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Cellular bases for the Brauer and Birman–Murakami–Wenzl algebras

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Abstract

An explicit combinatorial construction is given for cellular bases (in the sense of Graham and Lehrer) for the Birman–Murakami–Wenzl and Brauer algebra. We provide cell modules for the Birman–Murakami–Wenzl and Brauer algebras with bases index by certain bitableaux, generalising the Murphy basis for the Specht modules of the Iwahori–Hecke algebra of the symmetric group. The bases for the cell modules given here are constructed non-diagrammatically and hence are relatively amenable to computation.

Introduction

The Birman–Murakami–Wenzl algebras, defined independently by Birman and Wenzl in [1] and Murakami in [10], are finite dimensional algebras defined over a rational function field in two variables and can be considered as deformations of the Brauer algebras obtained by replacing the symmetric group algebras with the corresponding Iwahori–Hecke algebras.

The connections between the Birman–Murakami–Wenzl and the Iwahori–Hecke algebras have led several authors, notably, Halverson and Ram in [4], Fishel and Grojnowski in [2] and Xi in [13] in to determine analogues of results about the representations and characters of the Iwahori–Hecke algebra for the Birman–Murakami–Wenzl algebras. The

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present work has similar motivation, namely we exploit the fact that the Iwahori–Hecke algebra of the symmetric group is cellular in the sense of Graham and Lehrer, to investigate the representation theory of the Birman–Murakami–Wenzl algebra.

The axioms for a cellular algebra were first formulated by Graham and Lehrer in [3] where they showed that as a consequence of the axioms, a cellular algebra has certain naturally defined cell representations. They related these cell representations to the ideal structure of the algebra and obtained a general description of the irreducible representations of the cellular algebra together with a criterion for the cellular algebra to be semisimple. Graham and Lehrer also showed the Brauer algebras, the Ariki–Koike algebras (including the Iwahori–Hecke algebras), and the Temperley–Lieb algebras to be cellular and described the cell representations of the Brauer algebras.

Subsequently König and Xi in [7] have given a general construction which produces all cellular algebras and used this construction to show that the Brauer algebra and the Temperley–Lieb algebras are cellular.

Given that the Brauer algebras are cellular, one is naturally lead to ask whether the Birman–Murakami–Wenzl algebras are also cellular. Xi in [13] answers this question by showing that certain analogues of the Kazhdan–Lusztig basis for the Birman–Murakami–Wenzl algebras given by Fishel and Grojnowski in [2] (see also Morton and Traczyk [9]) are in fact cellular. The basis given by Xi in [13] is constructed by "blowing up," using certain diagrams called *dangles*, a basis of the Iwahori–Hecke algebras to obtain a basis for the Birman–Murakami–Wenzl algebras.

In this paper we also study the relation between the cellular bases of the Iwahori–Hecke algebras and the cellular bases of the Birman–Murakami–Wenzl Algebras. As in [13], we give an explicit construction showing that a cellular basis for the Iwahori–Hecke algebra gives rise to a cellular basis for the Birman–Murakami–Wenzl algebra. However, the method used here produces a basis indexed by certain *bitableaux* and thereby gives an explicit combinatorial description of the cellular basis and cell modules which is amenable to computation. In the special case where the cellular basis for the Iwahori–Hecke algebra is the Murphy basis [11], our is a natural analogue for the Birman–Murakami–Wenzl algebras of the Murphy basis. The cell modules which we construct using the analogue of the Murphy basis generalise the classical Specht modules of the symmetric group.

1. Preliminaries

We establish the basic notation and state some known results which will be used later. A reference for the material presented here is [8].

1.1. The symmetric group

Let $\mathfrak{S}_n$ denote the symmetric group acting on the integers $\{1, 2, \dots, n\}$ on the right. The elementary transpositions in $\mathfrak{S}_n$ are the elements

$$S = \big\{s_i = (i, i+1) \mid 1 \leqslant i < n\big\}.$$

The elementary transpositions, together with the relations

$$\begin{aligned} &s_i^2 = 1 && \text{for } 1 \leqslant i < n, \\ &s_i s_j = s_j s_i && \text{for } 2 \leqslant |i-j| \text{ and } 1 \leqslant i, j < n, \\ &s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} && \text{for } 1 \leqslant i < n-1 \end{aligned}$$

give a presentation for $\mathfrak{S}_n$ as a Coxeter group. Let w be a permutation in $\mathfrak{S}_n$. An expression $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ for w in terms of elementary transpositions is said to be *reduced* if w cannot be written as a proper sub-expression of $s_{i_1} s_{i_2} \cdots s_{i_k}$. In this case we say w is a permutation with *length* k and write $l(w) = k$. Note that while there are usually several reduced expressions for w, the length of w will not depend on this choice. The length function on $\mathfrak{S}_n$ is determined by the properties

$$l(s_i w) = \begin{cases} l(w) + 1 & \text{if } (i)w < (i+1)w, \\ l(w) - 1 & \text{otherwise;} \end{cases} \tag{1.1}$$

and

$$l(w s_i) = \begin{cases} l(w) + 1 & \text{if } (i)w^{-1} < (i+1)w^{-1}, \\ l(w) - 1 & \text{otherwise,} \end{cases} \tag{1.2}$$

together with the normalizing condition $l(1_{\mathfrak{S}_n}) = 0$.

1.2. Compositions and tableaux

Let $k \geqslant 0$ be an integer. A partition of k is a non-increasing sequence $\nu = (\nu_1, \nu_2, \ldots)$ of integers such satisfying $\sum_{i \geqslant 1} \nu_i = k$. We will write $\nu \vdash k$ to denote the fact that ν is a partition of k. If ν is a partition it will also be convenient to write $|\nu| = k$ whenever $\sum_{i \geqslant 1} \nu_i = k$. If μ, ν are partitions of k, then write $\mu \trianglerighteq \nu$ and say μ *dominates* ν, if

$$\sum_{i=1}^{j} \mu_k \geqslant \sum_{i=1}^{j} \nu_k \quad \text{for all } j \geqslant 0.$$

The fact that $\mu \trianglerighteq \nu$ and $\mu \neq \nu$ will be denoted by $\mu \triangleright \nu$.

The diagram of a partition $\nu \vdash k$ is the set of nodes

$$[\nu] = \left\{ (i, j) \mid 1 \leqslant j \leqslant \nu_i \text{ and } i \geqslant 1 \right\} \subset \mathbb{N} \times \mathbb{N}.$$

Let $\nu \vdash k$. A ν-tableau is a bijection $\mathfrak{t} : [\nu] \to \{1, 2, \ldots, k\}$; equivalently a ν-tableau $\mathfrak{t}$ may be regarded as a labeling of the nodes of $[\nu]$ by the integers $1, 2, \ldots, k$. For example, if $k = 7$ and $\nu = (4, 2, 1)$, then

$$\mathfrak{t} = \begin{array}{|c|c|c|c|}\hline 2 & 4 & 6 & 7 \\ \hline 1 & 3 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \tag{1.3}$$

is a ν-tableau. The *super-standard* tableau $\mathfrak{t}^\nu$ is the unique ν-tableau in which has as its entries the integers $1, 2, \ldots, k$ appearing in increasing sequence from left to right and top to bottom. In case $k = 7$ and $\nu = (4, 2, 1)$ we have

$$\mathfrak{t}^\nu = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}. \tag{1.4}$$

A ν-tableau $\mathfrak{t}$ is said to be *row standard* if the entries of each row of $\mathfrak{t}$ increase when read from left to right and a row standard ν-tableau $\mathfrak{t}$ is said to be *standard* if the entries of each column of $\mathfrak{t}$ increase when read from top to bottom. The tableau of (1.3) is row standard but not standard. We will denote by $\mathrm{Std}(\nu)$ the collection of standard ν-tableaux.

Let $\nu \vdash k$ be a partition. The symmetric group $\mathfrak{S}_k$ acts from the right on the set of ν-tableaux by permuting entries. Let, for example, $n = 5$ and $\nu = (3, 2)$; if $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$, then

$$\mathfrak{t}(1,2)(4,5) = \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 1 & 5 \\ \cline{1-2}\end{array}.$$

If $\mathfrak{t}$ is a ν-tableau, then $d(\mathfrak{t}) \in \mathfrak{S}_k$ is the permutation defined by the equation $\mathfrak{t}^\nu d(\mathfrak{t}) = \mathfrak{t}$. The Young subgroup of $\mathfrak{S}_\nu \cong \mathfrak{S}_{\nu_1} \times \mathfrak{S}_{\nu_2} \times \cdots \times \mathfrak{S}_{\nu_k}$ will be the row stabilizer of $\mathfrak{t}^\nu$ in $\mathfrak{S}_k$; that is

$$\mathfrak{S}_\nu = \langle s_i \mid i, i+1 \text{ are in the same row of } \mathfrak{t}^\nu \rangle.$$

For example, when $\nu = (4, 2, 1)$ and $\mathfrak{t}^\nu$ is given by (1.4), then $\mathfrak{S}_\nu = \langle s_1, s_2, s_3 \rangle \times \langle s_5 \rangle$.

1.3. *The Iwahori–Hecke algebra of the symmetric group*

Let R be a domain and q^2 be an invertible element in R. The Iwahori–Hecke algebra $\mathcal{H}_{R,n}(q^2)$ associated with $\mathfrak{S}_n$ is the unital associative R-algebra generated by the elements $\{X_i \mid 1 \leqslant i < n\}$ subject to the relations

$$\begin{aligned}
&(X_i - q^2)(X_i + 1) = 0 && \text{for } 1 \leqslant i < n, \\
&X_i X_{i+1} X_i = X_{i+1} X_i X_{i+1} && \text{for } 1 \leqslant i \leqslant n-2, \\
&X_i X_j = X_j X_i && \text{for } 2 \leqslant |i - j| \text{ and } 1 \leqslant i, j < n.
\end{aligned}$$

If w is a permutation in $\mathfrak{S}_n$ with reduced expression $w = s_{i_1} \cdots s_{i_k}$, the element X_w of $\mathcal{H}_{R,n}(q^2)$ is defined by

$$X_w = X_{i_1} \cdots X_{i_k}.$$

By Matsumoto's theorem (Theorem 1.8 of [8]), X_w is a well defined element of $\mathcal{H}_{R,n}(q^2)$. The next statement follows from (1.1) and (1.2) together with the defining relations for $\mathcal{H}_{R,n}(q^2)$.

Lemma 1.1. *If $w \in \mathfrak{S}_n$ and s is an elementary transposition, then*

$$X_w X_s = \begin{cases} X_{ws} & \text{if } l(ws) > l(w), \\ q^2 X_{ws} + (q^2 - 1) X_w & \text{if } l(ws) < l(w); \end{cases}$$

and

$$X_s X_w = \begin{cases} X_{sw} & \text{if } l(sw) > l(w), \\ q^2 X_{sw} + (q^2 - 1) X_w & \text{if } l(sw) < l(w). \end{cases}$$

The following result is well-known (Theorem 1.13 of [8]).

Theorem 1.1. *The Iwahori–Hecke algebra $\mathcal{H}_{R,n}(q^2)$ is free as an R-module , having for a basis the collection $\{X_w \mid w \in \mathfrak{S}_n\}$.*

The next statement is Lemma 2.3 of [11].

Lemma 1.2. *Let $*, \dagger, \#$ be the maps defined by*:

$$* : X_w \mapsto X_{w^{-1}},$$
$$\dagger : X_w \mapsto \left(-q^2\right)^{l(w)} X_w^{-1},$$
$$\sharp : X_w \mapsto \left(-q^2\right)^{l(w)} X_{w^{-1}}^{-1},$$

for each $w \in \mathfrak{S}_n$, extended to $\mathcal{H}_{R,n}(q^2)$ by linearity. Then $*$ and $\dagger$ are R-algebra anti-involutions of $\mathcal{H}_{R,n}(q^2)$ and $\sharp$ is an R-algebra automorphism of $\mathcal{H}_{R,n}(q^2)$.

1.4. The Murphy basis for the Iwahori–Hecke algebra

In [11] Murphy gives a nice basis for $\mathcal{H}_{R,n}(q^2)$ indexed by pairs of standard tableaux, a basis which allows him to define a filtration on $\mathcal{H}_{R,n}(q^2)$ by two-sided ideals and to describe the representations of $\mathcal{H}_{R,n}(q^2)$.

For a partition $\lambda \vdash n$, Murphy defines the element $m_\lambda \in \mathcal{H}_{R,n}(q^2)$ by

$$m_\lambda = \sum_{w \in \mathfrak{S}_\lambda} X_w,$$

and associates to each pair $\mathfrak{s}, \mathfrak{t}$ of standard λ-tableaux the element

$$m_{\mathfrak{s}\mathfrak{t}} = X^*_{d(\mathfrak{s})} m_\lambda X_{d(\mathfrak{t})}.$$

Let N^λ denote the R-submodule of $\mathcal{H}_{R,n}(q^2)$ generated by the elements

$$\left\{ m_{\mathfrak{s}\mathfrak{t}} = X^*_{d(\mathfrak{s})} m_\mu X_{d(\mathfrak{t})} \;\middle|\; \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\mu) \text{ and } \mu \trianglerighteq \lambda \right\}$$

and $\check{N}^\lambda$ be the R-submodule of N^λ generated by

$$\left\{ m_{\mathfrak{s}\mathfrak{t}} = X^*_{d(\mathfrak{s})} m_\mu X_{d(\mathfrak{t})} \;\middle|\; \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\mu) \text{ and } \mu \triangleright \lambda \right\}.$$

The following result is due to Murphy (Theorems 4.17 and 4.18 of [11] or Theorem 3.2 of [8]).

Theorem 1.2. *The Iwahori–Hecke algebra $\mathcal{H}_{R,n}(q^2)$ has a free R-basis*

$$\mathscr{M} = \left\{ m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda) \textit{ and } \lambda \vdash n \right\}.$$

Moreover, the following hold:

(1) *The R-linear map determined by $m_{\mathfrak{s}\mathfrak{t}} \mapsto m_{\mathfrak{t}\mathfrak{s}}$, for all $m_{\mathfrak{s}\mathfrak{t}} \in \mathscr{M}$, is an algebra anti-involution of $\mathcal{H}_{R,n}(q^2)$.*
(2) *Suppose that $h \in \mathcal{H}_{R,n}(q^2)$ and that $\mathfrak{t} \in \operatorname{Std}(\lambda)$. Then there exist $a_\mathfrak{v} \in R$, for $\mathfrak{v} \in \operatorname{Std}(\lambda)$, such that*

$$m_{\mathfrak{s}\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in \operatorname{Std}(\lambda)} a_\mathfrak{v} m_{\mathfrak{s}\mathfrak{v}} \mod \check{N}^\lambda \tag{1.5}$$

for all $\mathfrak{s} \in \operatorname{Std}(\lambda)$.

The crucial point about (1.5) is that the elements $\mathfrak{v}$ and $a_\mathfrak{v}$ depend on $\mathfrak{t}$ and h but not on $\mathfrak{s}$. Also, as a consequence of Theorem 1.2, both N^λ and $\check{N}^\lambda$ are two sided ideals of $\mathcal{H}_{R,n}(q^2)$ and the dominance order on partitions gives rise to a filtration of $\mathcal{H}_{R,n}(q^2)$ by two-sided ideals.

The right *Specht module* S^λ is defined to be the $\mathcal{H}_{R,n}(q^2)$-submodule of $N^\lambda/\check{N}^\lambda$ generated by the elements

$$\left\{ \check{N}^\lambda + m_{\mathfrak{t}^\lambda \mathfrak{t}} \;\middle|\; \mathfrak{t} \in \operatorname{Std}(\lambda) \right\}. \tag{1.6}$$

By the last item of Theorem 1.2, the set (1.6) is a free R-basis for S^λ. For $\mathfrak{s} \in \operatorname{Std}(\lambda)$, let $m_\mathfrak{s}$ denote the element $\check{N}^\lambda + m_{\mathfrak{t}^\lambda \mathfrak{s}} \in S^\lambda$. Murphy defines a symmetric bilinear form $\langle\, , \rangle : S^\lambda \times S^\lambda \to R$ by setting

$$\langle m_\mathfrak{s}, m_\mathfrak{t} \rangle m_\lambda \equiv m_{\mathfrak{t}^\lambda \mathfrak{s}} m^*_{\mathfrak{t}^\lambda \mathfrak{t}} \mod \check{N}^\lambda.$$

Since $\langle\,,\rangle$ satisfies the condition $\langle m_\mathfrak{s}, m_\mathfrak{t} h\rangle = \langle m_\mathfrak{s} h^*, m_\mathfrak{t}\rangle$ for all $h \in \mathcal{H}_{R,n}(q^2)$, it follows that the set $\operatorname{rad}(S^\lambda) = \{a \in S^\lambda \mid \langle a, b\rangle = 0 \text{ for all } b \in S^\lambda\}$ will be a right $\mathcal{H}_{R,n}(q^2)$-module. Consequently Murphy defines $D^\lambda = S^\lambda / \operatorname{rad}(S^\lambda)$. The first item below is Theorem 6.2 of [11] while the second item is Theorem 6.3 of [11].

Theorem 1.3. *Let R be a field. Then*

(1) *Then either $D^\lambda = 0$ or D^λ is an absolutely irreducible $\mathcal{H}_{R,n}(q^2)$-module.*
(2) *The collection $\{D^\lambda \mid \lambda \vdash n \text{ and } D^\lambda \neq 0\}$ is a complete set of pairwise non-isomorphic absolutely irreducible $\mathcal{H}_{R,n}(q^2)$-modules.*

1.5. Cellular algebras

In this section we state the main results of Graham and Lehrer [3] and refer the reader to the exposition in [8]. For an equivalent but basis free approach to the subject, the reader is referred to a work of König and Xi [6].

Definition 1.1. Let R be a domain and A a unital associative R algebra with a free R basis. Let Λ be a finite set with partial order $\leqslant$ and suppose that for each $\lambda \in \Lambda$ there is a finite index set $\mathcal{I}(\lambda)$ such that there exists a set

$$\mathscr{C} = \left\{ c^\lambda_{\mathfrak{v}\mathfrak{u}} \in A \;\middle|\; \mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\lambda) \text{ and } \lambda \in \Lambda \right\}$$

which is an R-basis for A. For $\lambda \in \Lambda$, let $\check{A}^\lambda$ denote the R-submodule of A generated by the elements

$$\left\{ c^\mu_{\mathfrak{v}\mathfrak{u}} \;\middle|\; \mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\mu) \text{ where } \mu \in \Lambda \text{ and } \lambda < \mu \right\}.$$

Then $(\Lambda, \mathscr{C})$ is a *cellular basis* and A a *cellular algebra* if

(1) the R-linear map $*: A \to A$ determined by $*: c^\lambda_{\mathfrak{v}\mathfrak{u}} \mapsto c^\lambda_{\mathfrak{u}\mathfrak{v}}$ for all $\lambda \in \Lambda$ and $\mathfrak{u}, \mathfrak{v} \in \mathcal{I}(\lambda)$ is an algebra anti-automorphism of A; and,
(2) if $\lambda \in \Lambda$, $\mathfrak{v} \in \mathcal{I}(\lambda)$ and $a \in A$, then there exist $\alpha_\mathfrak{t} \in R$, for $\mathfrak{t} \in \mathcal{I}(\lambda)$, such that

$$c^\lambda_{\mathfrak{u}\mathfrak{v}} a \equiv \sum_{\mathfrak{t} \in \mathcal{I}(\lambda)} \alpha_\mathfrak{t} c^\lambda_{\mathfrak{u}\mathfrak{t}} \mod \check{A}^\lambda \tag{1.7}$$

for all $\mathfrak{u} \in \mathcal{I}(\lambda)$.

The essential feature of the expression (1.7) is that the elements $\mathfrak{t} \in \mathcal{I}(\lambda)$ and the constants $\alpha_\mathfrak{t}$ are determined entirely by a and $\mathfrak{v}$ and are independent of $\mathfrak{u}$.

Examples of cellular algebras include Ariki–Koike algebras (including the Iwahori–Hecke algebras), the Brauer and Temperley–Lieb algebras (Theorems 4.10 and 6.7 of [3]) and the Birman–Murakami–Wenzl algebras (Theorem 3.11 of [13]). Note that a cellular algebra may have more than one cellular basis; the Murphy basis, for instance, makes the

Iwahori–Hecke algebra into a cellular algebra, as does the Kazhdan–Lusztig basis for the Iwahori–Hecke algebra (see, for example, Theorem 5.5 of [3]).

For $\lambda \in \Lambda$, denote by A^λ the R-submodule of A generated by the elements $c^\mu_{\mathfrak{v}\mathfrak{u}}$ where $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\mu)$ and $\mu \geqslant \lambda$. Observe that $\check{A}^\lambda \subseteq A^\lambda$ and that $A^\lambda/\check{A}^\lambda$ has an R-basis given by $\check{A}^\lambda + c^\lambda_{\mathfrak{v}\mathfrak{u}}$ where $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\lambda)$. The next statement is now a straightforward consequence of the definitions (Lemma 2.3 of [8]).

Lemma 1.3. *Let $(\mathscr{C}, \Lambda)$ be a cellular basis for A and λ be an element of Λ.*

(1) *Suppose that $\mathfrak{u} \in \mathcal{I}(\lambda)$ and that $a \in A$. Then for all $\mathfrak{v} \in \mathcal{I}(\lambda)$,*

$$a^* c^\lambda_{\mathfrak{u}\mathfrak{v}} \equiv \sum_{\mathfrak{t} \in \mathcal{I}(\lambda)} \alpha_{\mathfrak{t}} c^\lambda_{\mathfrak{t}\mathfrak{v}} \mod \check{A}^\lambda$$

where, for each $\mathfrak{t}$, $\alpha_{\mathfrak{t}}$ is the element of R determined by (1.7).

(2) *The R-modules A^λ and $\check{A}^\lambda$ are two-sided ideals of A.*

(3) *If $\mathfrak{s}, \mathfrak{t} \in \mathcal{I}(\lambda)$, then there are $\alpha_{\mathfrak{s}\mathfrak{t}} \in R$ such that for any $\mathfrak{u}, \mathfrak{v} \in \mathcal{I}(\lambda)$,*

$$c^\lambda_{\mathfrak{v}\mathfrak{s}} c^\lambda_{\mathfrak{t}\mathfrak{u}} \equiv \alpha_{\mathfrak{s}\mathfrak{t}} c^\lambda_{\mathfrak{v}\mathfrak{u}} \mod \check{A}^\lambda. \tag{1.8}$$

The second item of Lemma 1.3 shows that there is a filtration of A by the ideals A^λ. The third item shows that each of the quotients $A^\lambda/\check{A}^\lambda$ is equipped with a bilinear form; this bilinear form will be defined below.

Let $\lambda \in \Lambda$ be fixed. For $\mathfrak{v} \in \mathcal{I}(\lambda)$, define $C^\lambda_{\mathfrak{v}}$ to be the R-submodule of $A/\check{A}^\lambda$ generated by the elements $\{\check{A}^\lambda + c^\lambda_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{u} \in \mathcal{I}(\lambda)\}$. By (1.7), the algebra A has a well-defined action on $C^\lambda_{\mathfrak{v}}$ by right multiplication. Moreover, under this action $C^\lambda_{\mathfrak{v}} \cong C^\lambda_{\mathfrak{u}}$ whenever $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\lambda)$. Given the latter observation, the right *cell module* C^λ is defined to be the right A-module which is free as an R-module with basis $\{c^\lambda_{\mathfrak{v}} \mid \mathfrak{v} \in \mathcal{I}(\lambda)\}$ and right A-action given by

$$c^\lambda_{\mathfrak{v}} a = \sum_{\mathfrak{t}} \alpha_{\mathfrak{t}} c^\lambda_{\mathfrak{t}}$$

where the $\alpha_{\mathfrak{t}}$ are given by (1.7). Then the map $C^\lambda_{\mathfrak{u}} \to C^\lambda$ defined by $c^\lambda_{\mathfrak{u}\mathfrak{v}} + \check{A}^\lambda \mapsto c^\lambda_{\mathfrak{v}}$ is an isomorphism of right A-modules. The left cell module $C^{*\lambda}$ is defined to be the left A-module which is free as an R-module with basis $\{c^\lambda_{\mathfrak{v}} \mid \mathfrak{v} \in \mathcal{I}(\lambda)\}$ and left A-action given by

$$a^* c^\lambda_{\mathfrak{v}} = \sum_{\mathfrak{t}} \alpha_{\mathfrak{t}} c^\lambda_{\mathfrak{t}}$$

where $\alpha_{\mathfrak{t}}$ are once more determined by (1.7).

By Lemma 1.3 there is a bilinear form $\langle\,,\rangle : C^\lambda \times C^\lambda \to R$

$$\langle c^\lambda_{\mathfrak{s}}, c^\lambda_{\mathfrak{t}} \rangle = \alpha_{\mathfrak{s}\mathfrak{t}} \quad \text{for all } \mathfrak{s}, \mathfrak{t} \in \mathcal{I}(\lambda),$$

where $\alpha_{\mathfrak{s}\mathfrak{t}}$ are determined by (1.8). The following statements follow readily from the definitions (Proposition 2.9 of [8]).

Proposition 1.1. *Let $\lambda \in \Lambda$ and $a \in A$. Then*

(1) $\langle c^\lambda_{\mathfrak{u}}, c^\lambda_{\mathfrak{v}}\rangle = \langle c^\lambda_{\mathfrak{v}}, c^\lambda_{\mathfrak{u}}\rangle$ *for all* $\mathfrak{u}, \mathfrak{v} \in \mathcal{I}(\lambda)$.
(2) $\langle c^\lambda_{\mathfrak{u}} a, c^\lambda_{\mathfrak{v}}\rangle = \langle c^\lambda_{\mathfrak{v}}, c^\lambda_{\mathfrak{u}} a^*\rangle$ *for all* $\mathfrak{u}, \mathfrak{v} \in \mathcal{I}(\lambda)$.
(3) $bc^\lambda_{\mathfrak{u}\mathfrak{v}} = \langle b, c^\lambda_{\mathfrak{u}}\rangle c^\lambda_{\mathfrak{v}}$ *for all* $\mathfrak{u}, \mathfrak{v} \in \mathcal{I}(\lambda)$ *and* $b \in C^\lambda$.

The *radical* of the module C^λ is defined to be

$$\operatorname{rad}(C^\lambda) = \{a \in C^\lambda \mid \langle a, b\rangle = 0 \text{ for all } b \in C^\lambda\}. \tag{1.9}$$

By the second item of Proposition 1.1, $\operatorname{rad}(C^\lambda)$ is an A-submodule of C^λ, motivating the definition $D^\lambda = C^\lambda / \operatorname{rad}(C^\lambda)$.

Proposition 1.2. *Let R be a field and let $\lambda \in \Lambda$.*

(1) *If $D^\lambda \neq 0$, then $D^\lambda = 0$ or D^λ is absolutely irreducible.*
(2) *The intersection of the maximal submodules of C^λ is equal to* $\operatorname{rad}(C^\lambda)$.

In principle at least, the following theorem of Graham and Lehrer (Theorem 2.19 of [8]) allows us to classify the simple A-modules.

Theorem 1.4. *Suppose that R is a field. Then*

$$\{D^\lambda \mid \lambda \in \Lambda \text{ and } D^\lambda \neq 0\}$$

is a complete set of pairwise non-isomorphic irreducible A-modules.

Graham and Lehrer also give the following equivalences (Corollary 2.21 of [8]).

Theorem 1.5. *Suppose that R is a field. Then the following are equivalent.*

(1) *A is (split) semisimple.*
(2) $C^\lambda = D^\lambda$ *for all* $\lambda \in \Lambda$.
(3) $\operatorname{rad}(C^\lambda) = 0$ *for all* $\lambda \in \Lambda$.

2. The Birman–Murakami–Wenzl algebras

The Birman–Murakami–Wenzl (B-M-W) algebras as defined in [1] and [10] are associative algebras over a field $\kappa = \mathbb{C}(\hat{r}, \hat{q})$. In what follows, rather than working over κ, we will consider a generic version of the B-M-W algebras defined over an appropriate localization R of a polynomial ring over $\mathbb{Z}$. For these generic algebras we will produce cellular bases corresponding to cellular bases for the Iwahori–Hecke algebra and then obtain cellular bases for the B-M-W algebras over κ by appropriate specializations.

Let r, q be indeterminates over $\mathbb{Z}$ and $R = \mathbb{Z}[r^{\pm 1}, q^{\pm 1}, (q - q^{-1})^{-1}]$ be the localization of $\mathbb{Z}[r^{\pm 1}, q^{\pm 1}]$ at $(q - q^{-1})$ and define the element x in R to be

$$x = \frac{r - r^{-1}}{q - q^{-1}} + 1.$$

The generic B-M-W algebra $\mathcal{B}_n(r, q)$ over R is the unital associative R-algebra generated by the elements $\{T_i \mid 1 \leqslant i < n\}$ subject to the following relations:

$$\begin{aligned}
&\big(T_i - q^2\big)\big(T_i - qr^{-1}\big)(T_i + 1) = 0 && \text{for } 1 \leqslant i < n,\\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} && \text{for } 1 \leqslant i < n-1,\\
&T_i T_j = T_j T_i && \text{for } 2 \leqslant |i - j| \text{ and } 1 \leqslant i, j < n,\\
&E_{i+1} T_i^{\pm 1} E_{i+1} = (qr)^{\pm 1} E_{i+1} && \text{for } 1 \leqslant i < n-1,\\
&E_{i-1} T_i^{\pm 1} E_{i-1} = (qr)^{\pm 1} E_{i-1} && \text{for } 2 \leqslant i < n,\\
&E_i T_i = T_i E_i = qr^{-1} E_i && \text{for } 1 \leqslant i < n,
\end{aligned}$$

where E_i is defined by the equation

$$\big(q^2 - 1\big)(1 - E_i) = T_i - q^2 T_i^{-1} \quad \text{for } i \leqslant i < n.$$

In [12], Wenzl derives the following additional relations from the defining relations:

$$\begin{aligned}
&T_{i\pm 1} T_i E_{i\pm 1} = E_i T_{i\pm 1} T_i = q^2 E_i E_{i\pm 1},\\
&T_{i\pm 1}^{-1} E_i T_{i\pm 1} = T_i^{-1} E_{i\pm 1} T_i,\\
&T_{i\pm 1} E_i T_{i\pm 1}^{-1} = T_i E_{i\pm 1} T_i^{-1},\\
&T_{i\pm 1} E_i E_{i\pm 1} = q^2 T_i^{-1} E_{i\pm 1},\\
&E_{i\pm 1} E_i T_{i\pm 1} = q^2 E_{i\pm 1} T_i^{-1},\\
&E_i T_i^{\pm 1} = T_i^{\pm 1} E_i = q^{\pm 1} r^{\mp 1} E_i,\\
&T_i E_j = E_j T_i \quad \text{if } |i - j| \geqslant 2,\\
&E_i E_j = E_j E_i \quad \text{if } |i - j| \geqslant 2,\\
&T_i^2 = q^2 + \big(q^2 - 1\big)\big(T_i - qr^{-1} E_i\big),\\
&E_i^2 = x E_i.
\end{aligned}$$

In each case above, the indices i, j are chosen from all values $1 \leqslant i, j < n$ for which the given relation makes sense.

The algebra $\mathcal{B}_n(r, q)$ has been given a geometric formulation using Kauffman's tangle monoid (see [5]). Since this construction will not be used here, we refer the reader to [4] and [5] for details.

If $w \in \mathfrak{S}_n$ has a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_k}$, then

$$T_w = T_{i_1}T_{i_2}\cdots T_{i_k}$$

is a well defined element of $\mathcal{B}_n(r,q)$. The elements $\{T_w \mid w \in \mathfrak{S}_n\}$, though a set of algebra generators, do not generate $\mathcal{B}_n(r,q)$ as an R-module.

The next statement is now a straightforward consequence of the defining relations.

Lemma 2.1. *If $w \in \mathfrak{S}_n$ and s_k is an elementary transposition, then*

$$T_wT_k = \begin{cases} T_{ws_k} & \text{if } l(w) < l(ws_k), \\ q^2T_{ws_k} + (q^2-1)(T_w - qr^{-1}T_{ws_k}E_k) & \text{if } l(ws_k) < l(w); \end{cases}$$

and

$$T_kT_w = \begin{cases} T_{s_kw} & \text{if } l(w) < l(s_kw), \\ q^2T_{s_kw} + (q^2-1)(T_w - qr^{-1}E_kT_{s_kw}) & \text{if } l(s_kw) < l(w). \end{cases}$$

The map $*: B_n(r,q) \to B_n(r,q)$ defined by $*: T_w \mapsto T_{w^{-1}}$ and extended by linearity is an R-algebra anti-involution.

2.1. Specializations of the generic B-M-W algebras

Let $\hat{R}$ be a unital associative ring with unity. If $\phi: R \to \hat{R}$ is a ring homomorphism, then $\hat{R}$ is an R-module via the action $a \cdot \hat{a} = \phi(a)\hat{a}$ so the $\hat{R}$-algebra $\mathcal{B}_n(r,q) \otimes_R \hat{R}$ makes sense. The algebra $\mathcal{B}_n(r,q) \otimes_R \hat{R}$ will be called a specialization of $\mathcal{B}_n(r,q)$ to $\hat{R}$. Xi in [13] considers cellular bases for Birman–Murakami–Wenzl algebras over an arbitrary ring, but for many applications we will be interested in the case where $\hat{R}$ is replaced by the rational function field $\kappa = \mathbb{C}(\hat{r},\hat{q})$ and $\phi: R \to \kappa$ is the ring homomorphism determined by $\phi: r \mapsto \hat{r}$ and $\phi: q \mapsto \hat{q}$. By this specialization we recover the usual Birman–Murakami–Wenzl algebra $B_n(\hat{r},\hat{q})$ over κ.

Proposition 2.1. *Let $\kappa = \mathbb{C}(\hat{r},\hat{q})$ and $\phi: R \to \kappa$ be the ring homomorphism determined by $\phi: r \mapsto \hat{r}$ and $\phi: q \mapsto \hat{q}$. Then the homomorphism of κ-algebras $\sigma: \mathcal{B}_n(r,q) \otimes_R \kappa \to B_n(\hat{r},\hat{q})$ determined by $T_i \otimes 1 \mapsto T_i$ is a surjective map of κ-algebras.*

2.2. The Brauer algebras

An *n-Brauer diagram* is defined to be a graph with n-edges and $2n$ vertices arranged in two rows of n vertices and such that each vertex is incident to exactly one edge. An example of a 5-Brauer diagram is given in Fig. 1.

Let y be an indeterminate over $\mathbb{Z}$. The generic Brauer algebra $\mathcal{B}_n(y)$ is the $\mathbb{Z}[y]$-algebra which takes as a $\mathbb{Z}[y]$-basis the collection of n-Brauer diagrams which will be multiplied by a concatenation product defined as follows: given n-Brauer diagrams d_1 and d_2, the product d_1d_2 in $\mathcal{B}_n(y)$ is obtained by placing d_1 above d_2, identifying vertices in the bottom

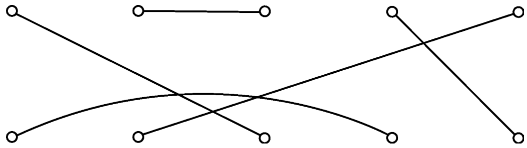

Fig. 1. An example of a 5-Brauer diagram.

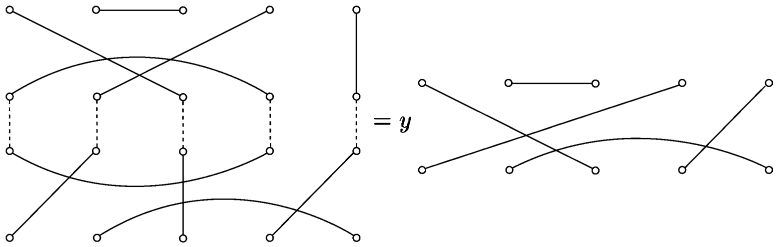

Fig. 2. The concatenation product on Brauer diagrams.

row of d_1 with the corresponding vertices in the top row of d_2, deleting any closed loops in the concatenation and multiplying the resulting graph by a factor of y for each deleted loop. In the example of Fig. 2, the two 5-diagrams are multiplied to give a figure containing a single closed loop.

The number of n-Brauer diagrams is $(2n-1)(2n-3)\cdots 3\cdot 1$ and is equal to the dimension of $\mathcal{B}_n(y)$ over $\mathbb{Z}[y]$. To recover the usual Brauer algebra $B_n(\hat{y})$ over $\mathbb{C}(\hat{y})$, let $\phi:\mathbb{Z}[y]\to\mathbb{C}(\hat{y})$ be the ring homomorphism determined by $y\mapsto\hat{y}$. Then, as in Proposition 2.1, $\mathbb{C}(\hat{y})$ becomes a $\mathbb{Z}[y]$-module and there is a surjective $\mathbb{C}(\hat{y})$-algebra homomorphism $\mathcal{B}(y)\otimes_{\mathbb{Z}[y]}\mathbb{C}(\hat{y})\to B_n(\hat{y})$ given by $d\otimes 1\mapsto d$ for each n-Brauer diagram d.

The next theorem (Theorem 3.5 of [12]) demonstrates, under certain generic restrictions, a close relation between the Wedderburn decomposition of the Brauer algebras and the Wedderburn decomposition of the B-M-W algebras.

The *numerical invariants* of a semi-simple algebra A are the dimensions of the simple A-modules.

Theorem 2.1. *Suppose that $\hat{r}$ is not an integral power of $\hat{q}$, that $\hat{q}$ is not a root of unity and that $\hat{y}$ is not an integer. Then the algebras $B_n(\hat{r},\hat{q})=\mathcal{B}_n(r,q)\otimes_R\mathbb{C}(\hat{r},\hat{q})$ and $B_n(\hat{y})=\mathcal{B}_n(y)\otimes_{\mathbb{Z}[y]}\mathbb{C}(\hat{y})$ are semi-simple and have the same numerical invariants. In particular,*

$$B_n(\hat{r},\hat{q})\cong\bigoplus_{f=0}^{[n/2]}\bigoplus_{\lambda\in\Gamma_f}C_{f,\lambda}$$

where $C_{f,\lambda}$ is a full matrix ring and $\Gamma_f=\{\lambda\mid\lambda\vdash n-2f\}$ for each $0\leqslant f\leqslant[n/2]$.

3. Cellular bases for the B-M-W algebras

The remainder of this paper is devoted to the construction of cellular bases and cell modules for the algebra $\mathcal{B}_n(r,q)$. We will, as in Theorem 2.1, let f denote an integer $0 \leqslant f \leqslant [n/2]$ and set $\Gamma_f = \{\lambda \mid \lambda \vdash n-2f\}$. For the present purpose, a bipartition ν of n will be an ordered pair of partitions $(\nu^{(1)}, \nu^{(2)})$ where $\nu^{(1)} = (2^f)$ and $\nu^{(2)} \in \Gamma_f$ for some fixed $0 \leqslant f \leqslant [n/2]$. The diagram $[\nu]$ is the ordered pair of diagrams $[\nu] = ([\nu^{(1)}], [\nu^{(2)}])$ and a ν-bitableau $\mathfrak{t}$ is a bijection $\mathfrak{t}: [\nu^{(1)}] \cup [\nu^{(2)}] \to \{1,\dots,n\}$. By way of example, if $n=12$, $f=3$ and ν is the multi-partition $\nu = ((2^3),(3,2,1))$, then

$$\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 7&8&9\\\hline 10&11\\\cline{1-2} 12\\\cline{1-1}\end{array}\right) \quad \text{and} \quad \left(\begin{array}{|c|c|}\hline 2&7\\\hline 3&9\\\hline 4&11\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&10&5\\\hline 8&12\\\cline{1-2} 6\\\cline{1-1}\end{array}\right)$$

are both ν-bitableaux. If $\mathfrak{t}$ is a ν-bitableau, we write $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$ where $\mathfrak{t}^{(1)}$ is identified with the labeled diagram $\mathfrak{t}[\nu^{(1)}]$ and $\mathfrak{t}^{(2)}$ is identified with the labeled diagram $\mathfrak{t}[\nu^{(2)}]$.

Let $\nu = (\nu^{(1)}, \nu^{(2)})$ be a bi-partition as above. The bitableau $\mathfrak{t}^{\nu} = (\mathfrak{t}^{\nu^{(1)}}, \mathfrak{t}^{\nu^{(2)}})$ is the ν-bitableau in which the integers $1,2,\dots,2f$ are entered in increasing order from left to right along the rows of $[\nu^{(1)}]$ and, along the rows of $[\nu^{(2)}]$, the integers $2f+1$, $2f+2,\dots,n$ are entered in increasing order from left to right. For example, if $n=7$ and $\nu^{(2)} = (2,1)$, then

$$\mathfrak{t}^{\nu} = \left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 5&6\\\hline 7\\\cline{1-1}\end{array}\right).$$

A ν-bitableau $\mathfrak{t}$ is *row standard* if the entries in each row of $\mathfrak{t}^{(i)}$ increase read from left to right for $i=1,2$. A row standard ν-bitableau $\mathfrak{t}$ is *standard* if $\mathfrak{t}^{(1)} = \mathfrak{t}^{\nu^{(1)}}$ and the entries in each column of $\mathfrak{t}^{(2)}$ is an increasing sequence read from top to bottom. The collection of standard ν-bitableaux will be denoted by $\mathrm{Std}(\nu)$.

The symmetric group $\mathfrak{S}_n$ acts from the right on the set of ν-bitableaux by permuting the entries of each bitableaux. For example, if $\mathfrak{t} = \left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 5&6\\\hline 7\\\cline{1-1}\end{array}\right)$, then

$$\mathfrak{t}(2,3)(4,7,6,5) = \left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&7\\\hline\end{array}, \begin{array}{|c|c|}\hline 4&5\\\hline 6\\\cline{1-1}\end{array}\right).$$

If $\mathfrak{t}$ is a ν-bitableau, we define $d(\mathfrak{t})$ to be the unique element of $\mathfrak{S}_n$ such that $\mathfrak{t}^{\nu} d(\mathfrak{t}) = \mathfrak{t}$.

For present purposes, the Young subgroup

$$\mathfrak{S}_{\nu} = \mathfrak{S}_{\nu_1^{(2)}} \times \dots \times \mathfrak{S}_{\nu_n^{(2)}}$$

of $\mathfrak{S}_n$ will be the row stabilizer of $\mathfrak{t}^{\nu^{(2)}}$ in $\mathfrak{S}_n$; that is to say

$$\mathfrak{S}_{\nu} = \left\langle s_i \;\middle|\; i, i+1 \text{ are in the same row of } \mathfrak{t}^{\nu^{(2)}} \right\rangle.$$

For $f > 0$ we define $\mathfrak{B}_f$ to be the subgroup of $\mathfrak{S}_n$

$$\mathfrak{B}_f = \langle s_1, s_{2i-1}s_{2i+1}s_{2i} \mid 1 \leqslant i < f \rangle$$

and set $\mathfrak{B}_f = \langle 1 \rangle$ when $f = 0$. For example, when $n = 7$ and $\nu^{(2)} = (2, 1)$ we have $\mathfrak{S}_\nu = \langle s_5 \rangle$ and $\mathfrak{B}_f = \langle s_1, s_2s_1s_3s_2 \rangle$. Note that if $\nu^{(2)} \vdash n - 2f$ and $f > 0$, then $\mathfrak{B}_f$ is the subgroup of $\mathfrak{S}_n$ which permutes the rows of $\mathfrak{t}^{\nu^{(1)}}$ and that $\mathfrak{B}_f$ is isomorphic to the hyperoctahedral group $\mathbb{Z}_2 \wr \mathfrak{S}_f$.

In the next proposition, we obtain a complete set $\mathscr{D}_\nu$ of coset representatives for $\mathfrak{B}_f\mathfrak{S}_\nu$ in $\mathfrak{S}_n$. In this statement, when $f = 0$ and $\nu \vdash n$, we recover the result that $\mathscr{D}_\nu$ is the usual set of distinguished coset representatives for the parabolic subgroup $\mathfrak{S}_\nu$ in $\mathfrak{S}_n$ (Proposition 3.3 of [8]).

Proposition 3.1. *Let $\nu^{(2)} \vdash n - 2f$ be a partition and $\mathfrak{t}^\nu = (\mathfrak{t}^{\nu(1)}, \mathfrak{t}^{\nu^{(2)}})$ and*

$$\mathscr{D}_\nu = \left\{ d \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)}) = \mathfrak{t}^\nu d \text{ is row standard and the first column of } \mathfrak{t}^{(1)} \\ \text{is an increasing sequence when read from top to bottom} \end{array} \right\}.$$

Then $\mathscr{D}_\nu$ is a complete set of right coset representatives for $\mathfrak{B}_f \times \mathfrak{S}_\nu$ in $\mathfrak{S}_n$. Moreover, if $d \in \mathscr{D}_\nu$, then $l(wd) = l(w) + l(d)$ for all $w \in \mathfrak{S}_\nu$.

Proof. Suppose that $\mathfrak{B}_f\mathfrak{S}_\nu v = \mathfrak{B}_f\mathfrak{S}_\nu w$ and let $\mathfrak{t}^\nu v = (\mathfrak{v}^{(1)}, \mathfrak{v}^{(2)})$ and $\mathfrak{t}^\nu w = (\mathfrak{u}^{(1)}, \mathfrak{u}^{(2)})$. Then $\mathfrak{v}^{(1)}$ and $\mathfrak{u}^{(1)}$ differ by a reordering of the entries of each row, and a permutation of the rows while $\mathfrak{v}^{(2)}$ and $\mathfrak{u}^{(2)}$ differ by a reordering of the entries of each row. Therefore $\mathscr{D}_\nu$ is a complete set of coset representatives for $\mathfrak{B}_f\mathfrak{S}_\nu$ in $\mathfrak{S}_n$. Now fix $d \in \mathscr{D}_\nu$; then $\mathfrak{t}^\nu d$ is row standard so, $(k)d < (k+1)d$ whenever k and $k+1$ are in the same row of $\mathfrak{t}^{\nu^{(2)}}$. In particular $l(s_kd) = l(d) + 1$ whenever $s_k \in \mathfrak{S}_\nu$.

Now suppose that $w \in \mathfrak{S}_\nu$ and that $l(w) > 1$. Then $w = s_kv$ and $l(w) = l(v) + 1$ for some $s_k \in \mathfrak{S}_\nu$; therefore $(k)v < (k+1)v$. Now $(k)v$ and $(k+1)v$ belong to the same row of $\mathfrak{t}^{\nu^{(2)}}$, so $(k)vd < (k+1)vd$ and hence $l(s_kvd) = l(vd) + 1$. By induction therefore, $l(wd) = l(s_kvd) = l(vd) + 1 = l(v) + l(d) + 1 = l(w) + l(d)$ as required. $\square$

We record the following useful fact for later reference.

Proposition 3.2. *Let $0 \leqslant f \leqslant [n/2]$, $\mu \vdash n - 2f$ and $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$ be a standard μ-bitableau. If ν is the bi-partition with $\nu^{(2)} = (n - 2f)$ and $w \in \mathscr{D}_\nu$, then $l(d(\mathfrak{t})w) = l(d(\mathfrak{t})) + l(w)$, $\mathfrak{t}^{(1)}w = \mathfrak{t}^{\nu^{(1)}}w$ and $d(\mathfrak{t})w \in \mathscr{D}_\mu$.*

Proof. That $l(d(\mathfrak{t})w) = l(d(\mathfrak{t})) + l(w)$ follows from the previous Proposition 3.1 in the following manner. We note that $\mathfrak{t}^\mu w$ must be row standard; therefore $l(vw) = l(v) + l(w)$ for all $v \in \mathfrak{S}_\nu$; in particular, since $d(\mathfrak{t}) \in \mathfrak{S}_\nu$ we have $l(d(\mathfrak{t})w) = l(d(\mathfrak{t})) + l(w)$. $\square$

Example 3.1. Let $n = 7$, $f = 2$, $\mu = (2, 2, 1)$. Then $\nu^{(2)} = (5)$ and if we take

$$\mathfrak{t} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 5 & 8 \\ \hline 6 & 9 \\ \hline 7 \\ \cline{1-1} \end{array} \right)$$

and $w = (1, 3, 5)(2, 9, 4, 8, 7)$, we observe that

$$\mathfrak{t}^{\nu} w = \left(\begin{array}{|c|c|}\hline 3 & 9 \\ \hline 5 & 8 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 6 & 7 \\ \hline \end{array} \right)$$

so $w \in \mathscr{D}_{\nu}$. Now, $l(d(\mathfrak{t})w) = l(d(\mathfrak{t})) + l(w)$ and

$$\mathfrak{t} w = \left(\begin{array}{|c|c|}\hline 3 & 9 \\ \hline 5 & 8 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 6 \\ \hline 2 & 7 \\ \hline 4 \\ \cline{1-1} \end{array} \right)$$

so $d(\mathfrak{t})w \in \mathscr{D}_{\mu}$.

For $0 \leqslant f < [n/2]$, we regard $\mathcal{H}_{R,n-2f}(q^2)$ as the subalgebra of $\mathcal{H}_{R,n}(q^2)$ generated by $\{T_i \mid 2f < i < n\}$ and set $\mathcal{H}_{R,n-2f}(q^2) = R$ when $f = [n/2]$. We therefore have a decreasing family of R-algebras

$$\mathcal{H}_{R,n}\big(q^2\big) \supset \mathcal{H}_{R,n-2}\big(q^2\big) \supset \cdots \supset R.$$

Similarly, regard $\mathcal{B}_{n-2f}(r, q)$ as the subalgebra of $\mathcal{B}_n(r, q)$ generated by $\{T_i \mid 2f < i < n\}$ and set $\mathcal{B}_{n-2f}(r, q) = R$ when $f = [n/2]$; thus,

$$\mathcal{B}_n(r, q) \supset \mathcal{B}_{n-2}(r, q) \supset \cdots \supset R.$$

For $0 \leqslant f < [n/2]$, let I_f be the two-sided ideal of $\mathcal{B}_{n-2f}(r, q)$ generated by E_{n-1} and let B^f be the two-sided ideal of $\mathcal{B}_n(r, q)$ generated by the element $E_1 E_3 \cdots E_{2f-1}$. If $f = [n/2]$, set $I_f = \{0\}$ and $B^f = \{0\}$. Then,

$$\mathcal{B}_n(r, q) = B^0 \supset B^1 \supset \cdots \supset \{0\} \tag{3.1}$$

is a filtration of $\mathcal{B}_n(r, q)$ by two sided ideals.

By Proposition 3.2 of [12], the map $\phi : \mathcal{H}_{R,n-2f}(q^2) \to \mathcal{B}_{n-2f}(r, q)/I_f$ defined on generators by

$$\phi : X_i \mapsto T_i + I_f, \quad 2f < i < n, \tag{3.2}$$

is an algebra isomorphism. Proposition 3.3 below shows that

$$E_1 E_3 \cdots E_{2f-1} I_f \subseteq B^{f+1}$$

from which it follows that the map defined on generators by

$$\iota : X_w \mapsto E_1 E_3 \cdots E_{2f-1} \cdot b + B^{f+1} \quad \text{where } \phi(X_w) = b + I_f, \tag{3.3}$$

is a well-defined R-module homomorphism, $\iota : \mathcal{H}_{R,n-2f}(q^2) \to B^f/B^{f+1}$. The map ι will allow us to produce, by passing to quotients, a cellular structure on B^f/B^{f+1} corresponding to a cellular structure on $\mathcal{H}_{R,n-2f}(q^2)$. The cellular structure on B^f/B^{f+1} will be used to refine the filtration (3.1), thereby obtaining a cellular basis for $\mathcal{B}_n(r,q)$.

The two propositions below are technical statements which will be important in later calculations.

Proposition 3.3. *Let* $0 \leqslant f < [n/2]$ *and* $b \in \mathcal{B}_{n-2f}(r,q)$. *If* $i \geqslant 2f+1$, *then*

$$E_1 E_3 \cdots E_{2f-1} b E_i \equiv 0 \mod B^{f+1}.$$

Proof. First note that

$$T_i T_{i+1} E_i T_{i+1} T_i = q^2 E_{i+1} E_i T_{i+1} T_i = q^4 E_{i+1} E_i E_{i+1} = q^4 E_{i+1},$$

from which it follows that $E_i = q^m T_v E_{2f+1} T_v^*$, where v has a reduced expression $v = s_{2f+2} s_{2f+1} s_{2f+3} s_{2f+2} \cdots s_{i-1} s_{i-2} s_i s_{i-1}$ and m is the integer $m = 4(2f+1-i)$. Since b, E_i and T_v commute with E_k whenever $k \leqslant 2f-1$,

$$E_1 E_3 \cdots E_{2f-1} b E_i = q^m b T_v^* E_1 E_3 \cdots E_{2f+1} T_v \equiv 0 \mod B^{f+1}$$

which proves the claim. □

Proposition 3.4. *Let* $w \in \mathfrak{S}_n$. *If* $(i)w = (k)$ *and* $(i+1)w = k+1$, *then* $E_i T_w = T_w E_k$.

Proof. For $j = 1, 2, \ldots, n$, let $a_j = (j)w$. In the first instance, suppose that $a_j < k+1$ for some $j > i+1$ and let $m = \min\{j \mid a_j < k+1 \text{ and } i < j\}$. Then $(m-1)w > (m)w$, so w has a reduced word beginning with s_{m-1}; in particular $w = uv$ where $u = s_{m-1} s_{m-2} s_{m-3} \cdots s_{i+2} s_{i+1} s_i$ and $l(w) = l(u) + l(v)$. Using the relation $E_i T_{i+1} T_i = T_{i+1} T_i E_{i+1}$ we have

$$\begin{aligned} E_i T_w = E_i T_u T_v &= E_i T_{m-1} T_{m-2} \cdots T_{i+2} T_{i+1} T_i T_v \\ &= T_{m-1} T_{m-2} \cdots T_{i+2} T_{i+1} T_i E_{i+1} T_v. \end{aligned}$$

Now $(i+1)v = k$, $(i+2)v = (k+1)$ and $l(v) < l(w)$ so, by induction it follows that $E_{i+1} T_v = T_v E_k$.

In the second instance, suppose that $a_j > k+1$ whenever $j > i+1$. If $a_j < k$ whenever $j < i$, then $k = i$ and $w = uv$ where u is a permutation on $\{1, 2, \ldots, k-1\}$ and v is a permutation on $\{k+2, k+3, \ldots, n\}$; in particular T_w commutes with E_k. Suppose therefore that $a_j > k$ for some $j < i$ and let $m = \max\{a_j \mid a_j > k \text{ and } i > j\}$. Then w has a reduced word ending in s_{m-1} and we can write $w = uv$ where $v = s_k s_{k+1} s_{i+2} \cdots s_{m-2} s_{m-1}$ and $l(w) = l(u) + l(v)$. By the relation $T_i T_{i+1} E_i = E_{i+1} T_i T_{i+1}$,

$$\begin{aligned} T_w E_k &= T_u T_v E_k = T_u T_k T_{k+1} T_{k+2} \cdots T_{m-2} T_{m-1} E_k \\ &= T_u E_{k+1} T_k T_{k+1} T_{k+2} \cdots T_{m-2} T_{m-1} \\ &= T_u E_{k+1} T_v = E_i T_u T_v \end{aligned}$$

where the last line follows by induction using the fact that $l(u) < l(w)$. □

We now take for each $0 \leqslant f < [n/2]$ a cellular basis $(\mathscr{C}_f, \Lambda_f)$ for $\mathcal{H}_{R,n-2f}(q^2)$; that is for each f, the collection $\mathscr{C}_f = \{c^\lambda_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda),\ \lambda \in \Lambda_f\}$ is a free R-basis for $\mathcal{H}_{R,n-2f}(q^2)$ satisfying the following conditions:

(1) the map $*: c^\lambda_{\mathfrak{v}\mathfrak{u}} \mapsto c^\lambda_{\mathfrak{u}\mathfrak{v}}$ is an anti-involution of $\mathcal{H}_{R,n-2f}(q^2)$; we assume that the map $*$ coincides with the anti-involution determined by $T_w \mapsto T_{w^{-1}}$ and;
(2) given $\lambda \in \Lambda_f$, $\mathfrak{u} \in \mathcal{I}_\lambda$ and $h \in \mathcal{H}_{R,n-2f}(q^2)$, there exist $a_\mathfrak{s}$, for $\mathfrak{s} \in \mathcal{I}_f(\lambda)$, such that

$$c^\lambda_{\mathfrak{v}\mathfrak{u}} h \equiv \sum_{\mathfrak{s}} a_\mathfrak{s} c^\lambda_{\mathfrak{v}\mathfrak{s}} \mod \check{A}^\lambda \tag{3.4}$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$, where $\check{A}^\lambda$ is the R-module generated by the elements $c^\mu_{\mathfrak{v}\mathfrak{u}}$ for $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda)$ and $\mu > \lambda$.

The Murphy basis for $\mathcal{H}_{R,n-2f}(q^2)$ given in Theorem 1.2 is one possible choice for $(\mathscr{C}_f, \Lambda_f)$. For $\lambda \in \Lambda_f$, we let A^λ be the R-submodule of $\mathcal{H}_{R,n-2f}(q^2)$ generated by the elements

$$\left\{ c^\mu_{\mathfrak{v}\mathfrak{u}} \;\middle|\; \mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\mu) \text{ and } \mu \geqslant \lambda \right\},$$

so that $\check{A}^\lambda = \sum_{\mu > \lambda} A^\mu$. To each $c_{\mathfrak{v}\mathfrak{u}} = c^\lambda_{\mathfrak{v}\mathfrak{u}}$ in $\mathscr{C}_f$ associate an element $b_{\mathfrak{v}\mathfrak{u}} = b^\lambda_{\mathfrak{v}\mathfrak{u}}$ in B^f / B^{f+1} as

$$b_{\mathfrak{v}\mathfrak{u}} = \iota(c_{\mathfrak{v}\mathfrak{u}}) = E_1 E_3 \cdots E_{2f-1} \cdot \tilde{c}_{\mathfrak{v}\mathfrak{u}} + B^{f+1}$$

where ι is defined by (3.3), so that $\tilde{c}_{\mathfrak{v}\mathfrak{u}}$ is a coset representative for $\phi(c_{\mathfrak{v}\mathfrak{u}})$ where ϕ is the isomorphism (3.2). Note that if $f = [n/2]$, then $\mathcal{H}_{R,n-2f}(q^2) = R$ can be given a formal cellular structure with $\Lambda_f = \{\lambda\}$, $\mathcal{I}_f(\lambda) = \{\mathfrak{v}\}$, $c_{\mathfrak{v}\mathfrak{v}} = 1_R$ and consequently,

$$b_{\mathfrak{v}\mathfrak{v}} := E_1 E_3 \cdots E_{2f-1}.$$

Since B^f / B^{f+1} is a invariant under the left and right actions of $\mathcal{B}_n(r, q)$, we define $B^\lambda \subseteq B^f / B^{f+1}$ to be the $\mathcal{B}_n(r, q)$-bimodule generated by the elements

$$\{b_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\mu) \text{ and } \mu \geqslant \lambda\}$$

and $\check{B}^\lambda \subseteq B^f/B^{f+1}$ to be the $\mathcal{B}_n(r,q)$-bimodule generated by the elements

$$\{b_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\mu) \text{ and } \mu > \lambda\}.$$

For a fixed $\mathfrak{v} \in \mathcal{I}_f(\lambda)$, we denote by $C^\lambda_{\mathfrak{v}}$ be the right $\mathcal{B}_n(r,q)$-submodule of $B^\lambda/\check{B}^\lambda$ generated by

$$\{b_{\mathfrak{v}\mathfrak{u}} + \check{B}^\lambda \mid \mathfrak{u} \in \mathcal{I}_f(\lambda)\}.$$

The following statements are easy consequences of the definitions.

Proposition 3.5. *Let* $0 \leqslant f \leqslant [n/2]$ *and* $\lambda \in \Lambda_f$. *Then*

(1) $B^f/B^{f+1} = \sum_{\lambda \in \Lambda_f} B^\lambda$;
(2) $\check{B}^\lambda \subseteq B^\lambda$;
(3) $\iota(A^\lambda) \subseteq B^\lambda$ *and* $\iota(\check{A}^\lambda) \subseteq \check{B}^\lambda$.

Proof. By assumption there are $c^\lambda_{\mathfrak{u}\mathfrak{v}} \in \mathscr{C}_f$ such that $1_{\mathcal{H}_{R,n-2f}(q^2)} = \sum c^\lambda_{\mathfrak{u}\mathfrak{v}}$, whence

$$E_1E_3\cdots E_{2f-1} + B^{f+1} = \sum \iota(c^\lambda_{\mathfrak{u}\mathfrak{v}}) = \sum b^\lambda_{\mathfrak{u}\mathfrak{v}}$$

which proves the first item. The statements (2) and (3) follow directly from the definitions. □

In a sequence of lemmas below we establish R-bases for each of the $\mathcal{B}_n(r,q)$-modules B^λ, $\check{B}^\lambda$ and $C^\lambda_{\mathfrak{v}}$ expressed, in each case, in terms of $\mathscr{C}_f$ and $\mathscr{D}_\nu$ where $\nu = (\nu^{(1)}, \nu^{(2)})$ is the bi-partition with $\nu^{(1)} = (2^f)$ and $\nu^{(2)} = (n-2f)$.

Proposition 3.6. *Suppose that* $0 \leqslant f < [n/2]$, $\lambda \in \Lambda_f$, $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda)$ *and let* ν *be the bi-partition with* $\nu^{(2)} = (n-2f)$. *If* $w \in \mathscr{D}_\nu$, $2f < (k)w^{-1}$ *and* $2f < (k+1)w^{-1}$, *then* $b_{\mathfrak{v}\mathfrak{u}}T_wE_k \equiv 0 \bmod B^{f+1}$.

Proof. Now $w \in \mathscr{D}_\nu$ so $(i)w = k$ and $(i+1)w = k+1$ for some i with $2f+1 \leqslant i < n$. By Proposition 3.4, we have $b_{\mathfrak{v}\mathfrak{u}}T_wE_k = b_{\mathfrak{v}\mathfrak{u}}E_iT_w$ so, from Proposition 3.3, it follows that the term $b_{\mathfrak{v}\mathfrak{u}}E_iT_w$ lies in the ideal B^{f+1}. □

Lemma 3.1. *Let* $0 \leqslant f \leqslant [n/2]$, *and* ν *be the bi-partition with* $\nu^{(2)} = (n-2f)$. *If* $\lambda \in \Lambda_f$, $\mathfrak{u} \in \mathcal{I}_f(\lambda)$ *and* $w \in \mathfrak{S}_\nu$, *then there exist* $\mathfrak{s} \in \mathcal{I}_f(\lambda)$ *and* $a_{\mathfrak{s}} \in R$ *determined by* (1.7) *such that*

$$b_{\mathfrak{v}\mathfrak{u}}T_w \equiv \sum_{\mathfrak{s}} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}} \mod \check{B}^\lambda$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$.

Proof. Observe that $T_w \in \mathcal{B}_{n-2f}(r,q) \subseteq \mathcal{B}_n(r,q)$. Let $\tilde{c}_{\mathfrak{v}\mathfrak{u}} \in \mathcal{B}_{n-2f}(r,q)$ be such that $\phi(c_{\mathfrak{v}\mathfrak{u}}) = \tilde{c}_{\mathfrak{v}\mathfrak{u}} + I_f$. Since $\phi(X_w) = T_w + I_f$ and ϕ is a homomorphism, we have

$$\begin{aligned} b_{\mathfrak{v}\mathfrak{u}} T_w &= \iota(c_{\mathfrak{v}\mathfrak{u}}) T_w = \big(E_1 \cdots E_{2f-1} \cdot \tilde{c}_{\mathfrak{v}\mathfrak{u}} + B^{f+1}\big) T_w \\ &= E_1 \cdots E_{2f-1} \cdot \tilde{c}_{\mathfrak{v}\mathfrak{u}} T_w + B^{f+1} = \iota(c_{\mathfrak{v}\mathfrak{u}} X_w). \end{aligned}$$

By (3.4), there exist $a_{\mathfrak{s}}$, for $\mathfrak{s} \in \mathcal{I}(\lambda)$, and $h \in \check{A}^\lambda$, such that for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$,

$$\iota(c_{\mathfrak{v}\mathfrak{u}} X_w) = \sum_{\mathfrak{s}} a_{\mathfrak{s}} \iota(c_{\mathfrak{v}\mathfrak{s}}) + \iota(h) = \sum_{\mathfrak{s}} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}} + \iota(h).$$

Therefore, by the inclusions (3) of Proposition 3.5, we have shown that for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$,

$$\iota(c_{\mathfrak{v}\mathfrak{u}} X_w) \equiv \sum_{\mathfrak{s}} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}} \mod \check{B}^\lambda.$$

This completes the proof of the lemma. □

Lemma 3.2. *Let $0 \leqslant f \leqslant [n/2]$, $\lambda \in \Lambda_f$ and ν be the bi-partition with $\nu^{(2)} = (n-2f)$. If $\mathfrak{u} \in \mathcal{I}_f(\lambda)$ and $w \in \mathfrak{S}_n$, then there exist elements $v \in \mathfrak{S}_n$, where $\mathfrak{t}^\nu v$ is row standard, together with $a_{\mathfrak{s}} \in R$, for $\mathfrak{s} \in \mathcal{I}_f(\lambda)$, such that*

$$b_{\mathfrak{v}\mathfrak{u}} T_w \equiv \sum_{\mathfrak{s}} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}} T_v \mod \check{B}^\lambda$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$.

Proof. Suppose that the tableau $\mathfrak{t}^{\nu^{(2)}}$ contains more than one entry. If $2f < j < n$ and $(j+1)w < (j)w$, then $b_{\mathfrak{v}\mathfrak{u}} T_w = b_{\mathfrak{v}\mathfrak{u}} T_j T_{s_j w}$ where $l(s_j w) + 1 = l(w)$. Repeating this procedure, by Lemma 3.1, we may write

$$b_{\mathfrak{v}\mathfrak{u}} T_w = b_{\mathfrak{v}\mathfrak{u}} T_{v'} T_{w'} \equiv \sum_{\mathfrak{s} \in \mathcal{I}(\lambda)} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}} T_{w'} \mod \check{B}^\lambda \tag{3.5}$$

where $(j)w' < (j+1)w'$ whenever $2f < j < n$, and $a_{\mathfrak{s}} \in R$ depends only on $\mathfrak{u}$ and v'.

Now suppose that $\mathfrak{t}^{\nu^{(1)}}$ is not empty. If $1 \leqslant j \leqslant f$ and $(2j-1)w' < (2j)w'$, then $l(s_{2j-1}w') + 1 = l(w')$. From the relation $E_{2i-1} T_{2i-1} = qr^{-1} E_{2i-1}$ it follows that

$$b_{\mathfrak{v}\mathfrak{s}} T_{w'} = b_{\mathfrak{v}\mathfrak{s}} T_{2i-1} T_{s_{2i-1}w'} = qr^{-1} b_{\mathfrak{v}\mathfrak{s}} T_{s_{2i-1}w'}.$$

Repeating this procedure if necessary, we may rewrite each summand on the right-hand side of (3.5) as

$$\sum_{\mathfrak{s}} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}}\, T_{w'} = q^k r^{-k} \sum_{\mathfrak{s}} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}} T_v$$

where $\mathfrak{t}^\nu v$ is row standard. This completes the proof of the lemma. □

The next statement is the first step in rewriting $b_{\mathfrak{v}\mathfrak{u}}T_w$ as an R-linear combination of terms $b_{\mathfrak{v}\mathfrak{u}}T_v$, where $v \in \mathscr{D}_\nu$.

Proposition 3.7. *Let $1 \leqslant j < f \leqslant [n/2]$, $\lambda \in \Lambda_f$, $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda)$ and $w \in \mathfrak{S}_n$. If ν is the bi-partition with $\nu^{(2)} = (n-2f)$ and $\mathfrak{t}^\nu w$ is row standard with $(2j+1)w < (2j-1)w$, then*

$$b_{\mathfrak{v}\mathfrak{u}}T_w = \begin{cases} b_{\mathfrak{v}\mathfrak{u}}T_u & \text{if } (2j+2)w > (2j)w > (2j-1)w; \\ q^4 b_{\mathfrak{v}\mathfrak{u}}T_u & \text{if } (2j)w > (2j-1)w > (2j+2)w; \\ q^2 b_{\mathfrak{v}\mathfrak{u}}T_u + (q^2-1)b_{\mathfrak{v}\mathfrak{u}}T_v & \\ \quad - q^2(q^2-1)b_{\mathfrak{v}\mathfrak{u}}T_{v'} & \text{if } (2j)w > (2j+2)w > (2j-1)w, \end{cases}$$

where u, v and v' are given by $u = s_{2j}s_{2j+1}s_{2j-1}s_{2j}w$, $v = s_{2j}s_{2j-1}s_{2j}w$ and $v' = s_{2j+1}s_{2j-1}s_{2j}w$. Moreover, $\mathfrak{t}^\nu u$, $\mathfrak{t}^\nu v$ and $\mathfrak{t}^\nu v'$ are row standard bitableaux.

Proof. Since $\mathfrak{t}^\nu w$ is row standard and $(2j-1)w > (2j+1)w$ we use (1.2) to observe that $l(s_{2j-1}s_{2j}w) = l(w) - 2$. Let $u' = s_{2j-1}s_{2j}w$ and consider the following three minor cases.

Case 1. Suppose that $(2j+2)w > (2j)w > (2j-1)w > (2j+1)w$; then it is verified using (1.1) that $l(s_{2j}s_{2j+1}u') = l(u') + 2$. From the relation $E_iT_{i\pm1}T_i = q^2E_iE_{i\pm1}$,

$$\begin{aligned} E_{2j-1}E_{2j+1}T_w &= E_{2j-1}E_{2j+1}T_{2j}T_{2j-1}T_{u'} = q^2E_{2j-1}E_{2j+1}E_{2j} \\ &= E_{2j-1}E_{2j+1}T_{2j}T_{2j+1}T_{u'} = E_{2j-1}E_{2j+1}T_u \end{aligned}$$

where $u = s_{2j}s_{2j+1}u'$ and $\mathfrak{t}^\nu u$ is row standard.

Case 2. Suppose that $(2j)w > (2j-1)w > (2j+2)w > (2j+1)w$; then it is verified using (1.1) that $l(s_{2j}s_{2j+1}u') = l(u') - 2$. Hence,

$$\begin{aligned} E_{2j-1}E_{2j+1}T_w &= E_{2j-1}E_{2j+1}T_{2j}T_{2j-1}T_{u'} = E_{2j-1}E_{2j+1}T_{2j}T_{2j-1}T_{2j+1}T_{2j}T_u \\ &= q^2E_{2j-1}E_{2j+1}E_{2j}T_{2j+1}T_{2j}T_u = q^4E_{2j-1}E_{2j+1}E_{2j}E_{2j+1}T_u \\ &= q^4E_{2j-1}E_{2j+1}T_u \end{aligned}$$

where $u = s_{2j}s_{2j+1}u'$ and $\mathfrak{t}^\nu u$ is row standard, and we have applied the relations $q^2E_iE_{i\pm1} = E_iT_{i\pm1}T_i$ and $E_iE_{i\pm1}E_i = E_i$.

Case 3. If $(2j)w > (2j+2)w > (2j-1)w > (2j+1)w$, then $l(s_{2j+1}u') = l(u') - 1$, $l(s_{2j}u') = l(u') + 1$ and $l(s_{2j}s_{2j+1}u') = l(s_{2j+1}u') + 1$. Thus

$$\begin{aligned}E_{2j-1}E_{2j+1}T_w &= E_{2j-1}E_{2j+1}T_{2j}T_{2j-1}T_{u'} = E_{2j-1}E_{2j+1}T_{2j}T_{2j+1}T_{u'}\\ &= E_{2j-1}E_{2j+1}T_{2j}\big[q^2T_{v'} + \big(q^2-1\big)T_{u'} - qr^{-1}\big(q^2-1\big)E_{2j+1}T_{v'}\big]\\ &= E_{2j-1}E_{2j+1}\big[q^2T_u + \big(q^2-1\big)T_v - q^2\big(q^2-1\big)T_{v'}\big]\end{aligned}$$

where $u = s_{2j}s_{2j+1}u'$, $v = s_{2j}u'$, $v' = s_{2j+1}u'$ and $\mathfrak{t}^\nu u$, $\mathfrak{t}^\nu v$ and $\mathfrak{t}^\nu v'$ are row standard. □

Corollary 3.1. *Let* $0 < f \leqslant [n/2]$ *and* ν *be the bi-partition with* $\nu^{(2)} = (n-2f)$. *If* $\lambda \in \Lambda_f$ *and* $w \in \mathfrak{S}_n$ *and* $\mathfrak{t}^\nu w$ *is row standard, then there exist elements* $a_v \in R$, *for* $v \in \mathscr{D}_\nu$, *such that*

$$b_{\mathfrak{v}\mathfrak{u}}T_w = \sum_v a_v b_{\mathfrak{v}\mathfrak{u}}T_v$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$.

Proof. Now the preceding Proposition 3.7 describes precisely an algorithm which allows us to reorder the entries of $\mathfrak{t}^\nu w$ while rewriting $b_{\mathfrak{v}\mathfrak{u}}T_w$ as a sum in the required form. □

Example 3.2. Let $n = 6$, $f = 3$ and $w = (1, 3, 2, 4, 6, 5)$. Then

$$\nu^{(2)} = (0), \qquad \mathfrak{t}^{\nu^{(1)}}w = \begin{array}{|c|c|}\hline 3&4\\\hline 2&6\\\hline 1&5\\\hline\end{array}$$

and $\mathfrak{t}^{\nu^{(2)}}$ is the empty tableau.

Now apply the procedure given in Proposition 3.7, with $j = 2$:

$$b_{\mathfrak{v}\mathfrak{v}}T_w = q^2 b_{\mathfrak{v}\mathfrak{v}}T_u + \big(q^2-1\big)b_{\mathfrak{v}\mathfrak{v}}T_v - q^2\big(q^2-1\big)b_{\mathfrak{v}\mathfrak{v}}T_{v'} \tag{3.6}$$

where

$$\mathfrak{t}^{\nu^{(1)}}u = \begin{array}{|c|c|}\hline 3&4\\\hline 1&5\\\hline 2&6\\\hline\end{array}, \qquad \mathfrak{t}^{\nu^{(1)}}v = \begin{array}{|c|c|}\hline 3&4\\\hline 1&6\\\hline 2&5\\\hline\end{array}, \qquad \mathfrak{t}^{\nu^{(1)}}v' = \begin{array}{|c|c|}\hline 3&4\\\hline 1&2\\\hline 5&6\\\hline\end{array}$$

and $\mathfrak{t}^{\nu^{(2)}}u$, $\mathfrak{t}^{\nu^{(2)}}v$, $\mathfrak{t}^{\nu^{(2)}}v'$ are the empty tableau.

To express $b_{\mathfrak{v}\mathfrak{v}}T_w$ as a sum $\sum_{v\in\mathscr{D}_\nu} b_{\mathfrak{v}\mathfrak{v}}T_v$, one repeatedly applies Proposition 3.7 to each of the terms on the right-hand side of (3.6), as in Example 3.3 below.

Note that in Corollary 3.1 we have expressed $b_{\mathfrak{v}\mathfrak{u}}T_w$ as a linear combination of $b_{\mathfrak{v}\mathfrak{u}}T_v$ where, for each v, the sequence $(1)v, (3)v, \ldots, (2f-1)v$ is increasing. While we could have equally specified, for instance, that the sequence $(1)v, (3)v, \ldots, (2f-1)v$ be decreasing, our convention is consistent with the choice of coset representatives made in Proposition 3.1.

The following special case of Proposition 3.7, which shows that the subalgebra of $\mathcal{B}_n(r,q)$ generated by the elements $\{T_w \mid w \in \mathfrak{B}_f\}$ stabilizes $b_{\mathfrak{u}\mathfrak{v}}$, may be of interest in its own right.

Corollary 3.2. *Let $1 \leqslant j < f \leqslant [n/2]$, $\lambda \in \Lambda_f$, $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda)$ and $w \in \mathfrak{S}_n$. If $\mathfrak{t}^\nu w$ is row standard, $(2j-1)w = k$, $(2j)w = k+1$ and $(2j+1)w < k$, then*

$$b_{\mathfrak{v}\mathfrak{u}} T_w = \begin{cases} b_{\mathfrak{v}\mathfrak{u}} T_u & \text{if } (2j+2)w > k; \\ q^4 b_{\mathfrak{v}\mathfrak{u}} T_u & \text{if } k > (2j+2)w, \end{cases}$$

where $u = s_{2j}s_{2j+1}s_{2j-1}s_{2j}w$, $\mathfrak{t}^\nu u$ is row standard and $k, k+1$ are in the same row of the bitableau $\mathfrak{t}^\nu u$.

Proof. Observe that if $(2j-1)w = k$ and $(2j)w = k+1$, the third case $(2j)w > (2j+2)w > (2j-1)w$ of Proposition 3.7 cannot occur. Observe also that $s_{2j}s_{2j+1}s_{2j-1}s_{2j}$ interchanges the j and $j+1$ rows of $\mathfrak{t}^{\nu^{(1)}}$. □

Example 3.3. Let $n = 6$, $f = 3$ and $w = (1,3,2,4,6,5)$ as in Example 3.2. Then Corollary 3.2, allows us to rewrite the expression (3.6) as

$$b_{\mathfrak{v}\mathfrak{v}} T_w = q^2 b_{\mathfrak{v}\mathfrak{v}} T_{u'} + \bigl(q^2-1\bigr) b_{\mathfrak{v}\mathfrak{v}} T_{w'} - q^6\bigl(q^2-1\bigr) b_{\mathfrak{v}\mathfrak{v}} T_{v''}$$

where

$$\mathfrak{t}^{\nu^{(1)}}u' = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 & 4 \\ \hline 2 & 6 \\ \hline\end{array}, \qquad \mathfrak{t}^{\nu^{(1)}}w' = \begin{array}{|c|c|}\hline 1 & 6 \\ \hline 3 & 4 \\ \hline 2 & 5 \\ \hline\end{array}, \qquad \mathfrak{t}^{\nu^{(1)}}v'' = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline\end{array}$$

and $\mathfrak{t}^{\nu^{(2)}}u'$, $\mathfrak{t}^{\nu^{(2)}}w'$, $\mathfrak{t}^{\nu^{(2)}}v''$ are the empty tableau.

Lemma 3.3. *Let $0 \leqslant f \leqslant [n/2]$ and ν be the bi-partition with $\nu^{(2)} = (n-2f)$. If $\lambda \in \Lambda_f$, $\mathfrak{u} \in \mathcal{I}_f(\lambda)$ and $w \in \mathfrak{S}_n$, then there exist elements $a_v, a_\mathfrak{s} \in R$, for $v \in \mathscr{D}_\nu$ and $\mathfrak{s} \in \mathcal{I}_f(\lambda)$, such that*

$$b_{\mathfrak{v}\mathfrak{u}} T_w \equiv \sum_v a_v \sum_\mathfrak{s} a_\mathfrak{s} b_{\mathfrak{v}\mathfrak{s}} T_v \mod \check{B}^\lambda$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$.

Proof. By Lemma 3.2 there are $a_\mathfrak{s} \in R$, for $\mathfrak{s} \in \mathcal{I}_f(\lambda)$, and $u \in \mathfrak{S}_\mathfrak{n}$, such that $\mathfrak{t}^\nu u$ is row standard and

$$b_{\mathfrak{v}\mathfrak{u}} T_w \equiv \sum_\mathfrak{s} a_\mathfrak{s} b_{\mathfrak{v}\mathfrak{s}} T_u \mod \check{B}^\lambda$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$. Now, if $0 < f$ then, Corollary 3.1 allows us to rewrite each $b_{\mathfrak{v}\mathfrak{s}} T_u$ as

$$b_{\mathfrak{v}\mathfrak{s}}T_u \equiv \sum_v a_v \sum_{\mathfrak{s}} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}} T_v \mod \check{B}^\lambda$$

where $v \in \mathscr{D}_\nu$, thus completing the proof of the lemma. □

The next two propositions are technical statements which will play an important part in establishing the multiplicative properties of our cellular basis.

Proposition 3.8. *Let* $0 \leqslant i < f \leqslant [n/2]$, $w \in \mathfrak{S}_n$ *and suppose that* $(2i-1)w = k$ *and* $(2i)w = k+1$. *If* $i < j \leqslant f$ *and* $\eta = \pm 1$, $\varepsilon \in \{0, \pm 1\}$, *then there exist* $a_v \in R$, *for* $v \in \mathfrak{S}_n$, *such that*

$$E_{2j-1}T_{2j}^{\eta}T_{2j-1}^{\varepsilon}T_w = \sum_v a_v E_{2j-1}T_v.$$

Moreover, the sum is over $v \in \mathfrak{S}_n$ *with* $(2i-1)v = k$ *and* $(2i)v = k+1$.

Proof. The proof of the proposition will then be by inspecting six possible cases and four possible subcases. We first suppose that $\varepsilon \neq 0$.

Case 1. If $\varepsilon = 1$ and $l(s_{2j-1}w) > l(w)$ or if $\varepsilon = -1$ and $l(s_{2j-1}w) < l(w)$, then $E_{2j-1}T_{2j}^{\eta}T_{2j-1}^{\varepsilon}T_w = E_{2j-1}T_{2j}^{\eta}T_v$ where $v = s_{2j-1}w$.

Case 2. If $\varepsilon = 1$ and $l(s_{2j-1}w) < l(w)$, then

$$\begin{aligned} E_{2j-1}T_{2j}^{\eta}T_{2j-1}T_w &= E_{2j-1}T_{2j}^{\eta}\big[q^2T_v + \big(q^2-1\big)T_w - qr^{-1}\big(q^2-1\big)E_{2j-1}T_v\big] \\ &= E_{2j-1}T_{2j}^{\eta}\big[q^2T_v + \big(q^2-1\big)T_w\big] - q^{1+\eta}r^{\eta-1}\big(q^2-1\big)E_{2j-1}T_v \end{aligned}$$

where $v = s_{2j-1}w$ and the last line is obtained from the relation $E_{2j-1}T_{2j}^{\eta}E_{2j-1} = (qr)^{\eta}E_{2j-1}$.

Case 3. If $\varepsilon = -1$ and $l(s_{2j-1}w) > l(w)$, then we express T_{2j-1}^{-1} in terms of T_{2j-1} and E_{2j-1} to obtain

$$\begin{aligned} E_{2j-1}T_{2j}^{\eta}T_{2j-1}^{-1}T_w &= q^{-2}E_{2j-1}T_{2j}^{\eta}\big[T_{2j-1} - \big(q^2-1\big) + \big(q^2-1\big)E_{2j-1}\big]T_w \\ &= q^{-2}E_{2j-1}T_{2j}^{\eta}\big[T_v - \big(q^2-1\big)T_w\big] + q^{\eta-2}r^{\eta}\big(q^2-1\big)E_{2j-1}T_w \end{aligned}$$

where $v = s_{2j-1}w$ and the last line is again obtained from the relation $E_{2j-1}T_{2j}^{\eta}E_{2j-1} = (qr)^{\eta}E_{2j-1}$.

In each case above we have demonstrated how the term T_{2j-1}^{ε} can be eliminated from the product. To complete the proof of the proposition, we show that the term T_{2j}^{η} can be similarly eliminated from the resulting expressions which are of the form $E_{2j-1}T_{2j}^{\eta}T_w$ where w satisfies $(2i-1)w = k$ and $(2i)w = k+1$.

Case 4. If $\eta = 1$ and $l(s_{2j}w) > l(w)$ or $\eta = -1$ and $l(s_{2j}w) < l(w)$, then, as in the first case, $E_{2j-1}T_{2j}T_w = E_{2j-1}T_v$ where $v = s_{2j}w$.

Case 5. If $\eta = 1$ and $l(s_{2j}w) < l(w)$, then

$$\begin{aligned} E_{2j-1}T_{2j}T_w &= E_{2j-1}\big[q^2T_v + \big(q^2-1\big)T_w - qr^{-1}\big(q^2-1\big)E_{2j}T_v\big] \\ &= E_{2j-1}\big[q^2T_v + \big(q^2-1\big)T_w - (qr)^{-1}\big(q^2-1\big)T_{2j}T_{2j-1}T_v\big] \end{aligned} \tag{3.7}$$

where $v = s_{2j}w$ and the last line follows from the relation

$$E_{2j-1}E_{2j} = q^{-2}E_{2j-1}T_{2j}T_{2j-1}.$$

Now from Case 5, two subcases arise.

Subcase 5A. If $l(s_{2j-1}v) > l(v)$, it follows that $l(s_{2j}s_{2j-1}v) > l(s_{2j-1}v)$, so (3.7) becomes

$$E_{2j-1}T_{2j}T_w = E_{2j-1}\big[q^2T_v + \big(q^2-1\big)T_w - (qr)^{-1}\big(q^2-1\big)T_{v'}\big]$$

where $v' = s_{2j}s_{2j-1}v$ and $l(v') = l(v) + 2$.

Subcase 5B. If $l(s_{2j-1}v) < l(v)$, then (3.7) becomes

$$\begin{aligned} E_{2j-1}T_{2j}T_w &= E_{2j-1}\big[q^2T_v + \big(q^2-1\big)T_w\big] - (qr)^{-1}\big(q^2-1\big)E_{2j-1}T_{2j} \\ &\quad \times \big[q^2T_{v'} + \big(q^2-1\big)T_v - qr^{-1}\big(q^2-1\big)E_{2j-1}T_{v'}\big] \\ &= E_{2j-1}\big[q^2T_v + \big(q^2-1\big)T_w\big] \\ &\quad - (qr)^{-1}\big(q^2-1\big)E_{2j-1}T_{2j}\big[q^2T_{v'} + \big(q^2-1\big)T_v\big] \\ &\quad + r^{-2}\big(q^2-1\big)^2E_{2j-1}T_{2j}E_{2j-1}T_{v'} \\ &= E_{2j-1}\big[q^2T_v + \big(q^2-1\big)T_w + qr^{-1}\big(q^2-1\big)^2T_{v'}\big] \\ &\quad - (qr)^{-1}\big(q^2-1\big)E_{2j-1}T_{2j}\big[q^2T_{v'} + \big(q^2-1\big)T_v\big] \\ &= E_{2j-1}\big[q^2T_v + \big(q^2-1\big)\big[1 - r^{-1}\big(q-q^{-1}\big)\big]T_w + qr^{-1}\big(q^2-1\big)^2T_{v'}\big] \\ &\quad - qr^{-1}\big(q^2-1\big)E_{2j-1}T_{2j}T_{v'} \end{aligned}$$

where $v' = s_{2j-1}v$ and we have exploited the relation $E_{2j-1}T_{2j}E_{2j-1} = (qr)E_{2j-1}$.

Now the only term in the last expression for $E_{2j-1}T_{2j}T_w$ above which contains T_{2j} is the product $E_{2j-1}T_{2j}T_{v'}$; we note that $l(s_{2j-1}v') > l(v')$ so this term can be disposed of by applying either Case 4 when $l(s_{2j}v') > l(v')$ or Subcase 5A when $l(v') > l(s_{2j}v')$.

Case 6. If $\eta = -1$ and $l(s_{2j}w) > l(w)$, then

$$E_{2j-1}T_{2j}^{-1}T_w = q^{-2}E_{2j-1}\big[T_{2j} - \big(q^2-1\big) + \big(q^2-1\big)E_{2j}\big]T_w$$
$$= q^{-2}E_{2j-1}\big[T_v - \big(q^2-1\big)T_w + q^{-2}\big(q^2-1\big)T_{2j}T_{2j-1}T_w\big] \qquad (3.8)$$

where v is given by $v = s_{2j}w$ and we have used the relation $E_{2j-1}E_{2j} = q^{-2}E_{2j-1}T_{2j}T_{2j-1}$. We again have two subcases.

Subcase 6A. If $l(s_{2j-1}w) > l(w)$, then $l(s_{2j}s_{2j-1}w) > l(s_{2j-1}w)$, by (1.1); thus (3.8) becomes

$$E_{2j-1}T_{2j}^{-1}T_w = q^{-2}E_{2j-1}\big[T_v - \big(q^2-1\big)T_w + q^{-2}\big(q^2-1\big)T_{v'}\big]$$

where $v' = s_{2j}s_{2j-1}w$.

Subcase 6B. If $l(s_{2j-1}w) < l(w)$, then (3.8) becomes

$$\begin{aligned} E_{2j-1}T_{2j}^{-1}T_w &= q^{-2}E_{2j-1}\big[T_v - \big(q^2-1\big)T_w\big] \\ &\quad + q^{-2}\big(q^2-1\big)T_{2j}\big[q^2T_u + \big(q^2-1\big)T_w - qr^{-1}\big(q^2-1\big)E_{2j-1}T_u\big] \\ &= q^{-2}E_{2j-1}\big[T_v - \big(q^2-1\big)T_w - q^2\big(q^2-1\big)^2T_u\big] \\ &\quad + q^{-2}\big(q^2-1\big)T_{2j}\big[q^2T_u + \big(q^2-1\big)T_w\big] \end{aligned}$$

where $u = s_{2j-1}w$. To complete the proof of the proposition, we note that $E_{2j-1}T_{2j}T_w$ and $E_{2j-1}T_{2j}T_u$ can be eliminated from last expression, as was done in Case 5, without reintroducing any terms involving T_{2j}^{-1}. $\square$

The next proposition is similar to Proposition 3.8 except that here we work modulo the ideal B^{f+1}.

Proposition 3.9. *Let $0 \leqslant i \leqslant f \leqslant [n/2]$, $w \in \mathfrak{S}_n$ and suppose that $(2i-1)w = k$ and $(2i)w = k+1$. If $2f < j < n$ and $\varepsilon = \pm 1$, then there exist $a_v \in R$, for $v \in \mathfrak{S}_n$, such that*

$$E_1E_3\cdots E_{2f-1}T_j^{\varepsilon}T_w \equiv \sum_v a_vE_1E_3\cdots E_{2f-1}T_v \mod B^{f+1}.$$

Moreover, the sum is over $v \in \mathfrak{S}_n$ with $(2i-1)v = k$ and $(2i)v = k+1$.

Proof.

Case 1. If $\varepsilon = 1$ and $l(s_jw) > l(w)$ or $\varepsilon = -1$ and $l(s_jw) < l(w)$, then $T_j^{\varepsilon}T_w = T_v$ where $v = s_jw$.

Case 2. If $\varepsilon = 1$ and $l(s_j w) < l(w)$, then

$$\begin{aligned} &E_1E_3 \cdots E_{2f-1}T_jT_w \\ &\quad = E_1E_3 \cdots E_{2f-1}\big[q^2T_v + \big(q^2-1\big)T_w - qr^{-1}\big(q^2-1\big)E_jT_v\big] \\ &\quad \equiv E_1E_3 \cdots E_{2f-1}\big[q^2T_v + \big(q^2-1\big)T_w\big] \mod B^{f+1} \end{aligned}$$

where $v = s_j w$ and the last line follows from Proposition 3.3.

Case 3. If $\varepsilon = -1$ and $l(s_j w) > l(w)$, then, by Proposition 3.3, we again have

$$\begin{aligned} &E_1E_3 \cdots E_{2f-1}T_j^{-1}T_w \\ &\quad = q^{-2}E_1E_3 \cdots E_{2f-1}\big[T_j - \big(q^2-1\big) + \big(q^2-1\big)E_j\big]T_w \\ &\quad \equiv q^{-2}E_1E_3 \cdots E_{2f-1}\big[T_v - \big(q^2-1\big)T_w\big] \mod B^{f+1} \end{aligned}$$

where $v = s_j w$. $\square$

We now consider terms $E_1E_3 \cdots E_{2f-1}T_wE_k$ for $(k)w^{-1} < (k+1)w^{-1} \leqslant 2f$.

Lemma 3.4. *Let* $0 < f \leqslant [n/2]$ *and* ν *be the bi-partition with* $\nu^{(2)} = (2f-1)$. *Suppose that* $\lambda \in \Lambda_f$, $\mathfrak{u} \in \mathcal{I}_f(\lambda)$, *and* $w \in \mathscr{D}_\nu$. *If* $(k)w^{-1} < (k+1)w^{-1} \leqslant 2f$, *then there exist* $a_v \in R$, *for* $v \in \mathscr{D}_\nu$, *such that*

$$b_{\mathfrak{v}\mathfrak{u}}T_wE_k = \sum_v a_v b_{\mathfrak{v}\mathfrak{u}}T_v \tag{3.9}$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$. *Moreover, in the expression* (3.9), $a_v = 0$ *whenever* k *and* $k+1$ *are not in the same row of* $\mathfrak{t}^\nu v$.

Proof. In the bitableau $\mathfrak{t} = \mathfrak{t}^\nu w$, k and $k+1$ occur as entries in the same row of $\mathfrak{t}^{(1)}$ precisely when $(2i-1)w = k$ and $(2i)w = k+1$ for some i with $1 \leqslant i \leqslant f$. Thus, when k and $k+1$ are in the same row of $\mathfrak{t}^{(1)}$, by Proposition 3.4, we have

$$b_{\mathfrak{v}\mathfrak{u}}T_wE_k = b_{\mathfrak{v}\mathfrak{u}}E_{2i-1}T_w = xb_{\mathfrak{v}\mathfrak{u}}T_w.$$

Now suppose that $(2i)w = k$ and $(p)w = k+1$ where $1 < 2i < p \leqslant 2f$. Then

$$b_{\mathfrak{v}\mathfrak{u}}T_wE_k = b_{\mathfrak{v}\mathfrak{u}}T_{p-1}^{\varepsilon_{p-1}}T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}}T_uE_k$$

where $u = s_{2i+1}s_{2i+2} \cdots s_{p-1}w$ and

$$\varepsilon_j = \begin{cases} 1 & \text{if } (j)w > k+1; \\ -1 & \text{otherwise} \end{cases}$$

for $j = 2i+1, 2i+2, \dots, p-1$. But since $(2i)u = k$ and $(2i+1)u = k+1$, Proposition 3.4 implies that

$$b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} T_u E_k = b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} E_{2i} T_u.$$

Now E_{2i-1} is a factor of $b_{\mathfrak{v}\mathfrak{u}}$ which commutes with T_j whenever $j = 2i+1$, $2i+2, \dots, p-1$, so using the relation $q^2 E_{2i-1} E_{2i} = E_{2i-1} T_{2i} T_{2i-1}$ we see that

$$\begin{aligned} b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} T_u E_k &= b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} E_{2i} T_u \\ &= q^{-2} b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} T_{2i} T_{2i-1} T_u. \end{aligned}$$

Since $(2i-1)u < k = (2i)$, it is verified using (1.1) that $l(s_{2i} s_{2i-1} u) = l(u) + 2$; thus

$$\begin{aligned} b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} E_{2i} T_u &= q^{-2} b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} T_{2i} T_{2i-1} T_u \\ &= q^{-2} b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} T_{u'} \end{aligned}$$

where $u' = s_{2i} s_{2i-1} u$ and $l(u') = l(u) + 2$. Since $(2i-1)u' = k$ and $(2i)u' = k+1$, Proposition 3.8 allows us to write

$$b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} T_{u'} = \sum_{v'} a_{v'} b_{\mathfrak{v}\mathfrak{u}} T_{v'}$$

where $v' \in \mathfrak{S}_n$, $k, k+1$ are entries in the same row of $\mathfrak{t}_1^{\nu} v'$ and $\mathfrak{t}^{\nu^{(2)}} v' = \mathfrak{t}^{\nu^{(2)}} w$ for each v'. Thus, we may use Corollary 3.1 to rewrite each of the terms $b_{\mathfrak{v}\mathfrak{u}} T_{v'}$ in the last expression as a sum

$$b_{\mathfrak{v}\mathfrak{u}} T_{v'} = \sum_{v} a_v b_{\mathfrak{v}\mathfrak{u}} T_v$$

where $v \in \mathscr{D}_\nu$ and $k, k+1$ are in the same row of $\mathfrak{t}^{\nu^{(1)}} v$.

To complete the proof of the lemma, suppose that $(2i-1)w = k$, and $(p)w = k+1$ where $0 < 2i < p \leqslant 2f$. Now since $w \in \mathscr{D}_\nu$, we must have $p = 2i+1$; therefore, $l(s_{2i} w) < l(w)$, and applying Proposition 3.4 together with the relation $E_{2i-1} T_{2i} E_{2i-1} = (qr) E_{2i-1}$,

$$b_{\mathfrak{v}\mathfrak{u}} T_w E_k = b_{\mathfrak{v}\mathfrak{u}} T_{2i} T_v E_k = b_{\mathfrak{v}\mathfrak{u}} T_{2i} E_{2i-1} T_v = (qr) b_{\mathfrak{v}\mathfrak{u}} T_v$$

where $v = s_{2i} w$, $(2i-1)v = k$ and $(2i)v = k+1$. Using Corollary 3.1, as before, the term $b_{\mathfrak{v}\mathfrak{u}} T_v$ can be rewritten as a sum of the form required by the lemma. □

In the next example we illustrate the above calculations for $n = 6$ and $f = 3$.

Example 3.4. Let $n=6$, $f=3$ and $w=s_2s_4$. Then

$$\nu^{(2)}=(0),\qquad w\in\mathscr{D}_\nu,\qquad \mathfrak{t}^{\nu^{(1)}}w=\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&6\\\hline\end{array}$$

and $\mathfrak{t}^{\nu^{(2)}}w$ is the empty tableau. We let $u=s_3s_4w$ and proceed as in the proof of Proposition 3.8;

$$\begin{aligned}b_{\mathfrak{v}\mathfrak{v}}T_wE_3&=E_1E_3E_5T_wE_3=E_1E_3E_5T_4T_3^{-1}T_uE_3\\&=E_1E_3E_5T_4T_3^{-1}E_2T_u=E_3E_5T_4T_3^{-1}E_1E_2T_u,\end{aligned}$$

and by the relation $E_kE_{k+1}=q^{-2}E_kT_{k+1}T_k$,

$$b_{\mathfrak{v}\mathfrak{v}}T_wE_3=q^{-2}E_1E_3E_5T_4T_3^{-1}T_2T_1T_u=q^{-2}E_1E_3E_5T_4T_3^{-1}T_{u'}$$

where

$$\mathfrak{t}^{\nu^{(1)}}u'=\begin{array}{|c|c|}\hline 3&4\\\hline 1&2\\\hline 5&6\\\hline\end{array}$$

and $\mathfrak{t}^{\nu^{(2)}}u'$ is the empty tableau. Now $l(s_3u')=l(u')+1$ so we use the relations

$$T_3^{-1}=q^{-2}T_3+q^{-2}\big(q^2-1\big)(E_3-1)\quad\text{and}\quad E_3T_3E_3=(qr)E_3$$

to obtain

$$\begin{aligned}b_{\mathfrak{v}\mathfrak{v}}T_wE_3&=q^{-4}b_{\mathfrak{v}\mathfrak{v}}\big[T_4T_3T_{u'}+\big(q^2-1\big)T_4E_3T_{u'}-\big(q^2-1\big)T_4T_{u'}\big]\\&=q^{-4}b_{\mathfrak{v}\mathfrak{v}}\big[T_4T_3T_{u'}+qr\big(q^2-1\big)T_{u'}-\big(q^2-1\big)T_4T_{u'}\big].\end{aligned}$$

Now, since $l(s_4s_3u')=l(u')+2$ and $l(s_4u')=l(u')+1$, we have shown that

$$b_{\mathfrak{v}\mathfrak{v}}T_wE_3=q^{-4}b_{\mathfrak{v}\mathfrak{v}}T_v+q^{-3}r\big(q^2-1\big)b_{\mathfrak{v}\mathfrak{v}}T_{u'}-q^{-4}\big(q^2-1\big)b_{\mathfrak{v}\mathfrak{v}}T_{v'}\tag{3.10}$$

where

$$\mathfrak{t}^{\nu^{(1)}}v=\begin{array}{|c|c|}\hline 3&4\\\hline 2&5\\\hline 1&6\\\hline\end{array},\qquad \mathfrak{t}^{\nu^{(1)}}v'=\begin{array}{|c|c|}\hline 3&4\\\hline 1&5\\\hline 2&6\\\hline\end{array}$$

and $\mathfrak{t}^{\nu^{(2)}}v,\mathfrak{t}^{\nu^{(2)}}v'$ are the empty tableau. Using Corollary 3.1, the right-hand side of the expression (3.10) may be rewritten as

$$b_{\mathfrak{v}\mathfrak{v}}T_wE_3 = q^{-4}b_{\mathfrak{v}\mathfrak{v}}T_{v''} + qr\bigl(q^2-1\bigr)b_{\mathfrak{v}\mathfrak{v}} - q^{-4}\bigl(q^2-1\bigr)b_{\mathfrak{v}\mathfrak{v}}T_{w'}$$

where

$$\mathfrak{t}^{\nu^{(1)}}v'' = \begin{array}{|c|c|}\hline 1 & 6 \\ \hline 2 & 5 \\ \hline 3 & 4 \\ \hline\end{array}, \qquad \mathfrak{t}^{\nu^{(1)}}w' = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 & 4 \\ \hline\end{array}$$

and $\mathfrak{t}^{\nu^{(2)}}v''$, $\mathfrak{t}^{\nu^{(2)}}w'$ are the empty tableau. We have therefore expressed the product $b_{\mathfrak{v}\mathfrak{v}}T_wE_3$ as a sum $\sum_v a_v b_{\mathfrak{v}\mathfrak{v}}T_v$ where $v \in \mathscr{D}_\nu$.

The next lemma is vacuous in case $f = [n/2]$ and n is even.

Lemma 3.5. *Let $0 \leqslant f \leqslant [n/2]$, and ν be the bi-partition with $\nu^{(2)} = (n-2f)$. Suppose that $\lambda \in \Lambda_f$, $\mathfrak{u} \in \mathcal{I}_f(\lambda)$ and $w \in \mathscr{D}_\nu$. If $(k)w^{-1} \leqslant 2f$ and $2f < (k+1)w^{-1}$, then there exist $a_v, a_{\mathfrak{s}} \in R$, for $\mathfrak{s} \in \mathcal{I}_f(\lambda)$ and $v \in \mathscr{D}_\nu$, such that*

$$b_{\mathfrak{v}\mathfrak{u}}T_wE_k \equiv \sum_v a_v \sum_{\mathfrak{s}} a_{\mathfrak{s}} b_{\mathfrak{v}\mathfrak{s}}T_v \mod \check{B}^\lambda \tag{3.11}$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$. Moreover, in the expression (3.11), *$a_v = 0$ whenever k and $k+1$ are not in the same row of $\mathfrak{t}^\nu v$.*

Proof. Suppose in the first instance that $(2i-1)w = k$ and that $(p)w = k+1$ where $1 \leqslant i \leqslant f$ and $2f < p \leqslant n$. Then

$$\begin{aligned} b_{\mathfrak{v}\mathfrak{u}}T_wE_k &= b_{\mathfrak{v}\mathfrak{u}}, T_{p-1}^{\varepsilon_{p-1}}T_{p-2}^{\varepsilon_{p-2}}\cdots T_{2i}^{\varepsilon_{2i}}T_uE_k \\ &= b_{\mathfrak{v}\mathfrak{u}}T_{p-1}^{\varepsilon_{p-1}}T_{p-2}^{\varepsilon_{p-2}}\cdots T_{2i}^{\varepsilon_{2i}}E_{2i-1}T_u \\ &= (qr)^{\varepsilon_{2i}}b_{\mathfrak{v}\mathfrak{u}}T_{p-1}^{\varepsilon_{p-1}}T_{p-2}^{\varepsilon_{p-2}}\cdots T_{2i-1}^{\varepsilon_{2i-1}}T_u \end{aligned}$$

where $u = s_{2i}s_{2i+1}\cdots s_{p-1}w$, $(2i-1)u = k$ and $(2i)u = k+1$. Now Proposition 3.8 allows us to rewrite the above expression as a sum

$$b_{\mathfrak{v}\mathfrak{u}}T_wE_k = \sum_{u'} a_{u'}b_{\mathfrak{v}\mathfrak{u}}T_{p-1}^{\varepsilon_{p-1}}T_{p-2}^{\varepsilon_{p-2}}\cdots T_{2f+1}^{\varepsilon_{2f+1}}T_{u'}$$

where $(2i-1)u' = k$ and $(2i)u' = k+1$ whenever $a_{u'} \neq 0$. By Proposition 3.9, each of the summands $b_{\mathfrak{v}\mathfrak{u}}T_{p-1}^{\varepsilon_{p-1}}T_{p-2}^{\varepsilon_{p-2}}\cdots T_{2f+1}^{\varepsilon_{2f+1}}T_{u'}$ in the preceding can in turn be rewritten, modulo the ideal B^{f+1}, as a sum

$$b_{\mathfrak{v}\mathfrak{u}}T_{p-1}^{\varepsilon_{p-1}}T_{p-2}^{\varepsilon_{p-2}}\cdots T_{2f+1}^{\varepsilon_{2f+1}}T_{u'} = \sum_{v'} a_{v'}b_{\mathfrak{v}\mathfrak{u}}T_{v'} \mod B^{f+1}$$

where $v' \in \mathfrak{S}_n$, $(2i-1)v' = k$ and $(2i)v' = k+1$. Finally, since $B^{f+1} \subseteq \check{B}^\lambda$, using Lemma 3.3 to rewrite each summand in the preceding expression, we have shown that

$$b_{\mathfrak{v}\mathfrak{u}} T_w E_k \equiv \sum_{v \in \mathscr{D}_\nu} a_v \sum_{\mathfrak{s} \in \mathcal{I}_f(\lambda)} a_\mathfrak{s} b_{\mathfrak{v}\mathfrak{s}} T_v \mod \check{B}^\lambda$$

where a_v and $a_\mathfrak{s}$, do not depend on $\mathfrak{v}$, and $a_v = 0$ whenever k, $k+1$ are not in the same row of $\mathfrak{t}^\nu v$.

In the second instance suppose that $(2i)w = k$ and $(p)w = k+1$ where $1 \leqslant i \leqslant f$ and $2f+1 \leqslant p \leqslant n$. Then, from the relation $q^2 E_{2i-1} E_{2i} = E_{2i-1} T_{2i} T_{2i-1}$, we have

$$\begin{aligned} b_{\mathfrak{v}\mathfrak{u}} T_w E_k &= b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} T_u E_k \\ &= b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} E_{2i} T_u \\ &= q^{-2} b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} T_{2i} T_{2i-1} T_u \end{aligned}$$

where $u = s_{2i+1} s_{2i+2} \cdots s_{p-1} w$. Let $u' = s_{2i} s_{2i-1} u$; using (1.1), it is verified that $l(u') = l(u) + 2$ and therefore,

$$b_{\mathfrak{v}\mathfrak{u}} T_w E_k = q^{-2} b_{\mathfrak{v}\mathfrak{u}} T_{p-1}^{\varepsilon_{p-1}} T_{p-2}^{\varepsilon_{p-2}} \cdots T_{2i+1}^{\varepsilon_{2i+1}} T_{u'}.$$

Since $(2i-1)u' = k$ and $(2i-1)u' = k+1$, we may write, using Propositions 3.8 and 3.9, and Lemma 3.3, the preceding as a sum

$$b_{\mathfrak{v}\mathfrak{u}} T_w E_k \equiv \sum_{v \in \mathscr{D}_\nu} a_v \sum_{\mathfrak{s} \in \mathcal{I}_f(\lambda)} a_\mathfrak{s} b_{\mathfrak{v}\mathfrak{s}} T_v \mod \check{B}^\lambda$$

of terms of the form required by the lemma. □

We now have the necessary ingredients to give B^f/B^{f+1} a filtration by $B_n(r,q)$-modules. Recall that $C_\mathfrak{v}^\lambda$ is the right $\mathcal{B}_n(r,q)$-submodule of $B^\lambda/\check{B}^\lambda$ generated by $b_{\mathfrak{v}\mathfrak{u}} + \check{B}^\lambda$ for $\mathfrak{u} \in \mathcal{I}_f(\lambda)$.

Proposition 3.10. *Let $0 \leqslant f \leqslant [n/2]$ and ν be the bi-partition with $\nu^{(2)} = (n-2f)$ and suppose that $\lambda \in \Lambda_f$.*

(1) *If $w \in \mathscr{D}_\nu$, $\mathfrak{u} \in \mathcal{I}_f(\lambda)$ and $b \in \mathcal{B}_n(r,q)$, then there exist $a_v, a_\mathfrak{s} \in R$, for $v \in \mathscr{D}_\nu$ and $\mathfrak{s} \in \mathcal{I}_f(\lambda)$ such that*

$$b_{\mathfrak{v}\mathfrak{u}} T_w b \equiv \sum_{v \in \mathscr{D}_\nu} a_v \sum_{\mathfrak{s} \in \mathcal{I}_f(\lambda)} a_\mathfrak{s} b_{\mathfrak{v}\mathfrak{s}} T_v \mod \check{B}^\lambda$$

for all $\mathfrak{v} \in \mathcal{I}_f(\lambda)$.

(2) *The elements*

$$\left\{b_{\mathfrak{v}\mathfrak{u}}T_w + \check{B}^\lambda \;\middle|\; \mathfrak{u} \in \mathcal{I}_f(\lambda),\, w \in \mathscr{D}_\nu\right\}$$

generate $C^\lambda_{\mathfrak{v}}$ as an R-module.

(3) *If $\mathfrak{v}, \mathfrak{s} \in \mathcal{I}_f(\lambda)$, then $C^\lambda_{\mathfrak{v}}$ and $C^\lambda_{\mathfrak{s}}$ are isomorphic as right $\mathcal{B}_n(r,q)$-modules.*

Proof. Let $w \in \mathscr{D}_\nu$ and $\mathfrak{t} = \mathfrak{t}^\nu w$. The preceding lemmas show that $b_{\mathfrak{v}\mathfrak{u}}T_w E_k$ and $b_{\mathfrak{v}\mathfrak{u}}T_w T_k$ can be expressed as a sum of the form stated in item (1) above whenever $(k)w^{-1} < (k+1)w^{-1}$. If $(k)w^{-1} > (k+1)w^{-1}$, then $k, k+1$ are not in the same row of $\mathfrak{t}$ or both in the first column of $\mathfrak{t}^{(1)}$. Thus $ws_k \in \mathscr{D}_\nu$ and by (1.1), $T_w = T_{ws_k}T_k$ so $b_{\mathfrak{v}\mathfrak{u}}T_w E_k = b_{\mathfrak{v}\mathfrak{u}}T_{ws_k}T_k E_k = qr^{-1}b_{\mathfrak{v}\mathfrak{u}}T_{ws_k}E_k$. Now, since $(k)ws_k < (k+1)ws_k$ and $ws_k \in \mathscr{D}_\nu$, we can express $b_{\mathfrak{v}\mathfrak{u}}T_{ws_k}E_k$ as a sum of the required form. Similarly, if $(k)w^{-1} > (k+1)w^{-1}$, then

$$b_{\mathfrak{v}\mathfrak{u}}T_w T_k = q^2 b_{\mathfrak{v}\mathfrak{u}}T_{ws_k} + \left(q^2-1\right)b_{\mathfrak{v}\mathfrak{u}}T_w - qr^{-1}\left(q^2-1\right)b_{\mathfrak{v}\mathfrak{u}}T_{ws_k}E_k$$

where we have shown that each term on the right-hand side can be expressed in the required form.

The second and third items of the lemma now follow directly from the first statement. □

Lemma 3.6. *Let $0 \leqslant f \leqslant [n/2]$ and ν be the bi-partition with $\nu^{(2)} = (n-2f)$. If $\lambda \in \Lambda_f$, then the set*

$$\left\{T_w^* b_{\mathfrak{v}\mathfrak{u}}T_v + \check{B}^\lambda \;\middle|\; \mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda) \text{ and } w, v \in \mathscr{D}_\nu\right\}$$

generates $B^\lambda/\check{B}^\lambda$ as an R-module.

Proof. Since $\mathcal{H}_{R,n-2f}(q^2)$ is finite dimensional, we initially take λ to be a minimal element in $(\Lambda_f, \leqslant)$, in which case $\check{B}^\lambda = \{0\}$. Now let $\mathfrak{v} \in \mathcal{I}_f(\lambda)$. Since

$$\left\{b_{\mathfrak{v}\mathfrak{u}}T_u + \check{B}^\lambda \;\middle|\; \mathfrak{u} \in \mathcal{I}_f(\lambda) \text{ and } u \in \mathscr{D}_\nu\right\}$$

generates $C^\lambda_{\mathfrak{v}}$ as a right $\mathcal{B}_n(r,q)$-module, whenever $b \in \mathcal{B}_n(r,q)$, we have

$$\left(bT_w^* b_{\mathfrak{v}\mathfrak{u}}T_v\right)^* = T_v^* b_{\mathfrak{u}\mathfrak{v}}T_w b^* = \sum_{u \in \mathscr{D}_\nu} a_u \sum_{\mathfrak{s} \in \mathcal{I}_f(\lambda)} a_{\mathfrak{s}} T_v^* b_{\mathfrak{u}\mathfrak{s}} T_u.$$

Therefore, using the anti-involution $*$ once more,

$$bT_w^* b_{\mathfrak{v}\mathfrak{u}}T_v = \sum_{u \in \mathscr{D}_\nu} a_u \sum_{\mathfrak{s} \in \mathcal{I}_f(\lambda)} a_{\mathfrak{s}} T_u^* b_{\mathfrak{s}\mathfrak{u}} T_v.$$

Having shown the lemma to hold true for any minimal element in $(\Lambda_f, \leqslant)$, using Proposition 3.10 and arguing by induction on $\leqslant$, the result now follows. □

It is a consequence of the proof of our next proposition that the R-algebra homomorphism σ of Proposition 2.1 is in fact an R-algebra isomorphism.

Proposition 3.11. *Let $0 \leqslant f \leqslant [n/2]$ and let ν be the bi-partition with $\nu^{(2)} = (n-2f)$. If $(\mathscr{C}_f, \Lambda_f)$ is a cellular basis for $\mathcal{H}_{R,n-2f}(q^2)$, then the collection*

$$\left\{T_w^* b_{\mathfrak{v}\mathfrak{u}} T_v \;\middle|\; \mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda), \lambda \in \Lambda_f \text{ and } w, v \in \mathscr{D}_\nu\right\}$$

is a free R-basis for B^f/B^{f+1}.

Proof. For $\lambda \in \Lambda_f$ and $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda)$, let $a_{\mathfrak{v}\mathfrak{u}}^w \in R$, for $w \in \mathfrak{S}_n$, be elements satisfying

$$c_{\mathfrak{v}\mathfrak{u}}^\lambda = \sum_{w \in \mathfrak{S}_n} a_{\mathfrak{v}\mathfrak{u}}^w X_w.$$

Then, for each $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda)$, the element $\hat{b}_{\mathfrak{v}\mathfrak{u}} \in B^f$ defined by

$$\hat{b}_{\mathfrak{v}\mathfrak{u}} = E_1 E_3 \cdots E_{2f-1} \sum_{w \in \mathfrak{S}_n} a_{\mathfrak{v}\mathfrak{u}}^w T_w \tag{3.12}$$

will satisfy $\iota(c_{\mathfrak{v}\mathfrak{u}}^\lambda) = \hat{b}_{\mathfrak{v}\mathfrak{u}} + B^{f+1}$; that is $\hat{b}_{\mathfrak{v}\mathfrak{u}}$ will be a coset representative for $b_{\mathfrak{v}\mathfrak{u}}$ in B^f.

Now observe that, since $B^f/B^{f+1} = \sum_{\lambda \in \Lambda_f} B^\lambda$, the collection

$$\left\{T_w^* b_{\mathfrak{v}\mathfrak{u}} T_v \;\middle|\; \mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda), \lambda \in \Lambda_f \text{ and } w, v \in \mathscr{D}_\nu\right\}$$

generates B^f/B^{f+1} as an R-module by Lemma 3.6. Therefore, the collection

$$\mathscr{C} = \left\{T_w \hat{b}_{\mathfrak{v}\mathfrak{u}} T_v \;\middle|\; \mathfrak{v}, \mathfrak{u} \in \mathcal{I}_f(\lambda), \lambda \in \Lambda_f,\ v, w \in \mathscr{D}_\nu,\ 0 \leqslant f \leqslant [n/2]\right\}$$

will generate $\mathcal{B}_n(r,q)$ as an R-module. To prove the Proposition, it will now suffice to show that the elements of $\mathscr{C}$ are linearly independent over R.

To this end, let $B_n(\hat{q}, \hat{r})$ denote the specialization of $\mathcal{B}_n(r,q)$ to $\kappa = \mathbb{C}(\hat{q}, \hat{r})$ via the specialization homomorphism σ defined in Proposition 2.1. Since $\mathscr{C}$ generates $\mathcal{B}_n(r,q)$ as an R-module and σ is surjective, it follows that $\sigma(\mathscr{C})$ generates $B_n(\hat{q}, \hat{r})$ as an κ-module. Thus, noting that $|\mathscr{C}| = |\sigma(\mathscr{C})|$, the linear independence of $\mathscr{C}$ over R will follow once we have shown that $\sigma(\mathscr{C})$ is linearly independent over κ. Counting we have,

$$\left|\sigma(\mathscr{C})\right| = \sum_{f=0}^{[n/2]} |\mathscr{D}_\nu|^2 \sum_{\lambda \in \Lambda_f} \left|\mathcal{I}_f(\lambda)\right|^2 = \sum_{f=0}^{[n/2]} \left(\frac{(2f)!}{2^f \cdot f!}\right)^2 \binom{n}{2f}^2 (n-2f)!$$

where, for $0 \leqslant f \leqslant [n/2]$, ν is the bi-partition with $\nu^{(2)} = (n-2f)$. Now each summand in the latter expression simply evaluates the number of Brauer diagrams with $2f$ horizontal bars. Thus, from Theorem 2.1 it follows that $|\sigma(\mathscr{C})| = \dim_\kappa(B_n(\hat{q},\hat{r}))$ which completes the proof of the lemma. □

In the course of proving the previous Proposition 3.11, we showed the collection

$$\mathscr{C} = \left\{ T_w \hat{b}_{\mathfrak{v}\mathfrak{u}} T_v \;\middle|\; \mathfrak{v},\mathfrak{u} \in \mathcal{I}_f(\lambda),\ \lambda \in \Lambda_f,\ v,w \in \mathscr{D}_\nu,\ 0 \leqslant f \leqslant [n/2] \right\}$$

to be a free R-basis for $\mathcal{B}_n(r,q)$. We now wish to show that, with an appropriate choice of index set and partial order, $\mathscr{C}$ is in fact a cellular basis for $\mathcal{B}_n(r,q)$ in the sense of Graham and Lehrer.

We set

$$\Lambda = \bigcup_{f=0}^{[n/2]} \Lambda_f$$

and give Λ a partial order by writing $\lambda \leqslant \mu$ in $(\Lambda, \leqslant)$ if either (i) $\lambda \in \Lambda_f$ and $\mu \in \Lambda_g$ where $f < g$ or, (ii) $\lambda, \mu \in \Lambda_f$ and $\lambda \leqslant \mu$ in $(\Lambda_f, \leqslant)$. For each $\lambda \in \Lambda_f$, $\mathcal{I}(\lambda)$ is the set of ordered pairs

$$\mathcal{I}(\lambda) = \left\{ (\mathfrak{v}, w) \mid \mathfrak{v} \in \mathcal{I}_f(\lambda) \text{ and } w \in \mathscr{D}_\nu \right\}$$

where ν is the bi-partition with $\nu^{(2)} = (n-2f)$. Write

$$\hat{b}_{(\mathfrak{v},w)(\mathfrak{s},u)} = T_w^* \hat{b}_{\mathfrak{v}\mathfrak{s}} T_u \tag{3.13}$$

for all $\mathfrak{v},\mathfrak{s} \in \mathcal{I}_f(\lambda)$ and $u,w \in \mathscr{D}_\nu$. Let $\check{\mathcal{B}}^\lambda$ be the R-module generated by the elements

$$\left\{ \hat{b}_{(\mathfrak{v},w)(\mathfrak{u},v)} \;\middle|\; \text{for } (\mathfrak{v},w), (\mathfrak{u},v) \in \mathcal{I}(\mu) \text{ and } \mu > \lambda \right\}.$$

We are now able to state and prove the main theorem of this section.

Theorem 3.1. *For $0 \leqslant f \leqslant [n/2]$, let $(\mathscr{C}_f, \Lambda_f)$ be a cellular basis for the Iwahori–Hecke algebra $\mathcal{H}_{R,n-2f}(q^2)$. Then the collection*

$$\mathscr{C} = \left\{ \hat{b}_{(\mathfrak{v},w)(\mathfrak{u},v)} \;\middle|\; \text{for } (\mathfrak{v},w), (\mathfrak{u},v) \in \mathcal{I}(\lambda) \text{ and } \lambda \in \Lambda \right\}$$

is a free R-basis for $\mathcal{B}_n(r,q)$. Furthermore, the following hold.

(1) *The R-linear map determined by $\hat{b}_{(\mathfrak{v},w)(\mathfrak{u},v)} \mapsto \hat{b}_{(\mathfrak{u},v)(\mathfrak{v},w)}$ for all $\hat{b}_{(\mathfrak{v},w)(\mathfrak{u},v)} \in \mathscr{C}$ is an anti-involution of $\mathcal{B}_n(r,q)$.*

(2) *If* $\lambda \in \Lambda$, $(\mathfrak{u}, v) \in I(\lambda)$ *and* $b \in \mathcal{B}_n(r, q)$, *then there exist* $a_{(\mathfrak{s},u)} \in R$, *for* $(\mathfrak{s}, u) \in \mathcal{I}(\lambda)$, *such that*

$$\big(\hat{b}_{(\mathfrak{v},w)(\mathfrak{u},v)}\big)b \equiv \sum_{(\mathfrak{s},u)\in\mathcal{I}(\lambda)} a_{(\mathfrak{s},u)}\hat{b}_{(\mathfrak{v},w)(\mathfrak{s},u)} \bmod \check{\mathcal{B}}^{\lambda}$$

for all $(\mathfrak{v}, w) \in \mathcal{I}(\lambda)$.

Consequently $(\mathscr{C}, \Lambda)$ *is a cellular basis for* $\mathcal{B}_n(r, q)$.

Proof. By Proposition 3.11, we know that $\mathscr{C}$ is an R-basis for $\mathcal{B}_n(r, q)$. Since $\hat{b}_{(\mathfrak{v},w)(\mathfrak{u},v)} = T_w^*\hat{b}_{\mathfrak{v}\mathfrak{u}}T_v$, we observe from the definition of $\hat{b}_{\mathfrak{v}\mathfrak{u}}$ given in (3.12), that the map defined by $T_w \mapsto T_{w^{-1}}$, $E_i \mapsto E_i$ is an algebra anti-involution of $\mathcal{B}_n(r, q)$ which, applied to the basis $\mathscr{C}$, maps $\hat{b}_{(\mathfrak{v},w)(\mathfrak{u},v)} \mapsto \hat{b}_{(\mathfrak{u},v)(\mathfrak{v},w)}$. Hence we have the first item.

The second item is now a simple restatement of Proposition 3.10. □

4. A Murphy basis for the B-M-W algebras

As an application of Theorem 3.1 we use the cellular basis constructed by Murphy in Theorem 1.2 to give an explicit cellular basis for $\mathcal{B}_n(r, q)$ indexed by bitableaux. If $0 \leqslant f < [n/2]$, we identify $\mathcal{H}_{R,n-2f}(q^2)$ with the subalgebra of $\mathcal{H}_{R,n}(q^2)$ generated by the elements $\{T_i \mid 2f < i < n\}$ and set $\mathcal{H}_{R,n-2f}(q^2) = R$ when $f = [n/2]$. Let $\Lambda_f = \{\lambda \mid \lambda \vdash n - 2f\}$ and $\mathcal{I}_f(\lambda) = \mathrm{Std}(\lambda)$. Write

$$\mathscr{M}_f = \big\{m_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\lambda)\big\},$$

where

$$m_{\mathfrak{u}\mathfrak{v}} = T^*_{d(\mathfrak{u})}m_\lambda T_{d(\mathfrak{v})}$$

and for $\lambda \in \mathcal{I}_f(\lambda)$, m_λ is the element

$$m_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w. \tag{4.1}$$

Then $(\mathscr{M}_f, \Lambda_f)$ is a cellular basis for $\mathcal{H}_{R,n-2f}(q^2)$. Define

$$\Lambda = \big\{\lambda \mid \lambda \vdash n - 2f \text{ for } f = 0, 1, \ldots, [n/2]\big\}$$

and extend the dominance order to Λ by writing $\lambda \trianglelefteq \mu$ if either (i) $\lambda \vdash n - 2f$ and $\mu \vdash n - 2g$ where $f < g$ or, (ii) μ, λ are both partitions of $n - 2f$ and $\sum_{i=1}^k \lambda_i \leqslant \sum_{i=1}^k \mu_i$ for all $k \geqslant 0$.

The cellular basis for $\mathcal{B}_n(r, q)$ will be indexed by ordered pairs $\mathcal{I}(\lambda) = \{(\mathfrak{v}, w) \mid \mathfrak{v} \in \mathcal{I}_f(\lambda)$ and $w \in \mathscr{D}_\nu\}$ where ν is the bi-partition with $\nu^{(2)} = (n - 2f)$. Each pair

$(\mathfrak{v}, w) \in \mathcal{I}(\lambda)$ corresponds to a unique λ-bitableau

$$(\mathfrak{v}, w) \leftrightarrow \mathfrak{t} = \mathfrak{t}^{\lambda} d(\mathfrak{v}) w$$

where $d(\mathfrak{t}) = d(\mathfrak{v})w \in \mathscr{D}_{\lambda}$ and each column of $\mathfrak{t}^{(2)}$ is an increasing sequence read from top to bottom (cf. Proposition 3.2 and Example 3.1). Thus, for a partition $\lambda \vdash n - 2f$, it will be convenient to identify $\mathcal{I}(\lambda)$ with the bitableaux

$$\mathcal{I}(\lambda) = \left\{ \begin{array}{c} \lambda\text{-bitableaux } \mathfrak{t} \text{ with } d(\mathfrak{t}) \in \mathscr{D}_{\lambda} \text{ and each column of } \mathfrak{t}^{(2)} \\ \text{an increasing sequence read from top to bottom} \end{array} \right\}.$$

We now set

$$\hat{b}_{\lambda} := \sum_{w \in \mathfrak{S}_{\lambda}} T_w \cdot E_1 E_3 \cdots E_{2f-1}$$

and for λ-bitableaux $\mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\lambda)$ define

$$\hat{b}_{\mathfrak{u}\mathfrak{v}} = T^{*}_{d(\mathfrak{u})} \hat{b}_{\lambda} T_{d(\mathfrak{v})}.$$

Let $\check{\mathcal{B}}^{\lambda}$ be the R-submodule of $\mathcal{B}_n(r, q)$ generated by $\{\hat{b}_{\mathfrak{v}\mathfrak{u}} \mid \mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\mu) \text{ for } \mu \rhd \lambda\}$.

Theorem 4.1. *For $0 \leqslant f \leqslant [n/2]$, let $(\mathscr{M}_f, \Lambda_f)$ be the Murphy basis for the Iwahori–Hecke algebra $\mathcal{H}_{R,n-2f}(q^2)$. Then the collection*

$$\mathscr{M} = \left\{ \hat{b}_{\mathfrak{v}\mathfrak{u}} \;\middle|\; \mathfrak{v}, \mathfrak{u} \in \mathcal{I}(\lambda) \text{ and } \lambda \in \Lambda \right\}$$

is a free R-basis for $\mathcal{B}_n(r, q)$. Furthermore, the following hold.

(1) *The R-linear map determined by*

$$\hat{b}_{\mathfrak{v}\mathfrak{u}} \mapsto \hat{b}_{\mathfrak{u}\mathfrak{v}}$$

for all $\hat{b}_{\mathfrak{v}\mathfrak{u}} \in \mathscr{M}$ is an anti-involution of $\mathcal{B}_n(r, q)$.

(2) *If $\lambda \in \Lambda$, $\mathfrak{u} \in I(\lambda)$ and $b \in \mathcal{B}_n(r, q)$, then there exist $a_{\mathfrak{s}} \in R$, for $\mathfrak{s} \in \mathcal{I}_f(\lambda)$, such that*

$$\hat{b}_{\mathfrak{v}\mathfrak{u}} b \equiv \sum_{\mathfrak{s} \in \mathcal{I}(\lambda)} a_{\mathfrak{s}} \hat{b}_{\mathfrak{v}\mathfrak{s}} \mod \check{\mathcal{B}}^{\lambda}$$

for all $\mathfrak{v} \in \mathcal{I}(\lambda)$.

Consequently $(\mathscr{M}, \Lambda)$ is a cellular basis for $\mathcal{B}_n(r, q)$.

5. Specht modules for the B-M-W algebras

In this section we specialize to the B-M-W algebra $B_n(\hat{r},\hat{q})$ over $\kappa = \mathbb{C}(\hat{r},\hat{q})$ and construct for each $\lambda \in \Lambda$, a module S^λ which will generalize the classical Specht module from the representation theory of the symmetric groups. Let

$$\mathscr{M} = \left\{ \hat{b}_{\mathfrak{v}\mathfrak{u}} \;\middle|\; \mathfrak{v},\mathfrak{u} \in \mathcal{I}(\lambda) \text{ and } \lambda \in \Lambda \right\}$$

be the specialization of the Murphy basis for $\mathcal{B}_n(r,q)$ given in Theorem 4.1. For $\lambda \in \Lambda_f$, let N^λ be the κ-module with basis

$$\left\{ \hat{b}_{\mathfrak{v}\mathfrak{u}} \;\middle|\; \mathfrak{v},\mathfrak{u} \in \mathcal{I}(\mu),\ \mu \trianglerighteq \lambda \right\}$$

and $\check{N}^\lambda = \sum_{\mu \rhd \lambda} N^\mu$. Define S^λ to be the right $B_n(\hat{r},\hat{q})$-submodule of $N^\lambda/\check{N}^\lambda$ generated by $\check{N}^\lambda + \hat{b}_\lambda$. Being isomorphic to the right cell module, S^λ has a κ-basis

$$\left\{ \check{N}^\lambda + \hat{b}_\lambda T_{d(\mathfrak{v})} \;\middle|\; \mathfrak{v} \in \mathcal{I}(\lambda) \right\}.$$

For $\mathfrak{v} \in \mathcal{I}(\lambda)$, let $\hat{b}_\mathfrak{v}$ denote the element $\check{N}^\lambda + m_\lambda T_{d(\mathfrak{v})}$ in S^λ. As in Lemma 1.3, there is a symmetric bilinear form $\langle\,,\rangle : S^\lambda \times S^\lambda \to R$ defined by

$$\langle \hat{b}_\mathfrak{v}, \hat{b}_\mathfrak{u} \rangle \hat{b}_\lambda \equiv \hat{b}_{\mathfrak{t}^\lambda \mathfrak{v}} \hat{b}_{\mathfrak{u}\mathfrak{t}^\lambda} \mod \check{N}^\lambda$$

for all bitableaux $\mathfrak{v},\mathfrak{u} \in \mathcal{I}(\lambda)$. Since $\langle\,,\rangle$ is associative,

$$\operatorname{rad} S^\lambda = \left\{ b \in S^\lambda \;\middle|\; \langle b, b' \rangle = 0 \text{ for all } b' \in S^\lambda \right\}$$

is a $B_n(\hat{r},\hat{q})$-submodule of S^λ. Naturally, we define D^λ to be the right $B_n(\hat{r},\hat{q})$-module $S^\lambda/\operatorname{rad} S^\lambda$. We now have the following consequences of Theorems 1.4 and 1.5 respectively (see also [13]).

Theorem 5.1. *The set*

$$\left\{ D^\lambda \;\middle|\; \lambda \text{ a partition of } n-2f \text{ such that } D^\lambda \neq 0 \right\}$$

is a complete set of non-isomorphic absolutely irreducible $B_n(\hat{r},\hat{q})$-modules.

Theorem 5.2. *The algebra $B_n(\hat{r},\hat{q})$ is semisimple if and only if $D^\lambda = S^\lambda$ for all $\lambda \in \Lambda$.*

Example 5.1. Let $n = 4$. Corresponding to $f = 2$ is the partition $\nu^{(2)} = (0)$. In this case the κ-module N^ν is spanned by the elements $E_1 E_3 T_{d(\mathfrak{u})}$ where $\mathfrak{u}$ is one of

$$\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, - \right), \qquad \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, - \right), \qquad \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array}, - \right).$$

Corresponding to $f = 1$ are the partitions $\lambda = (2)$ and $\mu = (1, 1)$. For $\lambda = (2)$, the κ-module N^λ is spanned by elements $E_1(1 + T_3)T_{d(\mathfrak{u})}$ where $\mathfrak{u}$ is one of the bitableaux

$$\left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 3&4\\\hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&4\\\hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&4\\\hline\end{array}\right),$$

$$\left(\begin{array}{|c|c|}\hline 1&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&3\\\hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline 2&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline 3&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right).$$

For the partition $\mu = (1, 1)$, N^μ is spanned by $E_1 T_{d(\mathfrak{u})}$ where $\mathfrak{u}$ is one of the bitableaux

$$\left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\right),$$

$$\left(\begin{array}{|c|c|}\hline 1&4\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline 2&4\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\right), \quad \left(\begin{array}{|c|c|}\hline 3&4\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right).$$

Since $\nu \rhd \lambda \rhd \mu$, we see that as $B_n(\hat{q}, \hat{r})$-modules, $S^\nu = N^\nu$, $S^\lambda = (N^\nu + N^\lambda)/N^\nu$ and $S^\mu = (N^\nu + N^\lambda + N^\mu)/(N^\nu + N^\lambda)$.

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References

[1] J. Birman, H. Wenzl, Braids, link polynomials and a new algebra, Trans. Amer. Math. Soc. 313 (1) (1989) 249–273.
[2] S. Fishel, I. Grojnowski, Canonical bases for the Brauer centralizer algebra, Math. Res. Lett. 2 (1) (1995) 15–26.
[3] J. Graham, G. Lehrer, Cellular algebras, Invent. Math. 123 (1) (1996) 1–34.
[4] T. Halverson, A. Ram, Characters of algebras containing a Jones basic construction: the Temperley–Lieb, Okada, Brauer, and Birman–Wenzl algebras, Adv. Math. 116 (2) (1995) 263–321.
[5] L. Kauffman, An invariant of regular isotopy, Trans. Amer. Math. Soc. 318 (2) (1990) 417–471.
[6] S. König, C.C. Xi, On the structure of cellular algebras, in: I. Reiten, S. Smalø, Ø. Solberg (Eds.), Algebras and Modules II, in: Canad. Math. Soc. Conf. Proc., vol. 24, 1998, pp. 365–386.
[7] S. König, C.C. Xi, Cellular algebras: inflations and Morita equivalences, J. London Math. Soc. (2) 60 (3) (1999) 700–722.
[8] A. Mathas, Iwahori–Hecke Algebras and Schur Algebras of the Symmetric Group, Univ. Lecture Ser., vol. 15, Amer. Math. Soc., Providence, RI, 1999.
[9] H. Morton, P. Traczyk, Knots and algebras, in: E. Martin, A. Rodez Usan (Eds.), Contribuciones en Homenaje al Professor D. Antonio Plans Sans de Bredmond, University of Zaragoza, 1990, pp. 201–220.
[10] J. Murakami, The Kauffman polynomial of links and representation theory, Osaka J. Math. 26 (4) (1987) 745–758.
[11] E. Murphy, The representations of Hecke algebras of type A_n, J. Algebra 173 (1) (1995) 97–121.
[12] H. Wenzl, Quantum groups and subfactors of type B, C, and D, Comm. Math. Phys. 133 (2) (1990) 342–383.
[13] C.C. Xi, On the quasi-heredity of Birman–Wenzl algebras, Adv. Math. 154 (2) (2000) 280–298.