Distances between critical points and midpoints of zeros of hyperbolic polynomials

Dimitar K. Dimitrov a,*, Vladimir P. Kostov b

a Departamento de Ciências de Computação e Estatística, IBILCE, Universidade Estadual Paulista,
15054-000 São José do Rio Preto, SP, Brazil
b Université de Nice, Laboratoire de Mathématiques, Parc Valrose, 06108 Nice, Cedex 2, France

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Abstract

Let \( p(x) \) be a polynomial of degree \( n \) with only real zeros \( x_1 \leq x_2 \leq \cdots \leq x_n \). Consider their midpoints \( z_k = (x_k + x_{k+1})/2 \) and the zeros \( \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n-1} \) of \( p'(z) \). Motivated by a question posed by D. Farmer and R. Rhoades, we compare the smallest and largest distances between consecutive \( \xi_k \) to the ones between consecutive \( z_k \). The corresponding problem for zeros and critical points of entire functions of order one from the Laguerre–Pólya class is also discussed.

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* Corresponding author.

E-mail addresses: dimitrov@ibilce.unesp.br (D.K. Dimitrov), kostov@math.unice.fr (V.P. Kostov).
1. Introduction

Let \( p_n(x) = a_0 + a_1x + \cdots + a_nx^n, \ a_n \neq 0, \) be an algebraic polynomial with real coefficients \( a_j \) whose zeros \( x_1, \ldots, x_n \) are all real. In what follows, such polynomials will be called \emph{hyperbolic}. Suppose that \( x_1 \leq x_2 \leq \cdots \leq x_n \) and let \( z_k = (x_k + x_{k+1})/2 \) be the midpoints of the zeros of \( p(x) \). Set \( \tilde{p}(x) := (x - z_1) \cdots (x - z_{n-1}) \). Let \( \xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n-1} \) be the zeros of \( p'(z) \), i.e. the critical points of \( p(x) \). Denote by \( m(p), m(\tilde{p}) \) and \( m(p') \) the smallest distances between consecutive terms of the sequences \( \{x_k\}, \{z_k\} \) and \( \{\xi_k\} \), respectively, that is

\[
m(p) = \min\{x_{k+1} - x_k : k = 1, \ldots, n - 1\}, \quad m(\tilde{p}) = \min\{z_{k+1} - z_k : k = 1, \ldots, n - 2\}, \quad m(p') = \min\{\xi_{k+1} - \xi_k : k = 1, \ldots, n - 2\}.
\]

Similarly, we denote by \( M(p), M(\tilde{p}) \) and \( M(p') \) the corresponding maximal distances between consecutive zeros of \( p, \tilde{p} \) and \( p' \). The same notation will be used for entire functions with only real zeros with the convention that, instead of minimums and maximums, we shall consider infimums and supremums, whenever they are well defined.

In this paper we study the relation between \( m(p') \) and \( m(\tilde{p}) \), as well as the one between \( M(p') \) and \( M(\tilde{p}) \). We are motivated by a classical result of Marcel Riesz and by a question concerning the latter quantities for real entire functions of order one which possess only real zeros, posed very recently by David Farmer and Robert Rhoades. We recall that the set of real entire functions of order one which possess only real zeros with the convention that, instead of minimums and maximums, we shall consider infimums and supremums, whenever they are well defined.

In order to state formally the conjecture of Farmer and Rhoades [4], recall that every real entire function \( \phi(x) \) from to the Laguerre–Pólya class, written \( \phi \in \mathcal{LP} \), can be represented in the form

\[
\phi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left( 1 + \frac{x}{x_k} \right) e^{-\frac{x}{x_k}},
\]

where \( c, \beta, x_k \in \mathbb{R}, \alpha \geq 0, m \in \mathbb{N} \cup \{0\}, \sum x_k^{-2} < \infty \). It is known that \( \mathcal{LP} \)-functions, and only these, are uniform limits on compact subsets of the complex plane (locally uniform limits) of polynomials with real zeros only.

The Laguerre–Pólya class consists of entire functions of order at most two. Motivated by the behaviour of the zeros of the Riemann \( \xi \)-function and of its derivatives, Farmer and Rhoades...
[4] extended Riesz’ result to entire functions which belong to the subclass \( \mathcal{L} \mathcal{P} 1 \subset \mathcal{L} \mathcal{P} \) of entire functions of order one. It was proved in [4] (see Theorem 2.3.1 there) that for any \( f \in \mathcal{L} \mathcal{P} 1 \) with zeros \( x_k \), arranged in increasing order, if \( \xi < \eta \) are consecutive zeros of \( f' + af \), \( a \in \mathbb{R} \), then

\[
\inf \{ x_{k+1} - x_k \} \leq \eta - \xi \leq \sup \{ x_{k+1} - x_k \}.
\]

Moreover, if the zeros of \( f \) are simple and equality holds for one of the inequalities, it also holds for the other, and \( f(z) = Ae^{Bz} \cos(Cz + D) \) for some real constants \( A, B, C, \) and \( D \). After establishing various interesting results on the distribution of zeros of entire functions from \( \mathcal{L} \mathcal{P} 1 \) and applying them to the Riemann \( \xi \)-function, the authors of [4] formulate Conjecture 5.1.1 concerning inequalities between \( m(p') \) and \( m(\tilde{p}) \). It reads as follows:

**Conjecture 1.** Suppose that \( f \in \mathcal{L} \mathcal{P} 1 \) and that its zeros \( x_k \) are listed in increasing order. If \( \xi < \eta \) are consecutive zeros of \( f' \), then

\[
\inf \left\{ \frac{x_{k+2} - x_k}{2} \right\} \leq \eta - \xi \leq \sup \left\{ \frac{x_{k+2} - x_k}{2} \right\}. \tag{2}
\]

Observe that the quantities \( (x_{k+2} - x_k)/2 \) coincide with the distances \( z_{k+1} - z_k \) between consecutive midpoints of zeros of \( f \). The results of the present paper are inspired by this conjecture. We discuss it both for hyperbolic polynomials and entire functions in \( \mathcal{L} \mathcal{P} 1 \) and prove that the left-hand side inequality holds for polynomials of degree three and four. However, for each of the two inequalities, we construct hyperbolic polynomials of degree higher than four and entire functions from \( \mathcal{L} \mathcal{P} 1 \) for which it fails. The right-hand side inequality fails also for polynomials of degree three and four. In the next two sections we discuss the polynomial case. Section 4 deals with examples for entire functions. In Section 5 we discuss other results related to Rolle’s theorem and the positions of critical points of polynomials and of the so-called pseudopolynomials.

2. Results concerning hyperbolic polynomials

The exhaustive answer to Conjecture 1 is given by the following two theorems. The first one concerns the left-hand side inequality.

**Theorem 1.**

1. For \( n = 3 \) and 4 one has \( m(p') \geq m(\tilde{p}) \) for every hyperbolic polynomial \( p \) without a triple root.
2. For each \( n \geq 5 \) there are examples of such polynomials \( p \) with \( m(p') > m(\tilde{p}) \) and examples with \( m(p') < m(\tilde{p}) \).

The second theorem concerns the right-hand side inequality. It tells also how the two inequalities can be combined in the following sense. We denote by \( L - R+ \) the case when the left inequality in (2) fails while the right one holds; in a similar way we define the cases \( L - R-, \) \( L + R- \) and \( L + R+ \).

**Theorem 2.**

1. For \( n = 3 \) the right inequality in (2) fails for every hyperbolic polynomial without a triple root.
For every $n \geq 4$ there are examples of strictly hyperbolic polynomials for which this inequality fails.

For every $n \geq 4$ there are examples of strictly hyperbolic polynomials for which this inequality holds.

For $n \geq 7$ there are examples of strictly hyperbolic polynomials realizing any of the four cases $L - R -$, $L - R +$, $L + R -$ and $L + R +$.

For $n = 3, 4, 5$ and $6$ the following table tells which cases are realizable by strictly hyperbolic polynomials and which are not, with the exception of two of them (indicated by interrogation marks) for which the answer is unknown.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$L + R+$</th>
<th>$L + R-$</th>
<th>$L - R+$</th>
<th>$L - R-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>4</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>5</td>
<td>yes</td>
<td>yes</td>
<td>?</td>
<td>yes</td>
</tr>
<tr>
<td>6</td>
<td>yes</td>
<td>yes</td>
<td>?</td>
<td>yes</td>
</tr>
</tbody>
</table>

The theorems are proved in Section 3.

Remark 1. For the degree 4 polynomial $s(x) := (x^2 - 1)^2$ one has $m(s') = m(\tilde{s}) = M(s') = M(\tilde{s}) = 1$. This example and Theorem 1 show that for $n = 4$ the exact lower bound of the ratio $\kappa := m(p')/m(\tilde{p})$ equals 1. To compute this lower bound for $n = 3$ it suffices to consider the one-parameter family of polynomials $t_a(x) := x(x - 1)(x - a), a \in [1, \infty)$, whose critical points are $(1 + a \pm \sqrt{1 - a + a^2})/3$. Thus

$$m(\tilde{t}_a) = M(\tilde{t}_a) = \frac{a}{2} < \frac{2\sqrt{1 - a + a^2}}{3} = m(t'_a) = M(t'_a).$$

The lower bound of the ratio $\kappa$ equals $2/\sqrt{3}$. It is attained only for $a = 2$. Its upper bound equals $4/3$ and is attained only for $a = 1$ and when $a \to \infty$.

Example 1. Consider the hyperbolic degree 5 polynomial

$$f(x) := x^2(x + 1)^2(x - 2) = x^5 - 3x^3 - 2x^2.$$

Its roots equal $-1, -1, 0, 0, 2$, its derivative is $f'(x) = 5x^4 - 9x^2 - 4x$, with roots $-1, 1$ and $(5 \pm \sqrt{45})/5$. Then

$$m(f') = \frac{1}{2} > \frac{\sqrt{45} - 5}{5} = m(f') \quad \text{and} \quad M(f') = 1 < \frac{5 + \sqrt{45}}{5} = M(f').$$

Hence, the polynomial $f$ realizes the case $L - R -$. One can perturb $f$ to obtain a strictly hyperbolic polynomial $u$ of degree 5 realizing the case $L - R -$.

Example 2. Consider the hyperbolic degree 5 polynomial

$$g(x) := x(x^2 - 1)^2 = x^5 - 2x^3 + x.$$

Its roots equal $-1, -1, 0, 1, 1$. Then $g'(x) = 5x^4 - 6x^2 + 1$, the roots of $g'(x)$ are $\pm 1, \pm 1/\sqrt{5}$. Thus,

$$m(g') = \frac{1}{2} < 1 - \frac{1}{\sqrt{5}} = m(g') \quad \text{and} \quad M(g') = 1 > \frac{2}{\sqrt{5}} = M(g').$$
Hence, the polynomial \( g \) realizes the case \( L + R^+ \). By perturbing \( g \) one can obtain a strictly hyperbolic polynomial \( v \) of degree 5 realizing the case \( L + R^+ \).

**Example 3.** Consider the strictly hyperbolic degree 6 polynomial

\[
d(x) := (x^2 - 9)(x^2 - 16)(x^2 - 25) = x^6 - 50x^4 + 769x^2 - 3600.
\]

The roots of \( d'(x) = 6x^5 - 200x^3 + 1538x \) equal 0, ±3.469272547 and ±4.614919428. One has \( m(\tilde{d}) = 1, M(\tilde{d}) = 7/2, m(d') > 1.14 > 1 = m(\tilde{d}), M(d') < 3.47 < M(\tilde{d}) \). This means that \( d \) realizes the case \( L + R^+ \) for \( n = 6 \).

**Example 4.** Next we study the zeros of the strictly hyperbolic polynomial

\[
g(x) := x(x + 1) \cdots (x + n).
\]

Obviously \( m(g) = M(g) = 1 \). Consider the critical points \( \xi_k \) of \( g(x) \), arranged in decreasing order, \( \xi_n < \xi_{n-1} < \cdots < \xi_1 < 0 \). Then

\[
g'(\xi_k) \over g(\xi_k) = 1 \over \xi_k + 1 \cdots \over \xi_k + n = 0, \quad k = 1, \ldots, n.
\]

Hence

\[
\over g'(\xi_k + 1) \over g(\xi_k + 1) = \over g'(\xi_k) \over g(\xi_k) - 1 \over \xi_k + 1 \over \xi_k + n + 1 = - n + 1 \over \xi_k (\xi_k + 1).
\]

Since \(-k < \xi_k < -k + 1 \), then \( \text{sign}(g(\xi_k + 1)) = (-1)^{k+1} \) which yields \( \text{sign}(g'(\xi_k + 1)) = (-1)^{k+1} \). The fact that, for each \( k = 2, \ldots, n \), we have \( \text{sign}(g'(x)) = (-1)^{k+1} \) in the interval \( (\xi_k, \xi_{k-1}) \) implies \( \xi_k + 1 < \xi_{k-1} \), for \( k = 2, \ldots, n \). In other words, the inequalities \( \xi_{k-1} - \xi_k > 1 \) are true for \( k = 2, \ldots, n \). Thus, \( M(g') > 1 = M(\tilde{g}) \) and \( m(g') > 1 = M(\tilde{g}) \). Therefore in this case the left-hand side inequality (2) holds while the right-hand side one fails for polynomials of degree greater than three. This already provides an example corresponding to the second column of the table in the statement of Theorem 2.

### 3. Proofs

**Proof of Theorem 1.** 1°. Prove part (2) first. For \( n = 5 \) the proof follows from Examples 1 and 2, see the polynomials \( u \) and \( v \) there. Consider the degree 6 polynomial \( u_1 := u(x)(x - b) \) (resp. \( v_1 := v(x)(x - b) \)). For \( b > 0 \) large enough it is strictly hyperbolic and one has \( m(u_1) = m(\tilde{u}) \) (resp. \( m(\tilde{v}) = m(\tilde{v}) \)), the four smallest roots of \( u'_1 \) (resp. \( v'_1 \)) are close to the respective roots of \( u' \) (resp. \( v' \)) while the fifth one is “large”. Therefore for \( b > 0 \) large enough the quantity \( m(u'_1) \) (resp. \( m(v'_1) \)) is defined by the four smallest roots and \( \lim_{b \to \infty} m(u'_1) = m(u'), \lim_{b \to \infty} m(v'_1) = m(v') \), hence, for large values of \( b \) one has \( m(u_1) > m(u'_1) \) (resp. \( m(\tilde{v}) < m(v'_1) \)). By replacing in the above reasoning \( u \) and \( v \) by \( u_1 \) and \( v_1 \), one constructs analogs of Examples 1 and 2 for \( n = 7 \), then in the same way for \( n = 8, 9, \ldots \).

2°. For \( n = 3 \) the proof of part (1) of the theorem is in fact contained in Remark 1. Indeed, up to an affine transformation of the coordinate on the \( x \)-axis every monic hyperbolic polynomial of degree 3 and different from \( x^3 \) (this case is vacuous) is contained in the family \( t_0 \).

3°. For \( n = 4 \) we consider only the generic case when \( p \) is strictly hyperbolic. For hyperbolic polynomials with at most double roots the result follows from the generic case by continuity.

4°. We present on Fig. 1 the graphs of \( p, p' \) and \( p'' \) (drawn one above the other). The points \( A, N, Q \) (resp. \( H, U, T \), resp. \( R, S, V \)) have \( x \)-coordinate \( \xi_3 \) (resp. \( \xi_2 \), resp. \( x_4 \),...).
resp. \( x_3 \), resp. \( x_2 \)). The \( x \)-coordinates of the points \( B, C, D \) are equal. The same is true for the points \( E, F, G \).

The point \( J \) is the inflection point and the center of symmetry of the graph of \( p' \). We consider in \( 6^\circ - 8^\circ \) the case when it is above the \( x \)-axis. In this case one has \( \| HA \| < \| KH \| \). The case when \( J \) is below the axis can be considered by analogy. If this point is on the \( x \)-axis, then \( x_4 - x_3 = x_2 - x_1 \), the polynomial \( p \) (up to a shift of the origin on the \( x \)-axis) is even and it suffices to consider the case when it is of the form \( p = (x^2 - 1)(x^2 - a^2), \ a \geq 1 \). By direct computation one finds that \( m(\tilde{p}) = (a + 1)/2 < \sqrt{(a^2 + 1)/2} = m(p') \). Notice that the converse is also true – if one has \( x_4 - x_3 = x_2 - x_1 \), then the polynomial \( p \) (up to a shift of the origin on the \( x \)-axis) is even.

\( 5^\circ \). Present for \( x \) close to \( \xi_3 \) the value of \( p \) as \( p(\xi_3) + \int_{\xi_3}^x p'(t) \, dt \). Geometrically the absolute value of the last integral is equal to the area \( \Sigma(ABD) \) or \( \Sigma(AFE) \) of the curvilinear triangle \( ABD \) or \( AFE \) depending on whether \( x > \xi_3 \) or \( x < \xi_3 \). Suppose that \( \| AB \| = \| EA \| \). Then \( S(ABD) > S(ABC) = S(AEG) > S(AEF) \) which follows from the convexity of the graph of \( p' \) for \( x > \xi_2 \). This means that a point which follows the graph of \( p \) starting from the point \( Q \), reaches the \( x \)-axis faster when going to the right than when going to the left. Hence,

\[
\| NR \| < \| SN \|. 
\] (3)

\( 6^\circ \). In a similar way one shows that

\[
\| TS \| > \| VT \|. 
\] (4)
To prove this, the convexity of the graph of $p'$ cannot be used because of the presence of the inflection point $J$. Nevertheless it is true that if the numbers $c \in (\xi_1, \xi_2)$ and $d \in (\xi_2, \xi_3)$ are such that $\xi_2 - c = d - \xi_2$, then
\[
|p'(c)| > |p'(d)|.
\] (5)

Indeed, denote by $\zeta$ the $x$-coordinate of the point $J$. The graph of $p''$ is a parabola symmetric with respect to the vertical line $(x = \zeta)$. Hence, the slope of the graph of $p'$ decreases for $x \leq \zeta$ and increases for $x \geq \zeta$. Therefore
\[
p''(c_1) < p''(d_1)
\] (6)
for any $c_1 \in (\xi_1, \xi_2)$ and $d_1 \in (\xi_2, \xi_3)$ such that $\xi_2 - c_1 = d_1 - \xi_2$. Integrating inequality (6) from $\xi_2$ to $c$ and from $\xi_2$ to $d$ implies (5). Hence, a point which follows the graph of $p$ starting from the point $U$, reaches the $x$-axis faster when moving to the left than when moving to the right. This proves (4).

7°. Inequalities (3) and (4) imply $\|TN\| > \|VR\|/2$, i.e. $\xi_3 - \xi_2 > (x_4 - x_2)/2$. On the other hand, $\xi_3 - \xi_2 < \xi_2 - \xi_1$. Hence $\xi_3 - \xi_2 = m(p')$. We prove in $8°$ that
\[
x_4 - x_2 < x_3 - x_1.
\] (7)
Thus, $(x_4 - x_2)/2 = m(\tilde{p}) < \xi_3 - \xi_2 = m(p')$. This proves part (1) for $n = 4$.

8°. There remains to prove inequality (7) which is equivalent to $x_4 - x_3 < x_2 - x_1$. When $\xi_2 = \xi_3$, we must also have $x_2 = x_3 = x_4$ and the inequality is evident. Recall that when the inflection point $J$ is not on the $x$-axis, then we cannot have $x_4 - x_3 = x_2 - x_1$, see $4°$. Vary continuously the values of $\xi_1, \xi_2, \xi_3$ and $p(\xi_2)$ starting from $\xi_2 = \xi_3$. Hence, the coefficients and the zeros of $P$ vary continuously. The sign of the inequality between $x_4 - x_3$ and $x_2 - x_1$ must remain the same as equality between them is impossible. Hence, this sign is the same as for $\xi_2 = \xi_3$, i.e. (7) holds. □

**Proof of Theorem 2.** 1°. To prove part (1) it suffices to consider the polynomial $t_n$ from Remark 1.

2°. Prove part (2). Consider for $n \geq 3$ the hyperbolic polynomial $w := x^{n-1}(x - n)$. Show that the right inequality in (2) fails for some strictly hyperbolic polynomial close to it. Indeed, the polynomial $w$ has two critical points at $0$ and at $n - 1$. Perturb it so that it become strictly hyperbolic, with $n - 1$ distinct roots (and $n - 2$ distinct critical points) close to $0$, a critical point close to $n - 1$ and a root close to $n$. The greatest of the quantities $(x_{k+2} - x_k)/2$ is obtained for $k = n - 2$ and it is close to $n/2$ whereas the greatest of the distances between two consecutive critical points is $\xi_{n-1} - \xi_{n-2}$ which is close to $n - 1 > n/2$.

For $n \geq 6$ the perturbation of $w$ can be chosen of the form $e^{n-1}u_0(x/e)(x - n)$ or $e^{n-1}v_0(x/e)(x - n)$ where the degree $n - 1$ polynomials $u_0$ and $v_0$ are the polynomials $u$ and $v$ from Examples 1 and 2 for $n = 6$ or the polynomials $u_1$ and $v_1$ constructed as in 1° of the proof of Theorem 1 for $n \geq 7$. This proves the realizability of the cases $L - R$ and $L + R$ for $n \geq 6$. In the same way one proves for $n = 4$ and $5$ the realizability of the case $L + R$ (choose $v_0$ to be any strictly hyperbolic polynomial of degree $3$ or $4$, by Theorem 1 the left inequality in (2) holds for it).

3°. Consider for $n = 4$ the polynomial $g := x^4 - 12x^3 + 35x^2 = x^2(x - 5)(x - 7)$. Then $M(\tilde{g}) = 7/2$. The roots of $g' = 4x^3 - 36x^2 + 70x$ equal $0$ and $(9 \pm \sqrt{11})/2$. Therefore $M(g') = \sqrt{11} < 7/2 = M(\tilde{g})$, i.e. the right inequality in (2) holds for $g$. This proves part (3) of the theorem for $n = 4$. 
4°. Prove part (3) of the theorem for \( n \geq 5 \). Consider the polynomial
\[
h(x) = x^k(x - 1)(x - 2) = x^{k+2} - 3x^{k+1} + 2x^k
\]
whose derivative is \( h'(x) = (k + 2)x^{k+1} - 3(k + 1)x^k + 2kx^{k-1} \). One has \( m(h) = 1/2, \ M(h) = 1 \).
The roots of \( h' \) equal 0 ((\( k - 1 \))-fold) and
\[
\xi_{\pm} := \frac{3(k + 1) \pm \sqrt{k^2 + 2k + 9}}{2(k + 2)}.
\]
For \( k \geq 3 \) one has \( M(h') = \xi_+ - \xi_- < 1 = M(h) \) which can be checked directly. There exists
a perturbation of \( h \) which is a strictly hyperbolic polynomial with \( M(h') < M(h) \). This proves part (3).

For \( k \geq 5 \) one can look for a perturbation \( h_1 \) of the form \( \varepsilon^k w(x/\varepsilon)(x - 1)(x - 2) \) where for
\( k = 5 \) \( w \) is one of the polynomials \( u \) or \( v \) from Examples 1 and 2 and for \( k \geq 6 \) it is one of
the polynomials \( u_1 \) or \( v_1 \) constructed in 1° of the proof of Theorem 1. For such a perturbation
(which is strictly hyperbolic) the quantities \( m(h_1) \) and \( m(h'_1) \) are realized by the roots of \( h_1 \) and \( h'_1 \) which are close to 0 when \( \varepsilon > 0 \) is small enough.

Such perturbations realize the cases \( L - R + \) and \( L + R + \) for \( n \geq 7 \). Recall that the realiz-
ability of the cases \( L - R - \) and \( L + R - \) was justified in 2°. This proves part (4) of the theorem.

5°. To prove part (5) consider the table in Theorem 2 line by line. The first line follows from
part (1) of Theorem 1 and part (1) of Theorem 2. The second line follows from part (1) of
Theorem 1 and parts (2) and (3) of Theorem 2. The third line follows from Examples 1 and 2
and from 2°. The fourth line follows from Example 3 and from 2°. \( \Box \)

4. Examples concerning \( \mathcal{L} \mathcal{P}1 \)-functions

In this section we deal with real entire functions of order one with real zeros only. The first
three of them are of finite type while the last one is of maximal type. We begin with an example
where both the left and the right-hand side inequalities (2) hold. Let
\[
\varphi(x) = x^2 \cos \frac{\pi x}{2}.
\]
The function \( \varphi(x) \) has a zero of multiplicity two at the origin and simple zeros at the odd integers.
Hence \( \varphi'(x) = \inf \{ z_{k+1} - z_k \} = 1/2 \). The derivative
\[
\varphi'(x) = \frac{x}{2} \left( 4 \cos \frac{\pi x}{2} - \pi x \sin \frac{\pi x}{2} \right)
\]
is an odd function with simple nonnegative zeros \( 0 = \xi_0 < \xi_1 < \xi_2 < \cdots \). It is easy to see that
\( \varphi'(x) > 0 \) for \( x \in (0, \xi_1) \). On the other hand, \( \varphi'(1/2) = \sqrt{2}(8 - \pi)/16 > 0 \). Thus, \( \xi_1 > 1/2 \), that
is \( \xi_1 - \xi_0 > 1/2 \). Computations show that \( \xi_1 \approx 0.685559 \).

Observe that for \( k \neq 0 \) the numbers \( \xi_k \) are the roots of the equation \( \cot(\pi x/2) = \pi x/4 \). Since
the line \( y = \pi x/4 \) crosses the positive parts of the branches of \( \cot(\pi x/2) \), then \( \xi_k \in (2k - 2, 2k - 1) \), for each \( k \in \mathbb{N} \). Therefore \( \xi_{k+1} - \xi_k > 1 \) for \( k = 1, 2, \ldots \), and \( m(\varphi') = \xi_1 - \xi_0 > 1/2 \).
Also, the fact that the linear function \( \pi x/2 \) increases and \( \cot(\pi x/2) \) decreases in every interval
\( (2k - 2, 2k) \), shows that the sequence \( \xi_{k+1} - \xi_k, \ k = 0, 1, \ldots \) is an increasing one, and that
it tends to 2 when \( k \) tends to infinity. Thus, the inequalities \( 1/2 = m(\tilde{p}) < m(\varphi') < M(\varphi') < M(\tilde{p}) = 2 \) hold for \( \varphi(x) \).
The same inequalities will obviously hold after splitting the double root at the origin by a small perturbation. In other words, the latter inequalities hold for the entire function \( \phi_\varepsilon(x) = (x^2 - \varepsilon) \cos(\pi x/2) \) for small positive \( \varepsilon \), and this function has only real and simple zeros.

Our next example shows a function for which the left-hand side inequality fails. Let us consider the even entire function

\[
\theta(x) = x(x^2 - 1) \sin(\pi x).
\]

It has double zeros at \(-1, 0, 1\) and simple ones at the other integers. Thus \( m(\tilde{\theta}) = 1/2 \). Its derivative has simple zeros at \(-1, 0, 1\) and in every open interval between two consecutive integers. Let \( 0 = \xi_0 < \xi_1 < \xi_2 = 1 \) be the first three nonnegative zeros of \( \theta'(x) \). Thus, if \( \xi_1 \neq 1/2 \), then at least one of the distances \( \xi_1 - \xi_0 \) or \( \xi_2 - \xi_1 \) will be less than \( 1/2 \). We have

\[
\theta'(x) = \pi x(x^2 - 1) \cos(\pi x) + (3x^2 - 1) \sin(\pi x)
\]

so that \( \theta'(<\delta) < 0 \) for sufficiently small positive \( <\delta \) and \( \theta'(1/2) = -1/4 \). Thus \( \xi_1 \in (1/2, 1) \) and \( \xi_2 - \xi_1 < 1/2 \). Numerically we have \( \xi_1 \approx 0.536254 \). Therefore \( m(\theta') < m(\tilde{\theta}) \), and again small perturbations of the double zeros at 0, \( 1 \) will provide an example of an entire function with simple zeros for which the same inequality between \( m(\theta') \) and \( m(\tilde{\theta}) \) holds.

The next example concerns the case when the right-hand side inequality (2) fails for an entire function with only real zeros of order one and of finite type. Consider the function

\[
\phi(x) = e^x \frac{\sin \pi x}{x}
\]

whose zeros are at the nonzero integers. Thus \( M(\phi) = 1/2 \). On the other hand, since

\[
\phi'(x) = e^x \left( \pi x \cos(\pi x) + (x - 1) \sin(\pi x) \right)
\]

and the smallest in absolute value negative and positive zeros \( \xi_- \) and \( \xi_+ \) are

\[
\xi_- \approx -1.33845 \quad \text{and} \quad \xi_+ \approx 0.287083,
\]

then \( \xi_+ - \xi_- \approx 1.62553 > M(\phi) \).

Finally, we show that the right-hand side inequality (2) fails for the function \( \eta(x) = 1/\Gamma(x) \), where \( \Gamma \) is the Gamma-function. It is well known that \( 1/\Gamma \) is an entire function of order one and of maximal type and its zeros are the nonpositive integers. Thus \( M(1/\Gamma) = 1 \). Recall that \( \eta'(x) = -\Gamma(x) \psi(x) \), where \( \psi(x) \) is the digamma-function. Suppose that the zeros \( \xi_k \) of \( \psi(x) \) are arranged in decreasing order, \( 0 > \xi_1 > \xi_2 \cdots \). Since \( \xi_1 \approx -0.504083 \) and \( \xi_2 \approx -1.5735 \), then \( \xi_1 - \xi_2 \approx 1.06942 \) which yields \( M((1/\Gamma)') > M(1/\Gamma) \).

5. Other related results

For a hyperbolic polynomial \( p \) the classical Rolle theorem tells only that the root \( \xi_j \) of \( p' \) lies between the roots \( x_j \) and \( x_{j+1} \) of \( p \), just like \( z_k \). There are many refinements of Rolle’s theorem about polynomials. A beautiful theorem of Tchakaloff [17] states that if \( p \) is a polynomial of degree \( 2n \) with \( p(-1) = p(1) \), then there exists a critical point of \( p(x) \) in the interval \((x_{n1}, x_{nn})\) between the smallest and the largest zeros of the Legendre polynomial of degree \( n \). Moreover, this is the smallest possible interval with this property in the sense that if \( (a, b) \) is a proper subinterval of \((x_{n1}, x_{nn})\), then there is a polynomial \( q \) of degree \( 2n \) such that \( q(-1) = q(1) \) and \( q'(x) \neq 0 \) for every \( x \in (a, b) \). Another result due to Andrews [2] aiming to make the Rolle theorem about hyperbolic polynomials more precise states that, for \( n \geq 2 \), one has

\[
\frac{1}{n - j + 1} < \frac{\xi_j - x_j}{x_{j+1} - x_j} < \frac{j}{j + 1}, \quad j = 1, \ldots, n - 1,
\]
and these inequalities are necessary, but not sufficient conditions concerning the position of $\xi_j$ with respect to $x_j$ and $x_{j+1}$. We refer also to Horwitz’ papers [5,6].

Boris and Michael Shapiro [15] studied the following generalization of a hyperbolic polynomial. A smooth real-valued function $f$ is a pseudopolynomial (or a polynomial-like function) of degree $n$ if $f^{(n)}$ vanishes nowhere. If $f$ has exactly $n$ real roots $x_1^{(0)} \leq \cdots \leq x_n^{(0)}$ (in this case we say that $f$ is hyperbolic), then $f^{(j)}$ has exactly $n-j$ real roots $x_1^{(j)} \leq \cdots \leq x_{n-j}^{(j)}$ such that $x_k^{(j)} \in (x_{k-1}^{(j)}, x_{k+1}^{(j)})$, $j = 1, \ldots, n-1$. As in the case of hyperbolic polynomials, if there holds one of the equalities $x_k^{(j+1)} = x_k^{(j)}$ or $x_{k+1}^{(j+1)} = x_k^{(j)}$, then the other one holds as well. Necessary and sufficient conditions are given upon the choice of the real numbers $x_1^{(0)}$, $x_2^{(0)}$, $x_3^{(0)}$, $x_1^{(1)}$, $x_2^{(1)}$, $x_1^{(2)}$ so that they could be the roots of a hyperbolic pseudopolynomial of degree 3 and of its derivatives are given in [15]. The case $n = 3$ is the first nontrivial one because for $n = 2$ any triple $x_1^{(0)} < x_1^{(1)} < x_2^{(0)}$ is the triple of roots of a hyperbolic pseudopolynomial of degree 2 and of its derivative.

The arrangement of the roots of a hyperbolic pseudopolynomial $f$ of degree $n$ and of its derivatives up to order $n-1$ is defined when all these $n(n+1)/2$ roots are written in a string in which any two consecutive roots are connected by the sign < or =. For $n \leq 3$ all arrangements compatible with the Rolle theorem are realizable by the roots of hyperbolic polynomials and of their derivatives. For $n = 4$ this is no longer like this, see [1], but all of them are realizable by the roots of hyperbolic pseudopolynomials and their derivatives, see [7]. For $n \geq 5$ the latter do not realize all such arrangements, even not all arrangements without equalities between roots, see [8]. For $n = 5$ the exhaustive answer to the question which arrangements without equalities are realizable by the roots of pseudopolynomials and of their derivatives is given in [9–11].

We finish recalling a classical result of Vladimir Markov [12] (see also [14, Lemma 2.7.1] and [3, Lemma 1, Corollary 2]). One if its equivalent formulations states that every zero of any derivative of a strictly hyperbolic polynomial is an increasing function of each zero of the polynomial itself.

References

[15] B. Shapiro, M. Shapiro, This strange and mysterious Rolle theorem, electronic preprint, math.CA/0302215.