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On the Kronecker Problem and related problems of Linear Algebra

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Abstract

We consider some classification problems of Linear Algebra related closely to the classical Kronecker Problem on pairs of linear maps between two finite-dimensional vector spaces. As shown by Djoković and Sergeichuk, the Kronecker's solution is extended to the cases of pairs of semilinear maps and (more generally) pseudolinear bundles respectively. Our objective is to deal with the semilinear case of the Kronecker Problem, especially with its applications. It is given a new short solution both to this case and to its contragredient variant. The biquadratic matrix problem is investigated and reduced in the homogeneous case (in characteristic $\neq 2$) to the semilinear Kronecker Problem. The integer matrix sequence Θ_n and Θ-transformation of polynomials are introduced and studied to get a simplified canonical form of indecomposables for the mentioned homogeneous problem. Some applications to the representation theory of posets with additional structures are presented.

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1. Introduction

We recall that the classical *Kronecker Problem* is the problem of classifying all pairs of linear maps between two finite-dimensional vector spaces over a field K, as may be visualized by the diagram

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$$U_1 \underset{\mathscr{B}}{\overset{\mathscr{A}}{\rightrightarrows}} U_2. \tag{1}$$

In other words, it is the problem of classifying indecomposable in the natural sense objects of the form $(U_1, U_2, \mathscr{A}, \mathscr{B})$, up to isomorphism. It was solved completely by Kronecker [18] long ago, meanwhile a solution to the partial regular case was given earlier by Weierstrass [25].

One can find several relatively recent simplified solutions, including those which present descriptions not solely of the mentioned objects but also of the naturally defined morphisms between them, i.e. dealing with the whole category formed by the objects $(U_1, U_2, \mathscr{A}, \mathscr{B})$. See in particular [2,22] for the categorical and homological approach, [10] for a simplified matrix classification of indecomposables, [5,24] for other linear-algebraic solutions. Though usually it is assumed in the classical situation that K is a (commutative) field, in fact all the main results and considerations (at least, concerning indecomposables) are valid also for any division ring K.

Moreover, the Kronecker's classification has been extended already to the case of pairs of semilinear maps by Djoković [6] (this case is named below the semilinear Kronecker Problem[1]), as well as to a more general pseudolinear bundle case by Sergeichuk [23]. It appeared, in both situations the discrete canonical forms of indecomposables are the same as in the classical case, while the continuous ones are consequences of the known facts of the theory of modules over skew polynomial rings (exposed for instance in [3]).

The aim of the present article is to examine thoroughly once more the semilinear Kronecker Problem, especially its applications to other problems of Linear Algebra. First of all, using a unified matrix approach, we obtain a concise solution both to this problem and to its contragredient variant. Furthermore, we consider an important biquadratic matrix problem reducing it in the homogeneous case (in characteristic $\neq 2$) to the semilinear Kronecker Problem and then obtaining two natural canonical forms of indecomposables, with applications. The second form arises on the base of developing some special Θ-transformation technique for polynomials. The involved for this integer matrix sequence Θ_n is interesting of itself as well.

For the convenience of the reader, initially we present in Section 2 a purely matrix description of the main considered in the paper linear-algebraic problems and the obtained canonical forms, accompanied by minimum of definitions necessary to understand the formulations. In particular, the placed here Theorems 3 and 4 describe the two main obtained canonical forms and are equivalent to the given in Section 7 (in more abstract terms) Theorems 15 and 17 respectively.

The systematic exposition of the material starts in Section 3. First we recall briefly in Sections 3 and 4 basic definitions and facts concerning semilinear maps and canonical forms under transformations of σ-similarity (with a base in the theory of non-commutative polynomials over division rings).

Then we give at once in Section 5 a new short proof of the result by Djoković (Theorem 1) solving the semilinear Kronecker Problem.[2] We continue by a parallel solution to the corresponding contragredient problem for semilinear maps (Theorem 2) extending the known one by

[1] The semilinear case is in fact the case of a pair consisting of one linear and one semilinear map, to which the case of two semilinear maps $\mathscr{A}$, $\mathscr{B}$ is trivially reduced.

[2] In fact, this proof works also in the pseudolinear bundle case (see Remark 9).

Dobrovol'skaja and Ponomarev [9] for a pair of counter-oriented linear mappings. We hope, our approach helps to clarify a bit more the essence and unity of the observed problems.

In Section 6, the biquadratic matrix problem is defined as the problem of classifying rectangular matrices over a bimodule $G_1 \underset{F}{\otimes} G_2$, where G_1, G_2 are quadratic extensions of a field F, by G_1-elementary transformations of their rows and G_2-elementary transformations of columns. This (still open in general) problem plays an important role in the representation theory. Its particular case (when $F = \mathbb{R}$ and $G_1 = G_2 = \mathbb{C}$) was solved completely by Dlab and Ringel in [8] and can be reduced (as shown in [6]) to the semilinear Kronecker Problem.

Beginning Section 7, we restrict our considerations by the case $\operatorname{char} F \neq 2$ and show under this assumption (Proposition 13 and Corollary 14) that the homogeneous biquadratic problem (for which $G_1 = G_2$ is the same field G) is reduced to the $(1, \sigma)$-pencil problem, i.e. to the semilinear Kronecker Problem determined by a pair of automorphisms $\{1, \sigma\}$ of the field G with σ being the natural conjugation. This leads to the first canonical form of indecomposables described by Theorem 15 (or equivalently by Theorem 3).

Since the first canonical form is not the best possible with respect to the standard base of the bimodule $G \underset{F}{\otimes} G$, we need to examine the question more defining in Section 8 an integer matrix sequence Θ_n which allows to improve the situation. Its main properties are being established along that section (for $n \leqslant 9$, the matrices Θ_n are shown explicitly in Appendix).

Using the sequence Θ_n, we introduce and study in Section 9 the Θ-transformation of polynomials establishing in the next section its important relationship with the $(1, \sigma)$-pencil representations (Theorems 36 and 41 and several corollaries).

Then at last we are able to obtain in Section 11 the desired simplified second canonical form of indecomposables for an arbitrary homogeneous biquadratic problem in characteristic $\neq 2$ (Theorem 44 and its consequence Theorem 17, the last one equivalent to Theorem 4).

Finally, in Section 12, some applications to the representation theory of posets with additional structures are observed. We outline the way of extending the main results from [26,28,29] (obtained there for the pair of fields $\mathbb{R} \subset \mathbb{C}$) to the case of an arbitrary quadratic extension $F \subset G$ in characteristic $\neq 2$ (Theorems 48 and 49). Also we correct a gap in the construction of the differentiation algorithm X in [28].

Remark that the restriction to the case $\operatorname{char} F \neq 2$ (in the investigation of the homogeneous biquadratic problem, beginning Section 7) was chosen with a purpose to simplify the situation and exposition of the material. It seems, in characteristic 2 one can obtain similar results but using a more deep combinatorics.

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2. Main matrix canonical forms

An important role in our considerations plays a canonical form for the *semilinear Kronecker Problem*. In the matrix language, this problem consists in classifying indecomposable pairs (A, B) of rectangular matrices of equal size over a division ring K with an automorphism σ with respect to transformations of the form

$$(A, B) \mapsto (X^{-1}AY, X^{-1}BY^{\sigma}), \tag{2}$$

with non-singular square matrices X, Y. Its solution was given by Djoković [6] and is described by the following Theorem 1 which extends the classical Kronecker's classification [18] from the linear case ($\sigma = 1$) to the semilinear one.

We use the following notations. If I_n is the identity matrix of order n, then $I_n^{\uparrow}$, $I_n^{\downarrow}$, $I_n^{\rightarrow}$, $I_n^{\leftarrow}$ are the matrices obtained from I_n by joining to it one additional zero-row or zero-column from the above, below, right or left respectively. It is convenient to admit the case $n = 0$, then $I_0^{\uparrow}$ and $I_0^{\downarrow}$ are "empty matrices" consisting formally of one row and zero columns each one (dually, $I_0^{\rightarrow}$ and $I_0^{\leftarrow}$ consist of zero rows and one column each one).

Let $J_n^{+}(\lambda)$ (respectively $J_n^{-}(\lambda)$) be the Jordan block of order n with the eigenvalue λ and entries 1 above (below) the main diagonal. Let $J_n(\lambda)$ be any of the mentioned blocks.

If $f(t) = a_0 + ta_1 + \cdots + t^{n-1}a_{n-1} + t^n$ is a monic non-constant polynomial over K, denote by $C(f)$ its standard *companion matrix* which is a square matrix of order n of the form

$$C(f) = \boxed{\begin{array}{ccccc} & & & & -a_0 \\ 1 & & & & -a_1 \\ & \ddots & & & \vdots \\ & & 1 & & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{array}} \tag{3}$$

(throughout the whole paper, the non-shown entries of matrices are supposed to be zeroes).

Denote by $\Im = \cup_n \Im_n$ the set of all indecomposable polynomials (with right coefficients) of the skew polynomial ring $R = K[t, \sigma]$ (see details in Section 4)[3] where $\Im_n$ is the subset of polynomials of degree n. Notice, if $f \in \Im$ then $\det C(f) = 0 \Leftrightarrow f(t) = 0 \Leftrightarrow f = t^n$.

For a subset $X \subset \Im$, let $\dot{X}$ be any fixed maximal subset of non-similar in pairs polynomials in X.

Now the result of Djoković ([6], Theorem 1) can be presented in the following form.

Theorem 1. *The indecomposable pairs of matrices (A, B) for the semilinear Kronecker Problem (2) over an arbitrary skew field K with an automorphism σ are exhausted, up to equivalence, by the pairs of the following types:*

$0 = 0^* : (I_n, C(f))$, *where* $f \in \dot{\Im}_n \backslash t^n$ *and* $n \geqslant 1$.
$1 = 1^* :$ (a) $(I_n, J_n(0))$ *and* (b) $(J_n(0), I_n)$, $n \geqslant 1$.
$2 = 3^* : (I_n^{\uparrow}, I_n^{\downarrow})$, $n \geqslant 0$.
$3 = 2^* : (I_n^{\rightarrow}, I_n^{\leftarrow})$, $n \geqslant 0$.

These pairs are non-equivalent two-by-two.

We place a new short proof of Theorem 1 in Section 5. As for the formulation above, one should take into account the following:

(a) If (A, B) is an indecomposable pair of type N, then N^* denotes the type of the "dual pair" (defined naturally in the invariant language, as in Section 5).
(b) Representations of type (1), given by $J_n^{+}(0)$ and $J_n^{-}(0)$, are equivalent (in each of the subcases (a) and (b)).
(c) Formal "empty" matrix representations $(I_0^{\rightarrow}, I_0^{\leftarrow})$ and $(I_0^{\uparrow}, I_0^{\downarrow})$ correspond to trivial representations $(K, 0, 0, 0)$ and $(0, K, 0, 0)$ respectively of the form $(U_1, U_2, \mathscr{A}, \mathscr{B})$ in the scheme (1).

[3] If $\sigma = 1$ and K is a field, then $\Im$ is precisely the set of all polynomials of the form $p^m(t)$ with irreducible $p(t) \in K[t]$ and $m \geqslant 1$ (these are the classical *elementary divisors* of the ordinary commutative polynomial ring $K[t]$).

We also investigate the *contragredient semilinear Kronecker Problem* which is rather like the previous one and can be interpreted as the problem of classifying pairs (A, B) of rectangular matrices over K (such that A and B^{T} are of equal size) by transformations of the form

$$(A, B) \mapsto (X^{-1}AY, Y^{-1}BX^{\sigma}), \tag{4}$$

with non-singular X, Y. The linear case $\sigma = 1$ is known as the *contragredient equivalence problem* for linear maps solved originally by Dobrovol'skaja and Ponomarev [9]. Their solution is extended to the semilinear case as follows.

Theorem 2. *The indecomposable pairs of matrices (A, B) corresponding to the contragredient semilinear Kronecker Problem (4) over an arbitrary skew field K with an automorphism σ are exhausted, up to equivalence, by the pairs of the following types:*

$0 = 0^* : (I_n, C(f))$, *where* $f \in \dot{\Im}_n \backslash t^n$ *and* $n \geqslant 1$.
$1 = 1^* :$ (a) $(I_n, J_n(0))$ *and* (b) $(J_n(0), I_n)$, $n \geqslant 1$.
$2 = 3^* : (I_n^{\uparrow}, I_n^{\rightarrow})$, $n \geqslant 0$.
$3 = 2^* : (I_n^{\rightarrow}, I_n^{\uparrow})$, $n \geqslant 0$.

These pairs are non-equivalent two-by-two.

Theorem 2 does not follow from Theorem 1. At the same time, since the formulations of the theorems are very similar, one can expect also very similar solutions. We confirm the truth of this in Section 5 proving the result.

One of the main objectives of the present paper is to obtain canonical forms for some matrix problem determined by an arbitrary quadratic field extension $F \subset G$ in characteristic $\neq 2$, called the homogeneous biquadratic problem (for the reason of terminology, see Section 6). The problem can be described in the following way.

Fix an element u in G having the minimal polynomial $t^2 + q$ over F. Then $G = F(u)$ and each element $x \in G$ is presented uniquely in the form $x = \alpha + \beta u$ $(\alpha, \beta \in F)$ where $\alpha = \mathrm{Re}x$ and $\beta = \mathrm{Im}x$ are called the *real* and *imaginary* parts of x (with respect to F). Thus *real* and *complex* matrices are understood below in the generalized sense, as matrices over F and over G respectively.

On the other hand, real $2m \times 2n$ matrices of the form $\begin{bmatrix} X & -qY \\ Y & X \end{bmatrix}$ are said to be *formally complex*. If $m = n$, the correspondence

$$\begin{bmatrix} X & -qY \\ Y & X \end{bmatrix} \mapsto X + Yu$$

is an isomorphism from the ring of such special $2n \times 2n$ matrices over F onto the ring of all $n \times n$ matrices over G.

Naturally, if $(F, G) = (\mathbb{R}, \mathbb{C})$, where $\mathbb{R}$ and $\mathbb{C}$ are the fields of ordinary real and complex numbers, and $q = 1$, then we stay with the standard real, complex and formally complex matrices.

The *homogeneous biquadratic problem* corresponding to the extension $F \subset G = F(u)$ consists in finding a canonical form for real (in the agreed generalized sense) matrices of even size $2m \times 2n$ with respect to formally complex transformations of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} X & -qY \\ Y & X \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Z & -qT \\ T & Z \end{bmatrix}, \tag{5}$$

where all matrices are over F, the blocks A, B, C, D are of equal size $m \times n$ and X, Y, Z, T are square blocks of suitable order such that both the transforming matrices in (5) are non-singular.

A particular case of this problem, for $(F, G) = (\mathbb{R}, \mathbb{C})$ and $q = 1$, is precisely the problem considered and solved by Dlab and Ringel in [8]. As shown in [6], it is reduced to the semilinear Kronecker Problem over $\mathbb{C}$.

The problem (5) also can be reduced analogously to the semilinear Kronecker Problem (2) over G, as we explain in Section 7 using suitable bases of linear spaces (Proposition 13). This leads immediately to the *first canonical form* for the problem (5) described by the following Theorem 3 (which is the matrix version of Theorem 15).

Let σ be the natural involution (*conjugation*) on $G = F(u)$ given by $\sigma(\alpha + \beta u) = \alpha - \beta u$, and $R = G[t, \sigma]$, $\Im = \cup_n \Im_n$ the same polynomial sets as above (for $K = G$).

Theorem 3 (The first matrix canonical form). *Let $F \subset F(u)$ be a quadratic field extension in characteristic $\neq 2$ with the minimal polynomial $t^2 + q$ of the element u. Then the indecomposable matrices $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ for the homogeneous biquadratic problem* (5) *are exhausted, up to equivalence, by the pairwise non-equivalent ones of the following types* (*corresponding to the types of Theorem 1, under the assumption $K = G$*):

$$(0)\quad \begin{array}{|c|c|}\hline I_n + \operatorname{Re} C(f) & q \operatorname{Im} C(f) \\ \hline \operatorname{Im} C(f) & I_n - \operatorname{Re} C(f) \\ \hline \end{array} \qquad f \in \dot{\Im}_n \backslash t^n,\ n \geqslant 1.$$

$$(1a)\quad \begin{array}{|c|c|}\hline I_n + J_n(0) & \\ \hline & I_n - J_n(0) \\ \hline \end{array} \qquad n \geqslant 1. \qquad (2)\quad \begin{array}{|c|c|}\hline I_n^{\uparrow} + I_n^{\downarrow} & \\ \hline & I_n^{\uparrow} - I_n^{\downarrow} \\ \hline \end{array} \qquad n \geqslant 0.$$

$$(1b)\quad \begin{array}{|c|c|}\hline J_n(0) + I_n & \\ \hline & J_n(0) - I_n \\ \hline \end{array} \qquad n \geqslant 1. \qquad (3)\quad \begin{array}{|c|c|}\hline I_n^{\rightarrow} + I_n^{\leftarrow} & \\ \hline & I_n^{\rightarrow} - I_n^{\leftarrow} \\ \hline \end{array} \qquad n \geqslant 0.$$

Notice, one can realize also a purely matrix reduction of the biquadratic homogeneous problem (5) to the semilinear Kronecker Problem (2). Namely, set for brevity

$$P = \begin{bmatrix} P_1 & -qP_2 \\ P_2 & P_1 \end{bmatrix}, \quad P^{\sigma} = \begin{bmatrix} P_1 & qP_2 \\ -P_2 & P_1 \end{bmatrix}, \quad P' = \begin{bmatrix} P_1 & qP_2 \\ P_2 & -P_1 \end{bmatrix}, \quad E = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix},$$

where P_i are some $m \times n$ blocks over F, and take the same convention (as for P) for another real $2m \times 2n$ matrix Q, as well as for square real matrices X and Y of order $2m$ and $2n$ respectively. In such notations, Theorem 1 ensures a solution of the canonical form problem

$$P \mapsto XPY, \quad Q \mapsto XQY^{\sigma} \tag{6}$$

for pairs (P, Q) of formally complex $2m \times 2n$ matrices. Then the second transformation in (6) implies $Q' = QE \mapsto XQEEY^{\sigma}E = XQ'Y$. Thus each $2m \times 2n$ matrix $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ over F (which is uniquely presented in the form $M = P + Q'$) is reduced precisely by the transformations of type (5). So, canonical pairs (P, Q) for (6) give canonical matrices $P + Q'$ for (5).

The first canonical form can be simplified more, on the base of using some special Θ-transformation of polynomials. For this, we define in Section 8 an integer matrix sequence $\Theta_n = [\Theta_n^{ij}]$ by the explicit formula

$$\Theta_n^{ij} = \sum_k (-1)^k \binom{j-1}{k} \binom{n-j}{i-k-1}$$

and establish its main properties[4] (in the sum above, one assumes $\binom{m}{l} = 0$ if the condition $0 \leqslant l \leqslant m$ is not satisfied).

For a subset $\Sigma \subset R$ and an element $a \in K$, let $\Sigma\langle a\rangle$ be the subset of all polynomials in Σ not vanishing in a (notice that $\dot{\Im}_n\langle 0\rangle = \dot{\Im}_n \backslash t^n$, this set figured in the previous theorems).

To each polynomial $f = a_0 + ta_1 + \cdots + t^n a_n$ in $R\langle -1\rangle$ of degree $n \geqslant 0$ with the vector-column of coefficients $[f] = (a_0, a_1, \ldots, a_n)^{\mathrm{T}}$, one can attach a polynomial $\Theta_{n+1} f$ (this is the "almost Θ-transform of f") given by the matrix product $[\Theta_{n+1} f] = \Theta_{n+1}[f]$. Then the *$\Theta$-transform* of f is a polynomial

$$\Theta f = (\Theta_{n+1} f)\,\frac{a_n}{f(-1)},$$

and the transformation Θ appears to be an involution on $R\langle -1\rangle$ preserving degrees and leading coefficients of polynomials (see Section 9). Observing its relationship with the semilinear Kronecker Problem, we obtain in Section 11 Theorem 44 ensuring the following result which is equivalent to Theorem 17 and represents in matrix terms the *second canonical form* for the homogeneous biquadratic problem.

Theorem 4 (The second matrix canonical form). *Under the assumptions of Theorem 3, the indecomposable matrices* $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *for the homogeneous biquadratic problem* (5) *are exhausted, up to equivalence, by the pairwise non-equivalent ones of the following types*:

$$(1)\ \underset{f \in \Theta(\dot{\Im}_n\langle -1\rangle) \cup (t+1)^n}{\begin{array}{|c|c|}\hline I_n & \\ \hline \operatorname{Im} C(f) & \operatorname{Re} C(f) \\ \hline\end{array}}\ \ n \geqslant 1 \qquad (2)\ \underset{n \geqslant 0}{\begin{array}{|c|c|}\hline I_n^{\uparrow} & \\ \hline & I_n^{\downarrow} \\ \hline\end{array}} \qquad (3)\ \underset{n \geqslant 0}{\begin{array}{|c|c|}\hline I_n^{\rightarrow} & \\ \hline & I_n^{\leftarrow} \\ \hline\end{array}}\ .$$

It should be remarked that if a given polynomial $f \in \dot{\Im}$ is real, then $\operatorname{Re} C(f) = C(f)$ and $\operatorname{Im} C(f) = 0$, so the corresponding to f matrix (1) is simpler. This fact can be used in particular in the case $(F, G) = (\mathbb{R}, \mathbb{C})$ where the whole set of indecomposable polynomials $\dot{\Im}$ can be formed of real ones only, say as in Corollary 7.

If so, then one can consider the mentioned in Theorem 4 special set of polynomials $\Theta(\dot{\Im}_n\langle -1\rangle) \cup (t+1)^n$ as the set $\mathfrak{S}_n$ of all elementary divisors of degree n of the ring $\mathbb{R}[t]$ having the constant term inside the segment $[-1, 1]$ (see Example 46). In other words, it suffices to deal only with the collection $\mathfrak{S} = \cup_n \mathfrak{S}_n$ formed by all polynomials of the form g^m where $m \geqslant 1$ and $g(t) = t + c$ or $g(t) = t^2 + bt + c$ is a real irreducible polynomial with the condition $|c| \leqslant 1$.

Therefore, it holds the following result (equivalent to Theorem 47) which describes one more simplified canonical form for the problem solved in [8].

Theorem 5. *In the case of the quadratic extension* $\mathbb{R} \subset \mathbb{C}$, *the indecomposable non-equivalent matrices* $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *corresponding to the homogeneous biquadratic problem* (5) (*with* $q = 1$) *are exhausted, up to equivalence, by the pairwise non-equivalent matrices of the types* (1)–(3) *shown in Theorem* 4, *with a new condition* $f \in \mathfrak{S}_n$ *for the type* (1) *matrices* (*which implies automatically* $\operatorname{Re} C(f) = C(f)$ *and* $\operatorname{Im} C(f) = 0$).

[4] There is also a simple recurrence definition of Θ_n (see Lemma 19 and Corollary 22, together with (23), and observe the matrices shown in Appendix).

3. Pairs of semilinear maps

Let K be a division ring with an automorphism α and U_1, U_2 be right finite-dimensional vector spaces over K. We remind (see [1,3,16]) that a map $\mathscr{A} : U_1 \longrightarrow U_2$ is called *semilinear* (with respect to the automorphism α) if it satisfies the conditions

$$(x+y)\mathscr{A} = x\mathscr{A} + y\mathscr{A}$$
$$(xa)\mathscr{A} = (x\mathscr{A})a^{\alpha}.$$

for all $x, y \in U_1$ and $a \in K$. Below we call briefly such maps α-*semilinear*. In particular, each linear map is 1-semilinear.

Given two automorphisms α, β of the division ring K, then the (α, β)-*pencil* is a quiver (i.e. an oriented graph) $\mathscr{P}$ of the form

$$\bigcirc \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \bigcirc \tag{7}$$

the arrows of which are marked by the automorphisms α, β. A *representation* of $\mathcal{P}$ over K is any collection $U = (U_1, U_2, \mathscr{A}, \mathscr{B})$ consisting of two right finite-dimensional K-spaces U_1, U_2 and two semilinear maps $\mathscr{A}, \mathscr{B}$ between them where $\mathscr{A}$ is α-semilinear and $\mathscr{B}$ is β-semilinear.

Representations are the objects of the *category of representations* $\operatorname{rep}\mathcal{P} = \operatorname{rep}(\mathcal{P}, K, \alpha, \beta)$ of the (α, β)-pencil $\mathcal{P}$, with a natural definition of morphisms. Namely, a *morphism* $\varphi : U \longrightarrow U'$ in $\operatorname{rep}\mathcal{P}$ is any pair $\varphi = (\varphi_1, \varphi_2)$ of K-linear maps $\varphi_i : U_i \longrightarrow U_i'$ such that $\mathscr{A}\varphi_2 = \varphi_1\mathscr{A}'$ and $\mathscr{B}\varphi_2 = \varphi_1\mathscr{B}'$.

It is clear that the category $\operatorname{rep}\mathcal{P}$ is additive. A morphism φ in $\operatorname{rep}\mathcal{P}$ is an *isomorphism* iff both φ_1 and φ_2 are ordinary isomorphisms between vector spaces. The direct sum of representations and indecomposability are defined in a standard way. The *dimension* of a representation U is a vector $d = \underline{\dim}\, U = (d_1, d_2)$ with $d_i = \dim_K U_i$.

The problem of classifying indecomposable representations of $\mathcal{P}$, up to isomorphism, is called in the present paper the (α, β)-*pencil problem*. It admits a natural matrix interpretation.

Let U be a representation of $\mathcal{P}$. If you fix some K-bases in U_1 and U_2, then the pair of semilinear maps $\mathscr{A}, \mathscr{B}$ is presented by a pair of rectangular matrices (A, B) of equal size over K called a *matrix representation* of $\mathcal{P}$. The pair (A, B) is transformed (when changing the bases) into another pair (A', B') as follows:

$$A' = X^{-1}AY^{\alpha}, \quad B' = X^{-1}BY^{\beta} \tag{8}$$

where X, Y are some non-singular square matrices. We write in this case $(A, B) \overset{(\alpha,\beta)}{\sim} (A', B')$ or simply $(A, B) \sim (A', B')$ and call the considered pairs (α, β)-*equivalent*.

One can apply to the pair of matrices (A, B) the following *admissible transformations* to obtain an (α, β)-equivalent matrix representation (A', B'):

(a) Simultaneous left[5] elementary row transformations in A and B over K.
(b) Simultaneous right α-elementary column transformations in A and β-elementary column transformations in B over K.[6]

[5] One should multiply rows by scalars to the left.
[6] These transformations are generated by simultaneous multiplications to the right the i-ph columns in A and B by x^{α} and x^{β} respectively ($x \in k^{\times} = k\backslash 0$) and by simultaneous additions the i-ph column to the j-ph column ($i \neq j$) in A and B with the right coefficients x^{α} and x^{β} respectively.

Setting $Y = Z^{\omega}$, where ω is any automorphism of K, you get immediately

$$A' = X^{-1}AZ^{\omega\alpha}, \quad B' = X^{-1}BZ^{\omega\beta}. \tag{9}$$

Hence, the (α, β)-pencil problem is equivalent to the $(\omega\alpha, \omega\beta)$-pencil problem, in particular, to the $(1, \sigma)$-pencil problem for $\sigma = \alpha^{-1}\beta$. In other words, the classification of pairs of semilinear maps of the form (1) is reduced to the classification of pairs consisting of one linear and one semilinear map.

Notice that the $(1, 1)$-pencil problem is nothing else but the classical Kronecker Problem.

Changing the orientation of one of the arrows in the diagram (7), you get the *contragredient* (α, β)*-pencil*

$$\bigcirc \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bigcirc \tag{10}$$

representations of which

$$U_1 \underset{\mathscr{B}}{\overset{\mathscr{A}}{\rightleftarrows}} U_2 \tag{11}$$

are defined analogously to the single-directed case. So, their classification leads to the *contragredient* (α, β)*-equivalence problem* consisting in classifying pairs (A, B) of rectangular matrices over K (with equal size of A and B^{T}) by transformations of the form

$$A' = X^{-1}AY^{\alpha}, \quad B' = Y^{-1}BX^{\beta} \tag{12}$$

with square regular X, Y. In other words, one can apply to the pair (A, B):

(a′) Simultaneous left elementary row transformations in A and right β-inverse[7] column transformations in B.
(b′) Simultaneous left row transformations in B and right α-inverse column transformations in A.

Here also it is enough to deal with the $(1, \sigma)$-case only, and the $(1, 1)$-case is nothing else but the known *contragredient equivalence problem* for linear maps solved originally in [9] (see also [14], in particular for applications).

4. Canonical form under σ-similarity

Suppose that $U = (U_1, U_2, \mathscr{A}, \mathscr{B})$ is a representation of the $(1, \sigma)$-pencil $\mathcal{P}$ such that $\mathscr{A}$ is an isomorphism. Then the spaces U_1 and U_2 are naturally identified with help of $\mathscr{A}$ (let it be $U_1 = U_2 = U$ and $\mathscr{A} = Id$), hence only one σ-semilinear operator $\mathscr{B} : U \longrightarrow U$ remains. Its matrix B is considered as a representation of the σ*-loop* $\mathscr{L}(\sigma)$ of the form

$$\begin{tikzcd} \circ \arrow[loop right, "\sigma"] \end{tikzcd}$$

and is transformed (accordingly to base changing in U with a matrix X) by the rule

$$B' = X^{-1}BX^{\sigma}. \tag{13}$$

We call such transformations the *transformations of* σ*-similarity* and write $B \overset{\sigma}{\sim} B'$. Their special cases are the transformations of ordinary similarity ($\sigma = 1$) and of *consimilarity* [13,19]

[7] One has to apply β to the inverse transformation matrix, before to multiply by it.

(when σ is the complex conjugation $^{-}$ in $K = \mathbb{C}$). The notation $A \overset{c}{\sim} B$ is used for consimilar complex matrices A, B (such that $A = X^{-1}B\overline{X}$).

A canonical form of a square matrix under σ-similarity is known, being obtained in fact as a direct consequence of the theory of finitely generated modules over principal ideal domains (not necessarily commutative). For the convenience of the reader, we recall now briefly the main scheme referring for more details and proofs to [1,3,15] (where even a more general type of similarity is considered corresponding to pseudolinear operators). As for a more deep exposition of the theory of non-commutative polynomials itself (used in the cited works), see also [17,20].

Let $R = K[t, \sigma] = \{a_0 + ta_1 + \cdots + t^n a_n : a_i \in K, n \geqslant 0\}$ be the skew polynomial ring of right polynomials over K in one variable t, with the permutation rule $at = ta^\sigma$ for any $a \in K$. Such a ring R is a (left and right) artinian principal ideal domain, and even an Euclidean domain [3]. Considering the space U as a right R-module, such that $ut = u\mathscr{B}$ for each $u \in U$, and taking some its K-base $u_1, \ldots, u_n$, we get $u_j t = u_j \mathscr{B} = \sum_i u_i b_{ij}$ where $B = [b_{ij}]$ is the matrix of $\mathscr{B}$ in this base.

Analogously to the case of ordinary commutative polynomial rings, two matrices B, B' over K are σ-similar if and only if

$$P(tI - B)Q = tI - B'$$

for some invertible matrices P, Q over R. Moreover, there exist invertible P, Q such that

$$P(tI - B)Q = \operatorname{diag}\{f_1, \ldots, f_n\},$$

where I is the identity matrix and the (monic) invariant factors $f_1, \ldots, f_n$ are such that each f_i ($i \leqslant n-1$) is both a left and right factor of f_{i+1} and $f_i R \supset J \supset f_{i+1}R$ (or equivalently $Rf_i \supset J \supset Rf_{i+1}$) for some two-sided ideal $J \subset R$ (see [15,16] or [3], Chapter 8).

These polynomials f_i are unique up to some special kind of *similarity* for non-commutative polynomials defined in R in such a way that the cyclic modules R/f_iR are unique up to isomorphism (more precisely: non-divisors of zero a, b of any ring R are called *similar*[8] if $R/aR \simeq R/bR$, which is equivalent to $R/Ra \simeq R/Rb$).

It follows that the right R-module U also admits a unique (up to isomorphism) decomposition

$$U = W_s \oplus \cdots \oplus W_n,$$

where $W_i \simeq R/f_iR$ are all the nonzero factors (corresponding to $f_i \neq 1$). The matrix canonical form of each given restriction $\mathscr{B}_i = \mathscr{B}|W_i$ coincides in general with the ordinary Rational Canonical Form.

Namely, supposing $W_i = uR$ for some fixed i and some $u \in W_i$, and taking the base of W_i in the form $e_0 = u, e_1 = u\mathscr{B}, \ldots, e_{m-1} = u\mathscr{B}^{m-1}$, with $u\mathscr{B}^m = -e_0a_0 - \cdots - e_{m-1}a_{m-1}$ being the first vector among $u\mathscr{B}^i$ depending on the previous ones, we see that the matrix of $\mathscr{B}_i$ is precisely the companion matrix $C(f_i)$ of type (3) of the monic polynomial $f_i(t) = a_0 + ta_1 + \cdots + t^{m-1}a_{m-1} + t^m$. And the polynomial f_i itself is a generator of the annihilator of W_i.

The R-modules W_i can be decomposed more $W_i = \bigoplus_j W_{ij}$ where W_{ij} are indecomposable cyclic R-modules. The unique (up to the mentioned similarity) invariant factors f_{ij} of W_{ij} are called the *elementary divisors* of the semilinear operator $\mathscr{B}$ or its matrix B. So, almost like in the case of classical commutative polynomial rings, two matrices B and B' over K are σ-similar iff their elementary divisors are similar in pairs ([15], Theorem 9).

The elementary divisors of all possible indecomposable cyclic R-modules are precisely the *indecomposable* polynomials of R in the sense of [3,15,20] where one can find some general

[8] This general definition of similarity is specified for the ring $R = K[t, \sigma]$ in a concrete polynomial form (see [15,20]).

definitions and conditions concerning these notions. It is a separate task and maybe an art to determine indecomposable polynomials for a given polynomial ring (compare with [1,11,12,15]). We need to recall the following result obtained in [1], Section 6 (see also [8], Lemma D). Set $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geqslant 0\}$.

Proposition 6. *Let $R = \mathbb{C}[t, {}^-]$ be the skew polynomial ring over the field of complex numbers $\mathbb{C}$ (with the automorphism σ being the usual complex conjugation ${}^-$). Then the indecomposable polynomials in R are exhausted, up to similarity, by the following pairwise non-similar ones $(n \geqslant 1)$:*

(a) $(t-\alpha)^n$, *for* $\alpha \in \mathbb{R}^+$,
(b) $(t^2-\xi)^n$, *for* $\xi \in (\mathbb{R}\backslash\mathbb{R}^+) \cup (\mathbb{C}\backslash\mathbb{R})$.

For some goals, it is convenient to change in (b) the complex polynomials by real ones. In two words, set $\xi = \omega^2$, where $\omega = \alpha + \mathrm{i}\beta (\alpha, \beta \in \mathbb{R})$ and $\beta > 0$, and also set $p = \pm 2\alpha, q = \alpha^2 + \beta^2$. Then $\overline{S}\begin{bmatrix} 0 & \xi \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -q \\ 1 & -p \end{bmatrix} S$, where $S = \begin{bmatrix} \pm\omega & -q \\ 1 & \mp\omega \end{bmatrix}$, hence $\begin{bmatrix} 0 & \xi \\ 1 & 0 \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 0 & -q \\ 1 & -p \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 0 & -q \\ 1 & p \end{bmatrix}$ and one can choose freely for instance $\alpha \geqslant 0$ and $p = -2\alpha$ getting the following equivalent form of Proposition 6.

Corollary 7. *Under the assumptions of Proposition 6, the indecomposable polynomials in R are exhausted, up to similarity, by the following pairwise non-similar ones:*

$$(t-\alpha)^n \quad \textit{and} \quad (t^2 - 2\alpha t + \alpha^2 + \beta^2)^n,$$

where $\alpha \geqslant 0$ and $\beta > 0$ are real numbers and $n \geqslant 1$.

Remark that Proposition 6 and Corollary 7 follow also from the known results on consimilarity of complex matrices, presented for instance in [13,19].

5. Solving the semilinear Kronecker Problem and its contragredient variant

Proof of Theorem 1. Let $U = (U_1, U_2, \mathscr{A}, \mathscr{B})$ be an indecomposable representation of the $(1,\sigma)$-pencil $\mathcal{P}$, with a matrix presentation (A, B) (with respect to some chosen bases in U_1 and U_2).

Case A. If the map $\mathscr{A}$ is an isomorphism, then one can assume $A = I_n (n \geqslant 1)$. Hence, the admissible transformations of B (not changing the block $A = I_n$) are precisely the transformations of σ-similarity $X^{-1}BX^\sigma$. Therefore (see Section 4), $X^{-1}BX^\sigma = C(f)$ for some X and $f \in \dot{\Im}_n$. So, we get either the type 0 (if $f(t) \neq t^n$), or the type $1(a)$ (if $f(t) = t^n$).

Case B. If the map $\mathscr{A}$ is not an isomorphism, we can suppose, up to duality[9], that $W = \mathrm{Im}\mathscr{A} \neq U_2$. We carry out the induction on $d = \dim_K U_2$. The case $d = 1$ is trivial and leads either to the type $1(b)$ with $n = 1$, or to the type 2 with $n = 0$.

Let $d > 1$ and $M = \boxed{A \mid B}$ be a matrix realization of U. Assuming that the chosen base of U_2 contains a base of the image W, present M in the form

[9] For $\lambda = \sigma^{-1}$ and any σ-semilinear map $\mathscr{B} : U_1 \longrightarrow U_2$ between right K-spaces, the *dual* λ-semilinear map $\mathscr{B}^* : U_2^* \longrightarrow U_1^*$ between left K-spaces $U_i^* = Hom_K(U_i, K)$ is defined by $(\mathscr{B}^* f)u = (f(u\mathscr{B}))^\lambda$ (where $f \in U_2^*, u \in U_1$). Then the natural identifications $U_i^{**} = U_i$ and $\mathscr{B}^{**} = \mathscr{B}$ hold.

$$M = \begin{array}{|c|c|}\hline & Z \\ \hline X & Y \\ \hline \end{array}\ \begin{array}{c} E \\ W \end{array},$$

where the rows of X are linearly independent (and E is any complement to W). Then the rows of Z are also independent, otherwise one can get a zero-row in the whole matrix M, hence (because of indecomposability) $U_2 = K$ in contradiction to $d > 1$.

Therefore, reducing the block Z to the form $Z = \begin{array}{|c|c|}\hline I_s & \\ \hline \end{array}$, $s \geqslant 1$, and making then (by row additions) zeroes below I_s, we transform M to the form

$$M = \begin{array}{|c|c|c|c|}\hline & & I_s & \\ \hline C & A' & & B' \\ \hline \end{array}. \tag{14}$$

Obviously, in (14), the pair (A', B') is again a matrix representation of the $(1, \sigma)$-pencil $\mathcal{P}$, and the block C admits in fact arbitrary independent column transformations since the satellite changes in I_s are corrected by row transformations. Moreover, one can make also independent column additions $A' \to C$ since the corresponding additions of columns of B' to the left are neutralized by additions of rows of I_s to the down. The following is true.

Lemma 8. *If a matrix representation $M = (A, B)$ of the form* (14) *is free, up to equivalence, of trivial direct summands* $(0, 1)$, *then (A, B) is indecomposable if and only if (A', B') is indecomposable.*

Proof. Suppose $(A', B') = (A'_1, B'_1) \oplus (A'_2, B'_2)$ is a non-trivial matrix direct decomposition corresponding to some (possibly trivial) vector space decomposition $W = W_1 \oplus W_2$. Since the rows of the block $X = C \cup A'$ are linearly independent, its columns generate the whole space W. Hence the matrix C can be reduced (by elementary transformations of its columns and by column additions $A' \to C$) to a new form $C' = \begin{array}{|c|c|}\hline C_1 & C_2 \\ \hline \end{array}$ such that the columns of each block $C_i \cup A'_i$ generate the whole space W_i ($i = 1, 2$). In other words $C' = C'_1 \oplus C'_2$ for some submatrices $C'_i \subset C_i$ and $(C', A', B') = (C'_1, A'_1, B'_1) \oplus (C'_2, A'_2, B'_2)$, which means decomposability of (A, B).

Conversely, if $(A, B) = (A_1, B_1) \oplus (A_2, B_2)$ is a non-trivial direct sum, then by the construction $(A', B') = (A'_1, B'_1) \oplus (A'_2, B'_2)$ where the summands to the right are non-zero because (A, B) is $(0, 1)$-free. $\square$

We are able now to finish the proof of the theorem. By the induction hypothesis, up to equivalence, the indecomposable pair (A', B') must be of type 0, 1, 2 or 3, and the rows of the matrix A' are linearly dependent (otherwise C is annihilated and indecomposability implies that $s = 1$, $(A, B) = (0, 1)$ and $d = 1$, a contradiction). Hence, the pair (A', B') is of type 1(b) or 2 only, A' has exactly one zero-row and all but one elements in C can be annulated. But then (due to indecomposability) C consists of one column only, and one can make sure trivially that M itself has the form 1(b) or 2 respectively (better to take $J_n(0) = J_n^-(0)$ in the case 1(b)).

So, all matrix indecomposables of $\mathcal{P}$ are exhausted (up to equivalence) by the types 0, 1, 2, 3 (they really are indecomposable by Lemma 8 and obviously pairwise non-isomorphic). The proof of Theorem 1 is complete. $\square$

Remark 9. In fact, the given proof can be applied (practically without modifications) to the pseudolinear bundle case affording an alternative proof to that one given in [23]. For, in [23], pairs of matrices (A, B) over K are reduced by the transformations

$$(A, B) \longmapsto (X^{-1}AY, X^{-1}(BY^{\sigma} + AY^{\delta})), \tag{15}$$

where X, Y are invertible matrices over K and δ is a right σ-derivation in K (satisfying the condition $(ab)^{\delta} = ab^{\delta} + a^{\delta}b^{\sigma}$). Here left multiplications by X^{-1} correspond to left K-elementary row transformations, while right column transformations are determined by the matrix Y. The reader may check easily that in fact all the steps of our proof (including Lemma 8) remain valid in the situation (15) and are not broken by the appearance of a new additional term AY^{δ}. Clearly, in Case A, a canonical form of type (3) for (σ, δ)-similarity $B \longmapsto X^{-1}BX^{\sigma} + X^{-1}X^{\delta}$ (see [3], §8.4) replaces that one for σ-similarity. The only moment has to be pointed out is that we use (to minimize combinatorics) the standard duality (see the footnote at the beginning of Case B), so one has to use now its more general version. Namely, given a pseudolinear bundle $(U_1, U_2, \mathscr{A}, \mathscr{B})$ in the sense of [23], i.e. a pair of additive maps $\mathscr{A}, \mathscr{B} : U_1 \longrightarrow U_2$ (between right K-spaces) such that $\mathscr{A}$ is linear and $\mathscr{B}$ satisfies the condition $(ua)\mathscr{B} = (u\mathscr{B})a^{\sigma} + (u\mathscr{A})a^{\delta}$, then (for $a \in K$, $u \in U_1$, $f \in U_2^*$) the dual map $\mathscr{B}^* : U_2^* \longrightarrow U_1^*$ is defined by the equality $(\mathscr{B}^* f)u = (f(u\mathscr{B}))^{\lambda} + (f(u\mathscr{A}))^{\partial}$ and possesses the property $\mathscr{B}^*(af) = a^{\lambda}\mathscr{B}^* f + a^{\partial}\mathscr{A}^* f$ where $\lambda = \sigma^{-1}$ and $\partial = -\delta\lambda$ is a left λ-derivation such that $(ab)^{\partial} = a^{\partial}b + a^{\lambda}b^{\partial}$ (notice that $a^{-\delta} = -a^{\delta}$). We omit exhaustive details because the pseudolinear case is out of the objectives of the present paper.

Proof of Theorem 2. Let (A, B) be a matrix realization of an indecomposable representation (11) of the contragredient pencil (10) with $(\alpha, \beta) = (1, \sigma)$. If the map $\mathscr{A}$ in (11) is an isomorphism, then one has simply to repeat precisely Case A of the previous proof for Theorem 1.

Assuming that $\mathscr{A}$ is not an isomorphism and (up to duality) $W = \mathrm{Im}\mathscr{A} \neq U_2$, we again carry out the induction on $d = \dim_K U_2$. The case $d = 1$ again leads either to the type 1(b) with $n = 1$, or to the type 2 with $n = 0$.

So, we suppose $d > 1$ and recall that the transformations (a$'$),(b$'$) of Section 3 (for $(\alpha, \beta) = (1, \sigma)$) are admissible for the matrices A, B.

Set $U_2 = W \oplus E$ for some complement E and include a base of W into the chosen base of U_2. Then the pair (A, B) takes the form

$$A = \begin{array}{c|c|c} \hline & & \mathrm{E} \\ \hline \mathrm{X} & & \mathrm{W} \\ \hline \end{array} \qquad B = \begin{array}{c} \begin{array}{|c|c|} \hline Y & Z \\ \hline \end{array} \\ E \quad W \end{array}$$

where the rows of X are linearly independent and arrows symbolize simultaneous row and column additions of type (a$'$) not changing the upper zero-block in A. Then the columns of Y are also independent (otherwise, by indecomposability, a zero-row in Y implies $U_2 = K$ in contradiction to $d > 1$).

Reducing the block Z to the form $Z = \begin{array}{|c|} \hline \\ \hline B' \\ \hline \end{array}$ with a non-singular in rows matrix B' and then annulling all the elements to the left of B', we get

$$A = \begin{array}{|c|c|} \hline & \\ \hline X' & A' \\ \hline \end{array} \qquad B = \begin{array}{|c|c|} \hline Y' & \\ \hline & B' \\ \hline \end{array}$$

where the matrix Y' is non-singular in columns and hence can be reduced to the form $Y' = \begin{array}{|c|} \hline \\ \hline I_s \\ \hline \end{array}$ for some $s \geqslant 1$ ($s > 0$ since $W \neq U_2$). Thus the pair (A, B) takes finally the form

$$A = \begin{array}{|c|c|c|}\hline & & \\ \hline D & C & A' \\ \hline \end{array} \qquad B = \begin{array}{|c|c|}\hline & \\ \hline I_s & \\ \hline & B' \\ \hline \end{array} \tag{16}$$

with the shown by arrows possible simultaneous additions of type (a′) and (b′) between different stripes in A and B. Here actually one gets independent column additions $A' \to C \to D$[10], and the columns of C are linearly independent modulo the columns of A' (otherwise a zero-column in C implies by indecomposability $(A, B) = (0, 1)$). Further, the pair (A', B') itself (if not to pay attention to changes in the blocks C, D) is reduced precisely by the transformations (a′) and (b′).

Moreover, Lemma 8 remains true in the considered situation without changes, except for the substitution the formula number (16) for (14) (its proof is entirely analogous to the previous case, the appearance of a three-block chain $A' \to C \to D$ instead of the two-block one $A' \to C$ is not essential, a verification of this is left to the reader as an easy exercise).

Therefore it remains, in the induction step, to consider in (16) the pair (A', B') of type 1(b) or 2 only (otherwise A' were non-singular in rows and the block C could be annulated, in contradiction to the independence of its $s \geqslant 1$ columns modulo A'). Thus A' contains precisely one zero-row, and one can annul by column additions $A' \to C \to D$ all but one elements in C and hence all in D. In such a case by indecomposability the block D in reality is empty (i.e. consists of 0 columns) but C consists of one column. So, one comes at once to the form 1(b) or 2 of the whole pair (A, B). This completes the proof of Theorem 2. □

Now pass to some applications of the $(1, \sigma)$-pencil problem.

6. Biquadratic matrix problem

If A, B are rings and W an (A, B)-bimodule, then the ${}_A W_B$*-problem* is a matrix problem on reducing to some canonical form one rectangular matrix M over W by elementary transformations of its rows over A and columns over B. It means that one can apply to M combinations of transformations of the following types:

(a) Multiplication a row to the left (a column to the right) by an invertible element from $A(B)$.
(b) Addition of one row (column) to another one with a left (right) coefficient from $A(B)$.

In particular, any permutations of rows or columns are admissible.

The matrix M itself is called a (matrix) *representation*[11] of the bimodule W, or briefly a W-*representation*. Representations M and M' are said to be (A, B)-*equivalent* (or simply *equivalent* or *isomorphic*), with a notation $M \overset{(A,B)}{\sim} M'$, if they are turned into each other by the mentioned elementary transformations over A and B. Direct sums and indecomposability of representations are defined naturally.

Remark 10. Due to formal reasonings, we accept (as representations of the bimodule W) abstract "empty matrices" with zero number of rows or columns, like for the $(1, \sigma)$-pencil problem. There

[10] For instance, column additions $A' \to C$ are simultaneous with the inverse additions of rows of the block I_s to the zero-block below which can be restored by suitable additions of columns of B' to the left (simultaneously with additions of zero-rows in A to the down).

[11] As for a matrix free definition of representations of bimodules, see for instance [21].

exist (up to equivalence) precisely two indecomposable matrices of such a kind: $M_{(1,0)}$ and $M_{(0,1)}$, having 1 row and 0 columns and 0 rows and 1 column respectively.

Our interest is to investigate one important particular case of the ${}_AW_B$-problem. Namely, let F be a field admitting quadratic extensions G_1, G_2 (which may coincide) in the algebraic closure $\overline{F}$ of F. Then one can consider a natural (G_1, G_2)-bimodule

$$W = G_1 \underset{F}{\otimes} G_2$$

and define the *biquadratic matrix problem* over the triple (G_1, F, G_2), being precisely the $G_1 \underset{F}{\otimes} G_2$-problem. We call the biquadratic problem *homogeneous* if $G_1 \simeq G_2$ over F, i.e. if it is in fact the $G \underset{F}{\otimes} G$-problem for a quadratic extension $F \subset G$.

In general, a complete solution to the biquadratic problem is still unknown. Among the most important situations with a complete or partial solutions one can mention the following ones.

(i) The $\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C}$-problem is equivalent (as we explain below) to the problem considered and solved completely in [8].

(ii) The so-called non-homogeneous indecomposable representations (of discrete nature, embracing all the non-regular and some special regular ones) are described in [7] in fact for an arbitrary $G_1 \underset{F}{\otimes} G_2$-problem.

In this connection, we consider below homogeneous biquadratic problems over a field F of characteristic $\neq 2$ (for instance, the situation (i) above is of such a kind) reducing them to the $(1, \sigma)$-pencil problem, with a suitable σ. Using the obtained reduction, we then establish a precise relationship between the corresponding matrix canonical forms.

The following simple reformulation of the general biquadratic $G_1 \underset{F}{\otimes} G_2$-problem may be found useful in various considerations.

Set $G_i = F(u_i)$ for some element $u_i \in \overline{F}$ having the minimal polynomial $t^2 + p_i t + q_i$ ($i = 1,2$). Then the initial fields G_1 and G_2 are canonically isomorphic to the matrix fields $\widetilde{G_1} = F\{I, U_1\}$ and $\widehat{G_2} = F\{I, U_2\}$ respectively, where

$$U_1 = \begin{bmatrix} 0 & -q_1 \\ 1 & -p_1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} -p_2 & -q_2 \\ 1 & 0 \end{bmatrix}$$

and I is the identity matrix of order 2.

Let M be a representation of the bimodule $W = G_1 \underset{F}{\otimes} G_2$, i.e. any $m \times n$ matrix over W which has to be reduced. Representing each element x of M in the form

$$x = a(1 \otimes u_2) + b(1 \otimes 1) + c(u_1 \otimes u_2) + d(u_1 \otimes 1)$$

(with $a, b, c, d \in F$) and then replacing it by a square 2×2 block $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we obtain a new matrix $\widetilde{M}$ over F of size $2m \times 2n$, which one can consider as a representation of the matrix $(\widetilde{G_1}, \widehat{G_2})$-bimodule $\widetilde{W} = M_2(F)$ formed by all 2×2 matrices over F.

Moreover, since left G_1-elementary (right G_2-elementary) transformations of rows (columns) of M correspond to left $\widetilde{G_1}$-elementary (right $\widehat{G_2}$-elementary) transformations of pairs of rows (columns) of $\widetilde{M}$, the following is true.

Proposition 11. *Each biquadratic $G_1 \underset{F}{\otimes} G_2$-problem is equivalent to the ${}_{\widetilde{G_1}}\widetilde{W}_{\widehat{G_2}}$-problem consisting in reducing to a canonical form one rectangular matrix $\widetilde{M}$ over F (of size $2m \times 2n$) by the transformations*

$$X\widetilde{M}Y,$$

where X and Y are invertible matrices over the matrix fields $\widetilde{G_1}$ and $\widehat{G_2}$.

Example 12. Considering the $\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C}$-problem, we have $G_1 = G_2 = \mathbb{C}$ and can take $u_1 = u_2 = \boldsymbol{i}$ (thus $p_1 = p_2 = 0$ and $q_1 = q_2 = 1$) getting matrix field

$$\widetilde{\mathbb{C}} = \widehat{\mathbb{C}} = \left\{ \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\} \tag{17}$$

isomorphic to $\mathbb{C}$. Hence, the $\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C}$-problem is equivalent to the ${}_{\widetilde{\mathbb{C}}}\widetilde{W}_{\widetilde{\mathbb{C}}}$-problem, for $\widetilde{W} = M_2(\mathbb{R})$. This is precisely the solved in [8] problem on reducing one real matrix (of even size) by formally complex transformations of pairs of its rows and columns. It will be observed more below (Example 46).

7. Homogeneous biquadratic problem in characteristic ≠ 2

From now on, F is a field of characteristic $\neq 2$.

Let $G = F(u)$ be a quadratic extension of F with the minimal polynomial $t^2 + q$ of the element u over F and the natural *conjugation* σ on G given by $\sigma(\alpha + \beta u) = \overline{\alpha + \beta u} = \alpha - \beta u$ ($\alpha, \beta \in F$). Consider a (G, G)-bimodule $W = G \underset{F}{\otimes} G$ with the canonical base

$$w_{11} = 1 \otimes 1, \quad w_{1u} = 1 \otimes u, \quad w_{u1} = u \otimes 1, \quad w_{uu} = u \otimes u \tag{18}$$

and four special elements

$$e_1 = w_{1u} + w_{u1}, \quad e_2 = w_{uu} - qw_{11}, \quad e_3 = w_{1u} - w_{u1}, \quad e_4 = w_{uu} + qw_{11} \tag{19}$$

which evidently are linearly independent over F (and therefore form another F-base of W) and satisfy the relations

$$\begin{array}{llll} ue_1 = e_2, & e_1u = e_2, & ue_3 = e_4, & e_3u = -e_4, \\ ue_2 = -qe_1, & e_2u = -qe_1, & ue_4 = -qe_3, & e_4u = qe_3. \end{array} \tag{20}$$

Let M be a representation of W, i.e. a matrix over W. Considering each its element x as the sum $x = x' + x''$ where $x' = \alpha e_1 + \beta e_2$ and $x'' = \gamma e_3 + \delta e_4$ ($\alpha, \beta, \gamma, \delta \in F$), one can attach to M a pair of matrices (M', M'') formed by the summands x', x''.

Identifying each entry $x' = \alpha e_1 + \beta e_2$ ($x'' = \gamma e_3 + \delta e_4$) with a formal expression $\alpha + \beta u$ ($\gamma + \delta u$), one gets a pair of matrices (M', M'') over G and comes immediately (taking into account the relations (20)) to the following conclusions:

(a) G-elementary transformations of rows of M correspond to simultaneous G-elementary transformations of rows in M' and M''.

(b) G-elementary transformations of columns of M correspond to simultaneous *conjugate* G-elementary transformations of columns in M' and M'', in the sense that a multiplication of a column in M' by some coefficient $\alpha + \beta u \in G$ (or an addition of one column of M' to another one with this coefficient) is simultaneous with the multiplication (addition) of the same columns in M'' with the conjugate coefficient $\alpha - \beta u$.

It means that in the considered case the pair of matrices (M', M'') can be transformed into any other pair of the form $(X^{-1}M'Y, X^{-1}M''Y^{\sigma})$ with non-singular matrices X, Y over G, i.e. one obtains the $(1, \sigma)$-pencil problem over G. We have established.

Proposition 13. *For $G = F(u)$ and σ as above, the biquadratic $G \underset{F}{\otimes} G$-problem is equivalent to the $(1, \sigma)$-pencil problem over G.*

As a consequence of Proposition 11 (applied to the case $G_1 = G_2 = G$) and Proposition 13, we get the following result.

Corollary 14. *Let $G = F(u)$ be a quadratic extension of a field F of characteristic $\neq 2$, with the minimal polynomial $t^2 + q$ of the element u. Then the following matrix problems are equivalent:*

(a) *The homogeneous biquadratic $G \underset{F}{\otimes} G$-problem.*
(b) *The $(1, \sigma)$-pencil problem over G, where $\sigma(\alpha + \beta u) = \alpha - \beta u$ is the natural conjugation on G.*
(c) *The ${}_{\widetilde{G}}\widetilde{W}_{\widetilde{G}}$-problem, where $\widetilde{W} = M_2(F)$, $\widetilde{G} = \left\{ \begin{bmatrix} \alpha & -q\beta \\ \beta & \alpha \end{bmatrix} : \alpha, \beta \in F \right\}$.*

Since canonical forms of indecomposables for the $(1, \sigma)$-pencil are known (Theorem 1), this yields at once the following complete classification of indecomposables for the $G \underset{F}{\otimes} G$-problem.

Theorem 15 (The first canonical form). *Let F and $G = F(u)$ be as in Corollary 14. Then the indecomposable representations of the bimodule $G \underset{F}{\otimes} G$ are exhausted, up to isomorphism, by the pairwise non-isomorphic ones of the following types:*

(0) $I_n e_1 + C(f)e_3 = e_1 I_n + e_3 C(f^{\sigma})$, *where* $f \in \dot{\mathfrak{I}}_n\langle 0\rangle$ *and* $n \geqslant 1$.
(1) (a) $I_n e_1 + J_n(0)e_3$ *and* (b) $J_n(0)e_1 + I_n e_3$, $n \geqslant 1$.
(2) $I_n^{\uparrow} e_1 + I_n^{\downarrow} e_3$, $n \geqslant 0$.
(3) $I_n^{\rightarrow} e_1 + I_n^{\leftarrow} e_3$, $n \geqslant 0$.

Remark 16. In the invariant language, the arguments above mean that $W = W' \oplus W''$ is a direct sum of two (G, G)-sub-bimodules

$$W' = F\{e_1, e_2\} \simeq {}_G G_G \quad \text{and} \quad W'' = F\{e_3, e_4\} \simeq {}_G G_{\overline{G}},$$

where ${}_G G_G$ is the natural bimodule G and ${}_G G_{\overline{G}}$ is the "twisted from the right" (G, G)-bimodule G (such that the right action of an element $\alpha + \beta u$ on it coincide with the multiplication by the conjugate element $\alpha - \beta u$).

We see that the homogeneous biquadratic problem is reduced easily to the $(1, \sigma)$-pencil problem. Nevertheless, the obtained first canonical form cannot be considered as a quite good one with respect to the standard base (18) of the bimodule W because of the knotty expressions for the special base (19) (remind that $e_1 = 1 \otimes u + u \otimes 1$ and $e_3 = 1 \otimes u - u \otimes 1$).

Thus our goal in subsequent sections is to obtain another canonical form for the same situation. It is expedient to present just now a general formulation of the result, still without its proof (given in Section 11).

Theorem 17 (The second canonical form). *Let F and $G = F(u)$ be as in Corollary 14. Then the indecomposable representations of the bimodule $G \underset{F}{\otimes} G$ are exhausted, up to isomorphism, by the pairwise non-isomorphic ones of the following types:*

(a) $(1 \otimes u)I_n + (u \otimes 1)C(g)$, *where* $g \in \Theta(\dot{\Im}_n\langle -1\rangle) \cup (t+1)^n$ *and* $n \geqslant 1$.
(b) $(1 \otimes u)I_n^{\uparrow} + (u \otimes 1)I_n^{\downarrow}$, $n \geqslant 0$.
(c) $(1 \otimes u)I_n^{\rightarrow} + (u \otimes 1)I_n^{\leftarrow}$, $n \geqslant 0$.

Notice that, in Theorems 3 and 4 (which are matrix versions of Theorems 15 and 17), the blocks A, B, C, D are formed by the coefficients a, b, c, d in the presentation $x = aw_{1u} + bw_{11} + cw_{uu} + dw_{u1}$ of elements of the bimodule.

Before to prove Theorem 17, we need to introduce an integer matrix sequence Θ_n which expresses in a perfect way a precise relationship between polynomial invariants for the $G \underset{F}{\otimes} G$-problem and the $(1, \sigma)$-pencil problem.

8. Integer matrix sequence Θ_n

Let $\binom{n}{k} = n!/k!(n-k)!$ be the ordinary binomial coefficients which usually are defined for integers $0 \leqslant k \leqslant n$. It is convenient for us to extend formally the range of definition of $\binom{n}{k}$ to all integers $k, n \in \mathbb{Z}$ setting $\binom{n}{k} = 0$ if the condition $0 \leqslant k \leqslant n$ is not satisfied. Then it holds, as one checks easily, $\binom{n}{k} = \binom{n}{n-k}$ for any $k, n \in \mathbb{Z}$, and $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ for any $k, n \in \mathbb{Z}$ except for the only case $k = n = -1$.

Denote by $\Theta_n = [\Theta_n^{ij}]$ a square matrix of order n whose elements are of the form

$$\Theta_n^{ij} = \sum_k (-1)^k \binom{j-1}{k}\binom{n-j}{i-k-1}, \tag{21}$$

where $k \in \mathbb{Z}$ and obviously almost all terms of the sum are zeroes. Since $i, j \in \{1, \dots, n\}$, possible non-zero summands in (21) can be met only if the inequalities $0 \leqslant k \leqslant j-1$ and $0 \leqslant i-1-k \leqslant n-j$ are satisfied, this is equivalent to the condition

$$\max\{0, i+j-n-1\} \leqslant k \leqslant \min\{i, j\} - 1. \tag{22}$$

Remark 18. The numbers Θ_n^{ij} are defined only for *admissible* values of indices: $1 \leqslant i, j \leqslant n$. So, any formula below containing any number Θ_n^{ij} is supposed to be valid only under this assumption.

The matrix Θ_n contains some known numbers. Using (21), you get at once all the entries of the border rows and columns

$$\Theta_n^{1j} = 1, \quad \Theta_n^{nj} = (-1)^{j-1}, \quad \Theta_n^{i1} = \binom{n-1}{i-1}, \quad \Theta_n^{in} = (-1)^{i-1}, \ \Theta_n^{i1} = \binom{n-1}{i-1} \tag{23}$$

meeting (in the first and the last columns) the ordinary binomial coefficients. If $j = 2$ and $n \geqslant 2$, you get the numbers

$$\Theta_n^{i2} = \binom{n-2}{i-1} - \binom{n-2}{i-2} \tag{24}$$

which, for a given $n \geqslant 2$ and all $i \leqslant m = \lfloor \frac{n}{2} \rfloor$ (where $\lfloor x \rfloor \leqslant x$ is the whole part of x), are precisely the coefficients $\Theta_n^{i2} = \alpha_{i-1}$ of the Catalan polynomial

$$C_{n-2}(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{m-1}x^{m-1}$$

determined also by the known Catalan triangle (see for instance [4], Section 5.3). In particular, for even $n = 2m$, you get the Catalan numbers

$$C_{m-1} = \Theta_{2m}^{m2} = \frac{1}{m}\binom{2m-2}{m-1}$$

which are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, ... for $m = 1, 2, 3, \ldots$.

The elements Θ_n^{ij} can be calculated in a simple recurrent way.

Lemma 19 (The recursion formulas). *The following holds for admissible indices*:

$$\text{(a)}\quad \Theta_n^{ij} + \Theta_n^{i+1,j} = \Theta_{n+1}^{i+1,j}, \qquad \text{(b)}\quad \Theta_n^{ij} - \Theta_n^{i-1,j} = \Theta_{n+1}^{i,j+1}.$$

Proof. (a) Since $\binom{n-j}{i-1-k} + \binom{n-j}{i-k} = \binom{(n+1)-j}{(i+1)-k-1}$, you get at once, using (21), the desired equality. (b) Shifting the summation index $k \to k-1$ in the expression of type (21) for $\Theta_n^{i-1,j}$, you get $\Theta_n^{ij} - \Theta_n^{i-1,j} = \sum_k (-1)^k \left(\binom{j-1}{k} + \binom{j-1}{k-1}\right)\binom{n-j}{i-1-k} = \Theta_{n+1}^{i,j+1}$. □

Obviously, for $j = 1$, the equality (a) coincides with the binomial property $\binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}$, while (b) represents in fact the equality (24).

The next property is a direct consequence of (b) and (23).

Corollary 20. *Let s_n^j be the sum of all elements of the j*th *column of Θ_n. Then $s_n^1 = 2^{n-1}$ and $s_n^j = 0$ for $j > 1$.*

Immediately from (a) and (b) we get the following relations valid for admissible indices (to verify them, use (a) (resp. (b)) for the first (resp. second) summands; for $i = 1$ the first equality is obvious):

$$2\Theta_n^{ij} = \Theta_{n+1}^{ij} + \Theta_{n+1}^{i,j+1} = \Theta_{n+1}^{i+1,j} - \Theta_{n+1}^{i+1,j+1}. \tag{25}$$

This double property is reformulated in the matrix language as follows.

Corollary 21. $\Theta_{n+1}(I_n^{\downarrow} + I_n^{\uparrow}) = 2I_n^{\downarrow}\Theta_n$ *and* $\Theta_{n+1}(I_n^{\downarrow} - I_n^{\uparrow}) = 2I_n^{\uparrow}\Theta_n$.

Taking into account (23) and applying any of the formulas (a) or (b), one can easily obtain Θ_{n+1} from Θ_n. On the other hand, substituting $j+1$ for j in (a) and $i+1$ for i in (b), we get

$$\Theta_{n+1}^{i+1,j+1} = \Theta_n^{i,j+1} + \Theta_n^{i+1,j+1} = \Theta_n^{i+1,j} - \Theta_n^{ij}, \tag{26}$$

in particular

$$\Theta_n^{ij} + \Theta_n^{i,j+1} + \Theta_n^{i+1,j+1} = \Theta_n^{i+1,j}. \tag{27}$$

We have the following consequence of (27) and Lemma 19.

Corollary 22. *Let* $\begin{array}{|c|c|}\hline a & b \\ \hline d & c \\ \hline\end{array}$ *be any square* 2×2 *block of the matrix* Θ_n (*belonging to adjacent rows and adjacent columns*). *Then* $a + b + c = d$, *in particular, if* $a = \Theta_n^{ij}$, *then* $b + c = d - a = \Theta_{n+1}^{i+1,j}$.

Since the border rows and columns of Θ_n are known (see (23)), the property $a + b + c = d$ allows to calculate directly Θ_n without knowing Θ_{n-1}. In Appendix, the matrices Θ_n are presented for $n \leqslant 9$.

Setting (for a given fixed n) $i^* = n - i + 1$, we note that the elements Θ_n^{ij} and $\Theta_n^{i^*j^*}$ occupy symmetric positions in Θ_n with respect to the center of Θ_n.

Lemma 23 (The $(\pm)$-symmetry of rows and columns). *Each odd* (*even*) *row or column of the matrix* Θ_n *is symmetric* (*antisymmetric*) *with respect to its own center*, *i.e.*

$$\Theta_n^{ij} = (-1)^{i+1}\Theta_n^{ij^*} = (-1)^{j+1}\Theta_n^{i^*j} = (-1)^{n+i+j+1}\Theta_n^{i^*j^*}. \tag{28}$$

In particular, *if n is odd*, *the central element of each even row or column is zero*.

Proof. Write the second and the third terms in the form (21) and then simply shift the summation index k to $i - 1 - k$ and $j - 1 - k$ respectively. You get the expression (21) for Θ_n^{ij}, up to changes of type $\binom{m}{l} = \binom{m}{m-l}$ for the third term. The last equality is a consequence of the previous ones. □

Considering a diagonal matrix $D_n = D_n^{-1} = \operatorname{diag}\{1, -1, \ldots, (-1)^{n-1}\}$ and a codiagonal permutation matrix $P_n = P_n^{-1} = \operatorname{codiag}\{1, 1, \ldots, 1\}$ of order n, we get an obvious consequence.

Corollary 24. $\Theta_n P_n = D_n \Theta_n$.

Denote by α_n^i and β_n^i the ith row and column respectively of the matrix Θ_n, considered as vector-rows of the standard euclidian space $\mathbb{R}^n$ $(1 \leqslant i \leqslant n)$. For any $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^m$, let $(\alpha, \beta) \in \mathbb{R}^{n+m}$ be a natural row-union in which β follows α. Let $0_n \in \mathbb{R}^n$ be the zero-row. Set by definition $\alpha_n^0 = \alpha_n^{n+1} = 0_n$. The following statement is a direct consequence of Lemma 19 and of the formulas (23).

Lemma 25

(a)

$$\begin{aligned}\alpha_{n+1}^i &= (\alpha_n^i, 0) + (\alpha_n^{i-1}, 0) + \left(0_n, (-1)^{i-1}\binom{n}{i-1}\right) \\ &= (0, \alpha_n^i) - (0, \alpha_n^{i-1}) + \left(\binom{n}{i-1}, 0_n\right) \quad \textit{for all } i \in \{1, \ldots, n+1\}.\end{aligned}$$

(b)

$$\begin{aligned}\beta_{n+1}^j &= (\beta_n^j, 0) + (0, \beta_n^j) \quad \textit{for } j \leqslant n, \\ \beta_{n+1}^j &= (\beta_n^{j-1}, 0) - (0, \beta_n^{j-1}) \quad \textit{for } j \geqslant 2.\end{aligned}$$

Lemma 26 (The square of Θ_n). *$\Theta_n^2 = 2^{n-1} I_n$, in particular, $\Theta_n^{-1} = 1/2^{n-1}\Theta_n$.*

Proof. Use induction on n. For $n = 1$ the formula is true. Suppose it is true for a given $n \geqslant 1$, i.e. $\alpha_n^i \cdot \beta_n^j = 2^{n-1}\delta_{ij}$, where $\cdot$ denotes the inner product in $\mathbb{R}^n$ and δ_{ij} is the ordinary Kronecker symbol. We have to prove $\alpha_{n+1}^i \cdot \beta_{n+1}^j = 2^n \delta_{ij}$. In view of Lemma 23, this is true if i and j are of different parity, so assume the contrary. Then α_{n+1}^i, β_n^j are both symmetric or both antisymmetric and (if $j \geqslant 2$) β_n^{j-1} is respectively antisymmetric or symmetric. Thus it holds $\alpha_{n+1}^i \cdot (\beta_n^j, 0) = \alpha_{n+1}^i \cdot (0, \beta_n^j)$ for $j \leqslant n$ and $\alpha_{n+1}^i \cdot (\beta_n^{j-1}, 0) = -\alpha_{n+1}^i \cdot (0, \beta_n^{j-1})$ for $j \geqslant 2$. Then by Lemma 25 you get (use the first presentation for α_{n+1}^i in (a) and take into account that $\alpha_n^{i-1} \cdot \beta_n^j = 0$, as well as $\alpha_n^i \cdot \beta_n^{j-1} = 0$, due to different parities)

$$\alpha_{n+1}^i \cdot \beta_{n+1}^j = \begin{cases} 2\alpha_{n+1}^i \cdot (\beta_n^j, 0) = 2\alpha_n^i \cdot \beta_n^j = 2^n \delta_{ij}, & \text{if } j \leqslant n, \\ 2\alpha_{n+1}^i \cdot (\beta_n^{j-1}, 0) = 2\alpha_n^{i-1} \cdot \beta_n^{j-1} = 2^n \delta_{ij}, & \text{if } j \geqslant 2. \end{cases}$$

The proof is complete. $\square$

Lemma 27. det $\Theta_n = (-2)^{\frac{n(n-1)}{2}}$.

Proof. Set $d_n = \det \Theta_n$. Due to (25), $\Theta_{n+1} J_{n+1}^{+}(-1) = \begin{bmatrix} -1 & 0_n \\ * & 2\Theta_n \end{bmatrix}$ (where $*$ denotes a column of arbitrary elements) and $d_{n+1}(-1)^{n+1} = -2^n d_n$, i.e. $d_{n+1} = (-2)^n d_n$. Since $d_1 = 1$, we get $d_{n+1} = (-2)^{1+2+\cdots+n} = (-2)^{\frac{n(n+1)}{2}}$. $\square$

Using the known Vandermonde convolution formula for non-negative integers (which follows easily from the equality $(1+x)^a(1+x)^b = (1+x)^{a+b}$)

$$\sum_i \binom{a}{i}\binom{b}{c-i} = \binom{a+b}{c}, \tag{29}$$

one can deduce directly from the definition (21) one more very useful relation.

Lemma 28. *The following holds for admissible indices*:

$$\Theta_{n+m-1}^{pq} = \sum_i \Theta_n^{ij}\Theta_m^{p-i+1,q-j+1}. \tag{30}$$

Proof. Denoting $s = n + m - 1$ and $k = k(i) = p - i + 1$, $l = l(i) = q - j + 1$, we get

$$\begin{aligned} A = \sum_i \Theta_n^{ij}\Theta_m^{kl} &\overset{(21)}{=} \sum_i \sum_\alpha (-1)^\alpha \binom{j-1}{\alpha}\binom{n-j}{i-1-\alpha} \sum_\beta (-1)^\beta \binom{l-1}{\beta}\binom{m-l}{k-1-\beta} \\ &= \sum_{\alpha,\beta} (-1)^{\alpha+\beta}\binom{j-1}{\alpha}\binom{l-1}{\beta} \sum_i \binom{n-j}{i-1-\alpha}\binom{m-l}{k-1-\beta} \\ &\overset{(29)}{=} \sum_{\alpha,\beta} (-1)^{\alpha+\beta}\binom{j-1}{\alpha}\binom{l-1}{\beta}\binom{s-q}{p-1-\alpha-\beta}. \end{aligned}$$

If $\alpha + \beta = \gamma$, then

$$A = \sum_{\gamma} (-1)^{\gamma} \sum_{\alpha} \binom{j-1}{\alpha} \binom{l-1}{\gamma-\alpha} \binom{s-q}{p-1-\gamma}$$
$$\overset{(29)}{=} \sum_{\gamma} (-1)^{\gamma} \binom{q-1}{\gamma} \binom{s-q}{p-1-\gamma} \overset{(21)}{=} \Theta_s^{pq}. \qquad \square$$

The dual sequence. To each matrix $\Theta_n = [\Theta_n^{ij}]$ one can naturally attach the *dual* one $\Theta_n^* = [\Theta_n^{*ij}]$ reflecting all the elements of Θ_n with respect to its center. In other words, the rows and columns of Θ_n have to be written in the inverse order, i.e. (see Lemma 23)

$$\Theta_n^{*ij} = \Theta_n^{i^* j^*} = (-1)^{n+i+j+1} \Theta_n^{ij}. \tag{31}$$

Hence, $\Theta_n^* = P_n \Theta_n P_n$ where P_n is the defined above codiagonal permutation matrix. The analog of (21) for Θ_n^* looks as follows:

$$\Theta_n^{*ij} = \sum_{k} (-1)^k \binom{n-j}{k} \binom{j-1}{n-i-k}, \tag{32}$$

where possible non-zero summands are met only on condition that

$$\max\{0, n+1-i-j\} \leqslant k \leqslant n - \max\{i, j\}. \tag{33}$$

Denoting $T_n = D_n P_n = \mathrm{codiag}\{1, -1, \dots, (-1)^{n-1}\}$ and taking into account Corollary 24 and Lemma 26, we get $\Theta_n \Theta_n^* = \Theta_n P_n \Theta_n P_n = D_n \Theta_n^2 P_n = 2^{n-1} D_n P_n = 2^{n-1} T_n$ and $\Theta_n^* \Theta_n = P_n \Theta_n P_n \Theta_n = P_n D_n \Theta_n^2 = 2^{n-1} P_n D_n = (-2)^{n-1} T_n$. Therefore, the following lemma holds.

Lemma 29. $\Theta_n \Theta_n^* = (-1)^{n-1} \Theta_n^* \Theta_n = 2^{n-1} T_n$.

One can easily reformulate for Θ_n^* the established above properties of Θ_n. Some of them are listed below (where, as before, all the formulas are supposed to be valid for admissible indices).

Corollary 30. *Each matrix Θ_n^* possesses the properties*:

(a) $\Theta_n^{*1j} = (-1)^{n-j}$, $\Theta_n^{*nj} = 1$, $\Theta_n^{*i1} = (-1)^{n-i} \binom{n-1}{i-1}$, $\Theta_n^{*in} = \binom{n-1}{i-1}$.
(b) $\Theta_n^{*ij} - \Theta_n^{*i+1,j} = \Theta_{n+1}^{*i+1,j}$ *and* $\Theta_n^{*ij} + \Theta_n^{*i-1,j} = \Theta_{n+1}^{*i,j+1}$.
(c) *If s_n^{*j} is the sum of all elements of the jth column of Θ_n^*, then $s_n^{*j} = 0$ for $j < n$ and* $s_n^{*n} = 2^{n-1}$.
(d) $2\Theta_n^{*ij} = \Theta_{n+1}^{*i,j+1} - \Theta_{n+1}^{*ij} = \Theta_{n+1}^{*i+1,j} + \Theta_{n+1}^{*i+1,j+1}$.
(e) $\Theta_{n+1}^*(I_n^{\uparrow} - I_n^{\downarrow}) = 2 I_n^{\downarrow} \Theta_n^*$ *and* $\Theta_{n+1}^*(I_n^{\uparrow} + I_n^{\downarrow}) = 2 I_n^{\uparrow} \Theta_n^*$.
(f) *For the same square 2×2 block as in Corollary 22 (considered with respect to Θ_n^*), it holds $a = b + c + d$.*
(g) *Each co-odd (co-even)*[12] *row or column of Θ_n^* is symmetric (antisymmetric) with respect to its own center.*

[12] I.e. being odd (even) when counting from the last one.

(h) $(\Theta_n^*)^2 = \Theta_n^2 = 2^{n-1} I_n$ *and* $\det \Theta_n^* = \det \Theta_n = (-2)^{\frac{n(n-1)}{2}}$.
(i) $\Theta_{n+m-1}^{*pq} = \sum_i \Theta_n^{*ij} \Theta_m^{*p-i+1,q-j+1}$.

9. Θ-transform of a polynomial

Let K be a field of characteristic $\neq 2$ with an automorphism σ, K^0 the subfield of all σ-invariant elements and $R = K[t, \sigma]$ the observed earlier in Section 4 skew polynomial ring. It is clear that the elements of K^0 are precisely those in K which are permutable with all polynomials in R. Set $R^0 = K^0[t]$, this is the maximal commutative polynomial subring in R.

We have a disjoint union $R = \cup_n R_n$ $(n \geqslant -1)$ where R_n is the subset of polynomials of degree n (notice that R_{-1} contains the only one zero polynomial having each element of K as its root).

For an arbitrary polynomial $f = a_0 + ta_1 + \cdots + t^n a_n$ in R_n $(n \geqslant 0)$, let $[f] = (a_0, a_1, \ldots, a_n)^{\mathrm{T}}$ be the vector-column of its coefficients. Denote by $\Theta_{n+1} f$ a polynomial in R given by the formula

$$[\Theta_{n+1} f] = \Theta_{n+1}[f], \tag{34}$$

with the matrix product to the right.

Obviously, the leading coefficient of $\Theta_{n+1} f$ is $a_0 - a_1 + \cdots + (-1)^n a_n = f(-1)$, hence $g = \Theta_{n+1} f \in R_n$ if $f \in R_n\langle -1\rangle$. Moreover, since by Lemma 26 $\Theta_{n+1}^2 = 2^n I_{n+1}$, the leading coefficient of the polynomial $\Theta_{n+1} g = \Theta_{n+1}^2 f = 2^n f$ is $g(-1) = 2^n a_n \neq 0$. So, $f \in R_n\langle -1\rangle$ if and only if $\Theta_{n+1} f \in R_n\langle -1\rangle$.

Thus one can define a function $\Theta : R\langle -1\rangle \longrightarrow R\langle -1\rangle$, which will be called the *Θ-transformation*, attaching to a polynomial $f = a_0 + ta_1 + \cdots + t^n a_n$ from $R_n\langle -1\rangle$ its *Θ-transform* Θf in the following way

$$\Theta f = (\Theta_{n+1} f)\frac{a_n}{f(-1)}. \tag{35}$$

It is clear that $\Theta(fc) = (\Theta f)c$ for any $c \in K^{\times}$.

The observations above imply the following fact.

Lemma 31. *The function Θ is an involution on $R\langle -1\rangle$ preserving degrees and leading coefficients of polynomials. If $f \in R_n\langle -1\rangle$ and $g = \Theta f$, then*

$$f(-1)g(-1) = 2^n a_n^2, \tag{36}$$

where a_n is the common leading coefficient of f and g.

Lemma 32. *If $f \in R^0\langle -1\rangle$ and $g \in R\langle -1\rangle$, then it holds*

$$\Theta(fg) = (\Theta f)(\Theta g).$$

Therefore, the restriction of the transformation Θ to the subset $R^0\langle -1\rangle$ is a multiplicative function. In particular, a polynomial $f \in R^0\langle -1\rangle$ is irreducible if and only if its transform Θf is irreducible.

Proof. By the assumption, the coefficients of f commute with t. Hence, for $f = a_0 + \cdots + t^n a_n$ and $g = b_0 + \cdots + t^m b_m$, we have $fg = c_0 + \cdots + t^{s-1} c_{s-1}$, where $s = n + m + 1$ and $c_{q-1} = \sum_{j=1}^q a_{j-1} b_{q-j}$. And (for admissible indices) $\Theta_{n+1} f \Theta_{m+1} g = \sum_{i,j} t^{i-1} \Theta_{n+1}^{ij} a_{j-1} \sum_{k,l} t^{k-1} \times \Theta_{m+1}^{kl} b_{l-1} = \sum_{i,j,k,l} t^{i+k-2} \Theta_{n+1}^{ij} \Theta_{m+1}^{kl} a_{j-1} b_{l-1} = A$. Denoting $p = i + k - 1$ and $q = j +$

$l-1$ $(p, q \in \{1, 2, \ldots, s\})$, we get $A = \sum_{p,l,j} \sum_i t^{p-1} \Theta_{n+1}^{ij} \Theta_{m+1}^{p-i+1,q-j+1} a_{j-1} b_{l-1} \overset{(30)}{=} \sum_{p,l,j} t^{p-1} \Theta_s^{pq} a_{j-1} b_{l-1} = \sum_{p,q} t^{p-1} \Theta_s^{pq} \sum_j a_{j-1} b_{q-j} = \sum_{p,q} t^{p-1} \Theta_s^{pq} c_{q-1} = \Theta_s(fg)$. □

Corollary 33. *If $\sigma = 1$, then Θ is a multiplicative involution.*

For any polynomial $f \in R_n$, we define the *dual* one f^* setting

$$f^*(t) = (-1)^n f(-t). \tag{37}$$

Obviously, if $f = a_0 + ta_1 + \cdots + t^n a_n$ $(a_n \neq 0)$ and $f^* = a_0^* + ta_1^* + \cdots + t^n a_n^*$, then $a_i^* = (-1)^{n-i} a_i$ and the correspondence $f \longmapsto f^*$ is an involution on R preserving the co-odd coefficients (in particular, the leading coefficient a_n) and changing signs of the co-even ones. For example, $((t-a)^n)^* = (t+a)^n$ for any $a \in K$. It holds also evidently:

$c^* = c$ for any $c \in K$,
$(fg)^* = f^* g^*$ for any polynomials f, g in R, in particular, $(fc)^* = f^* c$,
$(f+g)^* = f^* + g^*$ for polynomials f, g of equal parity degrees,
$(f^\alpha)^* = f^{*\alpha}$ for any automorphism α of K (where $f^\alpha = a_0^\alpha + ta_1^\alpha + \cdots + t^n a_n^\alpha$),
$f(a) = 0 \Leftrightarrow f^*(-a) = 0$.

Moreover, the following is true.

Lemma 34. *If $f = a_0 + ta_1 + \cdots + t^n a_n \in R_n\langle -1 \rangle$, then*

$$(\Theta_{n+1} f)^* = (-1)^n \Theta_{n+1}^* f^* \text{ or equivalently } \Theta_{n+1} f(t) = (\Theta_{n+1}^* f^*)(-t). \tag{38}$$

Proof. We have $\Theta_{n+1} f = \sum t^i b_i$ where $b_i = \sum_j \Theta_{n+1}^{i+1,j+1} a_j$ (with $i, j \in \{0, 1, \ldots, n\}$). On the other hand, if $\Theta_{n+1}^* f^* = \sum t^i c_i$, then (use (31)) $c_i = \sum_j \Theta_{n+1}^{*i+1,j+1} a_j^* = \sum_j (-1)^{n+i+j} \times \Theta_{n+1}^{i+1,j+1} (-1)^{n-j} a_j = (-1)^i \sum_j \Theta_{n+1}^{i+1,j+1} a_j = (-1)^i b_i$, i.e. $\Theta_{n+1} f(t) = (\Theta_{n+1}^* f^*)(-t)$. □

Lemma 34 and Corollaries 20 and 30(c) lead to some more consequences.

Corollary 35. *If $f = a_0 + ta_1 + \cdots + t^n a_n \in R_n\langle \pm 1 \rangle$, then*

(a) $(\Theta_{n+1} f)(-1) = (\Theta_{n+1}^* f^*)(1) = 2^n a_n$,
(b) $(\Theta_{n+1} f)(0) = (\Theta_{n+1}^* f^*)(0) = f(1) = (-1)^n f^*(-1)$,
(c) $(\Theta_{n+1} f)(1) = (\Theta_{n+1}^* f^*)(-1) = 2^n a_0 = 2^n f(0) = (-2)^n f^*(0)$.

Symmetrically to the Θ-transform (35), one can consider the dual Θ^*-*transform* of a polynomial $f = \sum t^i a_i \in R_n\langle 1 \rangle$ setting

$$\Theta^* f = (\Theta_{n+1}^* f) \frac{a_n}{f(1)}. \tag{39}$$

It follows from (38) and (39) that

$$\Theta^* f^* = (\Theta f)^*, \tag{40}$$

therefore complete analogs of Lemmas 31 and 32 for the function Θ^* hold. It means that $\Theta^* : R\langle 1 \rangle \longrightarrow R\langle 1 \rangle$ also is an involution preserving degrees and leading coefficients of polynomials, multiplicative on $R^0\langle 1 \rangle$, with the property

$$f(1)g(1) = 2^n a_n^2 \tag{41}$$

for each polynomial $f \in R_n\langle 1\rangle$ and $g = \Theta^* f$.

Remark that evidently $\Theta_2(t-a) = (1-a) - t(1+a)$ and $\Theta_2^*(t-a) = (1+a) + t(1-a)$ for any $a \in K$. Therefore, due to the multiplicative property of Θ (resp. Θ^*) with respect to polynomials in $R^0\langle -1\rangle$ (resp. $R^0\langle 1\rangle$), for any $n \geqslant 0$ and any $a \in K^0$ it holds

$$\begin{aligned}
&\text{(a) } \Theta(t-a)^n = \left(t - \frac{1-a}{1+a}\right)^n \quad \text{if } a \neq -1,\\
&\text{(b) } \Theta^*(t-a)^n = \left(t - \frac{a+1}{a-1}\right)^n \quad \text{if } a \neq 1,\\
&\text{(c) } \Theta(t-a)^n = \Theta^*(t+1/a)^n \quad \text{if } a \neq 0, -1.
\end{aligned} \tag{42}$$

In particular

$$\Theta(t-1)^n = \Theta^*(t+1)^n = t^n. \tag{43}$$

10. Θ-Transform and pencil representations

In this section, K is a field of characteristic $\neq 2$. Denote by $\mathfrak{M} = \cup_n \mathfrak{M}_n$ the set of all non-constant right monic polynomials over K (with $\mathfrak{M}_n$ being formed by polynomials of degree $n \geqslant 1$).

Let, as before, $C(f)$ be the companion matrix of type (3) of a polynomial $f \in \mathfrak{M}_n$. Since for the considered above diagonal matrix $D_n = \mathrm{diag}\{1, -1, \ldots, (-1)^{n-1}\}$ it holds $D_n C(f) D_n = -C(f^*)$, we have an ordinary similarity (over the prime subfield) of square matrices

$$C(f) \sim -C(f^*). \tag{44}$$

Let α, β be any automorphisms of K and $g = b_0 + tb_1 + \cdots + t^{n-1}b_{n-1} + t^n$ a polynomial in $\mathfrak{M}_n$. Denote by $\Omega(g) = \Omega(g, \alpha, \beta)$ a matrix representation of the (α, β)-pencil over K of the form

$$\Omega(g) = (I_n + C(g^\alpha), I_n - C(g^\beta))$$
$$= \left[\begin{array}{ccccc|c||ccccc|c}
1 & & & & & -b_0^\alpha & 1 & & & & & b_0^\beta \\
1 & 1 & & & & -b_1^\alpha & -1 & 1 & & & & b_1^\beta \\
 & \ddots & \ddots & & & \vdots & & \ddots & \ddots & & & \vdots \\
 & & & 1 & 1 & -b_{n-2}^\alpha & & & & -1 & 1 & b_{n-2}^\beta \\
 & & & & 1 & 1-b_{n-1}^\alpha & & & & & -1 & 1+b_{n-1}^\beta
\end{array}\right] \tag{45}$$

where the shown inside the blocks single vertical lines simply separate the last columns. Symmetrically set

$$\widetilde{\Omega}(g) = (I_n - C(g^\alpha), I_n + C(g^\beta))$$
$$= \left[\begin{array}{ccccc|c||ccccc|c}
1 & & & & & b_0^\alpha & 1 & & & & & -b_0^\beta \\
-1 & 1 & & & & b_1^\alpha & 1 & 1 & & & & -b_1^\beta \\
 & \ddots & \ddots & & & \vdots & & \ddots & \ddots & & & \vdots \\
 & & & -1 & 1 & b_{n-2}^\alpha & & & & 1 & 1 & -b_{n-2}^\beta \\
 & & & & -1 & 1+b_{n-1}^\alpha & & & & & 1 & 1-b_{n-1}^\beta
\end{array}\right] \tag{46}$$

Since (44) is a similarity over the prime subfield (which is stable under automorphisms of K), an equivalence of representations of the (α, β)-pencil holds

$$\Omega(g) \sim \widetilde{\Omega}(g^*). \tag{47}$$

Our key statement is as follows.

Theorem 36. *If $g = \Theta f$ is a polynomial in $\mathfrak{M}_n\langle -1\rangle$, then*

(a) $\Omega(g) \sim \widetilde{\Omega}(g^*) \sim (I_n, C(f^\beta)) \sim (I_n, -C(f^{*\beta}))$,
(b) $\widetilde{\Omega}(g) \sim \Omega(g^*) \sim (C(f^\alpha), I_n) \sim (-C(f^{*\alpha}), I_n)$.

Proof. Due to symmetry and (44), the formulas (a) and (b) are equivalent, hence it suffices to prove (a).

Set $g = b_0 + tb_1 + \cdots + t^{n-1}b_{n-1} + t^n$ and $g_1 = b_0 + tb_1 + \cdots + t^{n-1}b_{n-1}$. Clearly, one can rewrite the matrix $\Omega(g)$ as follows (the symbol | again symbolizes the shown in (45) separation of the last columns)

$$(A, B) = \Omega(g) = (I^{\downarrow}_{n-1} + I^{\uparrow}_{n-1}|e_n - [g_1^\alpha], I^{\downarrow}_{n-1} - I^{\uparrow}_{n-1}|e_n + [g_1^\beta]), \tag{48}$$

where $e_n = (0, \ldots, 0, 1)^{\mathrm{T}}$. Using the matrix $X_n = \begin{bmatrix} 2\Theta_{n-1} & 0^{\mathrm{T}}_{n-1} \\ 0_{n-1} & 1 \end{bmatrix}$ (with $X_1 = I_1$), we get by Corollary 21 (let ξ_n be the last column of Θ_n)

$$(\Theta_n A X_n^{-1}, \Theta_n B X_n^{-1}) = (A', B') = (I^{\downarrow}_{n-1}|\xi_n - v^\alpha, I^{\uparrow}_{n-1}|\xi_n + v^\beta),$$

where $v = (v_1, \ldots, v_n)^{\mathrm{T}} = \Theta_n[g_1]$, in particular $v_1 = b_0 + b_1 + \cdots + b_{n-1}$ and $v_n = b_0 - b_1 + \cdots + (-1)^{n-1}b_{n-1}$.

Let $Y_n = \begin{bmatrix} I_{n-1} & w \\ 0_{n-1} & 1 \end{bmatrix}$ where $w = (v_1 - \Theta_n^{1n}, \ldots, v_{n-1} - \Theta_n^{n-1,n})^{\mathrm{T}}$ (with $Y_1 = I_1$). Then

$$(A'Y_n^\alpha, B'Y_n^\beta) = \left[\begin{array}{ccccc|ccccc} 1 & & & & & & & & & c_0^\beta \\ & 1 & & & & 1 & & & & c_1^\beta \\ & & \ddots & & & & \ddots & & & \vdots \\ & & & 1 & & & & 1 & & c_{n-2}^\beta \\ & & & & -c_n^\alpha & & & & 1 & c_{n-1}^\beta \end{array}\right], \tag{49}$$

where, as one checks easily by the recursion formulas of Lemma 19, for the polynomial $f_1 = c_0 + tc_1 + \cdots + t^n c_n$ it holds precisely $f_1 = \Theta_{n+1} g$ and $c_n = g(-1)$. Since by Lemma 31 $g(-1) \neq 0$, the coefficients of the polynomial

$$f = f_1 \cdot (1/c_n) = (\Theta_{n+1} g) \cdot (1/g(-1)) = \Theta g$$

are $a_i = c_i/c_n$. Dividing the last columns of the blocks in (49) by $-c_n^\alpha$ and $-c_n^\beta$ respectively, we come finally to the equivalence

$$\Omega(g) \sim \left[\begin{array}{ccccc|ccccc} 1 & & & & & & & & & -a_0^\beta \\ & 1 & & & & 1 & & & & -a_1^\beta \\ & & \ddots & & & & \ddots & & & \vdots \\ & & & 1 & & & & 1 & & -a_{n-2}^\beta \\ & & & & 1 & & & & 1 & -a_{n-1}^\beta \end{array}\right] \tag{50}$$

In other words $\Omega(g) \sim (I_n, C(f^\beta))$. The proof is complete. $\square$

We shall write sometimes I instead of I_n, if no confusions.

Taking into account trivial facts that (for any square matrices A, B)

$$(I, A) \overset{(\alpha,\beta)}{\sim} (I, B) \Leftrightarrow A \sim \alpha^{-1}\beta B \quad \textit{and} \quad (A, I) \overset{(\alpha,\beta)}{\sim} (B, I) \Leftrightarrow A \overset{\beta^{-1}\alpha}{\sim} B$$

and also

$$A \overset{\lambda}{\sim} B \quad \Leftrightarrow \quad (A^{\mu}) \overset{\mu^{-1}\lambda\mu}{\sim} (B^{\mu}),$$

we get the following immediate consequence of Theorem 36.

Corollary 37. *Let α, β be arbitrary automorphisms of the field K.*

(a) *If $f = \Theta g$ and $f' = \Theta g'$ are polynomials in $\mathfrak{M}\langle -1\rangle$, then*
$\Omega(g) \overset{(\alpha,\beta)}{\sim} \Omega(g') \quad \Leftrightarrow \quad C(f) \overset{\beta\alpha^{-1}}{\sim} C(f'),$
$\widetilde{\Omega}(g) \overset{(\alpha,\beta)}{\sim} \widetilde{\Omega}(g') \quad \Leftrightarrow \quad C(f) \overset{\alpha\beta^{-1}}{\sim} C(f').$

(b) *If $f = \Theta^* g$ and $f' = \Theta^* g'$ are polynomials in $\mathfrak{M}\langle 1\rangle$, then*
$\Omega(g) \overset{(\alpha,\beta)}{\sim} \Omega(g') \quad \Leftrightarrow \quad C(f) \overset{\alpha\beta^{-1}}{\sim} C(f'),$
$\widetilde{\Omega}(g) \overset{(\alpha,\beta)}{\sim} \widetilde{\Omega}(g') \quad \Leftrightarrow \quad C(f) \overset{\beta\alpha^{-1}}{\sim} C(f').$

For an abstract set with an equivalence relation $(S, \sim)$, denote by $[x] = [x]_{\sim}$ the equivalence class of an element $x \in S$. For a subset $X \subset S$, set $[X] = [X]_{\sim} = \{[x] : x \in X\}$.

For a set of polynomials $X \subset \mathfrak{M}$, denote $C(X) = \{C(f) : f \in X\}$ and analogously define $\Omega(X)$ and $\widetilde{\Omega}(X)$. For an element $a \in K$, set for short $C\langle a\rangle = C(\mathfrak{M}\langle a\rangle)$ and similarly define $\Omega\langle a\rangle$ and $\widetilde{\Omega}\langle a\rangle$.

The main consequence of Theorem 36 and Corollary 37 is as follows.

Corollary 38. *The involutions Θ and Θ^* induce the bijections* (a) *and* (b) *between equivalence classes of representations*

(a) $[\Omega\langle -1\rangle]_{(\alpha,\beta)} \rightleftarrows [C\langle -1\rangle]_{\beta\alpha^{-1}}$ *and* $[\widetilde{\Omega}\langle -1\rangle]_{(\alpha,\beta)} \rightleftarrows [C\langle -1\rangle]_{\alpha\beta^{-1}}$,

(b) $[\widetilde{\Omega}\langle 1\rangle]_{(\alpha,\beta)} \rightleftarrows [C\langle 1\rangle]_{\beta\alpha^{-1}}$ *and* $[\Omega\langle 1\rangle]_{(\alpha,\beta)} \rightleftarrows [C\langle 1\rangle]_{\alpha\beta^{-1}}$,

as well as the bijections (a′) *and* (b′) *between equivalence classes of indecomposable representations*

(a′) $\mathrm{Ind}[\Omega\langle -1\rangle]_{(\alpha,\beta)} \rightleftarrows \mathrm{Ind}[C\langle -1\rangle]_{\beta\alpha^{-1}}$, $\mathrm{Ind}[\widetilde{\Omega}\langle -1\rangle]_{(\alpha,\beta)} \rightleftarrows \mathrm{Ind}[C\langle -1\rangle]_{\alpha\beta^{-1}}$,

(b′) $\mathrm{Ind}[\widetilde{\Omega}\langle 1\rangle]_{(\alpha,\beta)} \rightleftarrows \mathrm{Ind}[C\langle 1\rangle]_{\beta\alpha^{-1}}$ *and* $\mathrm{Ind}[\Omega\langle 1\rangle]_{(\alpha,\beta)} \rightleftarrows \mathrm{Ind}[C\langle 1\rangle]_{\alpha\beta^{-1}}$.

Next we want to investigate the structure of matrix indecomposables of the (α, β)-pencil $\mathcal{P}$. First pay attention to square block indecomposables.

Let σ be an automorphism of K and $K^0 \subset K$, $R = K[t, \sigma]$ and $R^0 = K^0[t, \sigma]$ the same as in Section 9.

For any $a \in K^0$, denote by $\mathfrak{I}_a = \mathfrak{I}\backslash\mathfrak{I}\langle a\rangle$ the subset of all polynomials in $\mathfrak{I}$ having a root a. Set

$$P_a = [(I, C(f)) : f \in \Im_a]_{(\alpha,\beta)}, \quad \widetilde{P}_a = [(C(f), I) : f \in \Im_a]_{(\alpha,\beta)}.$$

The structure of the sets P_0 and $\widetilde{P}_0$ is clear. Since each similar to t monic polynomial in R coincide with t (see for instance the equality (1.3.3) in [17]) and t is bounded in the sense of [3,15,16], in view of Theorem 6.5.4 in [3] we get $\Im_0 = \{t^n : n \geqslant 1\}$ and so

$$P_0 = [(I_n, J_n(0))] = [\Omega((t-1)^n)] = [\widetilde{\Omega}((t+1)^n)],$$
$$\widetilde{P}_0 = [(J_n(0), I_n)] = [\Omega((t+1)^n)] = [\widetilde{\Omega}((t-1)^n)],$$

where n runs through the positive integers inside the brackets.

Further, if $a \in K^0\backslash\{0\}$ then $\Im_a \cap \Im_0 = \emptyset$ and, for a polynomial $f \in \Im_a$ of degree n with the constant term c_0, the matrix $C(f)$ is non-singular. Then $(C(f))^{-1} = C^*(f') := (C(f'))^*$ where $f' = t^n f(1/t)c_0^{-1}$ and X^* denotes the dual matrix to a matrix X (defined just after Lemma 28 above for Θ_n).

Since an ordinary similarity $C(f') \sim C^*(f')$ over the prime subfield holds (realized by simultaneous row and column permutations), we get $(I, C(f)) \sim (C^*(f'), I) \sim (C(f'), I)$. Hence $P_a = \widetilde{P}_{1/a}$, so $P_{-1} = \widetilde{P}_{-1}$ and $P_1 = \widetilde{P}_1$.

Let $\mathrm{Ind}_{sq}\mathcal{P}(\alpha, \beta)$ be the set of all isomorphism classes of indecomposable matrix representations (A, B) of the (α, β)-pencil $\mathcal{P}$ with square blocks A, B. Since $\det(I_n \pm C(g^\alpha)) = 0$ if and only if $g(\mp 1) = 0$, Theorems 1 and 36 yield the following statement.

Corollary 39. *There hold equalities*

(a) $\mathrm{Ind}[\Omega\langle -1\rangle]_{(\alpha,\beta)} = \mathrm{Ind}[\widetilde{\Omega}\langle 1\rangle]_{(\alpha,\beta)}$,
(b) $\mathrm{Ind}_{sq}\mathcal{P}(\alpha, \beta) = \mathrm{Ind}[\Omega\langle -1\rangle] \cup P_{-1} \cup \widetilde{P}_0 = \mathrm{Ind}[\widetilde{\Omega}\langle 1\rangle] \cup \widetilde{P}_1 \cup P_0$,
(c) $(\mathrm{Ind}[\Omega\langle -1\rangle] \cup P_{-1}) \cap \widetilde{P}_0 = (\mathrm{Ind}[\widetilde{\Omega}\langle 1\rangle] \cup \widetilde{P}_1) \cap P_0 = \emptyset$.

It may happen $P_{-1} \subset \mathrm{Ind}[\Omega\langle -1\rangle]$ and $\widetilde{P}_1 \subset \mathrm{Ind}[\widetilde{\Omega}\langle 1\rangle]$ (see Section 11).

As for the non-square block indecomposables of the (α, β)-pencil (of type 2 and 3 in Theorem 1), they can be represented in some special form as well. Corollary 21 and the matrix transpose T imply

$$I_n^{\downarrow} = \Theta_{n+1}(I_n^{\downarrow} + I_n^{\uparrow})(2\Theta_n)^{-1}, \quad I_n^{\rightarrow} = ((2\Theta_n)^{\mathrm{T}})^{-1}(I_n^{\rightarrow} + I_n^{\leftarrow})\Theta_{n+1}^{\mathrm{T}},$$
$$I_n^{\uparrow} = \Theta_{n+1}(I_n^{\downarrow} - I_n^{\uparrow})(2\Theta_n)^{-1}, \quad I_n^{\leftarrow} = ((2\Theta_n)^{\mathrm{T}})^{-1}(I_n^{\rightarrow} - I_n^{\leftarrow})\Theta_{n+1}^{\mathrm{T}},$$

leading to the following fact (the permutation property of which is checked easily directly).

Lemma 40. *For an arbitrary (α, β)-pencil $\mathcal{P}$ it holds*

$$(I_n^{\downarrow}, I_n^{\uparrow}) \sim (I_n^{\downarrow} + I_n^{\uparrow}, I_n^{\downarrow} - I_n^{\uparrow}) \quad and \quad (I_n^{\rightarrow}, I_n^{\leftarrow}) \sim (I_n^{\rightarrow} + I_n^{\leftarrow}, I_n^{\rightarrow} - I_n^{\leftarrow}),$$

where one can permute the coordinates inside any bracket separately, as well as permute simultaneously the orientations of all arrows inside any bracket separately.

We resume the considerations above by the following statement.

Theorem 41. *For an arbitrary (α, β)-pencil $\mathcal{P}$, the set* $\mathrm{Ind}\,\mathcal{P}(\alpha, \beta)$ *is a disjoint union*

$$\mathrm{Ind}\,\mathcal{P}(\alpha, \beta) = E_0 \cup E^- \cup E^+,$$

where $E_0 = Ind_{sq}\mathcal{P}(\alpha, \beta)$ *is described in Corollary* 39 *and*

$$E^- = [(I_n^{\uparrow} + I_n^{\downarrow}, I_n^{\uparrow} - I_n^{\downarrow}) : n \geqslant 0], \quad E^+ = [(I_n^{\rightarrow} + I_n^{\leftarrow}, I_n^{\rightarrow} - I_n^{\leftarrow}) : n \geqslant 0].$$

11. Second canonical form

We return to the homogeneous biquadratic $G \underset{F}{\otimes} G$-problem, with a quadratic extension $G = F(u)$ of the basic field F of characteristic $\neq 2$, the minimal polynomial $t^2 + q$ of the element u and the conjugation $\sigma(x) = \overline{x}$ on G given by $\sigma(\alpha + \beta u) = \overline{\alpha + \beta u} = \alpha - \beta u$ (here $G^0 = F$).

We would like to obtain a simplified (in comparison with Theorem 15) canonical form for the $G \underset{F}{\otimes} G$-problem with respect to the canonical base (18) $w_{11}, \ldots, w_{uu}$ of the bimodule $W = G \underset{F}{\otimes} G$. Recall (see Section 6) that the notation $X \overset{(G,G)}{\sim} Y$ is used for equivalent $G \underset{F}{\otimes} G$-representations X, Y.

For any polynomial $g \in \mathfrak{M}_n$ over $K = G$, denote by $\Psi(g)$ a $G \underset{F}{\otimes} G$-representation of the form

$$\Psi(g) = w_{1u} I_n + w_{u1} C(g) = (1 \otimes u) I_n + (u \otimes 1) C(g). \tag{51}$$

Representing $C(g)$ in the form $C(g) = X + Yu$, where X, Y are over F, and taking into account the relations (19) and (20), we have

$$2\Psi(g) = (e_1 + e_3)I + (e_1 - e_3)(X + Yu) = e_1(I + X) + e_2 Y + e_3(I - X) + e_4 Y. \tag{52}$$

Denote $M = 2\Psi(g)$, then due to (52), for the attached pair of matrices (M', M'') (defined in Section 7 and corresponding to the $(1, \sigma)$-pencil problem over G), one gets an equivalence of $(1, \sigma)$-pencil representations

$$(M', M'') \sim (I + C(g), \quad I - C(\bar{g})) = \Omega(g). \tag{53}$$

Hence it holds.

Proposition 42. *The correspondence* $\Omega(g) \leftrightarrow \Psi(g)$ *induces mutually inverse bijections*

$$\mathrm{Ind}[\Omega(\mathfrak{M})]_{(1,\sigma)} \rightleftarrows \mathrm{Ind}[\Psi(\mathfrak{M})]_{(G,G)}.$$

As a consequence, owing to Corollary 38(a′), one obtains two pairs of mutually inverse bijections

$$\mathrm{Ind}[C\langle -1\rangle]_\sigma \rightleftarrows \mathrm{Ind}[\Omega\langle -1\rangle]_{(1,\sigma)} \rightleftarrows \mathrm{Ind}[\Psi\langle -1\rangle]_{(G,G)} \tag{54}$$

given by the correspondences

$$[C(f)] \leftrightarrow [\Omega(\Theta f)] \leftrightarrow [\Psi(\Theta f)].$$

The second pair of bijections in (54) is complemented by one more

$$\widetilde{P}_0 \rightleftarrows [\Psi((t+1)^n) : n \geqslant 1]_{(G,G)} \tag{55}$$

(clearly, $\Psi((t-a)^n) \overset{(G,G)}{\sim} (1 \otimes u) I_n + (u \otimes 1) J_n(a)$ for any $a \in F$).

Similarly, by Corollary 38(b′) we have bijections

$$\mathrm{Ind}[C\langle 1\rangle]_\sigma \rightleftarrows \mathrm{Ind}[\widetilde{\Omega}\langle 1\rangle]_{(1,\sigma)} \rightleftarrows \mathrm{Ind}[\Psi\langle 1\rangle]_{(G,G)} \tag{56}$$

induced by the correspondences

$$[C(f)] \leftrightarrow [\Omega(\Theta^* f)] \leftrightarrow [\Psi(\Theta^* f)].$$

The last pair of bijections in (56) is complemented by an additional one too (compare with Corollary 39)

$$P_0 \rightleftarrows [\Psi((t-1)^n) : n \geqslant 1]_{(G,G)}. \tag{57}$$

Further, since in the considered situation σ is an involution, for the matrix $H_n = uD_n = \mathrm{diag}\{u, -u, \dots, (-1)^{n-1}u\}$ we get $\overline{H_n^{-1}} = (u/q)D_n$. Hence, as one checks trivially, for any $f \in \mathfrak{M}_n$ it holds $H_nC(f)\overline{H_n^{-1}} = C(f^*)$ and thus $(I_n, C(f)) \overset{(1,\sigma)}{\sim} (I_n, C(f^*))$ as well as $(C(f), I_n) \overset{(1,\sigma)}{\sim} (C(f^*), I_n)$.

In particular, $P_a = P_{-a}$ for any $a \in F$. So, $P_1 = P_{-1}$ is a subset of both $\mathrm{Ind}[\Omega\langle -1\rangle]$ and $\mathrm{Ind}[\Omega\langle 1\rangle]$, due to the following fact.

Lemma 43. *If an automorphism σ of some field K is of finite order, then an indecomposable polynomial in $R = K[t, \sigma]$ cannot have different roots in K^0.*

Proof. By Theorem 15 of Chapter 3 in [16], each non-zero polynomial in R is bounded in the sense of [3,15,16]. Hence, by Corollary 2 of Proposition 6.5.7 in [3], each (left and right) non-constant factor of an indecomposable polynomial $f \in R$ is indecomposable.

Assume that $f(a) = f(b) = 0$ for different $a, b \in K^0$. Since a, b commute with t and $a \neq b$, we get $f(t) = g(t)(t-b) = h(t)(t-a)(t-b)$ obtaining a contradiction with the mentioned property since the factor $(t-a)(t-b)$ of f obviously is decomposable accordingly to the standard definitions in [3,15,16] (for $R(t-a) \cap R(t-b) = R(t-a)(t-b)$ and $R(t-a) + R(t-b) = R$). $\square$

We conclude (see Corollary 39(b)) that the set $\mathrm{Ind}_{sq}\mathcal{P}(\alpha, \beta)$ is presented in the considered case as disjoint unions

$$\mathrm{Ind}_{sq}\mathcal{P}(\alpha, \beta) = \mathrm{Ind}[\Omega\langle -1\rangle] \cup \widetilde{P}_0 = \mathrm{Ind}[\Omega\langle 1\rangle] \cup P_0. \tag{58}$$

Turning now to discrete representations, denote (for each $n \geqslant 0$)

$$\Delta_n^- = (1 \otimes u)I_n^{\uparrow} + (u \otimes 1)I_n^{\downarrow}, \quad \Delta_n^+ = (1 \otimes u)I_n^{\rightarrow} + (u \otimes 1)I_n^{\leftarrow} \tag{59}$$

and

$$\Delta^- = [\Delta_n^- : n \geqslant 0]_{(G,G)}, \qquad \Delta^+ = [\Delta_n^+ : n \geqslant 0]_{(G,G)}. \tag{60}$$

It can be easily shown that, for $M = 2\Delta_n^-$, the mentioned above attached pair of matrices (M', M'') takes the form $(M', M'') = (I_n^{\uparrow} + I_n^{\downarrow}, I_n^{\uparrow} - I_n^{\downarrow})$, i.e.

$$(M', M'') \sim (I_n^{\uparrow} + I_n^{\downarrow}, I_n^{\uparrow} - I_n^{\downarrow}).$$

Analogously, for $N = \Delta_n^+$, one gets

$$(N', N'') \sim (I_n^{\rightarrow} + I_n^{\leftarrow}, I_n^{\rightarrow} - I_n^{\leftarrow}).$$

Therefore, in the notations of Theorem 41, we have evident natural bijections

$$E^- \rightleftarrows \Delta^- \quad \text{and} \quad E^+ \rightleftarrows \Delta^+.$$

We obtain finally, as a consequence of Corollaries 14, 38 and 39, Proposition 42 and the formula (58), the desired (simplified with respect to the base (18)) classification of indecomposables for the considered $G \underset{F}{\otimes} G$-problem.

Theorem 44. *Let $G = F(u)$ be a quadratic extension of a field F in characteristic $\neq 2$, with $u^2 \in F$, and σ the natural involution on G given by $\sigma(a + bu) = a - bu$ $(a, b \in F)$. Then the set* $\operatorname{Ind} G \underset{F}{\otimes} G$ *of all isomorphism classes of indecomposable $G \underset{F}{\otimes} G$-representations is a disjoint union*

$$\operatorname{Ind} G \underset{F}{\otimes} G = \Delta_0 \cup \Delta^- \cup \Delta^+,$$

where Δ^-, Δ^+ are defined by (59), (60) *and Δ_0 is presented as disjoint unions*

$$\begin{aligned}\Delta_0 &= [\Psi(\Theta f) : f \in \Im\langle -1\rangle] \cup [\Psi((t+1)^n) : n \geqslant 1] \\ &= [\Psi(\Theta^* f) : f \in \Im\langle 1\rangle] \cup [\Psi((t-1)^n) : n \geqslant 1].\end{aligned}$$

We have just proved simultaneously Theorem 17 which obviously is a part of Theorem 44 (the last one contains also the dual variant of Theorem 17 with respect to the dual sequence Θ_n^*).

To present an example, we first repeat, for the convenience of the reader, the formulation and proof of one simple elementary fact from [27].

Lemma 45. *The relations*

$$\begin{cases} a = x^2 - y^2, \\ b = 2xy, \end{cases} \qquad \begin{cases} p = 2(x^2 + y^2 - 1)/((1+x)^2 + y^2), \\ q = ((1-x)^2 + y^2)/((1+x)^2 + y^2), \end{cases}$$

establish bijections between the values of three pairs of real parameters

$$(a, b) \rightleftarrows (x, y) \rightleftarrows (p, q)$$

satisfying respectively the restrictions (1) $b > 0$ *or* $b = 0$ *and* $a < 0$; (2) $x \geqslant 0$ *and* $y > 0$; (3) $p^2 - 4q < 0$ *and* $q \leqslant 1$.

Proof. Since $a + b\boldsymbol{i} = (x + y\boldsymbol{i})^2$, using polar coordinates $a = r\cos\varphi$, $b = r\sin\varphi$ $(0 < \varphi \leqslant \pi$ and $r > 0)$, we get $r = |a + b\boldsymbol{i}| = x^2 + y^2$, hence $p = 2(r-1)/(1+r+2x)$, $q = (1 + r - 2x)/(1 + r + 2x)$. One checks easily that $0 < q \leqslant 1$ and $p^2 - 4q < 0$ and $x = \sqrt{r}\cos(\varphi/2) = \sqrt{r(1+\cos\varphi)/2} = \sqrt{(r+a)/2}$, analogously $y = \sqrt{(r-a)/2}$. Using convenient notations $\alpha = 1 - q$, $\beta = \sqrt{4q - p^2}$, $\gamma = 1 + q - p$, one gets $a = (\alpha^2 - \beta^2)/\gamma^2$, $b = 2\alpha\beta/\gamma^2$, $x = \alpha/\gamma$ and $y = \beta/\gamma$ obtaining the desired bijections. Note that $\gamma > 0$, due to $\alpha \geqslant 0$ and $\beta > 0$. $\square$

Example 46. Consider the bimodule $\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C}$. Here σ is the ordinary complex conjugation, $u = \boldsymbol{i}$, and $\Delta_0 = [\Psi(\Theta f) : f \in \Im\langle -1\rangle] \cup [\Psi(t+1)^n : n \geqslant 1]$. To find Δ_0, one has to know the set $\Im\langle -1\rangle$ described by Proposition 6. Due to its equivalent form (Corollary 7), $\Delta_0 = \Delta_0' \cup \Delta_0'' \cup \Delta_0'''$ where

$$\begin{aligned}&\Delta_0' = [\Psi(\Theta(t-\alpha)^n) : \alpha \in \mathbb{R}^+], \quad \Delta_0'' = [\Psi(t+1)^n : n \geqslant 1], \\ &\Delta_0''' = [\Psi(\Theta(t^2 - 2xt + x^2 + y^2)^n) : x \geqslant 0, y > 0].\end{aligned}$$

From (42) we get $\Theta(t-\alpha)^n = (t+q)^n$ where $q = (\alpha - 1)/(\alpha + 1)$ and $q \in [-1, 1) \Leftrightarrow \alpha \in \mathbb{R}^+$. Hence $\Delta_0' = [\Psi((t+q)^n) : q \in [-1, 1), n \geqslant 1]$, so $\Delta_0' \cup \Delta_0'' = [\Psi((t+q)^n) : |q| \leqslant 1, n \geqslant 1]$.

Further, $\Theta(t^2 - 2xt + x^2 + y^2) = t^2 + pt + q$ where p, q are related with x, y as in Lemma 45. Thus $\Delta_0''' = [\Psi((t^2 + pt + q)^n) : n \geqslant 1, p^2 - 4q < 0, q \leqslant 1]$ and (since the condition $p^2 - 4q < 0$ is equivalent to irreducibility of a polynomial)

$$\Delta_0 = [\Psi(f) : f \in \mathfrak{S}],$$

where $\mathfrak{S}$ is the set of all polynomials of the form $f = g^n$, for $n \geqslant 1$ and g being monic real irreducible polynomial (having automatically the form $t + q$ or $t^2 + pt + q$) with the constant term q satisfying the restriction $|q| \leqslant 1$. We obtain the following result (its purely matrix version is Theorem 5).

Theorem 47. *The indecomposable* $\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C}$*-representations are exhausted, up to isomorphism, by the pairwise non-isomorphic representations of the following types corresponding to the sets* $\Delta_0, \Delta^-, \Delta^+$ *respectively*:

(a) $(1 \otimes \mathbf{i})I + (\mathbf{i} \otimes 1)C(f), \quad f \in \mathfrak{S},$
(b) $(1 \otimes \mathbf{i})I_n^{\downarrow} + (\mathbf{i} \otimes 1)I_n^{\uparrow}, \quad n \geqslant 0,$
(c) $(1 \otimes \mathbf{i})I_n^{\leftarrow} + (\mathbf{i} \otimes 1)I_n^{\rightarrow}, \quad n \geqslant 0.$

Notice, the case $n = 0$ in (b) and (c) corresponds to the formal "empty matrices" $M_{(1,0)}$ and $M_{(0,1)}$ mentioned in Remark 10.

Theorem 47 is equivalent to the main theorem in [8], as demonstrated earlier by the author partially (for the main type (a)) in [27] using some technical arguments (based on the results from [8]).

Recall that there were obtained in [8] the four types (i)–(iv) of matrix indecomposables for the $\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C}$-problem, and it appeared (as shown in [27]) that the types (iii) and (iv) can be united and (taken together) are equivalent to the type (a) of Theorem 47. Meanwhile the types (i) and (ii) correspond to the types (b) and (c).

While it was used originally in [8] a special functorial approach (based in particular on the results and methods developed in [7,21]), we have just managed (through the Θ-transformation and solution to the semilinear Kronecker Problem) to deduce Theorem 47 from the available description of indecomposable non-commutative polynomials over the ring $\mathbb{C}[t, ^-]$ (Proposition 6 and Corollary 7). Moreover, in view of (54), (56), there exists a possibility to realize the inverse procedure (and in more general situations) if one knows the indecomposables for the given $G \underset{F}{\otimes} G$-problem.

12. Applications to the representation theory

We are going to sketch now briefly, how the main results on representations of equipped posets (and also of equipped posets with involution) over the pair $(\mathbb{R}, \mathbb{C})$, obtained in [26,28,29], can be extended to the case of an arbitrary quadratic extension in characteristic $\neq 2$.

Let $F \subset F(u) = G$ be the same quadratic field extension as in Section 11, with the conjugation $\sigma(\alpha + \beta u) = \alpha - \beta u$, where $u^2 + q = 0$. Notice that the fields F, G automatically are infinite ($\operatorname{char} F \neq 2$).

Let $\mathcal{P}$ be an equipped poset in the sense of [28] (with single and double points and weak and strong order relations between them). Then one can define representations of $\mathcal{P}$ over the pair (F, G) in fact precisely as in [28], substituting the pair (F, G) for $(\mathbb{R}, \mathbb{C})$ (see in [28] the matrix definition on page 391 and the invariant one in Section 2).

Further, since in the proof of the main results in [28], a wider notion of representations of equipped posets with involution (in particular, with primitive involution) over the pair $(\mathbb{R}, \mathbb{C})$ is used, one can extend it naturally and analogously to [28] to the considered pair (F, G).

Then the definitions of equipped posets and equipped posets with involution of various representation types (tame, wild, of finite growth, m-parametric, etc.) are introduced analogously to [26,28,29] by formal substituting the pair (F, G) for $(\mathbb{R}, \mathbb{C})$ (in particular real and complex series of indecomposables are defined as in [28]).

Analyzing the system of the main differentiation algorithms VII–XVII for nontrivially equipped posets with involution over the pair $(\mathbb{R}, \mathbb{C})$ (used in [28] together with several more simple additional operations), as well as the main results obtained in [28,29] with help of them, one can conclude that they remain valid in the (F, G)-situation as well, with inessential modifications. This is confirmed by the following arguments:

- Various standard linear-algebraic considerations over the pair $(\mathbb{R}, \mathbb{C})$ are extended easily (actually without changes) to the case (F, G), if one substitutes F, G, u for $\mathbb{R}, \mathbb{C}, \boldsymbol{i}$ respectively.
- Each non-singular Jordan block over $\mathbb{C}$, involved in certain matrix considerations, has to be replaced by the non-singular companion matrix of type (3), while Jordan blocks of type $J_n(0)$ can be left without changes.
- The available classification of representations of the critical equipped poset $M_1 = \{\otimes\otimes\}$ over the pair $(\mathbb{R}, \mathbb{C})$ (used essentially in the construction of the algorithms XI, XIII, XV, XVI) is reduced easily to the $\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C}$-problem and therefore, as shown above, to the $(1, \sigma)$-pencil problem over the complex field $\mathbb{C}$. Analogously to this, one can get like the same classification of representations of M_1 over the generalized pair (F, G), by reducing it to the $(1, \sigma)$-pencil problem over the field G, or equivalently, to the problem of type (c) in Corollary 14. Hence there exists a firm base for extending the algorithms XI, XIII, XV, XVI to the case (F, G).
- The notion of the classical consimilarity over $\mathbb{C}$, used in the description of the algorithm XVII, is replaced by the generalized consimilarity over G, i.e. the task is reduced in fact to the $(1, \sigma)$-pencil problem over G.
- As for the rest of the algorithms VII–XVII, they are based (besides of standard linear-algebraic and matrix constructions) on using at most the ordinary pencil representation classification (for instance, representations of the critical equipped poset $L_1 = \{\otimes \circ \circ\}$, figured in the description of the algorithm XII, are reduced to representations of the ordinary pencil). Clearly, that classification is available over an arbitrary field.

Following the outlined scheme, one can obtain, verifying details, natural generalizations of the main Theorems A–D from [28] to the case of the pair (F, G) (actually without changing their formulations, due to a completely similar to the case $(\mathbb{R}, \mathbb{C})$ chosen system of definitions, as explained above). The combined result may be presented in the following way (supposing to be known the main definitions and notations in [28]).

Theorem 48. *Let $F \subset F(u) = G$ be a quadratic field extension in characteristic $\neq 2$. Then the following holds*:

(a) *An equipped poset $\mathcal{P}$ is tame* (*wild*) *over the pair* (F, G) *if its evolvent $\widehat{\mathcal{P}}$ is tame* (*wild*).

(b) *A reduced equipped poset with primitive involution $\mathcal{P}$ is tame* (*wild*) *over the pair* (F, G) *if its evolvent $\widehat{\mathcal{P}}$* (*with respect to the subset of all small points $\mathcal{P}^0$*) *is tame (wild).*

(c) *A reduced equipped poset with involution $\mathcal{P}$ is tame over the pair (F, G) if it satisfies two conditions*:

(1) *each its bidouble point is comparable with all other points*;

(2) *the evolvent $\widehat{\mathcal{P}_{\text{prim}}}$ of the subset $\mathcal{P}_{\text{prim}}$ (with respect to the subset of its small points) is tame*.

Otherwise $\mathcal{P}$ is wild over (F, G).

(d) *An equipped poset with involution $\mathcal{P}$ is tame over the pair (F, G) if each its bidouble point is comparable with all other points and the subset $\mathcal{P}_{\text{prim}}$ is tame. Otherwise $\mathcal{P}$ is wild over (F, G).*

Analogously, the main results from [29] (Theorems 1.1–1.3) and [26] (Theorems 1–3) can be extended to the case (F, G). In particular, the following result is expected.

Theorem 49. *Let $F \subset F(u) = G$ be a quadratic field extension in characteristic $\neq 2$ and $\mathcal{P}$ an equipped poset. Then $\mathcal{P}$ is of finite growth (m-parameter) over the pair (F, G) if and only if its evolvent $\widehat{\mathcal{P}}$ (with respect to the subset of small points) is so.*

It should be mentioned that there are two inessential (for the whole proof) gaps in the description of the differentiation algorithms X and XI in [28]. While a correction to the algorithm XI was given in [29], we would like, using an opportunity, to correct now the algorithm X.

Correction to Differentiation X. We outline here a modified corrected version of Differentiation X. Let $(\mathcal{P}, \Theta)$ be an equipped poset with involution. It means (see [28] for basic definitions and notations) that a poset $\mathcal{P}$ is given, and on the set of all its points an involution * is defined, with Θ being the set of the equivalence classes of points with respect to this involution.

A pair of incomparable points (a, b) in $\mathcal{P}$, where a is big and b is double, is called *X-suitable* (i.e. suitable for Differentiation X) if $\mathcal{P} = a^\triangledown + b_\vartriangle$. The *derived* equipped poset with involution $(\mathcal{P}', \Theta')$, with respect to the pair (a, b), is obtained from $(\mathcal{P}, \Theta)$ in the following way:

(a) the point a^* is replaced by a three-point chain $a^* < q < a_0$ where a^*, a_0 are big points and q is double;

(b) the point b is replaced by a two-point chain $b_0 < b$ where b_0 is big and b is double;

(c) an order relation $a < b_0$ is added;

(d) Θ' is obtained from Θ by adding two new classes: a non-trivial one $\{a_0, b_0\}$ and a trivial one $\{q\}$.

Naturally, all the order relations induced by those in $\mathcal{P}$ and by the mentioned above are added as well.

Set $A = a^\triangledown \backslash a$, $B = b_\vartriangle \backslash b$ in $\mathcal{P}$ and $\widehat{a} = a^\triangledown \backslash a$, $B' = \mathcal{P}' \backslash a^\triangledown$ in $\mathcal{P}'$. Let $U = (U_0; U_K : K \in \Theta)$ be a representation of the set $(\mathcal{P}, \Theta)$, where U_0 is a finite-dimensional $\mathbb{R}$-space. Considering an ordered direct sum $U_0^2 = U_0 \oplus U_0$, one can define (similarly to the definitions in [28]) the *coupling* of a sequence of n subspaces $X_1, \ldots, X_n \subset U_0^2$ being a subspace in U_0^2 of the form

$$[X_1 - X_2 - \cdots - X_n] = \{(t_0, t_n) : (t_{i-1}, t_i) \in X_i \text{ for some } t_i\}.$$

Denote $\mathscr{R} = \{(\mathcal{P}, \Theta) - sp : a^+ \subset b^+, b^- = B^+\}$ (in the conditional notations of [28]) and $\mathscr{R}' = \{(\mathcal{P}', \Theta') - sp : a^+ \subset (B')^+, a_0^- = q^+, b^- = b_0^+\}$. The action of the differentiation functor $\prime : \mathscr{R} \longrightarrow \mathscr{R}'$ is then defined for an object $U \in \mathscr{R}$ by the formulas

$$\begin{aligned}
&U'_0 = U_0, \quad U'_b = U_b + \widetilde{U_a^+},\\
&U'_{(a_0,b_0)} = [U_{(a^*,a)} - U_b] + (0, U_a^+),\\
&U'_q = [U_{(a^*,a)} - U_b - U_{(a,a^*)}],\\
&U'_{(a,a^*)} = U_{(a,a^*)} \cap (U_B^+, U_0),\\
&U'_K = U_K \text{ for the remaining classes } K \in \Theta',
\end{aligned} \tag{61}$$

and also by the equality $\varphi' = \varphi$ for any morphism $\varphi : U_0 \to V_0$ (considered as a linear map).

If (E_0, W_0) is a complementing pair of subspaces in U_0, with respect to the pair (U_a^+, U_B^+), then the *reduced derived* representation $U^\downarrow$ is defined (uniquely up to isomorphism) by the equality $U' = U^\downarrow \oplus P^m(\widehat{a})$, where $m = \dim E_0 = \dim(U_a^+ + U_B^+)/U_B^+$. Its evident form is $U^\downarrow = W$, with W_0 taken from the complementing pair, and $W_K = U'_K \cap W_0^K$.

Obviously $G'_1(b, a) = P(\widehat{a}) \oplus P(b_0)$ and $G'_2(b, a) = P^2(\widehat{a})$, hence $G_1^\downarrow(b, a) = P(b_0)$ and $G_2^\downarrow(b, a) = 0$.

Let W be an object in $\mathscr{R}'$. To construct the *primitive* object $W^\uparrow \in \mathscr{R}'$, represent the spaces $W_{(a_0,b_0)}, W_q, W_b$ respectively in the form

$$\begin{aligned}
&W_{(a_0,b_0)} = \underline{W}_{(a_0,b_0)} \oplus F_1, \quad F_1 = \{(f_{11}, f'_{11}), \ldots, (f_{1p_1}, f'_{1p_1})\},\\
&W_q = \widetilde{W_{a^*}^+} \oplus F_2, \quad F_2 = \{(f_{21}, f'_{21}), \ldots, (f_{2p_2}, f'_{2p_2})\},\\
&W_b = \widetilde{W_{b_0}^+} \oplus H,
\end{aligned} \tag{62}$$

where F_i, H are some complements with the shown bases for F_i. Consider a new $\mathbb{R}$-space E_0 with a base $\{e_{11}, \ldots, e_{1p_1}\} \cup \{e_{21}, e'_{21}, \ldots, e_{2p_2}, e'_{2p_2}\}$ of dimension $m = p_1 + 2p_2$. Then set $W^\uparrow = U$ where

$$\begin{aligned}
&U_0 = W_0 \oplus E_0,\\
&\dot{U}_K = W_K \oplus E_0^{K\cap A} \quad for\ K \neq \{a, a^*\}, \{b\},\\
&\dot{U}_{(a,a^*)} = W_{(a,a^*)} + \{(e_{11}, f_{11}), \ldots, (e_{1p_1}, f_{1p_1})\}\\
&\qquad + \{(e_{2j}, f_{2j}), (e'_{2j}, f'_{2j}) : j = 1, \ldots, p_2\},\\
&\dot{U}_b = \widetilde{W_{B'}^+} + \{(e_{11}, f'_{11}), \ldots, (e_{1p_1}, f'_{1p_1})\} + H.
\end{aligned} \tag{63}$$

The desired isomorphisms $(U^\downarrow)^\uparrow \simeq U$ for a *reduced* object $U \in \mathscr{R}$ (not having direct summands $G_2(b, a)$) and $(W^\uparrow)^\downarrow \simeq W$ for a *reduced* object $W \in \mathscr{R}'$ (not having direct summands $P(\widehat{a})$) can be verified by standard linear-algebraic considerations analogous to those used for other algorithms in [28]. When verifying, one can be helped by the matrix interpretation of the algorithm (in the style of [28]) obtained by complete reducing the horizontal stripe, corresponding to some complement to U_B^+, at points a, b (that matrix form reflects a decomposition of an arbitrary representation of the subset $\{a, b\} = \{\circ\otimes\}$, satisfying the restrictions of the category $\mathscr{R}$, into a direct sum of possible indecomposables of types $P(\varnothing), D(b), G_1(b, a), G_2(b, a)$).

This way leads to the following result (which corrects and improves Theorem 7.1 in [28]).

Theorem 50. *In the case of Differentiation X, the operations $\downarrow$ and $\uparrow$ induce mutually inverse bijections*

$$\operatorname{Ind} \mathscr{R}\backslash G_2(b, a)\} \rightleftarrows \operatorname{Ind} \mathscr{R}'\backslash P(\widehat{a}).$$

Notice that the described correction has none negative after-effects to the rest of considerations in [28], due to Lemmas 16.4 and 16.5 there (compare with the combinatorial definition (a)–(d) of the derived poset $(\mathcal{P}', \Theta')$ above).

Appendix A. Matrices Θ_n for $n \leqslant 9$

1

1

2

1	1
1	−1

3

1	1	1
2	0	−2
1	−1	1

4

1	1	1	1
3	1	−1	−3
3	−1	−1	3
1	−1	1	−1

5

1	1	1	1	1
4	2	0	−2	−4
6	0	−2	0	6
4	−2	0	2	−4
1	−1	1	−1	1

6

1	1	1	1	1	1
5	3	1	−1	−3	−5
10	2	−2	−2	2	10
10	−2	−2	2	2	−10
5	−3	1	1	−3	5
1	−1	1	−1	1	−1

7

1	1	1	1	1	1	1
6	4	2	0	−2	−4	−6
15	5	−1	−3	−1	5	15
20	0	−4	0	4	0	−20
15	−5	−1	3	−1	−5	15
6	−4	2	0	−2	4	−6
1	−1	1	−1	1	−1	1

8

1	1	1	1	1	1	1	1
7	5	3	1	−1	−3	−5	−7
21	9	1	−3	−3	1	9	21
35	5	−5	−3	3	5	−5	−35
35	−5	−5	3	3	−5	−5	35
21	−9	1	3	−3	−1	9	−21
7	−5	3	−1	−1	3	−5	7
1	−1	1	−1	1	−1	1	−1

9

1	1	1	1	1	1	1	1	1
8	6	4	2	0	−2	−4	−6	−8
28	14	4	−2	−4	−2	4	14	28
56	14	−4	−6	0	6	4	−14	−56
70	0	−10	0	6	0	−10	0	70
56	−14	−4	6	0	−6	4	14	−56
28	−14	4	2	−4	2	4	−14	28
8	−6	4	−2	0	2	−4	6	−8
1	−1	1	−1	1	−1	1	−1	1

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