is a Zariski-dense subgroup. Replacing Γ with a finite index subgroup, if necessary, we may also assume that $\mathbf{H}_\Gamma$ is connected.

Proposition A.5. *There exists a* $\mathbb{Q}$*-defined linear algebraic group* $I(\mathbf{H}_\Gamma,\Delta)$ *which contains* $\mathbf{H}_\Gamma$, *and an embedding* $\psi:\Delta\to I(\mathbf{H}_\Gamma,\Delta)_{\mathbb{Q}}$ *which is the identity on* Γ, *such that* $I(\mathbf{H}_\Gamma,\Delta)=\mathbf{H}_\Gamma\psi(\Delta)$ *and* $\psi(\Delta)\cap\mathbf{H}_\Gamma=\psi(\Gamma)$.

Proof. Let $\Delta=\Gamma r_1\cup\cdots\cup\Gamma r_m$ be a decomposition of Δ into left cosets. By the rigidity of the algebraic hull (Proposition A.2), conjugation with r_i on Γ extends to $\mathbb{Q}$-defined morphisms f_i of $\mathbf{H}$ which satisfy (A.1). The results follows then from Lemma A.4, putting $\mathbf{I}(\mathbf{H}_\Gamma,\Delta):=\mathbf{I}(\mathbf{H}_\Gamma,\Gamma,\Delta)$. □

We need some more notations. Let $\mathbf{G}$ be an algebraic group. We let $\mathbf{G}^0$ denote its connected component of identity. If g is an element of $\mathbf{G}$ then $g=g_ug_s$ denotes the Jordan-decomposition of g (i.e., g_u is unipotent, g_s is semisimple and $[g_s,g_u]=1$). If M is a subset then let $M_u=\{g_u\mid g\in M\}$, $M_s=\{g_s\mid g\in M\}$.

If G is a subgroup then $\mathrm{u}(G)$ denotes the unipotent radical of G, i.e., the maximal normal subgroup of G which consists of unipotent elements. We will use the following facts (cf. [7, Section 10]): If G is a nilpotent subgroup then G_u and G_s are subgroups of $\mathbf{G}$, and $G\leqslant G_u\times G_s$. If $\mathbf{G}^0$ is solvable then $\mathbf{G}_u=\mathrm{u}(\mathbf{G})$. In particular, for any subgroup G of $\mathbf{G}$, $\mathrm{u}(G)=G\cap G_u$.

To construct the algebraic hull for Δ we have to further refine Proposition A.5.

Proof of Theorem A.1. Let $\mathbf{U}$ denote the unipotent radical of $\mathbf{H}_\Gamma$. By Proposition A.5, we may assume that $\Delta\leqslant\mathbf{G}_{\mathbb{Q}}$ is a Zariski-dense subgroup of a $\mathbb{Q}$-defined linear algebraic group $\mathbf{G}$ so that $\mathbf{G}=\mathbf{H}_\Gamma\Delta$, $\mathrm{u}(\mathbf{G})=\mathbf{U}$ and $\Delta\cap\mathbf{H}_\Gamma=\Gamma$. Since $\mathbf{H}_\Gamma$ is an algebraic hull, $\mathrm{Fitt}(\Gamma)\leqslant\mathbf{U}$, see Proposition A.7. Since $\mathrm{Fitt}(\Gamma)$ is a subgroup of finite index in $\mathrm{Fitt}(\Delta)$ the group $\mu=\{\gamma_s\mid\gamma\in\mathrm{Fitt}(\Delta)\}\leqslant\mathbf{G}$ is finite. Since $\mathrm{Fitt}(\Delta)$ is a normal subgroup of Δ, μ is normalized by Γ. Hence, the centralizer of μ in $\mathbf{G}$ contains a finite index subgroup of Γ. Since the centralizer of μ is a Zariski-closed subgroup of $\mathbf{G}$ it contains $(\mathbf{H}_\Gamma)^0=\mathbf{H}_\Gamma$. In particular, μ centralizes Γ. We consider now the homomorphism ψ_u: $\mathrm{Fitt}(\Delta)\to\mathbf{U}_{\mathbb{Q}}$ which is given by $\gamma\mapsto\gamma_u$. The kernel of ψ_u is contained in the finite group μ. Since $\mathrm{Fitt}(\Delta)$ is torsion-free, ψ_u is injective. Assigning $\psi:\delta\gamma\mapsto\psi_u(\delta)\gamma$ defines an injective homomorphism ψ: $\mathrm{Fitt}(\Delta)\Gamma\to(\mathbf{H}_\Gamma)_{\mathbb{Q}}$. (To see that ψ is injective suppose that $1=\psi(\delta\gamma)$, for $\delta\in\mathrm{Fitt}(\Delta)$, $\gamma\in\Gamma$. Then $\gamma=\psi_u(\delta)^{-1}$ is unipotent, i.e., $\gamma\in\mathrm{u}(\Gamma)\leqslant\mathrm{Fitt}(\Gamma)$. Therefore $\delta\gamma\in\mathrm{Fitt}(\Delta)$, and $\psi(\delta\gamma)=\psi_u(\delta\gamma)$, and hence $\delta\gamma=1$.) Clearly, the homomorphism ψ is the identity on Γ. Let us put $\Gamma^*=\psi(\mathrm{Fitt}(\Delta)\Gamma)$. Then $\mathbf{H}_\Gamma$ is an algebraic hull for Γ^*. We consider now the extension $\Gamma^*\leqslant\Delta$. By Proposition A.5, there exist an algebraic group $I^*(\mathbf{H}_\Gamma,\Delta)$ and an embedding $\psi^*:\Delta\to\mathbf{I}^*(\mathbf{H}_\Gamma,\Delta)_{\mathbb{Q}}$

so that $\psi^*(\Delta)\cap\mathbf{I}^*(\mathbf{H}_\Gamma,\Delta)=\Gamma^*$. By construction, the group $\mathbf{I}^*(\mathbf{H}_\Gamma,\Delta)$ satisfies (i) and (iii) of Theorem A.1 with respect to ψ^* and Δ. Moreover, by our construction $\psi^*(\mathrm{Fitt}(\Delta)) \leqslant \mathbf{U} = \mathrm{u}(\mathbf{I}^*(\mathbf{H}_\Gamma,\Delta))$.

We consider now the centralizer $\mathrm{Z}_{I^*(\mathbf{H}_\Gamma,\Delta)}(\mathbf{U})$ of $\mathbf{U}$ in $I^*(\mathbf{H}_\Gamma,\Delta)$. It is a $\mathbb{Q}$-defined algebraic subgroup of the solvable by finite group $I^*(\mathbf{H}_\Gamma,\Delta)$. Therefore $\mathrm{Z}_{I^*(\mathbf{H}_\Gamma,\Delta)}(\mathbf{U}) = \mathrm{Z}(\mathbf{U})S$, where S is a $\mathbb{Q}$-defined subgroup which consist of semisimple elements. We remark that $S = \mathrm{Z}_{I^*(\mathbf{H},\Delta)}(\mathbf{U})_S$ is normal in $I^*(\mathbf{H},\Delta)$. Since $S\cap\psi^*(\Delta)$ centralizes $\mathrm{Fitt}(\Delta) \leqslant \mathbf{U}$, the assumption that Δ has a strong Fitting subgroup implies that $S\cap\psi^*(\Delta)=\{1\}$. Let $\pi: I^*(\mathbf{H},\Delta)\to\mathbf{H}_\Delta = I^*(\mathbf{H},\Delta)/S$ be the projection homomorphism. Since $\mathbf{H}_\Delta$ is $\mathbb{Q}$-defined with a strong unipotent radical, $\mathbf{H}_\Delta$ with the embedding $\pi\psi^*:\Delta\to(\mathbf{H}_\Delta)_\mathbb{Q}$ is an algebraic hull for Δ. □

A.3. Properties of the algebraic hull

Let Γ be a virtually polycyclic group. We assume that Γ admits an algebraic hull $\mathbf{H}_\Gamma$. The next proposition implies Proposition A.2, and Corollary A.3.

Proposition A.6. *Let k be a subfield of $\mathbb{C}$, and $\mathbf{G}$ a k-defined linear algebraic group with a strong unipotent radical. Let $\rho:\Gamma\to\mathbf{G}$ be a homomorphism so that $\rho(\Gamma)$ is Zariski-dense in $\mathbf{G}$. Then ρ extends uniquely to a morphism of algebraic groups $\rho_{\mathbf{H}_\Gamma}:\mathbf{H}_\Gamma\to\mathbf{G}$. If $\rho(\Gamma)\leqslant\mathbf{G}_k$ then $\rho_{\mathbf{H}_\Gamma}$ is defined over k.*

Proof. We will use the diagonal argument. Therefore we consider the subgroup

$$D = \{(\gamma,\rho(\gamma)) \mid \gamma\in\Gamma\} \leqslant \mathbf{H}_\Gamma\times\mathbf{G}.$$

Let π_1, π_2 denote the projection morphisms onto the factors of the product $\mathbf{H}_\Gamma\times\mathbf{G}$. Let $\mathbf{D}$ be the Zariski-closure of D, and $\mathbf{U}=\mathrm{u}(\mathbf{D})$ the unipotent radical of $\mathbf{D}$. The group $\mathbf{D}$ is a solvable by finite linear algebraic group, and $\mathbf{D}$ is defined over k if $\rho(\Gamma)\leqslant\mathbf{G}_k$. Let $\alpha=\pi_1|_\mathbf{D}:\mathbf{D}\to\mathbf{H}_\Gamma$. Since Γ is Zariski-dense in $\mathbf{H}_\Gamma$, α is onto. In particular, a maps $\mathbf{U}=\mathbf{D}_u$ onto $\mathrm{u}(\mathbf{H}_\Gamma)$. By [28, Lemma 4.36] we have $\dim\mathbf{U}\leqslant\mathrm{rank}\,\Gamma=\dim\mathrm{u}(\mathbf{H}_\Gamma)$, and hence $\dim\mathbf{U}=\dim\mathrm{u}(\mathbf{H}_\Gamma)$. In particular, it follows that the restriction $\alpha:\mathbf{U}\to\mathrm{u}(\mathbf{H}_\Gamma)$ is an isomorphism. Thus the kernel of α consists only of semi-simple elements. Let $x\in\ker\alpha$. Then x centralizes $\mathbf{U}$. Since $\pi_2(\mathbf{U})=\mathrm{u}(\mathbf{G})$, $\pi_2(x)$ centralizes $\mathrm{u}(\mathbf{G})$. Since $\mathbf{G}$ has a strong unipotent radical, x is in the kernel of π_2, hence $x=1$. It follows that the morphism α is an isomorphism of groups. It is also an isomorphism of algebraic groups. If $\rho(\Gamma)\leqslant\mathbf{G}_k$ then α is k-defined. One can also show that α^{-1} is k-defined. (Compare e.g. [15, Lemma 2.3].) We put $\rho_{\mathbf{H}_\Gamma}=\pi_2\circ\alpha^{-1}$ to get the required unique extension. $\rho_{\mathbf{H}_\Gamma}$ is k-defined if $\rho(\Gamma)\leqslant\mathbf{G}_k$. □

Remark. The proposition shows that the condition that $\overline{\rho(\Gamma)}$ has a strong unipotent radical forces the homomorphism ρ to be well behaved. For example, ρ must be unipotent on the Fitting subgroup of Γ. See Proposition A.7 below.

We study some further properties of the algebraic hull. In particular, we characterize the abstract virtually polycyclic groups which admit an algebraic hull in the sense of Theorem A.1. Let us assume that Γ is a Zariski-dense subgroup of a linear algebraic group $\mathbf{H}$ with a strong unipotent radical.

Proposition A.7. *We have* $\mathrm{Fitt}(\Gamma)\leqslant\mathrm{u}(\mathbf{H})$. *In particular,* $\mathrm{u}(\Gamma)=\mathrm{Fitt}(\Gamma)$ *and* $\mathrm{Fitt}(\Gamma)$ *is torsion-free.*

Proof. Let F be the maximal nilpotent normal subgroup of $\mathbf{H}$. Clearly, $F = \mathbf{F}$ is a Zariski-closed subgroup. Therefore $\mathrm{u}(\mathbf{F}) = \mathrm{u}(\mathbf{H})$. Now since $\mathbf{F}$ is nilpotent, $\mathbf{F}_s$ is a subgroup, $\mathbf{F}_u = \mathrm{u}(\mathbf{F})$ and $\mathbf{F} = \mathbf{F}_s \cdot \mathrm{u}(\mathbf{F})$ is a direct product of groups. Since $\mathbf{H}$ has a strong unipotent radical $\mathbf{F}_s$ must be trivial, and it follows that $\mathbf{F} = \mathrm{u}(\mathbf{H})$. The Zariski-closure of $\mathrm{Fitt}(\Gamma)$ is a nilpotent normal subgroup of $\mathbf{H}$ and therefore $\mathrm{Fitt}(\Gamma)$ is contained in $\mathbf{F}$. Hence $\mathrm{Fitt}(\Gamma) \leqslant \mathrm{u}(\mathbf{H})$. □

Let $\mathbf{N} = \overline{\mathrm{Fitt}(\Gamma)}$ be the Zariski-closure of $\mathrm{Fitt}(\Gamma)$ in $\mathbf{H}$. We just proved $\mathbf{N} \leqslant \mathrm{u}(\mathbf{H})$.

Proposition A.8. *Let* $\mathbf{X} = \mathrm{Z}_{\mathbf{H}}(\mathbf{N})$ *be the centralizer of* $\mathbf{N}$ *in* $\mathbf{H}$. *Let* $\mathbf{X}^0$ *be its component of identity. Then* $\mathbf{X}^0$ *is a nilpotent normal subgroup of* $\mathbf{H}$, *and* $\mathbf{X}^0 \leqslant \mathrm{u}(\mathbf{H})$. *Moreover*, $\mathrm{Z}_\Gamma(\mathrm{Fitt}(\Gamma)) \leqslant \mathrm{Fitt}(\Gamma)$.

Proof. Since $\mathbf{X}^0$ is a connected solvable algebraic group, $\mathbf{X}^0 = \mathbf{U} \cdot \mathbf{T}$, where $\mathbf{U}$ is a connected unipotent group and $\mathbf{T}$ is a maximal torus in $\mathbf{X}^0$. Let $\mathbf{X}_1 = [\mathbf{X}^0, \mathbf{X}^0]$ and define $\Gamma_0 = \Gamma \cap \mathbf{H}^0$. Then Γ_0 is a polycyclic normal subgroup of Γ, and $\mathrm{Fitt}(\Gamma_0) = \mathrm{Fitt}(\Gamma)$. Since $\Gamma_0 \leqslant \mathbf{H}^0$, it follows that $[\Gamma_0, \Gamma_0] \leqslant \mathrm{u}(\Gamma_0) \leqslant \mathrm{Fitt}(\Gamma_0)$. This implies $[\mathbf{X}^0, \mathbf{X}^0] \leqslant [\mathbf{H}^0, \mathbf{H}^0] = \overline{[\Gamma_0, \Gamma_0]} \leqslant \mathbf{N}$. We deduce $[\mathbf{X}^0, \mathbf{X}_1] = [\mathbf{T}, \mathbf{X}_1] = \{1\}$. On the other hand, by Borel [7, Section 10.6] all maximal tori in $\mathbf{X}^0$ are conjugate by an element of $\mathbf{X}_1$. Hence $\mathbf{T}$ must be an invariant subgroup of $\mathbf{X}^0$. In particular, $\mathbf{T}$ is a normal abelian subgroup of $\mathbf{H}$. Therefore, by the proof of the previous proposition, $\mathbf{T}$ is contained in $\mathrm{u}(\mathbf{H})$. Since $\mathbf{T}$ consists of semisimple elements, $\mathbf{T} = \{1\}$. Hence, $\mathbf{X}^0 = \mathrm{Z}(\mathrm{u}(\mathbf{H}))$.

Next put $X = \mathrm{Z}_\Gamma(\mathrm{Fitt}(\Gamma))$ and $X_0 = X \cap \mathbf{X}^0$. Then X_0 is of finite index in X, nilpotent, and a normal subgroup of Γ. The latter implies that $X_0 \leqslant \mathrm{Fitt}(\Gamma)$. It follows that X_0 is centralized by X. Since X_0 is of finite index in X the commutator subgroup $[X, X]$ must be finite. Since $[X, X]$ is normal in Γ it follows that $\mathrm{u}(\mathbf{H}_\Gamma)$ centralizes $[X, X]$. Since $\mathbf{H}$ has a strong unipotent radical $[X, X] = \{1\}$. It follows that X is an abelian normal subgroup of Γ, and hence $X \leqslant \mathrm{Fitt}(\Gamma)$. □

Recall that Γ is said to have a *strong Fitting subgroup* if $\mathrm{Fitt}(\Gamma)$ is torsion-free and contains its centralizer. We summarize:

Corollary A.9. *Let* Γ *be virtually polycyclic. Then* Γ *admits an algebraic hull* $\mathbf{H}$ *if and only if* Γ *has a strong Fitting subgroup.*

We remark that this condition holds if and only if every finite normal subgroup of Γ is trivial. In fact, let us assume that the maximal normal finite subgroup of Γ is trivial. Then $\mathrm{Fitt}(\Gamma)$ is torsion-free since its elements of finite order form a finite normal subgroup of Γ. Now put $X = \mathrm{Z}_\Gamma(\mathrm{Fitt}(\Gamma))$, and let X_0 be a polycyclic normal subgroup of finite index in X which is nilpotent-by-abelian. From $[X_0, X_0] \leqslant \mathrm{Fitt}(X) \leqslant \mathrm{Fitt}(\Gamma)$ we deduce that $X_0 \leqslant \mathrm{Fitt}(\Gamma)$. Therefore $[X, X]$ must be finite, and it follows from our assumption that $[X, X] = \{1\}$. Hence, $X \leqslant \mathrm{Fitt}(\Gamma)$.

References

[1] L. Auslander, An exposition of the structure of solvmanifolds. I. Algebraic theory, Bull. Am. Math. Soc. 79 (2) (1973) 262–285.

[2] L. Auslander, F.E.A. Johnson, On a conjecture of C.T.C. Wall, J. London Math. Soc. 14 (2) (1976) 331–332.

[3] L. Auslander, R. Tolimieri, On a conjecture of G.D. Mostow and the structure of solvmanifolds, Bull. Am. Math. Soc. 75 (1969) 1330–1333.

[4] O. Baues, Finite extensions and unipotent shadows of affine crystallographic groups, C.R. Acad. Sci. Paris Ser. I 335 (2002) 785–788.

[5] Y. Benoist, K. Dekimpe, The uniqueness of polynomial crystallographic actions, Math. Ann. 322 (3) (2002) 563–571.

[6] L. Bieberbach, Über die Bewegungsgruppen der Euklidischen Räume II, Math. Ann. 72 (1912) 400–412.

[7] A. Borel, Linear algebraic groups, Second Edition, Graduate Texts in Mathematics, Vol. 126, Springer, Berlin, 1991.

[8] A. Borel, J.-P. Serre, Théorèmes de finitude en cohomologie galoisienne, Comment. Math. Helv. 39 (1964) 111–164.

[9] W. Browder, On the action of $\Theta^n(\partial\pi)$, Differential and Combinatorial Topology, A Symposium in Honor of Marston Morse, Princeton University Press, Princeton, NJ, 1965, pp. 23–36.

[10] K. Dekimpe, P. Igodt, Polycyclic-by-finite groups admit a bounded-degree polynomial structure, Invent. Math. 129 (1) (1997) 121–140.

[11] F.T. Farrell, L.E. Jones, The surgery L-groups of poly-(finite or cyclic) groups, Invent. Math. 91 (3) (1988) 559–586.

[12] F.T. Farrell, L.E. Jones, Compact infrasolvmanifolds are smoothly rigid, Geometry from the Pacific Rim (Singapore, 1994), de Gruyter, Berlin, 1997, pp. 85–97.

[13] D. Fried, W.M. Goldman, Three-dimensional affine crystallographic groups, Adv. Math. 47 (1) (1983) 1–49.

[14] W.M. Goldman, On the polynomial cohomology of affine manifolds, Invent. Math. 65 (3) (1981/1982) 453–457.

[15] F. Grunewald, V. Platonov, Solvable arithmetic groups and arithmeticity problems, Int. J. Math. 10 (3) (1999) 327–366.

[16] F. Grunewald, D. Segal, On affine crystallographic groups, J. Differential Geometry 40 (3) (1994) 563–594.

[17] G. Hochschild, Cohomology of algebraic linear groups, Ill. J. Math. 5 (1961) 492–519.

[18] R.C. Kirby, L.C. Siebenmann, Foundational Essays on Topological Manifolds, Smoothings, and Triangulations, Annals of Mathematics Studies, Vol. 88, Princeton University Press, Princeton, NJ, University of Tokyo Press, Tokyo, 1977.

[19] K.B. Lee, F. Raymond, Rigidity of almost crystallographic groups, Combinatorial methods in topology and algebraic geometry (Rochester, N.Y., 1982), Contemporary Mathematicians, Vol. 44, American Mathematical Society, Providence, RI, 1985, pp. 73–78.

[20] K.B. Lee, F. Raymond, Maximal torus actions on solvmanifolds and double coset spaces, Int. J. Math. 2 (1) (1991) 67–76.

[21] A.I. Malcev, On a class of homogeneous spaces, Am. Math. Soc. Translation 39 (1951).

[22] G.D. Mostow, Factor spaces of solvable groups, Ann. Math. 60 (2) (1954) 1–27.

[23] G.D. Mostow, On the fundamental group of a homogeneous space, Ann. Math. 66 (2) (1957) 249–255.

[24] G.D. Mostow, Representative functions on discrete groups and solvable arithmetic subgroups, Am. J. Math. 92 (1970) 1–32.

[25] G.D. Mostow, Some applications of representative functions to solvmanifolds, Am. J. Math. 93 (1971) 11–32.

[26] G.D. Mostow, Strong rigidity of locally symmetric spaces, in: Annals of Mathematics Studies, Vol. 78, Princeton University Press, Princeton, NJ, 1973.

[27] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, Ann. Math. 59 (2) (1954) 531–538.

[28] M.S. Raghunathan, Discrete subgroups of Lie groups, Ergebnisse der Mathematik und ihrer Grenzgebiete 68, Springer, Berlin, 1972.

[29] D. Segal, Polycyclic Groups, Cambridge University Press, London, 1983.

[30] W. Tuschmann, Collapsing, solvmanifolds and infrahomogeneous spaces, Differential Geom. Appl. 7 (3) (1997) 251–264.

[31] B. Wilking, Rigidity of group actions on solvable Lie groups, Math. Ann. 317 (2) (2000) 195–237.