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Fixed-Point Iterations for Asymptotically Nonexpansive Mappings in Banach Spaces

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In this paper, we suggest and analyze a three-step iterative scheme for asymptotically nonexpansive mappings in Banach spaces. The new iterative scheme includes Ishikawa-type and Mann-type interations as special cases. The results obtained in this paper represent an extension as well as refinement of previous known results.
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Key Words: three-step iterations; asymptotically nonexpansive mappings; uniformly convex Banach spaces; Mann-type and Ishikawa-type iterations.

1. INTRODUCTION

In recent years, one-step and two-step iterative schemes (including Mann iteration and Ishikawa iteration processes as the most important cases) have

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been studied extensively by many authors to solve the nonlinear operator equations as well as variational inequalities in Hilbert spaces and Banach spaces; see [1–4, 6–20, 22] and the references therein. Noor [11, 12] introduced and analyzed three-step iterative methods to study the approximate solutions of variational inclusions (inequalities) in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. A similar idea goes back to the so-called θ-schemes introduced by Glowinski and Le Tallec [4] to find a zero of sum of two (or more) maximal monotone operators by using the Lagrangian multiplier. Glowinski and Le Tallec [4] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation, and they showed that three-step approximations perform better numerically. Haubruge et al. [6] studied the convergence analysis of three-step schemes of Glowinski and Le Tallec [4] and applied these schemes to obtain new splitting-type algorithms for solving variational inequalities, separable convex programming, and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Inspired and motivated by these facts, we suggest a new class of three-step iterative schemes for solving the nonlinear equation $Tx = x$ for asymptotically nonexpansive mappings in Banach space in this paper. Our schemes can be viewed as an extension for three-step and two-step iterative schemes of Glowinski and Le Tallec [4], Noor [11–13], and Ishikawa [8].

ALGORITHM 1.1. Let D be a nonempty subset of normed space X and let $T\colon D \to D$ be a mapping. For a given $x_0 \in D$, compute sequences $\{z_n\}$, $\{y_n\}$, and $\{x_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= \gamma_n T^n x_n + (1 - \gamma_n)x_n \\ y_n &= \beta_n T^n z_n + (1 - \beta_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \qquad n \geq 0, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real numbers in [0, 1].

If $\gamma \equiv 0$, then Algorithm 1.1 reduces to

ALGORITHM 1.2. Let D be a nonempty subset of normed space X and let $T\colon D \to D$ be a mapping. For a given $x_0 \in D$, compute the sequences $\{y_n\}$ and $\{x_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= \beta_n T^n x_n + (1 - \beta_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \qquad n \geq 0, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real numbers in [0, 1]. Algorithm 1.2 is called an Ishikawa-type iterative process.

For $\beta_n \equiv 0$ and $\gamma_n \equiv 0$, Algorithm 1.1 reduces to

ALGORITHM 1.3. Let D be a nonempty subset of normed space X and let $T: D \to D$ be a mapping. For a given $x_0 \in D$, compute the sequence $\{x_n\}$ by the iterative schemes

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n) x_n, \qquad n \geq 0,$$

where $\{\alpha_n\}$ are real numbers in [0, 1]. Algorithm 1.3 is known as a Mann-type iterative process.

For a suitable choice of α_n, β_n, and γ_n, one can obtain a number of new and known iterative schemes for solving nonlinear equations in Banach space and Hilbert space; see [9, 14, 17, 18] and the references therein.

2. MAIN RESULTS

In this section, we study the convergence properties of the three-step iterative schemes. First of all, we recall the well known concepts and results.

DEFINITION 2.1. Let X be normed space and let D be a nonempty subset of X. A mapping $T: D \to D$ is said to be asymptotically nonexpansive on D if there exists a sequence k_n, $k_n \geq 1$ with $\lim k_n = 1$, such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for each x, y in D and each $n \geq 1$. If $k_n \equiv 1$, then T is known as a nonexpansive mapping.

LEMMA 2.1 [21, Theorem 2]. *Let $p > 1, r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function g: $[0, +\infty) \to [0, \infty)$, $g(0) = 0$, such that*

$$\|\lambda x + (1 - \lambda) y\|^p \leq \lambda \|x\|^p + (1 - \lambda)\|y\|^p - w_p(\lambda) g(\|x - y\|),$$

for all x, y in $B_r = \{x \in X : \|x\| \leq r\}$, $\lambda \in [0, 1]$, where $w_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

We also need the following result.

LEMMA 2.2. *Let X be a uniformly convex Banach space, and let D be a nonempty closed, bounded, and convex subset of X. Let T be an asymptotically nonexpansive self-map of D with $\{k_n\}$ satisfying $k_n \geq 1$ and*

$\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *be real numbers in* $[0, 1]$. *For a given* $x_0 \in D$, *consider the sequences* $\{x_n\}, \{y_n\}$, *and* $\{z_n\}$ *defined by*

$$\begin{aligned} z_n &= \gamma_n T^n x_n + (1 - \gamma_n)x_n \\ y_n &= \beta_n T^n z_n + (1 - \beta_n)x_n \qquad n \geq 0 \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n. \end{aligned}$$

(i) *If* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, *then*

$$\lim_{n\to\infty} \|T^n y^n - x_n\| = 0.$$

(ii) *If* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \alpha_n > 0$, *then*

$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0.$$

Proof. From [5, Theorem 1], T has a fixed point x^* in D. Choose an arbitrary number $p > 0$ and a number $r > 0$ such that $D \subseteq B_r$ and $D - D \subseteq B_r$. It follows from Lemma 2.1 that

$$\begin{aligned} \|z_n - x^*\|^p &= \|\gamma_n(T^n x_n - x^*) + (1-\gamma_n)(x_n - x^*)\|^p \\ &\leq \gamma_n\|T^n x_n - x^*\|^p + (1-\gamma_n)\|x_n - x^*\|^p - w_p(\gamma_n)g(\|T^n x_n - x_n\|) \\ &\leq \gamma_n k_n^p\|x_n - x^*\|^p + (1-\gamma_n)\|x_n - x^*\|^p \\ &\leq (1 + \gamma_n k_n^p - \gamma_n)\|x_n - x^*\|^p. \end{aligned}$$

Also

$$\begin{aligned} \|y_n - x^*\|^p &\leq \|\beta_n(T^n z_n - x^*) + (1-\beta_n)(x_n - x^*)\|^p \\ &\leq \beta_n\|T^n z_n - x^*\|^p + (1-\beta_n)\|x_n - x^*\|^p - w_p(\beta_n)g(\|T^n z_n - x_n\|) \\ &\leq \beta_n k_n^p\|z_n - x^*\|^p + (1-\beta_n)\|x_n - x^*\|^p - w_p(\beta_n)g(\|T^n z_n - x_n\|). \end{aligned}$$

Thus

$$\begin{aligned} \|x_{n+1} - x^*\|^p &= \|\alpha_n(T^n y_n - x^*) + (1-\alpha_n)(x_n - x^*)\|^p \\ &\leq \alpha_n\|T^n y_n - x^*\|^p + (1-\alpha_n)\|x_n - x^*\|^p - w_p(\alpha_n)g(\|T^n y_n - x_n\|) \\ &\leq \alpha_n k_n^p\|y_n - x^*\| + (1-\alpha_n)\|x_n - x^*\|^p - w_p(\alpha_n)g(\|T^n y_n - x_n\|) \\ &\leq \alpha_n k_n^p(\beta_n k_n^p\|z_n - x^*\|^p + (1-\beta_n)\|x_n - x^*\|^p \\ &\quad - w_p(\beta_n)g(\|T^n z_n - x_n\|)) + (1-\alpha_n)\|x_n - x^*\|^p \\ &\quad - w_p(\alpha_n)g(\|T^n y_n - x_n\|) \\ &\leq \alpha_n k_n^p \cdot \beta_n k_n^p(1 + \gamma_n k_n^p - \gamma_n)\|x_n - x^*\|^p \end{aligned}$$

$$
\begin{aligned}
&+\alpha_n k_n^p(1-\beta_n)\|x_n-x^*\|^p \\
&-\alpha_n k_n^p w_p(\beta_n)g(\|T^n z_n-x_n\|)+(1-\alpha_n)\|x_n-x^*\|^p \\
&-w_p(\alpha_n)g(\|T^n y_n-x_n\|) \\
=&\|x_n-x^*\|^p+(\alpha_n\beta_n\gamma_n(k_n^p)^2+\alpha_n\beta_n k_n^p+\alpha_n)(k_n^p-1)\|x_n-x^*\|^p \\
&-\alpha_n k_n^p w_p(\beta_n)g(\|T_n z_n-x_n\|)-w_p(\alpha_n)g(\|T^n y_n-x_n\|) \\
\le&\|x_n-x^*\|^p+((k_n^p)^2+k_n^p+1)(k_n^p-1)\|x_n-x^*\|^p \\
&-\alpha_n w_p(\beta_n)g(\|T_n z_n-x_n\|)-w_p(\alpha_n)g(\|T^n y_n-x_n\|). \qquad (1)
\end{aligned}
$$

The convergence of $\{k_n\}$ and the bounded property of D imply that there exists a constant $M > 0$ such that $((k_n^p)^2 + k_n^p + 1)\|x_n - x^*\|^p \le M$. Then from (1) we obtain

$$
w_p(\alpha_n)g(\|T^n y_n - x_n\|) \le \|x_n - x^*\|^p - \|x_{n+1} - x^*\|^p + M(k_n^p - 1) \qquad (2)
$$

and

$$
\alpha_n w_p(\beta_n)g(\|T_n z_n - x_n\|) \le \|x_n - x^*\|^p - \|x_{n+1} - x^*\|^p + M(k_n^p - 1). \qquad (3)
$$

(i) If $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, there exists some real number $\delta > 0$ and a natural number N_0, such that

$$
w_p(\alpha_n) = \alpha_n(1-\alpha_n)^p + \alpha_n^p(1-\alpha_n) \ge \delta > 0, \qquad \forall\, n > N_0.
$$

It follows from inequality (2) that for any natural number $m > N_0$

$$
\begin{aligned}
\sum_{n=N_0}^{m} g(\|T^n y_n - x_n\|) &\le \sum_{n=N_0}^{m} w_p(\alpha_n)g(\|T^n y_n - x_n\|) \\
&\le \|x_{N_0} - x^*\|^p - \|x_{m+1} - x^*\|^p + M\sum_{n=N_0}^{m}(k_n^p - 1) \\
&\le \|x_{N_0} - x^*\|^p + M\sum_{n=N_0}^{m}(k_n^p - 1). \qquad (4)
\end{aligned}
$$

It is easy to verify that $t^p - 1 \le pt^{p-1}(t-1)$ for $t \ge 1$ by the application of the Lagrange mean value theorem. This together with the assumption $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ implies that $\sum_{n=1}^{\infty}(k_n^p - 1) < \infty$. Let $m \to \infty$ in inequality (4); we get $\sum_{n=N_0}^{\infty} g(\|T^n y_n - x_n\|) < +\infty$, and therefore $\lim_{n\to\infty} g(\|T^n y_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T^n y_n - x_n\| = 0$.

(ii) If $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} < 1$ and $\liminf_{n\to\infty} \alpha_n > 0$, using a similar method, together with inequality (3), it can be proved that $\lim_{n\to\infty} \|T^n z_n - x_n\| = 0$. ∎

We now state and prove the main result of this paper and this is the main motivation of our next result.

THEOREM 2.1. *Let X be a uniformly convex Banach space, and let D be a nonempty closed, bounded, and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of D with $\{k_n\}$ satisfying $k_n \ge 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real numbers in* $[0, 1]$ *satisfying*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, *and*

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

For a given $x_0 \in D$, define

$$\begin{aligned} z_n &= \gamma_n T^n x_n + (1-\gamma_n)x_n \\ y_n &= \beta_n T^n z_n + (1-\beta_n)x_n \qquad n \ge 0 \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n. \end{aligned}$$

Then $\{x_n\}, \{y_n\}$, and $\{z_n\}$ converge strongly to a fixed point of T.

Proof. From (2), we have

$$\|x_{n+1} - x^*\|^p \le \|x_n - x^*\|^p + M(k_n^p - 1). \tag{5}$$

As in the proof of Lemma 2.2, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ implies that $\sum_{n=1}^{\infty}(k_n^p - 1) < \infty$. It follows from [19, Lemma 1] that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. From Lemma 2.2,

$$\begin{aligned} \lim_{n\to\infty} \|T^n y_n - x_n\| &= 0 \\ \lim_{n\to\infty} \|T^n z_n - x_n\| &= 0, \end{aligned} \tag{6}$$

from which we have

$$\begin{aligned} \|T^n x_n - x_n\| &= \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\ &\le k_n\|x_n - y_n\| + \|T^n y_n - x_n\| \\ &\le k_n\beta_n\|T^n z_n - x_n\| + \|T^n y_n - x_n\| \to 0 \qquad (\text{as } n \to \infty), \end{aligned}$$

so

$$\begin{aligned} \|x_{n+1} - T^n x_{n+1}\| &= \|x_{n+1} - x_n\| + \|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \\ &\le \|x_{n+1} - x_n\| + k_n\|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \\ &\le \alpha_n(1+k_n)\|T^n y_n - x_n\| + \|T^n x_n - x_n\| \to 0 \qquad (\text{as } n \to \infty). \end{aligned}$$

Thus

$$\|x_{n+1}-Tx_{n+1}\| \le \|x_{n+1}-T^{n+1}x_{n+1}\|+\|Tx_{n+1}-T^{n+1}x_{n+1}\|$$
$$\le \|x_{n+1}-T^{n+1}x_{n+1}\|+k_1\|x_{n+1}-T^n x_{n+1}\| \to 0 \quad (\text{as } n\to\infty),$$

which implies

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{7}$$

Since T is completely continuous and $\{x_n\} \subset D$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore, from (7), $\{x_{n_k}\}$ converges. Let $\lim_{k\to\infty} x_{n_k} = q$. It follows from the continuity of T and (7) that $q = Tq$; i.e., q is a fixed point of T. Let $x^* = q$ in the inequality (5); we know that $\lim_{n\to\infty} \|x_n - q\|$ exists. But $\lim_{n\to\infty} \|x_{n_k} - q\| = 0$. Then $\lim_{n\to\infty} \|x_n - q\| = 0$; that is, $\lim_{n\to\infty} x_n = q$. Since

$$\|y_n - x_n\| = \beta_n\|T^n z_n - x_n\| \to 0 \qquad (\text{as } n \to \infty)$$
$$\|z_n - x_n\| = \gamma_n\|T^n x_n - x_n\| \to 0 \qquad (\text{as } n \to \infty)$$

so $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. The proof is completed. ∎

For $\gamma_n \equiv 0$ in Theorem 2.1, we can obtain Ishikawa-type convergence result which is an generalization of Theorem 3 in [14]. Unfortunately, just as in [14], it cannot directly deduce the Mann-type convergence theorem for the condition $\liminf_{n\to\infty} \beta_n > 0$ (similar to the condition $1 - \beta_n < 1 - \epsilon, \epsilon > 0$ in [14]). In the following, by using Lemma 2.2, we can remove the restriction $\liminf_{n\to\infty} \beta_n > 0$ (or $1 - \beta_n < 1 - \epsilon, \epsilon > 0$) for Ishikawa-type iteration and then refine the result further and unify the proofs of Ishikawa-type and Mann-type convergence.

THEOREM 2.2. *Let X be an uniformly convex Banach space, and let D be a nonempty closed, bounded, and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of D with $\{k_n\}$ satisfying $k_n \ge 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ be real numbers in* [0, 1] *satisfying*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, *and*

(ii) $\limsup_{n\to\infty} \beta_n < 1$.

For a given $x_0 \in D$, define

$$y_n = \beta_n T^n x_n + (1 - \beta_n)x_n$$
$$x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \qquad n \ge 0.$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of T.

Proof. From Lemma 2.2(i) (with $\gamma_n \equiv 0$), we have

$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0. \tag{8}$$

On the other hand

$$\begin{aligned} \|x_n - y_n\| &= \|\beta_n(T^n x_n - x_n)\| \\ &\leq \beta_n[\|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\|] \\ &\leq \beta_n k_n \|x_n - y_n\| + \beta_n \|T^n y_n - x_n\|. \end{aligned} \tag{9}$$

Since $\lim_{n\to\infty} k_n = 1$, $\limsup_{n\to\infty} \beta_n < 1$, then

$$\begin{aligned} \liminf_{n\to\infty}(1 - \beta_n k_n) &= 1 - \limsup_{n\to\infty}(\beta_n k_n) \\ &= 1 - \limsup_{n\to\infty} \beta_n > 0. \end{aligned}$$

This, together with equality (8) and inequality (9), shows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{10}$$

Now

$$\begin{aligned} \|T^n x_n - x_n\| &\leq \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\ &\leq k_n \|x_n - y_n\| + \|T^n y_n - x_n\|, \end{aligned}$$

which implies

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{11}$$

The rest of the proof is exactly the same as in the proof of Theorem 2.1. ∎

For $\beta_n \equiv 0$, Theorem 2.2 reduces to the following Mann-type convergence result, which is a generalization and refinement of Theorem 2 in [14], Theorem 1.5 in [17], and Theorem 2.2 in [18].

THEOREM 2.3. *Let X be a uniformly convex Banach space, and let D be a nonempty closed, bounded, and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of D with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ be real numbers in $[0, 1]$ satisfying $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. For a given $x_0 \in D$, define*

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n) x_n, \qquad n \geq 0.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 2.1. Since $k_n \geq 1$, it is obvious that for any $p > 0$, $\sum_{n=1}^{\infty}(k_n^p - 1) < \infty$ implies $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. So in this paper, we use the more concise condition $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ instead of $\sum_{n=1}^{\infty}(k_n^p - 1) < \infty (p > 0)$ (see [14, 17]).

Remark 2.2. In this paper, we have suggested and analyzed a new three-step iterative algorithm for completely continuous asymptotically nonexpansive mappings. By using the same ideas and techniques, we can also discuss the weak convergence for general asymptotically nonexpansive mappings and thereby improve the results obtained by Bose [1], Chang [2], Rhoades [14], Schu [17, 18], and Tan and Xu [19, 20].

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