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# An Error Bound for the MAOR Method

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**Abstract**—Suppose $Ax = b$ is a system of linear equations where the matrix $A$ is symmetric positive definite and consistently ordered. A bound for the norm of the errors $\varepsilon_k = x - x^k$ of the MAOR method in terms of the norms of $\delta_k = x^k - x^{k-1}$ and $\delta_{k+1} = x^{k+1} - x^k$ and their inner product is derived,

$$\|\varepsilon_k\|_2^2 \le \frac{1}{\alpha^2}\left\{\left(|(\omega_1-1)(\omega_2-1)| + |\omega_1(\gamma-\omega_2)|\mu_1^2\right)^2 \|\delta_k\|_2^2 - 2(\omega_1-1)(\omega_2-1)\langle\delta_k,\delta_{k+1}\rangle \right.$$
$$\left. + 2|\omega_1(\gamma-\omega_2)|\mu_1^2\|\delta_k\|_2\|\delta_{k+1}\|_2 + \|\delta_{k+1}\|_2^2\right\}.$$



**Keywords**—Linear systems, MAOR method, Error bound.

## 1. INTRODUCTION

In order to solve linear systems

$$Ax = b, \tag{1.1}$$

where $A$ is an $n\times n$ real nonsingular matrix, the modified accelerated relaxation (MAOR) method was proposed. If the diagonal elements of the matrix $A$ are nonzero, let the matrix $A$ have the splitting

$$A = D - C_L - C_U = D(I - L - U),$$

where $L = D^{-1}C_L$, $U = D^{-1}C_U$, $D = \mathrm{diag}(A)$, $C_L$ and $C_U$ are strictly lower and upper triangular matrices of $A$, respectively. In [1], a class of the MAOR method was defined whenever the matrix $A$ is a GCO $(p,q)$-matrix. For the two-cyclic matrix $A$, the iterative scheme of the MAOR method is defined by

$$x^{k+1} = L_{\Omega,\Gamma}x^k + \Phi_{\Omega,\Gamma,b}, \qquad k = 0,1,2,\dots, \tag{1.2}$$

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where $\Omega = \operatorname{diag}(\omega_1 I_1, \omega_2 I_2)$, $\omega_1, \omega_2 \neq 0$, $\Gamma = \operatorname{diag}(\gamma_1 I_1, \gamma_2 I_2)$, and the MAOR iterative matrix $L_{\Omega,\Gamma}$ is defined by

$$\begin{aligned} L_{\Omega,\Gamma} &= (D - \Gamma C_L)^{-1}[(I-\Omega)D + (\Omega - \Gamma)C_L + \Omega C_U] \\ &= (I - \Gamma L)^{-1}[I - \Omega + (\Omega - \Gamma)C_L + \Omega C_U], \\ \Phi_{\Omega,\Gamma,b} &= (D - \Gamma C_L)^{-1}\Omega b = (I - \Gamma L)^{-1} D^{-1} \Omega b. \end{aligned}$$

It is easy to show that the MAOR iteration is independent of $\gamma_1$ so that we can denote the iterative matrix by $I_{\omega_1,\omega_2,\gamma}$, i.e., the MAOR method can be defined by

$$x^{k+1} = L_{\omega_1,\omega_2,\gamma} x^k + \Phi_{\omega_1,\omega_2,\gamma,b}, \qquad k = 0, 1, 2, \dots,$$

where the iterative matrix $L_{\omega_1,\omega_2,\gamma}$ is defined by

$$L_{\omega_1,\omega_2,\gamma} = (I - \gamma L)^{-1}[I - \Omega + (\omega_2 - \gamma)L + \omega_1 U],$$

and

$$\Phi_{\omega_1,\omega_2,\gamma,b} = (I - \gamma L)^{-1} D^{-1} \Omega b.$$

When the parameter $\gamma$ equals $\omega_2$, the MAOR method reduces to the MSOR method (see [2, Chapter 8] and the iterative matrix is denoted by $L_{\omega_1,\omega_2}$; i.e.,

$$L_{\omega_1,\omega_2} = (I - \omega_2 L)^{-1}[I - \Omega + \omega_1 U].$$

Let $B = L + U$, where $B$ is the Jacobi iterative matrix, let $x$ be the solution of (1.1), and let

$$\varepsilon_k = x - x^k, \qquad \delta_k = x^k - x^{k-1}.$$

Then,

$$\varepsilon_{k+1} = L_{\omega_1,\omega_2,\gamma}\varepsilon_k, \qquad \delta_{k+1} = L_{\omega_1,\omega_2,\gamma}\delta_k, \qquad \varepsilon_k = \left(I - L_{\omega_1,\omega_2,\gamma}\right)^{-1} L_{\omega_1,\omega_2,\gamma}\delta_k.$$

Assume that the matrices $A$ and $B$ satisfy the two conditions:

(A1) $A$ is symmetric and positive definite;

(A2) $A$ is consistently ordered and $B$ is symmetric.

In view of (A1) and (A2), we can assume that the matrix $A$ has the form

$$\begin{bmatrix} I & -S^\top \\ -S & I \end{bmatrix}.$$

The corresponding Jacobi iterative matrix $B$ is

$$\begin{bmatrix} 0 & S^\top \\ S & 0 \end{bmatrix}.$$

We suppose that matrix $A$ satisfies the conditions (A1) and (A2), and denote the eigenvalues of $B$ by $\mu_i$, $i = 1, \dots, n$. If all $\mu_i$ are real, set

$$\underline{\mu} = \min_{1 \le i \le n}\{\mu_i\}, \qquad \overline{\mu} = \max_{1 \le i \le n}\{\mu_i\}.$$

Obviously, if $A$ is positive definite, then $\mu_i$ are real, and $\underline{\mu} < 0 < \overline{\mu}$.

Now, we state some results of the MAOR method.

LEMMA 1.1. *(See [1].) Let $A$ be a Hermite positive definite matrix. Then, the MAOR method converges if the parameters $\omega_1$, $\omega_2$, $\gamma$ satisfy either*

$$0 < \omega_1 \le \omega_2 \le \gamma \le 2, \qquad \omega_2 < 2,$$

*or*

$$0 < \omega_2 \le \omega_1 < 2, \qquad \omega_2 \le \gamma \le \frac{(2\omega_2)}{\omega_1}.$$

LEMMA 1.2. *(See [3].) Let eigenvalues of $L_{\omega_1,\omega_2,\gamma}$ and the corresponding Jacobi iterative matrix $B$ be, respectively, $\{\lambda\}$ and $\{\mu\}$. Then, $(\lambda + \omega_1 - 1)(\gamma + \omega_2 - 1) = \omega_1(\omega_2 - \gamma + \gamma\lambda)\mu^2$; i.e.,*

$$\lambda^2 - \left(2 - \omega_1 - \omega_2 + \gamma\omega_1\mu^2\right)\lambda + (\omega_1 - 1)(\omega_2 - 1) + \omega_1(\gamma - \omega_2)\mu^2 = 0.$$

## 2. EIGENVALUES AND EIGENVECTORS OF $L_{\omega_1,\omega_2,\gamma}$

From this section to the end, we only suppose that the assumptions (A1) and (A2) are satisfied and the MAOR method is convergent. Further, without loss of generality, we can assume that $S$ is a nonsingular matrix of order $m = n/2$. From [4] it is known that the eigenvalues of $B$ are related by

$$-1 < -\mu_1 \le -\mu_2 \le \cdots \le -\mu_m < 0 < \mu_m \le \cdots \le \mu_2 \le \mu_1 < 1.$$

Let

$$z_i = \begin{pmatrix} z_i^{(1)} \\ z_i^{(2)} \end{pmatrix}, \qquad i = 1, 2, \ldots, m,$$

be the eigenvectors of $B$ corresponding to $\mu_i$. Then,

$$\bar{z}_i = \begin{pmatrix} z_i^{(1)} \\ -z_i^{(2)} \end{pmatrix}, \qquad i = 1, 2, \ldots, m,$$

are the eigenvectors of $B$ corresponding to $-\mu_i$.

The following lemma is given in [4].

LEMMA 2.1. *(See [4].)*

$$\left\langle z_i^{(1)}, z_i^{(1)} \right\rangle = \left\langle z_i^{(2)}, z_i^{(2)} \right\rangle = \frac{1}{2}, \qquad i = 1, 2, \ldots, m,$$

*and*

$$\left\langle z_i^{(1)}, z_j^{(1)} \right\rangle = \left\langle z_i^{(2)}, z_j^{(2)} \right\rangle = 0, \qquad i \ne j, \quad i, j = 1, 2, \ldots, m.$$

By Lemma 1.2, we have

$$\lambda_i = \frac{1}{2}\left(\gamma\omega_1\mu_i^2 - \omega_1 - \omega_2 + 2 + \sqrt{R_i}\right), \qquad i = 1, \ldots, m,$$
$$\bar{\lambda}_i = \frac{1}{2}\left(\gamma\omega_1\mu_i^2 - \omega_1 - \omega_2 + 2 - \sqrt{R_i}\right), \qquad i = 1, \ldots, m,$$

where $R_i = (\gamma\omega_1\mu_i + \omega_1 - \omega_2)^2 - 4\omega_1\mu_i^2(\gamma\omega_1 - \omega_2)$, $i = 1, \ldots, m$. Now, we construct the eigenvalues and eigenvectors of $L_{\omega_1,\omega_2,\gamma}$. Let

$$U_i = \sqrt{2}\begin{pmatrix} z_i^{(1)} \\ \alpha_i z_i^{(2)} \end{pmatrix}, \quad V_i = \sqrt{2}\begin{pmatrix} z_i^{(1)} \\ \bar{\alpha}_i z_i^{(2)} \end{pmatrix}, \qquad \text{if } R_i \ne 0, \tag{2.1}$$

where

$$\alpha_i = \frac{\lambda_i + \omega_1 - 1}{\omega_1\mu_i}, \quad \bar{\alpha}_i = \frac{\bar{\lambda}_i + \omega_1 - 1}{\omega_1\mu_i}, \qquad i = 1, \ldots, m,$$

or

$$U_i = \sqrt{2}\begin{pmatrix} z_i^{(1)} \\ \beta_i z_i^{(2)} \end{pmatrix}, \quad V_i = \sqrt{2}\begin{pmatrix} 0 \\ \Delta_i z_i^{(2)} \end{pmatrix}, \qquad \text{if } R_i = 0, \tag{2.2}$$

where

$$\beta_i = \frac{\gamma\omega_1\mu_i^2 + \omega_1 - \omega_2}{2\omega_1\mu_i}, \quad \Delta_i = \frac{1}{2\beta_i} = \frac{\omega_1\mu_i}{\gamma\omega_1\mu_i^2 + \omega_1 - \omega_2}, \qquad i = 1, \ldots, m.$$

When $B$ satisfies Assumption (A2), $L_{\omega_1,\omega_2,\gamma}$ is given by

$$L_{\omega_1,\omega_2,\gamma} = \begin{pmatrix} (1-\omega_1)I & \omega_1 S^\top \\ (\omega_2 - \gamma\omega_1)S & (1-\omega_2)I + \gamma\omega_1 S S^\top \end{pmatrix}.$$

By direct calculation, it is easy to prove the following statements.

LEMMA 2.2. *For $j = 1, \dots, m$, there holds*

$$L_{\omega_1,\omega_2,\gamma} U_j = \lambda_j U_j, \quad L_{\omega_1,\omega_2,\gamma} V_j = \bar{\lambda}_j V_j, \qquad \text{if } R_j \neq 0,$$

*or*

$$L_{\omega_1,\omega_2,\gamma} U_j = \lambda_j U_j, \quad L_{\omega_1,\omega_2,\gamma} V_j = \lambda_j V_j + \frac{\omega_1^2 \mu_j^2}{\gamma \omega_1 \mu_j^2 + \omega_1 - \omega_2} U_j, \qquad \text{if } R_j = 0.$$

LEMMA 2.3. *Let the definitions of $U_j$ and $V_j$ ($j = 1, \dots, m$) be the same as those in (2.1) and (2.2). Then, the set of vectors $\{U_j, V_j\}$ ($j = 1, \dots, m$) is a basis for $C^n$. Furthermore,*

$$\langle U_i, U_j \rangle = \langle U_i, V_j \rangle = \langle V_i, U_j \rangle = \langle V_i, V_j \rangle = 0, \qquad \text{if } i \neq j;$$

*if $R_j > 0$, then*

$$\langle U_j, U_j \rangle = 1 + \alpha_j^2, \qquad \langle V_j, V_j \rangle = 1 + \bar{\alpha}_j^2,$$
$$\langle U_j, V_j \rangle = \langle V_j, U_j \rangle = 1 + \gamma - \frac{\omega_2}{\omega_1};$$

*if $R_j = 0$, then*

$$\langle U_j, U_j \rangle = 1 + \gamma - \frac{\omega_2}{\omega_1}, \qquad \langle V_j, V_j \rangle = \frac{\omega_1}{4(\gamma\omega_1 - \omega_2)},$$
$$\langle U_j, V_j \rangle = \langle V_j, U_j \rangle = \frac{1}{2};$$

*if $R_j < 0$, then*

$$\langle U_j, U_j \rangle = \langle V_j, V_j \rangle = 1 + \gamma - \frac{\omega_2}{\omega_1},$$
$$\langle U_j, V_j \rangle = 1 + \alpha_j^2, \qquad \langle V_j, U_j \rangle = 1 + \bar{\alpha}_j^2.$$

LEMMA 2.4. *Let $U_j$ and $V_j$ ($j = 1, \dots, m$) be defined as above in (2.1) and (2.2). If $a_j U_j + b_j V_j$ and $C_j U_j + d_j V_j$ are real vectors, then*

$$\langle a_i U_i + b_i V_i,\ c_j U_j + d_j V_j \rangle = 0, \qquad \text{if } i \neq j;$$

*also, if $R_j \neq 0$,*

$$\langle a_j U_j + b_j V_j,\ c_j U_j + d_j V_j \rangle = a_j c_j \left(1 + \alpha_j^2\right) + b_j d_j \left(1 + \bar{\alpha}_j^2\right) + (a_j d_j + b_j c_j) \left(1 + \gamma - \frac{\omega_2}{\omega_1}\right), \quad (2.3)$$

*and if $R_j = 0$,*

$$\langle a_j U_j + b_j V_j,\ c_j U_j + d_j V_j \rangle = a_j c_j \left(1 + \gamma - \frac{\omega_2}{\omega_1}\right) + b_j d_j \frac{\omega_1}{4(\gamma\omega_1 - \omega_2)} + \frac{1}{2}(a_j d_j + b_j c_j). \quad (2.4)$$

PROOF. It is obvious for $i \neq j$. By direct computation and using Lemma 2.3, we can get (2.4) for $R_j = 0$ and (2.3) for $R_j > 0$.

In the case $R_j < 0$, $U_j$ and $V_j$ are complex conjugates, and it follows that $b_j$ and $d_j$ must be complex conjugates of $a_j$ and $c_j$. Thus,

$$\begin{aligned}
\langle a_j U_j + b_j V_j, c_j U_j + d_j V_j \rangle &= a_j \bar{c}_j \langle U_j, U_j \rangle + a_j \bar{d}_j \langle U_j, V_j \rangle + b_j \bar{c}_j \langle V_j, U_j \rangle + b_j \bar{d}_j \langle V_j, V_j \rangle \\
&= a_j d_j \langle U_j, U_j \rangle + a_j c_j \langle U_j, V_j \rangle + b_j d_j \langle V_j, U_j \rangle + b_j c_j \langle V_j, V_j \rangle \\
&= a_j d_j \left(1 + \gamma - \frac{\omega_2}{\omega_1}\right) + a_j c_j \left(1 + \alpha_j^2\right) + b_j d_j \left(1 + \bar{\alpha}_j^2\right) \\
&\quad + b_j c_j \left(1 + \gamma - \frac{\omega_2}{\omega_1}\right) \\
&= a_j c_j \left(1 + \alpha_j^2\right) + b_j d_j \left(1 + \bar{\alpha}_j^2\right) + (a_j d_j + b_j c_j) \left(1 + \gamma - \frac{\omega_2}{\omega_1}\right). \quad \blacksquare
\end{aligned}$$

Now, we expand $\varepsilon_k$, $\delta_k$, etc., in terms of the basis $\{U_j, V_j\}$, $j = 1, \dots, m$. That is, for some complex numbers $a_j$ and $b_j$, $j = 1, \dots, m$,

$$\delta_k = \sum_{j=1}^{m} (a_j U_j + b_j V_j) = \sum_{j=1}^{m} \xi_j,$$

where $\xi_j = a_j U_j + b_j V_j$ is real. Thus,

$$\delta_{k+1} = \sum_{j=1}^{m} L_{\omega_1,\omega_2,\gamma} \xi_j, \qquad \xi_k = \sum_{j=1}^{m} \left(I + L_{\omega_1,\omega_2,\gamma}\right)^{-1} L_{\omega_1,\omega_2,\gamma} \xi_j.$$

Since $\langle \xi_i, \xi_j \rangle = 0$ for $i \neq j$, it follows that

$$\|\delta_k\|_2^2 = \sum_{j=1}^{m} A_j, \qquad \langle \delta_k, \delta_{k+1} \rangle = \sum_{j=1}^{m} B_j,$$
$$\|\delta_{k+1}\|_2^2 = \sum_{j=1}^{m} C_j, \qquad \|\xi_k\|_2^2 = \sum_{j=1}^{m} E_j,$$

where

$$A_j = \|\xi_j\|_2^2, \qquad B_j = \langle \xi, L_{\omega_1,\omega_2,\gamma} \xi_j \rangle,$$
$$C_j = \|L_{\omega_1,\omega_2,\gamma} \xi_j\|_2^2, \qquad E_j = \left\| \left(I - L_{\omega_1,\omega_2,\gamma}\right)^{-1} L_{\omega_1,\omega_2,\gamma} \xi_j \right\|_2^2.$$

## 3. AN ERROR BOUND

LEMMA 3.1. *Let $A_j$, $B_j$, $C_j$, and $E_j$ be as above and $D_j = \omega_1 \omega_2 (1 - \mu_j)$. Then*

$$E_j D_j^2 = \lambda_j^2 \bar{\lambda}_j^2 A_j - 2\lambda_j \bar{\lambda}_j + C_j.$$

PROOF.

CASE 1. For $R_j \neq 0$,

$$\begin{aligned} A_j &= \langle a_j U_j + b_j V_j,\ a_j U_j + b_j V_j \rangle = a_j^2 \left(1 + \alpha_j^2\right) + b_j^2 \left(1 + \bar{\alpha}_j^2\right) + 2 a_j b_j \left(1 + \gamma - \frac{\omega_2}{\omega_1}\right), \\ B_j &= \left\langle a_j U_j + b_j V_j,\ a_j \lambda_j U_j + b_j \bar{\lambda}_j V_j \right\rangle \\ &= a_j^2 \lambda_j \left(1 + \alpha_j^2\right) + b_j^2 \bar{\lambda}_j \left(1 + \bar{\alpha}_j^2\right) + a_j b_j \left(\lambda_j + \bar{\lambda}_j\right) \left(1 + \gamma - \frac{\omega_2}{\omega_1}\right), \\ C_j &= \left\langle a_j \lambda_j U_j + b_j \bar{\lambda}_j V_j,\ a_j \lambda_j U_j + b_j \bar{\lambda}_j V_j \right\rangle \\ &= a_j^2 \lambda_j^2 \left(1 + \alpha_j^2\right) + b_j^2 \bar{\lambda}_j^2 \left(1 + \bar{\alpha}_j^2\right) + 2 a_j b_j \lambda_j \bar{\lambda}_j \left(1 + \gamma - \frac{\omega_2}{\omega_1}\right), \\ E_j &= \left\| \frac{a_j \lambda_j U_j}{1 - \lambda_j} + \frac{b_j \bar{\lambda}_j V_j}{1 - \bar{\lambda}_j} \right\|_2^2 = \frac{1}{(1 - \lambda_j)^2 \left(1 - \bar{\lambda}_j\right)^2} \left[ a_j^2 \lambda_j^2 \left(1 + \alpha_j^2\right) \left(1 - \bar{\lambda}_j\right)^2 \right. \\ &\quad \left. + b_j^2 \bar{\lambda}_j^2 \left(1 + \bar{\alpha}_j^2\right) (1 - \lambda_j)^2 + 2 a_j b_j \lambda_j \bar{\lambda}_j (1 - \lambda_j) \left(1 - \bar{\lambda}_j\right) \left(1 + \gamma - \frac{\omega_2}{\omega_1}\right) \right]. \end{aligned}$$

It is easy to verify that

$$\begin{aligned} \lambda_j \bar{\lambda}_j &= (\omega_1 - 1)(\omega_2 - 1) + \omega_1 \mu_j^2 (\gamma - \omega_2), \\ \lambda_j + \bar{\lambda}_j &= \gamma \omega_1 \mu_j^2 - \omega_1 - \omega_2 + 2, \end{aligned}$$

and

$$\begin{aligned}(1-\lambda_j)\left(1-\bar{\lambda}_j\right) &= \omega_1\omega_2\left(1-\mu_j^2\right) = D_j,\\ E_jD_j^2 &= a_j^2\lambda_j^2\left(1+\alpha_j^2\right)\left(1-\bar{\lambda}_j\right)^2 + b_j^2\bar{\lambda}_j^2\left(1+\bar{\alpha}_j^2\right)(1-\lambda_j)^2\\ &\quad + 2a_jb_j\lambda_j\bar{\lambda}_j(1-\lambda_j)\left(1-\bar{\lambda}_j\right)\left(1+\gamma-\frac{\omega_2}{\omega_1}\right)\\ &= a_j^2\left(1+\alpha_j^2\right)\left[\lambda_j^2\left(1-\bar{\lambda}_j\right)^2\right] + b_j^2\left(1+\bar{\alpha}_j^2\right)\left[\bar{\lambda}_j^2(1-\lambda_j)^2\right]\\ &\quad + a_jb_j\left(1+\gamma-\frac{\omega_2}{\omega_1}\right)\left[2\lambda_j\bar{\lambda}_j(1-\lambda_j)\left(1-\bar{\lambda}_j\right)\right]\\ &= a_jb_j\left(1+\gamma-\frac{\omega_2}{\omega_1}\right)\left[2\lambda_j^2\bar{\lambda}_j^2 - 2\lambda_1\bar{\lambda}_j\left(\lambda_j+\bar{\lambda}_j\right) + 2\lambda_j\bar{\lambda}_j\right]\\ &\quad + a_j^2\left(1+\alpha_j^2\right)\left(\lambda_j^2\bar{\lambda}_j^2 - 2\lambda_j^2\bar{\lambda}_j + \lambda_j^2\right) + b_j^2\left(1+\bar{\alpha}_j^2\right)\left(\lambda_j^2\bar{\lambda}_j^2 - 2\bar{\lambda}_j^2\lambda_j + \bar{\lambda}_j^2\right)\\ &= \lambda_j^2\bar{\lambda}_j^2A_j - 2\lambda_j\bar{\lambda}_jB_j + C_j.\end{aligned}$$

CASE 2. For $R_j = 0$, in this case,

$$\begin{aligned}A_j &= a_j^2\left(1+\gamma-\frac{\omega_2}{\omega_1}\right) + b_j^2\frac{\omega_1}{4(\gamma\omega_1-\omega_2)} + a_jb_j,\\ B_j &= \left\langle a_jU_j + b_jV_j, \left(a_j\lambda_j + \frac{b_j\omega_1^2\mu_j^2}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2}\right)U_j + b_j\lambda_jV_j\right\rangle\\ &= \left(a_j^2\lambda_j + \frac{a_jb_j\omega_1^2\mu_j^2}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2}\right)\left(1+\gamma-\frac{\omega_2}{\omega_1}\right) + b_j^2\lambda_j\frac{\omega_1}{4(\gamma\omega_1-\omega_2)}\\ &\quad + \frac{1}{2}\left(a_jb_j\lambda_j + \frac{b_j^2\omega_1^2\mu_j^2}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2} + a_jb_j\lambda_j\right)\\ &= a_j^2\lambda_j\left(1+\gamma-\frac{\omega_2}{\omega_1}\right) + a_jb_j\left[\lambda_j + \frac{\omega_1\mu_j^2(\omega_1+\gamma\omega_1-\omega_2)}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2}\right]\\ &\quad + b_j^2\left(\frac{\omega_1\lambda_j}{4(\gamma\omega_1-\omega_2)} + \frac{\omega_1^2\mu_j^2}{2\left(\gamma\omega_1\mu_j^2+\omega_1-\omega_2\right)}\right),\\ C_j &= \left\|\left(a_j\lambda_j + \frac{b_j\omega_1^2\mu_j^2}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2}\right)U_j + b_j\lambda_jV_j\right\|_2^2\\ &= \left(a_j\lambda_j + \frac{b_j\omega_1^2\mu_j^2}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2}\right)^2\left(1+\gamma-\frac{\omega_2}{\omega_1}\right) + b_j^2\lambda_j^2\frac{\omega_1}{4(\gamma\omega_1-\omega_2)}\\ &\quad + \left(a_j\lambda_j + \frac{b_j\omega_1^2\mu_j^2}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2}\right)b_j\lambda_j\\ &= a_j^2\lambda_j^2\left(1+\gamma-\frac{\omega_2}{\omega_1}\right) + a_jb_j\lambda_j\left[\frac{2\omega_1\mu_j^2(\omega_1+\gamma\omega_1-\omega_2)}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2} + \lambda_j\right]\\ &\quad + b_j^2\left[\frac{\omega_1\lambda_j^2}{4(\gamma\omega_1-\omega_2)} + \frac{\lambda_j\omega_1^2\mu_j^2}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2} + \frac{\omega_1^2\mu_j^2(\gamma\omega_1+\omega_1-\omega_2)}{4(\gamma\omega_1-\omega_2)}\right],\\ E_j &= \left\|\left[a_j\lambda_j(1-\lambda_j) + \frac{b_j\omega_1^2\mu_j^2}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2}\right]\frac{1}{(1-\lambda_j)^2}U_j + \frac{b_j\lambda_j}{1-\lambda_j}V_j\right\|_2^2\end{aligned}$$

$$= \left[a_j\lambda_j(1-\lambda_j) + \frac{b_j\omega_1^2\mu_j^2}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2}\right]^2 \frac{\omega_1+\gamma\omega_1-\omega_2}{(1-\lambda_j)^4\omega_1}$$
$$+ \frac{b_j^2\lambda_j^2}{(1-\lambda_j)^2}\frac{\omega_1}{4(\gamma\omega_1-\omega_2)} + \left[a_j\lambda_j(1-\lambda_j) + \frac{b_j\omega_1^2\mu_j^2}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2}\right]\frac{b_j\lambda_j}{(1-\lambda_j)^3}.$$

Since $R_j = 0$, $\lambda_j = \bar{\lambda}_j$, and $D_j = (1-\lambda_j)^2$, it follows that

$$E_jD_j^2 = \left[a_j\lambda_j(1-\lambda_j) + \frac{b_j\omega_1\mu_j^2}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2}\right]^2\left(1+\gamma-\frac{\omega_2}{\omega_1}\right)$$
$$+ \left[a_j\lambda_j(1-\lambda_j) + \frac{b_j\omega_1^2\mu_j^2}{\gamma\omega_1\mu_j^2+\omega_1-\omega_2}\right] b_j\lambda_j(1-\lambda_j) + \frac{\omega_1}{4(\gamma\omega_1-\omega_2)}b_j^2\lambda_j^2(1-\lambda_j)^2$$
$$= \lambda_j^4A_j - 2\lambda_j^2B_j + C_j. \qquad \blacksquare$$

By Lemma 3.1, we can get the following error bound.

THEOREM 3.2. *Let Assumptions (A1) and (A2) be satisfied by a matrix $A$ and its associated Jacobi iterative matrix $B$. Then, the error $\varepsilon_k$ of the MAOR method satisfies*

$$\|\varepsilon_k\|_2^2 \le \frac{1}{\alpha^2}\left\{\left[|(\omega_1-1)(\omega_2-1)| + |\omega_1(\gamma-\omega_2)|\mu_1^2\right]^2\|\delta_k\|_2^2 - 2(\omega_1-1)(\omega_2-1)\langle\delta_k,\delta_{k+1}\rangle + 2|\omega_1(\gamma-\omega_2)|\mu_1^2\|\delta_k\|_2\|\delta_{k+1}\|_2 + \|\delta_{k+1}\|_2^2\right\}, \tag{3.1}$$

*where $\alpha = \omega_1\omega_2(1-\mu_1^2)$.*

PROOF. Since $C_j \ge 0$ and $D_j \ge \alpha > 0$, then by Lemma 3.1, we have

$$\alpha^2E_j \le \lambda_j^2\bar{\lambda}_j^2A_j - 2\lambda_j\bar{\lambda}_jB_j + C_j, \qquad j = 1,\dots,m.$$

By direct calculation, we can get

$$\lambda_j\bar{\lambda}_j = (\omega_1-1)(\omega_2-1) + \omega_1\mu_j^2(\gamma-\omega_2), \tag{3.2}$$

and

$$\begin{aligned}\alpha^2E_j &\le \left((\omega_1-1)(\omega_2-1)+\omega_1\mu_j^2(\gamma-\omega_2)\right)^2A_j - 2(\omega_1-1)(\omega_2-1)B_j \\ &\quad - 2\omega_1\mu_j^2(\gamma-\omega_2)B_j + C_j \\ &\le \left(|(\omega_1-1)(\omega_2-1)| + \left|\omega_1\mu_1^2(\gamma-\omega_2)\right|\right)^2A_j - 2(\omega_1-1)(\omega_2-1)B_j \\ &\quad + 2\left|\omega_1\mu_1^2(\gamma-\omega_2)\right||B_j| + C_j.\end{aligned} \tag{3.3}$$

Notice that

$$|B_j| = |\langle\xi_j, L_{\omega_1,\omega_2,\gamma}\xi_j\rangle| \le \|\xi_j\|_2\left\|L_{\omega_1,\omega_2,\gamma}\xi_j\right\|_2,$$

and by the Cauchy-Schwarz inequality,

$$\sum_{j=1}^m|B_j| \le \sum_{j=1}^m\|\xi_j\|_2\left\|L_{\omega_1,\omega_2,\gamma}\xi_j\right\|_2 \le \left(\sum_{j=1}^mA_j\right)^{1/2}\left(\sum_{j=1}^mC_j\right)^{1/2} = \|\delta_k\|_2\|\delta_{k+1}\|_2. \tag{3.4}$$

Now, from (3.3) and (3.4) we derive

$$\begin{aligned}\alpha^2\|\varepsilon_k\|_2^2 = \alpha^2\sum_{j=1}^mE_j &\le \left[|(\omega_1-1)(\omega_2-1)| + \left|\omega_1\mu_1^2(\gamma-\omega_2)\right|\right]^2\sum_{j=1}^mA_j \\ &\quad - 2(\omega_1-1)(\omega_2-1)\sum_{j=1}^mB_j + 2\left|\omega_1\mu_j^2(\gamma-\omega_2)\right|\sum_{j=1}^m|B_j| + \sum_{j=1}^mC_j \\ &\le \left\{\left(|(\omega_1-1)(\omega_2-1)| + \left|\omega_1\mu_1^2(\gamma-\omega_2)\right|\right)^2\|\delta_k\|_2^2 - 2(\omega_1-1)(\omega_2-1)\langle\delta_k,\delta_{k+1}\rangle \right. \\ &\quad \left. + 2|\omega_1(\gamma-\omega_2)|\mu_1^2\|\delta_k\|_2\|\delta_{k+1}\|_2 + \|\delta_{k+1}\|_2^2\right\},\end{aligned}$$

and inequality (3.1) follows. $\blacksquare$

REMARK 1. From inequality (3.1), one can get the following results.

(1) For $\omega_1 = \omega_2 = \omega$, $\gamma_1 = \gamma_2 = \gamma$, the AOR case, the error bound reduces to

$$\|\varepsilon_k\|_2^2 \le \frac{1}{\alpha^2}\left\{\left[(\omega-1)^2 + |\omega(\gamma-\omega)|\mu_1^2\right)^2 \|\delta_k\|_2^2 - 2(\omega-1)^2\langle\delta_k,\delta_{k+1}\rangle \right.$$
$$\left. + 2|\omega(\gamma-\omega)|\mu_1^2\|\delta_k\|_2\|\delta_{k+1}\|_2 + \|\delta_{k+1}\|_2^2\right\},$$

which is given by Song [5].

(2) For $\omega_1 = \omega_2 = \gamma_1 = \gamma_2$, in the SOR case, the error bound reduces to

$$\|\varepsilon_k\|_2^2 \le \frac{1}{\alpha^2}\left\{(\omega-1)^4\|\delta_k\|_2^2 - 2(\omega-1)^2\langle\delta_k,\delta_{k+1}\rangle + \|\delta_{k+1}\|_2^2\right\},$$

by Hatcher [4].

(3) If there is a norm $\|\bullet\|$ such that $\|B\| < 1$, then in the error bound $\mu_1$ can be replaced by $\|B\|$.

## 4. EXAMPLE

For the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (x,y) \in \Omega,$$
$$u|_{\partial\Omega} = f(x,y), \qquad (x,y) \in \partial\Omega,$$

where $\Omega = [0,2] \times [0,1]$. We use the five-point difference scheme, and take the region $\Omega$. A red/black ordering is defined by the mesh point numbering given in Figure 1. The discretized equation is $Ax = b$, where

$$A = \begin{bmatrix} D_R & C^\top \\ C & D_B \end{bmatrix}, \qquad x = \begin{bmatrix} u_R \\ u_B \end{bmatrix},$$
$$b = [4,4,4,4,0,0,0,4,4,0,0,0,4,4,4,4,4,4,4,4,0,0,0,0,0,0,4,4,4,4,4]^\top,$$

$D_R = D_B = 4I$, $u_R = [u_{R1}, u_{R2}, \dots, u_{R16}]^\top$, $u_B = [u_{B1}, u_{B2}, \dots, u_{B16}]^\top$, $I$ is a $16 \times 16$ identity matrix. The matrix $A$ satisfied the assumptions (A1) and (A2). The spectral radius of $B$ is $\mu_1 = 8.743548075804281\text{E}{-001}$.

Using the MAOR method with different parameter pairs $(\omega_1, \omega_2, \gamma)$ to solve this equation, Table 1 gives the result comparing $\|\varepsilon_k\|$ with the bound given by Theorem 3.2 and with

$$e_k = \frac{\|\delta_k\|}{|\|\delta_{k-1}\|/\|\delta_k\| - 1|}$$

(which is provided by Wachspress).

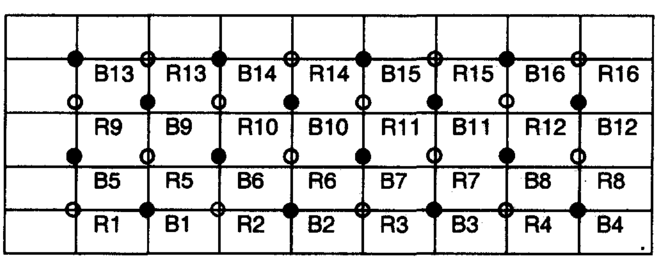


Figure 1.

Table 1.

| $\omega_1$ | $\omega_2$ | $\gamma$ | $k$ | $\varphi_k$ | $\|\varepsilon_k\|_2$ | $e_k$ |
|---|---|---|---|---|---|---|
| 1.5 | 1.6 | 1.8 | 32 | 9.661418514226472E−004 | 1.972223250677176E−004 | 9.380811409190035E−003 |
| 0.9 | 1.1 | 1.9 | 99 | 1.621728080257447E−012 | 1.087720711701143E−012 | 1.345302384370324E−013 |
| 1.3 | 1.4 | 1.5 | 56 | 7.455631475077494E−015 | 7.337557957613285E−015 | 5.669751483773890E−015 |
| 0.7 | 0.8 | 0.9 | 219 | 1.136331716265224E−014 | 4.593694506022584E−015 | $\infty$ |
| 1.0 | 1.3 | 1.6 | 14 | 2.866156766036221E−003 | 1.439279701643883E−003 | 1.416927067107923E−002 |
| 0.9 | 1.08 | 1.7 | 95 | 6.415331030290821E−015 | 3.233018248352212E−015 | $\infty$ |
| 0.8 | 1.0 | 1.6 | 28 | 4.215267798028027E−005 | 2.123019870848327E−005 | 2.233041839322302E−004 |
| 0.7 | 1.0 | 1.2 | 172 | 8.993324862699989E−015 | 3.922089704712238E−015 | $\infty$ |

Table 2.

| $\omega_1$ | $\omega_2$ | $\gamma$ | $1.0 \times 10^{-4}$ | | | $1.0 \times 10^{-6}$ | | | $1.0 \times 10^{-8}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $k_{(\varphi)}$ | $k_{(e)}$ | $k_{(\varepsilon)}$ | $k_{(\varphi)}$ | $k_{(e)}$ | $k_{(\varepsilon)}$ | $k_{(\varphi)}$ | $k_{(e)}$ | $k_{(\varepsilon)}$ |
| 1.5 | 1.6 | 1.8 | 40 | 34 | 36 | 54 | 49 | 51 | 69 | 67 | 65 |
| 0.9 | 1.1 | 1.9 | 44 | 33 | 37 | 58 | 52 | 55 | 72 | 66 | 69 |
| 1.3 | 1.4 | 1.5 | 17 | 16 | 16 | 23 | 22 | 21 | 30 | 28 | 27 |
| 0.7 | 0.8 | 0.9 | 75 | 74 | 74 | 103 | 102 | 102 | 131 | 130 | 130 |
| 1.0 | 1.3 | 1.6 | 20 | 19 | 18 | 23 | 23 | 22 | 32 | 31 | 31 |
| 0.9 | 1.08 | 1.7 | 30 | 29 | 28 | 42 | 38 | 37 | 52 | 46 | 50 |
| 0.8 | 1.0 | 1.6 | 26 | 25 | 24 | 37 | 35 | 35 | 47 | 40 | 44 |
| 0.7 | 1.0 | 1.2 | 60 | 58 | 58 | 82 | 80 | 80 | 103 | 102 | 102 |

In Table 1,

$$\varphi_k = \frac{1}{\alpha}\left\{\left(|(\omega_1-1)(\omega_2-1)| + |\omega_1(\gamma-\omega_2)|\mu_1^2|\right)^2 \|\delta_k\|_2^2 - 2(\omega_1-1)(\omega_2-1)\langle\delta_k,\delta_{k+1}\rangle \right. \\ \left. + 2|\omega_1(\gamma-\omega_2)|\mu_1^2\|\delta_k\|_2\|\delta_{k+1}\|_2 + \|\delta_{k+1}\|_2^2\right\}^{1/2}.$$

With given values, Table 2 gives iterative times which are generated, respectively, by $\varphi_k$, $e_k$, $\|\varepsilon_k\|$ used as stopping criteria. $k_{(\varphi)}$, $k_{(e)}$, $k_{(\varepsilon)}$ denote the number of iterative times which are generated by $\varphi_k$, $e_k$, $\|\varepsilon_k\|_2$ used as stopping criteria, respectively.

REMARK 2. Since $\|\varepsilon_k\|_2$ is an accurate error, then $k_{(e)}$ should not be less than $k_{(\varepsilon)}$ with a given accurate value. Table 2 shows that it does not depend on $e_k$ used as a stopping criterion.

EDITOR'S REMARK. The numerical results clearly demonstrate the correctness of the analysis and the proximity of the bound to the true error. This is a valuable generalization of the SOR (Hatcher) and AOR (Song) bounds. However, for use as a stopping criterion the bound is only as accurate as the estimate of the difference from unity of the spectral radius of the Jacobi matrix. This is often not easily found, nor is an upper bound on a norm of this matrix adequate. Therefore, the simple estimate that I suggested or some alternative may be useful. Realizing that my estimate is not a true bound, I have required that it be less than the prescribed error norm three iterations in succession. This has been reasonably successful when the error is not so small that roundoff is significant.

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