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Linearization of third-order ordinary differential equations by point and contact transformations

Nail H. Ibragimov [a,*], Sergey V. Meleshko [b]

[a] *Department of Mathematics and Science, Research Centre ALGA: Advances in Lie Group Analysis, Blekinge Institute of Technology, SE-371 79 Karlskrona, Sweden*
[b] *School of Mathematics, Suranaree University of Technology, Nakhon Ratchasima, 3000, Thailand*

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Abstract

We present here the solution of the problem on linearization of third-order ordinary differential equations by means of point and contact transformations. We provide, in explicit forms, the necessary and sufficient conditions for linearization, the equations for determining the linearizing point and contact transformations as well as the coefficients of the resulting linear equations.

Keywords: Nonlinear equations; Candidates for linearization; Contact transformations; Relative invariants; Linearization test

1. Introduction

Sophus Lie showed [7, Chapter 3, §3, p. 85], that the third-order ordinary differential equations connected by contact transformations with the simplest linear equation $u''' = 0$ are at most cubic in the second-order derivative. Lie himself did not investigate further the problem of linearization of third-order equations neither by contact nor by point transformations. We owe to Shiing-shen Chern [1,2] the first significant result toward the solution

* Corresponding author.
E-mail address: nib@bth.se (N.H. Ibragimov).

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of the problem on linearization of third-order equations by means of contact transformations. Using Cartan's method, he obtained conditions for the equivalence with the equations $u''' = 0$ and $u''' + u = 0$. In his work, the conditions for linearization are given in terms of geometric invariants of contact transformations and do not provide practical methods for determining linearizing transformations. Likewise, the conditions for equivalence with an arbitrary linear equation announced in [8] (see also [3]) are not given explicitly. Guy Grebot [4] studied the linearization of third-order equations by means of a restricted class of point transformations, namely $t = \varphi(x),\ u = \psi(x, y)$. However, the problem was not completely solved (see further Remark 2.4). Recently, we have solved [6] the problem on linearization by general point transformations.

In the calculations presented here, we used computer algebra packages. The final results were checked by comparing with theoretical results on invariants as well as by applying to numerous known and new examples of linearization.

2. Point transformations of third-order equations

This section is dedicated to linearization of the third-order equations

$$y''' = f(x, y, y', y'') \tag{2.1}$$

by means of point transformations

$$t = \varphi(x, y), \qquad u = \psi(x, y). \tag{2.2}$$

We first investigate the necessary conditions for linearization and find the general form of Eq. (2.1) that can be obtained from linear equations by any point transformation. In consequence, we arrive at two classes of equations providing the candidates for linearization by point transformations. The first class contains equations that are linear in the second-order derivative and at most cubic in the first derivative, while the second class is quadratic in the second derivative with a specific dependence on the first derivative. We write the general linear third-order equation in Laguerre's form

$$u''' + \alpha(t)u = 0. \tag{2.3}$$

2.1. The candidates for linearization

Using the common rules for the transformations of derivatives under the change of variables (2.2) and singling out the terms with y''' and $(y'')^2$, we have

$$u''' = \frac{D_x(Q)}{D_x(\varphi)} = \frac{\Delta}{(\varphi_x + y'\varphi_y)^5}\left[(\varphi_x + y'\varphi_y)y''' - 3\varphi_y(y'')^2\right] + \cdots, \tag{2.4}$$

the omitted terms being at most linear in y''. The subscripts x and y denote the differentiations in x and y, respectively, and

$$\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0 \tag{2.5}$$

is the Jacobian of the change of variables (2.2). It is manifest from Eq. (2.4) that the transformations (2.2) with $\varphi_y = 0$ and $\varphi_y \neq 0$, respectively, provide two distinctly different candidates for linearization (cf. Eq. (5.5)).

If $\varphi_y = 0$, we work out the missing terms in (2.4), substitute the resulting expression in (2.3) and obtain the following equation:

$$y''' + (A_1 y' + A_0)y'' + B_3 y'^3 + B_2 y'^2 + B_1 y' + B_0 = 0, \tag{2.6}$$

where

$$A_1 = 3\psi_y^{-1}\psi_{yy}, \qquad A_0 = 3(\varphi_x\psi_y)^{-1}(\varphi_x\psi_{xy} - \psi_y\varphi_{xx}), \tag{2.7}$$

$$B_3 = \psi_y^{-1}\psi_{yyy}, \qquad B_2 = 3(\varphi_x\psi_y)^{-1}(\varphi_x\psi_{xyy} - \psi_{yy}\varphi_{xx}), \tag{2.8}$$

$$B_1 = \left(\varphi_x^2\psi_y\right)^{-1}\left(3\varphi_{xx}^2\psi_y - \varphi_{xxx}\varphi_x\psi_y - 6\varphi_{xx}\varphi_x\psi_{xy} + 3\varphi_x^2\psi_{xxy}\right), \tag{2.9}$$

$$B_0 = \left(\varphi_x^2\psi_y\right)^{-1}\left(3\varphi_{xx}^2\psi_x - \varphi_{xxx}\varphi_x\psi_x - 3\varphi_{xx}\varphi_x\psi_{xx} + \varphi_x^2\psi_{xxx} + \alpha\psi\varphi_x^5\right). \tag{2.10}$$

If $\varphi_y \neq 0$, we set $r(x, y) = \varphi_x/\varphi_y$ and arrive at the following equation:

$$\begin{aligned} y''' &+ \frac{1}{y' + r}\Big[-3(y'')^2 + \left(C_2 y'^2 + C_1 y' + C_0\right)y'' \\ &+ D_5 y'^5 + D_4 y'^4 + D_3 y'^3 + D_2 y'^2 + D_1 y' + D_0\Big] = 0, \end{aligned} \tag{2.11}$$

where

$$C_2 = 3\left[\frac{\partial\varphi}{\partial y}\Delta\right]^{-1}\left\{\frac{\partial\varphi}{\partial y}\frac{\partial\Delta}{\partial y} - 2\frac{\partial^2\varphi}{\partial y^2}\Delta\right\}, \tag{2.12}$$

$$C_1 = 3\left[\frac{\partial\varphi}{\partial y}\Delta\right]^{-1}\left\{\left(r\frac{\partial\Delta}{\partial y} + \frac{\partial\Delta}{\partial x}\right)\frac{\partial\varphi}{\partial y} - 4\Delta\cdot\frac{\partial}{\partial y}\left(r\frac{\partial\varphi}{\partial y}\right)\right\}, \tag{2.13}$$

$$C_0 = 3\left[\frac{\partial\varphi}{\partial y}\Delta\right]^{-1}\left\{r\frac{\partial\varphi}{\partial y}\frac{\partial\Delta}{\partial x} - 2\frac{\partial^2\varphi}{\partial y^2}r^2\Delta - 2\frac{\partial\varphi}{\partial y}\frac{\partial r}{\partial x}\Delta - 2r\frac{\partial\varphi}{\partial y}\frac{\partial r}{\partial y}\Delta\right\}, \tag{2.14}$$

$$\begin{aligned} D_5 = \left[\frac{\partial\varphi}{\partial y}\Delta\right]^{-1}&\left\{\frac{\partial\varphi}{\partial y}\left(\frac{\partial\varphi}{\partial y}\frac{\partial^3\psi}{\partial y^3} - \frac{\partial^3\varphi}{\partial y^3}\frac{\partial\psi}{\partial y}\right)\right. \\ &\left. + 3\frac{\partial^2\varphi}{\partial y^2}\left(\frac{\partial^2\varphi}{\partial y^2}\frac{\partial\psi}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial^2\psi}{\partial y^2}\right) + \alpha\psi\left(\frac{\partial\varphi}{\partial y}\right)^5\right\}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} D_4 = \left[\frac{\partial\varphi}{\partial y}\Delta\right]^{-1}&\left\{\left[4\Delta - 5r\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}\right]\frac{\partial^3\varphi}{\partial y^3}\right. \\ &+ 3\left[5r\frac{\partial^2\varphi}{\partial y^2}\frac{\partial\psi}{\partial y} - 5\Delta\frac{\partial^2\varphi}{\partial y^2}\left(\frac{\partial\varphi}{\partial y}\right)^{-1} - 5r\frac{\partial\varphi}{\partial y}\frac{\partial^2\psi}{\partial y^2} + 4\frac{\partial\Delta}{\partial y}\right]\frac{\partial^2\varphi}{\partial y^2} \\ &\left. + 5r\left(\frac{\partial\varphi}{\partial y}\right)^2\frac{\partial^3\psi}{\partial y^3} - 3\frac{\partial\varphi}{\partial y}\frac{\partial^2\Delta}{\partial y^2} + 5\alpha\psi r\left(\frac{\partial\varphi}{\partial y}\right)^5\right\}, \end{aligned} \tag{2.16}$$

$$\begin{aligned} D_3 = \left[\frac{\partial\varphi}{\partial y}\Delta\right]^{-1}&\left\{2r\left[8\Delta - 5r\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}\right]\frac{\partial^3\varphi}{\partial y^3} + \left[6\left(\frac{\partial\Delta}{\partial x} + 7r\frac{\partial\Delta}{\partial y} - 3\Delta\frac{\partial r}{\partial y}\right)\right.\right. \\ &\left. + 30r\left(r\frac{\partial^2\varphi}{\partial y^2}\frac{\partial\psi}{\partial y} - 2\Delta\frac{\partial^2\varphi}{\partial y^2}\left(\frac{\partial\varphi}{\partial y}\right)^{-1} - r\frac{\partial\varphi}{\partial y}\frac{\partial^2\psi}{\partial y^2}\right)\right]\frac{\partial^2\varphi}{\partial y^2} \end{aligned}$$

$$+10r^2\left(\frac{\partial\varphi}{\partial y}\right)^2\frac{\partial^3\psi}{\partial y^3}+3\left[2\Delta\frac{\partial^2 r}{\partial y^2}+4\frac{\partial r}{\partial y}\frac{\partial\Delta}{\partial y}-\frac{\partial^2\Delta}{\partial x\partial y}-3r\frac{\partial^2\Delta}{\partial y^2}\right]\frac{\partial\varphi}{\partial y}$$
$$+10\alpha\psi r^2\left(\frac{\partial\varphi}{\partial y}\right)^5\Bigg\},\tag{2.17}$$

$$D_2=\left[\frac{\partial\varphi}{\partial y}\Delta\right]^{-1}\Bigg\{2r^2\left[12\Delta-5r\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}\right]\frac{\partial^3\varphi}{\partial y^3}+10r^3\left(\frac{\partial\varphi}{\partial y}\right)^2\frac{\partial^3\psi}{\partial y^3}$$
$$+6\left[3r\frac{\partial\Delta}{\partial x}+9r^2\frac{\partial\Delta}{\partial y}-5r^3\frac{\partial\varphi}{\partial y}\frac{\partial^2\psi}{\partial y^2}-\Delta\frac{\partial r}{\partial x}-8r\Delta\frac{\partial r}{\partial y}\right]\frac{\partial^2\varphi}{\partial y^2}$$
$$+30r^2\left[r\frac{\partial\psi}{\partial y}-3\Delta\left(\frac{\partial\varphi}{\partial y}\right)^{-1}\right]\left(\frac{\partial^2\varphi}{\partial y^2}\right)^2+10\alpha\psi r^3\left(\frac{\partial\varphi}{\partial y}\right)^5$$
$$+\left[4\frac{\partial}{\partial y}\left(\Delta\frac{\partial r}{\partial x}\right)+14r\Delta\frac{\partial^2 r}{\partial y^2}-16\Delta\left(\frac{\partial r}{\partial y}\right)^2+8\frac{\partial r}{\partial y}\frac{\partial\Delta}{\partial x}\right.$$
$$\left.+24r\frac{\partial r}{\partial y}\frac{\partial\Delta}{\partial y}-7r\frac{\partial^2\Delta}{\partial x\partial y}-\frac{\partial^2\Delta}{\partial x^2}-10r^2\frac{\partial^2\Delta}{\partial y^2}\right]\frac{\partial\varphi}{\partial y}\Bigg\},\tag{2.18}$$

$$D_1=\left[\frac{\partial\varphi}{\partial y}\Delta\right]^{-1}\Bigg\{r^3\left[16\Delta-5r\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}\right]\frac{\partial^3\varphi}{\partial y^3}+5r^4\left(\frac{\partial\varphi}{\partial y}\right)^2\left[\frac{\partial^3\psi}{\partial y^3}+\alpha\psi\left(\frac{\partial\varphi}{\partial y}\right)^3\right]$$
$$+3r\left[6r\frac{\partial\Delta}{\partial x}+10r^2\frac{\partial\Delta}{\partial y}-5r^3\frac{\partial\varphi}{\partial y}\frac{\partial^2\psi}{\partial y^2}-4\Delta\frac{\partial r}{\partial x}-14r\Delta\frac{\partial r}{\partial y}\right]\frac{\partial^2\varphi}{\partial y^2}$$
$$+15r^3\left[r\frac{\partial\psi}{\partial y}-4\Delta\left(\frac{\partial\varphi}{\partial y}\right)^{-1}\right]\left(\frac{\partial^2\varphi}{\partial y^2}\right)^2+\left[\left(\frac{\partial^2 r}{\partial x^2}+6r\frac{\partial^2 r}{\partial x\partial y}+11r^2\frac{\partial^2 r}{\partial y^2}\right.\right.$$
$$\left.-19r\left(\frac{\partial r}{\partial y}\right)^2-13\frac{\partial r}{\partial x}\frac{\partial r}{\partial y}\right)\Delta+3\frac{\partial r}{\partial x}\frac{\partial\Delta}{\partial x}+5r\frac{\partial r}{\partial x}\frac{\partial\Delta}{\partial y}+13r\frac{\partial r}{\partial y}\frac{\partial\Delta}{\partial x}$$
$$\left.+15r^2\frac{\partial r}{\partial y}\frac{\partial\Delta}{\partial y}-5r^2\frac{\partial^2\Delta}{\partial x\partial y}-2r\frac{\partial^2\Delta}{\partial x^2}-5r^3\frac{\partial^2\Delta}{\partial y^2}\right]\frac{\partial\varphi}{\partial y}\Bigg\},\tag{2.19}$$

$$D_0=\left[\frac{\partial\varphi}{\partial y}\Delta\right]^{-1}\Bigg\{r^4\left[4\Delta-r\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}\right]\frac{\partial^3\varphi}{\partial y^3}+r^5\left(\frac{\partial\varphi}{\partial y}\right)^2\left[\frac{\partial^3\psi}{\partial y^3}+\alpha\psi\left(\frac{\partial\varphi}{\partial y}\right)^3\right]$$
$$+3r^2\left[2r\frac{\partial\Delta}{\partial x}+2r^2\frac{\partial\Delta}{\partial y}-r^3\frac{\partial^2\psi}{\partial y^2}-2\Delta\frac{\partial r}{\partial x}-4r\Delta\frac{\partial r}{\partial y}\right]\frac{\partial^2\varphi}{\partial y^2}$$
$$+3r^4\left[r\frac{\partial\psi}{\partial y}-5\Delta\left(\frac{\partial\varphi}{\partial y}\right)^{-1}\right]\left(\frac{\partial^2\varphi}{\partial y^2}\right)^2+\left[\left(3r^3\frac{\partial^2 r}{\partial y^2}+2r^2\frac{\partial^2 r}{\partial x\partial y}+r\frac{\partial^2 r}{\partial x^2}\right.\right.$$
$$\left.-6r^2\left(\frac{\partial r}{\partial y}\right)^2-3\left(\frac{\partial r}{\partial x}\right)^2-7r\frac{\partial r}{\partial x}\frac{\partial r}{\partial y}\right)\Delta+r\left(3\frac{\partial r}{\partial x}\frac{\partial\Delta}{\partial x}+r\frac{\partial r}{\partial x}\frac{\partial\Delta}{\partial y}\right.$$
$$\left.\left.+5r\frac{\partial r}{\partial y}\frac{\partial\Delta}{\partial x}-r\frac{\partial^2\Delta}{\partial x^2}+3r^2\frac{\partial r}{\partial y}\frac{\partial\Delta}{\partial y}-r^2\frac{\partial^2\Delta}{\partial x\partial y}-r^3\frac{\partial^2\Delta}{\partial y^2}\right)\right]\frac{\partial\varphi}{\partial y}\Bigg\}.\tag{2.20}$$

Here $\Delta=\varphi_x\psi_y-\varphi_y\psi_x$ is the Jacobian (2.5).

Definition 2.1. Equations (2.6) and (2.11) provide two different *candidates for linearization*.

Thus, every linearizable third-order equation belongs either to the class of equations (2.6) with linear dependence on the second derivative y'' or to the class of equations (2.11) that are at most quadratic in y'' with a specific dependence on y'. In Sections 2.2 and 2.3, we formulate the main theorems containing necessary and sufficient conditions for linearization as well as the methods for constructing the linearizing point transformations for each candidate. The proofs and illustrative examples are provided in the subsequent sections.

2.2. The linearization test for Eq. (2.6)

Consider the first candidate for linearization, i.e., Eq. (2.6). In this case, the linearizing transformations (2.2) have the form

$$t=\varphi(x), \qquad u=\psi(x,y). \tag{2.21}$$

Theorem 2.1. *Equation* (2.6),

$$y'''+(A_1y'+A_0)y''+B_3y'^3+B_2y'^2+B_1y'+B_0=0, \tag{2.6}$$

is linearizable if and only if its coefficients obey the following five equations:

$$A_{0y}-A_{1x}=0, \qquad \left(3B_1-A_0^2-3A_{0x}\right)_y=0, \tag{2.22}$$

$$3A_{1x}+A_0A_1-3B_2=0, \qquad 3A_{1y}+A_1^2-9B_3=0, \tag{2.23}$$

$$\left(9B_1-6A_{0x}-2A_0^2\right)A_{1x}+9(B_{1x}-A_1B_0)_y+3B_{1y}A_0-27B_{0yy}=0. \tag{2.24}$$

Provided that the conditions (2.22)–(2.24) *are satisfied, the linearizing transformation* (2.21) *is defined by a third-order ordinary differential equation for the function* $\varphi(x)$, *namely by the Riccati equation*

$$6\frac{d\chi}{dx}-3\chi^2=3B_1-A_0^2-3A_{0x} \tag{2.25}$$

for

$$\chi=\frac{\varphi_{xx}}{\varphi_x}, \tag{2.26}$$

and by the following integrable system of partial differential equations for $\psi(x,y)$:

$$3\psi_{yy}=A_1\psi_y, \qquad 3\psi_{xy}=(3\chi+A_0)\psi_y, \tag{2.27}$$

$$\psi_{xxx}=3\chi\psi_{xx}+B_0\psi_y-\frac{1}{6}\left(3A_{0x}+A_0^2-3B_1+9\chi^2\right)\psi_x-\Omega\psi, \tag{2.28}$$

where χ *is given by* (2.26) *and* Ω *is the following expression:*

$$\Omega=\frac{1}{54}\left(9A_{0xx}+18A_{0x}A_0+54B_{0y}-27B_{1x}+4A_0^3-18A_0B_1+18A_1B_0\right). \tag{2.29}$$

Finally, the coefficient α *of the resulting linear equation* (2.3) *is given by*

$$\alpha=\Omega\varphi_x^{-3}. \tag{2.30}$$

Remark 2.1. Since the system of Eqs. (2.22)–(2.24) provides the necessary and sufficient conditions for linearization, it is invariant with respect to the transformations (2.21). It means that the left-hand sides of Eqs. (2.22)–(2.24) are *relative invariants* (of the second-order) for the equivalence group (2.21).

Remark 2.2. Using (2.22) and (2.23), one can replace Eq. (2.24) by

$$\Omega_y = 0. \tag{2.31}$$

Equation (2.31) follows from (2.30) since $\varphi_y = 0$ and hence $(\alpha(t))_y = (\alpha(\varphi(x))_y = 0$.

Remark 2.3. Let us assume that $\Omega \neq 0$. Then one can find the expressions for φ_{xx} and φ_{xxx} from Eq. (2.30) and substitute them in (2.25) to obtain

$$\alpha^{-8/3}\Omega^{8/3}\big(6\alpha\alpha'' - 7(\alpha')^2\big) = 6\Omega_{xx}\Omega - 7\Omega_x^2 - 9\beta\Omega^2, \tag{2.32}$$

where $\beta = (3B_1 - A_0^2 - 3A_{0x})/3$ and α' is the derivative of the function $\alpha(t)$.

Remark 2.4. If $\Omega = 0$, the necessary and sufficient conditions for linearization given by our four equations (2.22), (2.23) together with the fifth equation $\Omega = 0$ are equivalent to the five equations given by (22), (23), (24) and (21) from [4].

If $\Omega \neq 0$, the conditions (22), (23), (26) and (21) are given in [4] as the necessary and sufficient conditions for linearization. However, upon examining simple examples (e.g., the equation $y''' + y^2 = 0$, see [6]) this statement appears false. To complete the linearization test, our Eq. (2.24) or the equivalent Eq. (2.31) should be added. Note that our Eq. (2.32) is equivalent to the equation (26) from [4] after substituting the factor 7 which is missing in [4].

2.3. The linearization test for Eq. (2.11)

Theorem 2.2. *Equation* (2.11) *is linearizable if and only if its coefficients obey the following equations*:

$$C_0 = 6r\frac{\partial r}{\partial y} - 6\frac{\partial r}{\partial x} + rC_1 - r^2C_2, \tag{2.33}$$

$$6\frac{\partial^2 r}{\partial y^2} = \frac{\partial C_2}{\partial x} - \frac{\partial C_1}{\partial y} + r\frac{\partial C_2}{\partial y} + C_2\frac{\partial r}{\partial y}, \tag{2.34}$$

$$\begin{aligned}18D_0 = {}& 3r^2\left[r\frac{\partial C_1}{\partial y} - 2\frac{\partial C_1}{\partial x} - r\frac{\partial C_2}{\partial x} + 3r^2\frac{\partial C_2}{\partial y} - 12\frac{\partial^2 r}{\partial x\partial y}\right] - 54\left(\frac{\partial r}{\partial x}\right)^2\\ &+ 6r\left[3\frac{\partial^2 r}{\partial x^2} + 15\frac{\partial r}{\partial x}\frac{\partial r}{\partial y} - 6r\left(\frac{\partial r}{\partial y}\right)^2 + (3C_1 - rC_2)\frac{\partial r}{\partial x}\right]\\ &+ r^2\left[9(rC_2 - 2C_1)\frac{\partial r}{\partial y} - 2C_1^2 + 2rC_1C_2 + 4r^2C_2^2 + 18r^2D_4 - 72r^3D_5\right],\end{aligned} \tag{2.35}$$

$$18D_1 = 9r^2\frac{\partial C_1}{\partial y} - 12r\frac{\partial C_1}{\partial x} - 27r^2\frac{\partial C_2}{\partial x} + 33r^3\frac{\partial C_2}{\partial y} - 36r\frac{\partial^2 r}{\partial x\partial y}$$

$$+ 18\frac{\partial^2 r}{\partial x^2} + 6(3C_1 + 4rC_2)\frac{\partial r}{\partial x} - 3r(6C_1 + 7rC_2)\frac{\partial r}{\partial y} + 18r\left(\frac{\partial r}{\partial y}\right)^2$$
$$- 18\frac{\partial r}{\partial x}\frac{\partial r}{\partial y} - 4rC_1^2 - 2r^2C_1C_2 + 20r^3C_2^2 + 72r^3D_4 - 270r^4D_5, \qquad (2.36)$$

$$9D_2 = 3r\frac{\partial C_1}{\partial y} - 3\frac{\partial C_1}{\partial x} - 21r\frac{\partial C_2}{\partial x} + 21r^2\frac{\partial C_2}{\partial y} + 15C_2\frac{\partial r}{\partial x}$$
$$- 15rC_2\frac{\partial r}{\partial y} - C_1^2 - 5rC_1C_2 + 14r^2C_2^2 + 54r^2D_4 - 180r^3D_5, \qquad (2.37)$$

$$3D_3 = 3r\frac{\partial C_2}{\partial y} - 3\frac{\partial C_2}{\partial x} - C_1C_2 + 2rC_2^2 + 12rD_4 - 30r^2D_5, \qquad (2.38)$$

$$54\frac{\partial D_4}{\partial x} = 18\frac{\partial^2 C_1}{\partial y^2} + 3C_2\frac{\partial C_1}{\partial y} - 72\frac{\partial^2 C_2}{\partial x\partial y} - 39C_2\frac{\partial C_2}{\partial x}$$
$$+ 18r\frac{\partial^2 C_2}{\partial y^2} - 3rC_2\frac{\partial C_2}{\partial y} + \left(72\frac{\partial C_2}{\partial y} + 33C_2^2\right)\frac{\partial r}{\partial y} + 108D_4\frac{\partial r}{\partial y}$$
$$+ 270D_5\frac{\partial r}{\partial x} + 378r\frac{\partial D_5}{\partial x} - 108r^2\frac{\partial D_5}{\partial y} - 540rD_5\frac{\partial r}{\partial y}$$
$$+ 36rC_1D_5 - 8rC_2^3 - 36rC_2D_4 + 108r^2C_2D_5 + 54rH, \qquad (2.39)$$

and

$$\frac{\partial H}{\partial x} = 3H\frac{\partial r}{\partial y} + r\frac{\partial H}{\partial y}, \qquad (2.40)$$

where

$$H = \frac{\partial D_4}{\partial y} - 2\frac{\partial D_5}{\partial x} - 3r\frac{\partial D_5}{\partial y} - 5D_5\frac{\partial r}{\partial y} - 2rC_2D_5$$
$$+ \frac{1}{3}\left[\frac{\partial^2 C_2}{\partial y^2} + 2C_2\frac{\partial C_2}{\partial y} - 2C_1D_5 + 2C_2D_4\right] + \frac{4}{27}C_2^3. \qquad (2.41)$$

Provided that the conditions (2.33)–(2.40) *are satisfied, the transformation* (2.2) *mapping Eq.* (2.11) *to a linear equation* (2.3) *is obtained by solving the following compatible system of equations for the functions* $\varphi(x, y)$ *and* $\psi(x, y)$:

$$\frac{\partial\varphi}{\partial x} = r\frac{\partial\varphi}{\partial y}, \qquad \frac{\partial\psi}{\partial x} = -\frac{\partial\varphi}{\partial y}W + r\frac{\partial\psi}{\partial y}, \qquad (2.42)$$

$$6\frac{\partial\varphi}{\partial y}\frac{\partial^3\varphi}{\partial y^3} = 9\left(\frac{\partial^2\varphi}{\partial y^2}\right)^2 + \left[15rD_5 - 3D_4 - C_2^2 - 3\frac{\partial C_2}{\partial y}\right]\left(\frac{\partial\varphi}{\partial y}\right)^2, \qquad (2.43)$$

$$\frac{\partial^3\psi}{\partial y^3} = WD_5\frac{\partial\varphi}{\partial y} + \frac{1}{6}\left[15rD_5 - C_2^2 - 3D_4 - 3\frac{\partial C_2}{\partial y}\right]\frac{\partial\psi}{\partial y}$$
$$- \frac{1}{2}H\psi + 3\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2\psi}{\partial y^2}\left(\frac{\partial\varphi}{\partial y}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2\varphi}{\partial y^2}\right)^2\frac{\partial\psi}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right)^{-2}, \qquad (2.44)$$

where the function W *is defined by the equations*

$$3\frac{\partial W}{\partial x} = \left[C_1 - rC_2 + 6\frac{\partial r}{\partial y}\right]W, \qquad 3\frac{\partial W}{\partial y} = C_2W. \qquad (2.45)$$

The coefficient α *of the resulting linear equation* (2.3) *is given by* (*cf.* (2.30))

$$\alpha = \frac{H}{2(\varphi_y)^3}, \tag{2.46}$$

where H *is the function defined in* (2.41).

Remark 2.5. Equations (2.33)–(2.40) define eight *relative invariants* of the second-order for the general point transformation group (2.2) (cf. Remark 2.5).

3. Proof of the linearization theorems

The proof of the linearization theorems formulated above requires investigation of integrability conditions for the equations given in Section 2.1. We will consider the problem for the candidates (2.6) and (2.11) separately. The problem is formulated as follows. Given the coefficients $A_i(x, y)$, $B_i(x, y)$ and $C_i(x, y)$, $D_i(x, y)$ of Eqs. (2.6) and (2.11), respectively, find the integrability conditions of the respective equations for the functions φ and ψ.

3.1. *Proof of Theorem 2.1*

Let us turn to the proof of Theorem 2.1 on linearization of Eq. (2.6). Namely, given the coefficients $A_i(x, y)$, $B_i(x, y)$ of Eq. (2.6), we have to find the necessary and sufficient conditions for integrability of the over-determined system (2.7)–(2.10) for the unknown functions $\varphi(x)$ and $\psi(x, y)$.

We first rewrite the expressions (2.7) for A_1 and A_0 in the following integrable form (cf. (3.14)):

$$A_0 = 3\frac{W_x}{W}, \qquad A_1 = 3\frac{H_y}{H},$$

where $W = \psi_y/\varphi_x$ and

$$H = \psi_y. \tag{3.1}$$

Equation (3.1) and the definition of W yield:

$$\varphi_x = HW^{-1}. \tag{3.2}$$

Differentiation of Eq. (3.2) with respect to y yields:

$$W_y = H_y W H^{-1}. \tag{3.3}$$

Now Eqs. (2.8) and (2.9) are written in the form

$$B_3 = H^{-1}H_{yy}, \qquad B_2 = 3W^{-1}H^{-2}(H_{xy}HW - H_x H_y W + H_y W_x H)$$

and

$$B_1 = (HW)^{-2}\big(2H_{xx}HW^2 - 3H_x^2 W^2 + 2H_x W_x HW + W_{xx}H^2 W + W_x^2 H^2\big),$$

respectively. Furthermore, Eq. (2.10) for B_0 becomes

$$S \equiv \left(H^2WW_{xx} - HW^2H_{xx} + 3W^2H_x^2 - 4HWH_xW_x + H^2W_x^2\right)W\psi_x + 3(HW_x - WH_x)HW^2\psi_{xx} + H^2W^3\psi_{xxx} - H^3W^3B_0 + \alpha H^5\psi = 0. \quad (3.4)$$

One can determine α from Eq. (3.4). Namely, the reckoning shows that it is convenient to use, instead of $S = 0$, the equation

$$HS_y - 5H_yS = 0.$$

It follows:

$$\alpha = \frac{W}{H^6}\big[H^3W^2B_{0y} - H_{xxx}H^2W^2 + 4H_{xx}H_xHW^2 - 3H_{xx}W_xH^2W - 3H_x^3W^2 + 4H_x^2W_xHW - H_xW_{xx}H^2W - H_xW_x^2H^2 + H_yB_0H^2W^2\big]. \quad (3.5)$$

Since $\varphi = \varphi(x)$, we have $\alpha_y = 0$ and Eq. (3.5) yields:

$$H^3B_{0yy} + H^2H_yB_{0y} + H^3\left(H^{-1}H_y\right)_yB_0 - \big[3H_{xy}H_x^2 - HH_{xy}H_{xx} + H^2H_{xxxy} - HH_{xxx}H_y - 3HH_{xxy}H_x + 4H_{xx}H_xH_y - 3H^{-1}H_x^3H_y\big] - W^{-1}\big[4H_x^2H_yW_x - HH_xH_yW_{xx} - 4HH_{xy}H_xW_x - 3HH_{xx}H_yW_x + H^2H_{xy}W_{xx} + 3H^2H_{xxy}W_x\big] - HW^{-2}W_x^2[HH_{xy} - H_yH_x] = 0. \quad (3.6)$$

Rewriting Eq. (3.2) in the form

$$H = W\varphi_x \quad (3.7)$$

and invoking that $\varphi = \varphi(x)$, the representations for B_2 and B_3 can be written as

$$B_2 = 3W^{-1}W_{xy}, \qquad B_3 = W^{-1}W_{yy}.$$

The representation for B_1, upon denoting $\chi = \varphi_x^{-1}\varphi_{xx}$, leads to Eq. (2.25):

$$3\left(2\chi' - \chi^2\right) = 3B_1 - 3\frac{\partial A_0}{\partial y} - A_0^2. \quad (3.8)$$

Using Eq. (3.3) and the expressions for A_0 and A_1, one determines the first-order derivatives of W:

$$W_x = \frac{1}{3}WA_0, \qquad W_y = \frac{1}{3}WA_1. \quad (3.9)$$

Hence, Eqs. (2.23):

$$3B_2 = 3\frac{\partial A_1}{\partial x} + A_0A_1, \qquad 9B_3 = 3\frac{\partial A_1}{\partial y} + A_1^2.$$

Equating the mixed derivatives W_{xy} and W_{yx} obtained from Eqs. (3.9), one arrives at the first equation (2.22):

$$\frac{\partial A_0}{\partial y} = \frac{\partial A_1}{\partial x}. \quad (3.10)$$

Since φ, and hence χ does not depend on y, differentiation of Eq. (3.8) with respect to y yields the second equation (2.22):

$$3\frac{\partial^2 A_1}{\partial x^2} + 2A_0\frac{\partial A_1}{\partial x} - 3\frac{\partial B_1}{\partial y} = 0. \quad (3.11)$$

Furthermore, invoking Eqs. (3.1), (3.9), we eliminate H and W, together with their derivatives, from Eq. (3.6) and arrive at Eq. (2.24).

Equations (2.27) are provided by (3.9) whereas Eq. (2.28) is obtained from Eq. (3.4). Thus, we can obtain all third-order derivatives of ψ. Namely Eq. (2.28) gives ψ_{xxx}, and the remaining derivatives ψ_{xxy}, ψ_{xyy} and ψ_{yyy} are obtained from Eqs. (2.27) by differentiating. The reckoning shows that all mixed fourth-order derivatives found from these different expressions for the third-order derivatives are equal. It means that Eqs. (2.27)–(2.28) for $\psi(x, y)$ are in involution. Finally, we obtain Eqs. (2.29)–(2.30) from (3.5) and complete the proof of Theorem 2.1.

3.2. *Proof of Theorem 2.2*

The problem is formulated as follows. Given the coefficients $C_i(x, y)$, $D_i(x, y)$ of Eq. (2.11), find the necessary and sufficient conditions for integrability of the over-determined system (2.12)–(2.20) for the unknown functions $\varphi(x, y)$ and $\psi(x, y)$. Recall that, according to our notation, the following equations hold:

$$\varphi_x = r\varphi_y, \qquad \psi_x = \frac{\psi_y \varphi_x - \Delta}{\varphi_y} \tag{3.12}$$

and

$$\alpha_x = \frac{\varphi_x}{\varphi_y}\alpha_y. \tag{3.13}$$

Let us simplify the expressions (2.12)–(2.14) for the coefficients C_i. We rewrite the right-hand side of Eq. (2.12) in the form

$$\frac{3}{\Delta\varphi_y}(\Delta_y\varphi_y - 2\Delta\varphi_{yy}) = \frac{3}{\Delta\varphi_y} \cdot \varphi_y^3 \frac{\partial}{\partial y}\left(\frac{\Delta}{\varphi_y^2}\right) = 3\frac{\varphi_y^2}{\Delta} \cdot \frac{\partial}{\partial y}\left(\frac{\Delta}{\varphi_y^2}\right)$$

and, setting

$$C_2 = 3\frac{W_y}{W}, \tag{3.14}$$

rewrite Eq. (2.12) as follows:

$$\frac{W_y}{W} = \frac{\varphi_y^2}{\Delta} \cdot \frac{\partial}{\partial y}\left(\frac{\Delta}{\varphi_y^2}\right).$$

The integration yields

$$W(x, y) = h(x)\frac{\Delta}{\varphi_y^2}.$$

Since the coefficient $h(x)$ will not appear in the expression (3.14) for C_2, we will let $h = 1$ without loss of generality and get

$$\Delta = W\varphi_y^2. \tag{3.15}$$

Then Eqs. (3.14), (2.13) and (2.14) yield:

$$C_2 = 3\frac{W_y}{W}, \qquad C_1 = 3\frac{W_x + rW_y}{W} - 6r_y, \qquad C_0 = 3\frac{rW_x - 2Wr_x}{W}. \tag{3.16}$$

Substituting the expression (3.15) for Δ in (2.15) and (2.16), one arrives at the following equations:

$$\varphi_y^2\psi_{yyy} = \left(\varphi_{yy}\varphi_{yyy}\psi_y - 3\varphi_{yy}^2\psi_y + 3\varphi_y\varphi_{yy}\psi_{yy} - \alpha\varphi_y^5\psi + W\varphi_y^3 D_5\right), \tag{3.17}$$

$$2W\varphi_y\varphi_{yyy} = \left(3W\varphi_{yy}^2 - 3\varphi_y^2 W_{yy} - W\varphi_y^2 D_4 + 5\varphi_y^2 rWD_5\right). \tag{3.18}$$

Furthermore, the expressions (2.17)–(2.20) for $D_3, \ldots, D_0$ become:

$$D_3 = W^{-1}\left[6\frac{\partial r}{\partial y}\frac{\partial W}{\partial y} - 3\frac{\partial^2 W}{\partial x\partial y} + 3\frac{\partial^2 W}{\partial y^2}r + 4D_4 rW - 10D_5 r^2 W\right], \tag{3.19}$$

$$\begin{aligned} D_2 = W^{-1}\Bigg[&2\frac{\partial^2 r}{\partial x\partial y}W + 4\frac{\partial r}{\partial x}\frac{\partial W}{\partial y} - 2\frac{\partial^2 r}{\partial y^2}rW - 4\left(\frac{\partial r}{\partial y}\right)^2 W + 4\frac{\partial r}{\partial y}\frac{\partial W}{\partial x} \\ &+ 10\frac{\partial r}{\partial y}\frac{\partial W}{\partial y}r - 7\frac{\partial^2 W}{\partial x\partial y}r - \frac{\partial^2 W}{\partial x^2} + 8\frac{\partial^2 W}{\partial y^2}r^2 + 6D_4 r^2 W - 20D_5 r^3 W\Bigg], \end{aligned} \tag{3.20}$$

$$\begin{aligned} D_1 = W^{-1}\Bigg[&2\frac{\partial^2 r}{\partial x\partial y}rW + \frac{\partial^2 r}{\partial x^2}W - 7\frac{\partial r}{\partial x}\frac{\partial r}{\partial y}W + 3\frac{\partial r}{\partial x}\frac{\partial W}{\partial x} \\ &+ 5\frac{\partial r}{\partial x}\frac{\partial W}{\partial y}r - 3\frac{\partial^2 r}{\partial y^2}r^2 W - \left(\frac{\partial r}{\partial y}\right)^2 rW + 5\frac{\partial r}{\partial y}\frac{\partial W}{\partial x}r + 5\frac{\partial r}{\partial y}\frac{\partial W}{\partial y}r^2 \\ &- 5\frac{\partial^2 W}{\partial x\partial y}r^2 - 2\frac{\partial^2 W}{\partial x^2}r + 7\frac{\partial^2 W}{\partial y^2}r^3 + 4D_4 r^3 W - 15D_5 r^4 W\Bigg], \end{aligned} \tag{3.21}$$

$$\begin{aligned} D_0 = W^{-1}\Bigg[&\frac{\partial^2 r}{\partial x^2}rW - 3\left(\frac{\partial r}{\partial x}\right)^2 W - \frac{\partial r}{\partial x}\frac{\partial r}{\partial y}rW + 3\frac{\partial r}{\partial x}\frac{\partial W}{\partial x}r \\ &+ \frac{\partial r}{\partial x}\frac{\partial W}{\partial y}r^2 - \frac{\partial^2 r}{\partial y^2}r^3 W + \frac{\partial r}{\partial y}\frac{\partial W}{\partial x}r^2 + \frac{\partial r}{\partial y}\frac{\partial W}{\partial y}r^3 \\ &- \frac{\partial^2 W}{\partial x\partial y}r^3 - \frac{\partial^2 W}{\partial x^2}r^2 + 2\frac{\partial^2 W}{\partial y^2}r^4 + D_4 r^4 W - 4D_5 r^5 W\Bigg]. \end{aligned} \tag{3.22}$$

Let us turn now to the integrability problem. One can find all third-order derivatives of the functions φ and ψ by using Eqs. (3.12), (3.17) and (3.18). Then, calculating the cross derivatives, one obtains from the equation $(\varphi_{xyy})_y = (\varphi_{yyy})_x$:

$$\begin{aligned} \frac{\partial D_4}{\partial x} = {}& r\frac{\partial D_4}{\partial y} + 5r\frac{\partial D_5}{\partial x} - 5r^2\frac{\partial D_5}{\partial y} + 5D_5\left(\frac{\partial r}{\partial x} - 3r\frac{\partial r}{\partial y}\right) - 2\frac{\partial^3 r}{\partial y^3} + 2D_4\frac{\partial r}{\partial y} \\ &+ 3W^{-1}\left[2\frac{\partial r}{\partial y}\frac{\partial^2 W}{\partial y^2} - \frac{\partial^3 W}{\partial x\partial y^2} + r\frac{\partial^3 W}{\partial y^3}\right] + 3W^{-2}\frac{\partial^2 W}{\partial y^2}\left[\frac{\partial W}{\partial x} - r\frac{\partial W}{\partial y}\right]. \end{aligned} \tag{3.23}$$

Furthermore, we consider the equation $(\psi_{xyy})_y = (\psi_{yyy})_x$ and write it in the form

$$S \equiv 2\,\alpha\varphi_y^3 - H = 0, \tag{3.24}$$

where (cf. (2.41))

$$H = \frac{\partial D_4}{\partial y} - 2\frac{\partial D_5}{\partial x} - 3r\frac{\partial D_5}{\partial y} - D_5\frac{\partial r}{\partial y} + W^{-1}\left[\frac{\partial^3 W}{\partial y^3} - 2D_5\frac{\partial W}{\partial x} + \left(2D_4 - 8rD_5 + 3W^{-1}\frac{\partial^2 W}{\partial y^2}\right)\frac{\partial W}{\partial y}\right].$$

Since $\varphi_y \neq 0$, Eq. (3.24) yields (2.46):

$$\alpha = \frac{H}{2\varphi_y^3}.$$

Now the equation $\alpha_x - r\alpha_y = 0$ leads to Eq. (2.40):

$$\frac{\partial H}{\partial x} - r\frac{\partial H}{\partial y} - 3H\frac{\partial r}{\partial y} = 0.$$

The reckoning shows that the above equations for the functions $\psi(x, y)$ and $\varphi(x, y)$ are in involution. Namely, all mixed fourth-order derivatives found from different equations are equal. Eliminating W from the above relations, one arrives at the linearization conditions summarized in Section 2.3. For example, using the expressions for C_1 and C_2 given in (3.16) one can find the first derivatives of W:

$$W_y = \frac{1}{3}WC_2, \qquad W_x = \frac{1}{3}W(C_1 - rC_2 + 6r_y). \tag{3.25}$$

Equating the mixed derivatives W_{xy} and W_{yx}, one obtains (2.35):

$$(C_2)_x - (C_1)_y + C_2 r_y + r(C_2)_y - 6r_{yy} = 0.$$

Other equations from Section 2.3 are obtained by invoking Eqs. (3.25) in the expressions for the functions D_0, D_1, D_2, D_3, $(D_4)_x$, and $(D_4)_y$. This completes the proof of Theorem 2.2.

4. Illustration of the linearization theorems

4.1. Examples on Theorem 2.1

Example 4.1. The equation

$$y''' - \left(\frac{6}{y}y' + \frac{3}{x}\right)y'' + \frac{6}{y^2}y'^3 + \frac{6}{xy}y'^2 + \frac{6}{x^2}y' + 6\frac{y}{x^3} = 0 \tag{4.1}$$

is an equation of the form (2.6) with the coefficients

$$A_1 = -\frac{6}{y}, \quad A_0 = -\frac{3}{x}, \quad B_3 = \frac{6}{y^2}, \quad B_2 = \frac{6}{xy}, \quad B_1 = \frac{6}{x^2}, \quad B_0 = \frac{6y}{x^3}. \tag{4.2}$$

One can readily verify that the coefficients (4.2) obey the conditions (2.22)–(2.24). We have

$$3B_1 - A_0^2 - 3A_{0x} = 0, \tag{4.3}$$

and Eq. (2.25) is written

$$2\frac{d\chi}{dx} - \chi^2 = 0.$$

Let us take its simplest solution $\chi = 0$. Then, invoking (2.26), we let $\varphi = x$. Now Eqs. (2.27) are written

$$\frac{\partial \ln|\psi_y|}{\partial y} = -\frac{2}{y}, \qquad \frac{\partial \ln|\psi_y|}{\partial x} = -\frac{1}{x}$$

and yield

$$\psi_y = \frac{K}{xy^2}, \qquad K = \text{const.}$$

Hence

$$\psi = -\frac{K}{xy} + f(x).$$

Since one can use any particular solution, we set $K = -1$, $f(x) = 0$ and take

$$\psi = \frac{1}{xy}.$$

Invoking (4.3) and noting that (2.29) yields $\Omega = 0$, one can readily verify that the function $\psi = 1/(xy)$ solves Eq. (2.28) as well. Since $\Omega = 0$, Eq. (2.30) gives $\alpha = 0$. Hence, the transformation

$$t = x, \qquad u = \frac{1}{xy} \tag{4.4}$$

maps Eq. (4.1) to the linear equation

$$u''' = 0.$$

Example 4.2. Consider the following equation of the form (2.6):

$$y''' + \frac{3}{y}y'y'' - 3y'' - \frac{3}{y}y'^2 + 2y' - y = 0. \tag{4.5}$$

Its coefficients

$$A_1 = \frac{3}{y}, \quad A_0 = -3, \quad B_3 = 0, \quad B_2 = -\frac{3}{y}, \quad B_1 = 2, \quad B_0 = -y$$

obey the linearization conditions (2.22)–(2.24). Furthermore,

$$3B_1 - A_0^2 - 3A_{0x} = -3$$

and Eq. (2.25) is written

$$6\frac{d\chi}{dx} - 3\chi^2 = -3.$$

We take its evident solution $\chi = 1$ and obtain from (2.26) the equation $\varphi'' = \varphi'$, whence

$$\varphi = \mathrm{e}^x.$$

Equations (2.27) have the form

$$\frac{\partial \ln|\psi_y|}{\partial y} = \frac{1}{y}, \qquad \psi_{xy} = 0$$

and can be readily solved. We take the simplest solution $\psi = y^2$ and obtain the following change of variables (2.21):

$$t = \mathrm{e}^x, \qquad u = y^2. \tag{4.6}$$

Substituting $\Omega = -2$ and $\varphi_x = \mathrm{e}^x = t$ in (2.30), we obtain $\alpha(t) = -2t^{-3}$. Thus, Eq. (4.5) is mapped by the transformation (4.6) to the linear equation

$$u''' - \frac{2}{t^3}u = 0. \tag{4.7}$$

Remark 4.1. In the previous examples, the calculations for determining the linearizing transformations were confined to particular solutions of Eqs. (2.25)–(2.28). The reason was that, by considering the general solutions to Eqs. (2.25)–(2.28), we would add only the symmetry and equivalence transformations for the original and linearized equations, respectively.

4.2. An example on Theorem 2.2

Consider the nonlinear equation

$$y''' + \frac{1}{y'}\left[-3y''^2 - xy'^5\right] = 0. \tag{4.8}$$

It has the form (2.11) with the following coefficients:

$$\begin{aligned} &r = 0, \qquad C_0 = C_1 = C_2 = 0, \\ &D_0 = D_1 = D_2 = D_3 = D_4 = 0, \qquad D_5 = -x. \end{aligned} \tag{4.9}$$

Let us test Eq. (4.8) for linearization by using Theorem 2.2. It is manifest that the coefficients (4.9) satisfy Eqs. (2.33)–(2.39). Furthermore, Eq. (2.40) also holds since (2.41) yields

$$H = 2. \tag{4.10}$$

Thus, Eq. (4.8) is linearizable, and we can proceed further. Equations (2.45) are written

$$\frac{\partial W}{\partial x} = 0, \qquad \frac{\partial W}{\partial y} = 0$$

and yield $W = \text{const}$. Therefore, Eqs. (2.42) have the form

$$\frac{\partial \varphi}{\partial x} = 0, \qquad \frac{\partial \psi}{\partial x} = -W\frac{\partial \varphi}{\partial y}$$

and hence:

$$\varphi = \varphi(y), \qquad \psi = -Wx\varphi'(y) + \omega(y). \tag{4.11}$$

Now the third-order equations (2.43) and (2.44) yield the ordinary differential equation

$$\varphi''' = \frac{3}{2}\frac{\varphi''^2}{\varphi'} \tag{4.12}$$

for $\varphi(y)$ and the partial differential equation

$$\frac{\partial^3\psi}{\partial y^3} = 3\frac{\varphi''}{\varphi'}\frac{\partial^2\psi}{\partial y^2} - \frac{3}{2}\frac{\varphi''^2}{\varphi'^2}\frac{\partial\psi}{\partial y} - \psi - Wx\varphi' \tag{4.13}$$

for $\psi(x, y)$, respectively. Using the expression for ψ given in (4.11) and Eq. (4.12) for φ, we reduce Eq. (4.13) to

$$3\frac{\varphi''}{\varphi'}\omega'' - \frac{3}{2}\frac{\varphi''^2}{\varphi'^2}\omega' - \omega - \omega''' = 0.$$

Hence, one can satisfy Eq. (4.13) by letting $\omega(y) = 0$. Then the construction of the linearizing transformation requires integration of Eq. (4.12) known in the literature as the Schwarzian equation. Its general solution is provided by the straight lines

$$\varphi = ky + l, \quad k, l = \text{const.}, \tag{4.14}$$

and the hyperbolas

$$\varphi = a + \frac{1}{b - cy}, \quad a, b, c = \text{const.} \tag{4.15}$$

Let us take the simplest solution $\varphi = y$ of the form (4.14). Then (2.46) yields $\alpha = 1$. Now we set $W = -1, \omega = 0$ in (4.11) and arrive at the change of variables

$$t = y, \qquad u = x, \tag{4.16}$$

reducing (4.8) to the following linear equation:

$$u''' + u = 0. \tag{4.17}$$

Taking the solution to Eq. (4.12) in the form (4.15), one obtains from (2.46):

$$\alpha = \frac{(b - cy)^6}{c^3}.$$

Thus, eliminating $b - cy$ by using Eq. (4.15) written as $t = a + (b - cy)^{-1}$, one obtains $\alpha(t) = [c(t - a)^2]^{-3}$. Hence, the change of variables

$$t = a + \frac{1}{b - cy}, \qquad u = x \tag{4.18}$$

maps Eq. (4.8) to the following alternative linear equation:

$$u''' + \frac{u}{c^3(t - a)^6} = 0. \tag{4.19}$$

It is known, however, that the two equations (4.17) and (4.19) are equivalent. Therefore, Eq. (4.17) and the change of variables (4.16) can be regarded as a *standard linearization* of Eq. (4.8).

5. Linearization by contact transformations

A transformation

$$t = \varphi(x, y, p), \qquad u = \psi(x, y, p), \qquad q = g(x, y, p) \tag{5.1}$$

of the variables x, y and $p = y' \equiv dy/dx$ is called a *contact transformation* if it obeys the *contact condition* $q = u' = du/dt$, i.e., if

$$g(x, y, p) = \frac{D_x\psi(x, y, p)}{D_x\varphi(x, y, p)}. \tag{5.2}$$

Equation (5.2) implies that the functions φ, ψ and g are related by

$$\psi_p = g\varphi_p, \qquad \psi_x + p\psi_y = (\varphi_x + p\varphi_y)g. \tag{5.3}$$

Furthermore, the functions φ, ψ and g should be functionally independent, i.e., should have the nonvanishing Jacobian. The latter condition, invoking (5.3), is written as follows:

$$(\varphi_y g - \psi_y)\left[(g_x + pg_y)\varphi_p - (\varphi_x + p\varphi_y)g_p\right] \neq 0.$$

It follows, in particular, that if $\varphi_p = 0$ then $\psi_p = 0$, and hence (5.1) is a point transformation considered in the previous sections. Therefore, we assume in what follows that $\varphi_p \neq 0$.

One can verify that if one applies the contact transformation (5.1) to the general linear equation, e.g., taken in Laguerre's form

$$u''' + \alpha(t)u = 0, \tag{5.4}$$

one arrives at nonlinear equations that are at most cubic in the second-order derivative, i.e., at the following equations indicated by Lie (see Introduction):

$$y''' + a(x, y, y')y''^3 + b(x, y, y')y''^2 + c(x, y, y')y'' + d(x, y, y') = 0. \tag{5.5}$$

5.1. Second-order relative invariants of contact transformations

It was mentioned in the previous sections (Remarks 2.1 and 2.5) that the second-order relative invariants of point transformations play a central part in the linearization problem. Therefore, let us investigate relative invariants for Eq. (5.5) with respect to the contact transformations. The search for second-order invariants of the equivalence group leads to the following result.

Lemma 5.1. *The contact transformations* (5.1) *have two distinctly different systems of invariant equations. The first system has the form*

$$J_1 = 0, \qquad J_2 = 0, \qquad J_3 = 0, \qquad J_4 = 0, \tag{5.6}$$

where J_1, J_2, J_3, J_4 are the second-order relative invariants defined by

$$J_1 = 27a_{px} + 27a_{py}p - 18a_pc - 18a_xb - 18a_ybp + 81a_y + 18b_pb - 9b_{pp} + 18b_xa + 18b_yap - 36c_pa - 54a^2d + 18abc - 4b^3,$$

$$J_2 = -18a_pd - 18a_xc + 9a_{xx} - 18a_ycp + 18a_{yx}p + 9a_{yy}p^2 + 6b_pc + 3b_{px}$$

$$
\begin{aligned}
&\quad + 3b_{py}p + 6b_xb + 6b_ybp + 24b_y - 6c_{pp} - 36d_pa - 18abd + 12ac^2 - 2b^2c,\\
J_3 &= 36a_xd + 36a_ydp - 6b_{xx} - 12b_{yx}p - 6b_{yy}p^2 - 6c_pc + 3c_{px} + 3c_{py}p - 6c_xb\\
&\quad - 6c_ybp - 21c_y + 18d_pb + 9d_{pp} + 18d_xa + 18d_yap - 18acd\\
&\quad + 12b^2d - 2bc^2,\\
J_4 &= -36b_xd - 36b_ydp + 18c_pd + 18c_xc + 9c_{xx} + 18c_ycp + 18c_{yx}p + 9c_{yy}p^2\\
&\quad - 18d_pc - 27d_{px} - 27d_{py}p - 18d_xb - 18d_ybp + 54d_y + 54ad^2\\
&\quad - 18bcd + 4c^3.
\end{aligned}
$$

The second invariant system has the form

$$
J_5 = 0, \qquad J_6 = 0, \tag{5.7}
$$

where J_5, J_6 *are the following second-order relative invariants*:

$$
J_5 = \frac{1}{3}\left(J_2 - \frac{J_3^2}{J_4}\right), \qquad J_6 = -\left(\frac{J_1}{9} + \frac{J_5J_3}{J_4} + \frac{J_3^3}{9J_4^2}\right).
$$

Proof. The generator of the group of contact transformations has the form

$$
X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial p} + \zeta^a\frac{\partial}{\partial a} + \zeta^b\frac{\partial}{\partial b} + \zeta^c\frac{\partial}{\partial c} + \zeta^d\frac{\partial}{\partial d},
$$

where

$$
\xi = -W_p, \qquad \eta = (W - pW_p), \qquad \zeta = (W_x + pW_y)
$$

and

$$
\begin{aligned}
\zeta^a &= -3W_{py}pa - 3W_{px}a - W_{ppp} + W_{pp}b - 2W_ya,\\
\zeta^b &= -W_{py}pb - 3W_{py} - W_{px}b - 3W_{ppy}p - 3W_{ppx} + 2W_{pp}c - 6W_{yx}pa\\
&\quad - 3W_{yy}p^2a - W_yb - 3W_{xx}a,\\
\zeta^c &= -6W_{pyx}p - 3W_{pyy}p^2 + W_{py}pc - 3W_{pxx} + W_{px}c + 3W_{pp}d - 4W_{yx}pb\\
&\quad - 3W_{yx} - 2W_{yy}p^2b - 3W_{yy}p - 2W_{xx}b,\\
\zeta^d &= 3W_{py}pd + 3W_{px}d - 3W_{yxx}p - 2W_{yx}pc - W_{yyy}p^3 - 3W_{yyx}p^2\\
&\quad - W_{yy}p^2c + W_yd - W_{xxx} - W_{xx}c.
\end{aligned}
$$

Here $W(x, y, p)$ is an arbitrary function known as the characteristic function of the contact transformation group. The reckoning shows that the invariant test yields the following equations:

$$
\begin{aligned}
S_1(F) &\equiv -3J_2\frac{\partial F}{\partial J_1} + J_4\frac{\partial F}{\partial J_3} + 2J_3\frac{\partial F}{\partial J_2} = 0,\\
S_2(F) &\equiv 3J_1\frac{\partial F}{\partial J_1} + J_3\frac{\partial F}{\partial J_3} + 2J_2\frac{\partial F}{\partial J_2} = 0,\\
S_3(F) &\equiv 3J_4\frac{\partial F}{\partial J_4} + 2J_3\frac{\partial F}{\partial J_3} + J_2\frac{\partial F}{\partial J_2} = 0,
\end{aligned}
$$

$$S_4(F) \equiv 3J_3\frac{\partial F}{\partial J_4} + 2J_2\frac{\partial F}{\partial J_3} - J_1\frac{\partial F}{\partial J_2} = 0,$$

where J_1, J_2, J_3 and J_4 are the quantities used in (5.6). The above equations have only the trivial solution $F = \text{const.}$, and hence there are no absolute invariants. Therefore, we look for relative invariants defined by equations

$$F_i(J_1, J_2, J_3, J_4) = 0, \quad i = 1, \ldots, k, \tag{Φ}$$

such that

$$S_j(F_i)|_{(\Phi)} = 0, \quad j = 1, 2, 3, 4; \; i = 1, \ldots, k.$$

Let $J_4|_{(\Phi)} = 0$. Then, setting $F_1 = J_4$ and considering the equation $S_4(F) = 0$, one obtains $J_3 = 0$. This implies $S_4(J_3) = 2J_2 = 0$, whence $S_4(J_2) = -J_1 = 0$. Hence, the equation $J_4 = 0$ implies $J_1 = 0$, $J_2 = 0$, $J_3 = 0$. Likewise, $J_3 = 0$ implies $J_1 = 0$, $J_2 = 0$, $J_4 = 0$. Thus, we arrive at the invariant system (5.6).

Now we let $J_4|_{(\Phi)} \neq 0$. In this case, the general solution to the equation $S_1 = 0$ has the form

$$F = F(J_4, J_5, J_6)$$

with J_5 and J_6 used in (5.7). The remaining equations become:

$$S_2(F) \equiv 3\frac{\partial F}{\partial J_6}J_6 + 2\frac{\partial F}{\partial J_5}J_5 = 0, \qquad S_3(F) \equiv \frac{\partial F}{\partial J_5}J_5 + 3\frac{\partial F}{\partial J_4}J_4 = 0,$$

$$J_4S_4(F) + J_3S_2(F) - J_3S_3(F) \equiv 3\left(J_4J_6\frac{\partial F}{\partial J_5} - 2J_5^2\frac{\partial F}{\partial J_6}\right) = 0.$$

The equation $S_3(F) = 0$ yields

$$F = F(J_7, J_6),$$

where $J_7 = 9J_4^{-1}J_5^3$. The remaining two equations become:

$$\frac{\partial F}{\partial J_6}J_6 + 2\frac{\partial F}{\partial J_7}J_7 = 0, \qquad J_5^2\left(-2\frac{\partial F}{\partial J_6} + 27\frac{\partial F}{\partial J_7}J_7\right) = 0.$$

If $J_5 \neq 0$, then $J_7 \neq 0$ and there are no invariant equations. On the other hand, if $J_5 = 0$ the invariance conditions reduce to one equation, namely:

$$J_6\frac{\partial J}{\partial J_6} = 0.$$

Thus, we arrive at the invariant system (5.7) thus completing the proof. □

5.2. *Equations equivalent to* $u''' = 0$

In Sections 5.2 and 5.3, we use the following operator of *semi-total differentiation*:

$$\hat{D} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y}. \tag{5.8}$$

In the case $\alpha = 0$, after running the program, we obtain the following result.

Theorem 5.1. *Equation* (5.5) *is linearizable to the equation* $u''' = 0$ *if and only if its coefficients obey Eqs.* (5.6):

$$J_1 = 0, \qquad J_2 = 0, \qquad J_3 = 0, \qquad J_4 = 0. \tag{5.6}$$

Provided that the four conditions (5.6) *are satisfied, the transformation* (5.1) *linearizing Eq.* (5.5) *to* $u''' = 0$ *is obtained by solving the following compatible system of equations* $\varphi(x, y, p)$, $\psi(x, y, p)$, $g(x, y, p)$, $k(x, y, p)$ *and* $H(x, y, p)$:

$$\begin{aligned}
&\hat{D}\psi = Hg, \qquad \psi_y = \varphi_y g + k, \qquad \psi_p = \varphi_p g,\\
&\varphi_p \hat{D} g = H g_p - k, \qquad 3 g_y \varphi_p^2 = 3\varphi_{pp} k + 3aHk + \varphi_p(3\varphi_y g_p - bk),\\
&\varphi_p g_{pp} = (\varphi_{pp} g_p + ak),\\
&3\varphi_p^2 \hat{D} k = 3\varphi_{pp} Hk + 3aH^2 k - H\varphi_p bk + 3\varphi_p \varphi_y k,\\
&9\varphi_p^2 k_y = 18\varphi_{py}\varphi_p k - 9\varphi_{pp}\varphi_y k + 3aHk(2\varphi_p c - 3\varphi_y) - 9a\varphi_p^2 dk\\
&\qquad - 9\hat{D}aHk\varphi_p + 3\hat{D}bk\varphi_p^2 + H\varphi_p k\big(3b_p - 2b^2\big)\\
&\qquad + \varphi_p k(-3c_p\varphi_p + \varphi_p bc + 3\varphi_y b),\\
&3\varphi_p k_p = 3\varphi_{pp} k + 3aHk - \varphi_p bk,\\
&54\varphi_p^2 \varphi_{yy} = 108\varphi_{py}\varphi_p\varphi_y - 54\varphi_{pp}\varphi_y^2 + 6aH\big(3\varphi_p^2 bd - 2\varphi_p^2 c^2 + 3\varphi_p^2 \hat{D}c\\
&\qquad + 6\varphi_p\varphi_y c - 9\varphi_y^2\big) + 18a\varphi_p^2(2\varphi_p cd - 3\varphi_p \hat{D}d - 3\varphi_y d)\\
&\qquad + 18\hat{D}aH\varphi_p(2\varphi_p c - 3\varphi_y) - 108\hat{D}a\varphi_p^3 d - 18\hat{D}bH\varphi_p^2 b\\
&\qquad + 6\hat{D}b\varphi_p^2(\varphi_p c + 3\varphi_y) + H\varphi_p\big(-6b_p\varphi_p c + 18b_p\varphi_y - 27b_y\varphi p\\
&\qquad + 6c_p\varphi_p b + 9(\hat{D}b)_p\varphi_p + 2\varphi_p b^2 c - 27\varphi_p \hat{D}^2 a - 12\varphi_y b^2\big)\\
&\qquad + 2\varphi_p\big(9b_p\varphi_p^2 d - 9c_p\varphi_p\varphi_y + 27c_y\varphi_p^2 - 9(\hat{D}c)_p\varphi_p^2 - 9d_p\varphi_p^2 b\\
&\qquad - 12\varphi_p^2 b^2 d + 2\varphi_p^2 bc^2 + 6\varphi_p^2 b\hat{D}c + 9\varphi_p^2 \hat{D}^2 b + 3\varphi_p\varphi_y bc + 9\varphi_y^2 b\big),\\
&54\varphi_p^2\varphi_{ppy} = 108\varphi_{py}\varphi_{pp}\varphi_p + 54\varphi_{py} aH\varphi_p - 18\varphi_{py}\varphi_p^2 b - 27\varphi_{pp}^2\varphi_y\\
&\qquad - 54\varphi_{pp} aH\varphi_y + 18\varphi_{pp}\varphi_p\varphi_y b + 9a^2H^2(-2\varphi_p c - 3\varphi_y) + 54a^2H\varphi_p^2 d\\
&\qquad + 27a\hat{D}aH^2\varphi_p - 18a\hat{D}bH\varphi_p^2 + 3aH^2\varphi_p\big(-3b_p + 2b^2\big)\\
&\qquad + 6aH\varphi_p(3c_p\varphi_p - \varphi_p bc + 3\varphi_y b) + 9a\varphi_p\big(-3d_p\varphi_p^2 - 4\varphi_p^2 bd + 2\varphi_p^2 c^2\\
&\qquad - \varphi_p^2\hat{D}c - 2\varphi_p\varphi_y c - 3\varphi_y^2\big) + 9\hat{D}a\varphi_p^2(-4\varphi_p c + 3\varphi_y) + 18\hat{D}b\varphi_p^3 b\\
&\qquad - 54Ha_y\varphi_p^2 + \varphi_p^2\big(6b_p\varphi_p c - 9b_p\varphi_y + 45b_y\varphi_p - 6c_p\varphi_p b - 9(\hat{D}b)_p\varphi_p\\
&\qquad - 2\varphi_p b^2 c + 27\varphi_p\hat{D}^2 a + 3\varphi_y b^2\big),\\
&6\varphi_p\varphi_{ppp} = 9\varphi_{pp}^2 - 9a^2H^2 + 6aH\varphi_p b - 12a\varphi_p\varphi_y - 3\hat{D}a\varphi_p^2\\
&\qquad - 6Ha_p\varphi_p + \varphi_p^2\big(3b_p - b^2\big),\\
&3\varphi_p^2\hat{D}H = 3\varphi_{pp}H^2 + 3aH^3 - H^2\varphi_p b + H\varphi_p(-\varphi_p c + 3\varphi_y) + 3\varphi_p^3 d,\\
&18\varphi_p H_y = \big(18\varphi_{py}H + 6aH^2 c - 18aH\varphi_p d - 9\hat{D}aH^2
\end{aligned}$$

$$+6\hat{D}bH\varphi_p + H^2\big(3b_p - 2b^2\big) + 2H\varphi_p(-3c_p + bc) + 9d_p\varphi_p^2$$
$$+6\varphi_p^2 bd - 2\varphi_p^2 c^2 - 3\varphi_p^2\hat{D}c + 9\varphi_y^2\big),$$
$$3\varphi_p H_p = 3\varphi_{pp}H + 3aH^2 - 2H\varphi_p b + \varphi_p(\varphi_p c + 3\varphi_y),$$

where

$$H = \varphi_x + p\varphi_y, \qquad k = \varphi_p g_x + \varphi_p g_y p - \varphi_x g_p - \varphi_y g_p p \neq 0.$$

5.3. Equations equivalent to $u''' + \alpha(t)u = 0$

The test for linearization by contact transformation by reducing to equations of the form $u''' + \alpha(t)u = 0$ with $\alpha(t) \neq 0$ is given by the following theorem.

Theorem 5.2. *Equation* (5.5) *is reducible to a linear equation $u''' + \alpha(t)u = 0$ with $\alpha(t) \neq 0$ if and only if $J_1 \neq 0$ and the following eight equations hold*:

$$J_3 = -J_2^2/J_1, \quad J_4 = J_2^3/J_1^2,$$
$$J_1\hat{D}J_1 = J_{1p}J_2 + 3aJ_2^2 - 2bJ_1J_2 + J_1^2c,$$
$$J_1^2\hat{D}J_2 = J_{2p}J_1J_2 + 2aJ_2^3 - bJ_1J_2^2 + J_1^3d,$$
$$9J_1^2J_{2pp} = 9a_pJ_1J_2^2 - 6b_pJ_1^2J_2 + 9J_{1pp}J_1J_2 - 15J_{1p}^2J_2 + 15J_{1p}J_{2p}J_1$$
$$+ J_{1p}\big(-15aJ_2^2 + 4bJ_1J_2 + J_1^2c\big) + 3J_{1y}J_1^2$$
$$+ 6J_{2p}J_1(3aJ_2 - bJ_1) + 3c_pJ_1^3,$$
$$18J_1^3J_{2x} = -3b_pJ_1^2J_2^2p - 9J_{1p}^2J_2^2p + 18J_{1p}J_{2p}J_1J_2p$$
$$+ 6J_{1p}J_2p\big(-3aJ_2^2 + 2bJ_1J_2 - J_1^2c\big) - 18J_{1y}J_1^2J_2p - 9J_{2p}^2J_1^2p$$
$$+ 6J_{2p}J_1\big(3aJ_2^2p - 2bJ_1J_2p + J_1^2cp + 3J_1J_2\big) - 9d_pJ_1^4p$$
$$+ 6c_pJ_1^3J_2p - 9a^2J_2^4p + 12abJ_1J_2^3p + 18aJ_1^3J_2dp - 12aJ_1^2J_2^2cp$$
$$+ 36aJ_1J_2^3 - 2b^2J_1^2J_2^2p - 6bJ_1^4dp + 2bJ_1^3J_2cp - 18bJ_1^2J_2^2 + J_1^4c^2p$$
$$+ 3J_1^4\hat{D}cp + 18J_1^4d - 6J_1^3J_2\hat{D}bp + 9J_1^2J_2^2\hat{D}ap,$$
$$18J_1^3J_{2y} = 3b_pJ_1^2J_2^2 + 9J_{1p}^2J_2^2 - 18J_{1p}J_{2p}J_1J_2 + 6J_{1p}J_2\big(3aJ_2^2 - 2bJ_1J_2 + J_1^2c\big)$$
$$+ 18J_{1y}J_1^2J_2 + 9J_{2p}^2J_1^2 + 6J_{2p}J_1\big(-3aJ_2^2 + 2bJ_1J_2 - J_1^2c\big)$$
$$+ 9d_pJ_1^4 - 6c_pJ_1^3J_2 + 9a^2J_2^4 - 12abJ_1J_2^3 - 18aJ_1^3J_2d$$
$$+ 12aJ_1^2J_2^2c + 2b^2J_1^2J_2^2 + 6bJ_1^4d - 2bJ_1^3J_2c - J_1^4c^2 - 3J_1^4\hat{D}c$$
$$+ 6J_1^3J_2\hat{D}b - 9J_1^2J_2^2\hat{D}a,$$
$$54a_pJ_{1p}J_1J_2^2 - 54a_pJ_{2p}J_1^2J_2 + 18a_pJ_1J_2\big(3aJ_2^2 - 2bJ_1J_2 + J_1^2c\big)$$
$$- 36b_pJ_{1p}J_1^2J_2 + 36b_pJ_{2p}J_1^3 + 3b_pJ_1^2\big(-9aJ_2^2 + 8bJ_1J_2 - 6J_1^2c\big)$$
$$+ 18J_{1pp}J_{1p}J_1J_2 - 18J_{1pp}J_{2p}J_1^2 + 6J_{1pp}J_1\big(3aJ_2^2 - 2bJ_1J_2 + J_1^2c\big)$$
$$- 24J_{1p}^3J_2 + 24J_{1p}^2J_{2p}J_1 + J_{1p}^2\big(-123aJ_2^2 + 22bJ_1J_2 - 8J_1^2c\big)$$

$$
\begin{aligned}
&- 24J_{1p}J_{1y}J_1^2 + 6J_{1p}J_{2p}J_1(30aJ_2 - bJ_1) + 2J_{1p}\big(-63a^2J_2^3 + 45abJ_1J_2^2\\
&- 39aJ_1^2J_2c + 4b^2J_1^2J_2 + bJ_1^3c + 27J_1^2J_2\hat{D}a\big) + 6J_{1y}J_1^2(-3aJ_2 + bJ_1)\\
&- 81J_{2p}^2aJ_1^2 + 6J_{2p}J_1\big(18a^2J_2^2 - 12abJ_1J_2 + 12aJ_1^2c - 2b^2J_1^2 - 9J_1^2\hat{D}a\big)\\
&+ 27d_paJ_1^4 + 54a_yJ_1^3J_2 - 45b_yJ_1^4 + 18J_{1py}J_1^3 - 18c_paJ_1^3J_2 + 6c_pbJ_1^4\\
&+ 9(\hat{D}b)_pJ_1^4 - 27a^3J_2^4 + 36a^2bJ_1J_2^3 - 54a^2J_1^3J_2d - 36a^2J_1^2J_2^2c\\
&- 6ab^2J_1^2J_2^2 + 36abJ_1^4d + 42abJ_1^3J_2c - 33aJ_1^4c^2 + 9aJ_1^4\hat{D}c\\
&+ 18aJ_1^3J_2\hat{D}b + 27aJ_1^2J_2^2\hat{D}a - 8b^3J_1^3J_2 + 6b^2J_1^4c - 18bJ_1^4\hat{D}b\\
&- 36bJ_1^3J_2\hat{D}a + J_1^4J_2 + 54J_1^4c\hat{D}a - 27J_1^4\hat{D}^2a = 0,
\end{aligned}
$$

$$
\begin{aligned}
&810a_pJ_{1p}J_1J_2^2d - 810a_pJ_{2p}J_1^2J_2d + 270a_pJ_1J_2d\big(3aJ_2^2 - 2bJ_1J_2 + J_1^2c\big)\\
&- 540b_pJ_{1p}J_1^2J_2d + 540b_pJ_{2p}J_1^3d + 9b_pJ_1^2\big(-45aJ_2^2d + 40bJ_1J_2d\\
&- 30J_1^2cd - 6J_1^2\hat{D}d\big) + 270J_{1pp}J_{1p}J_1J_2d - 270J_{1pp}J_{2p}J_1^2d\\
&+ 90J_{1pp}J_1d\big(3aJ_2^2 - 2bJ_1J_2 + J_1^2c\big) - 360J_{1p}^3J_2d + 360J_{1p}^2J_{2p}J_1d\\
&+ 15J_{1p}^2d\big(-123aJ_2^2 + 22bJ_1J_2 - 8J_1^2c\big) - 360J_{1p}J_{1y}J_1^2d\\
&+ 90J_{1p}J_{2p}J_1d(30aJ_2 - bJ_1) + 3J_{1p}\big(-630a^2J_2^3d + 450abJ_1J_2^2d\\
&- 390aJ_1^2J_2cd + 40b^2J_1^2J_2d + 10bJ_1^3cd + 270J_1^2J_2\hat{D}ad - J_1J_2^3\big)\\
&+ 90J_{1y}J_1^2d(-3aJ_2 + bJ_1) - 1215J_{2p}^2aJ_1^2d + 9J_{2p}J_1\big(180a^2J_2^2d\\
&- 120abJ_1J_2d + 120aJ_1^2cd - 20b^2J_1^2d - 90J_1^2\hat{D}ad + J_1J_2^2\big)\\
&+ 9d_pJ_1^4(45ad - 2bc + 6\hat{D}b) + 810a_yJ_1^3J_2d - 486b_yJ_1^4d + 270J_{1py}J_1^3d\\
&- 270c_paJ_1^3J_2d + 108c_pbJ_1^4d - 90c_yJ_1^4c + 54(\hat{D}b)_pJ_1^4d + 18(\hat{D}c)_pJ_1^4c\\
&- 243(\hat{D}c)_yJ_1^4 + 54(\hat{D}^2c)_pJ_1^4 + 81d_{py}J_1^4 + 162d_ybJ_1^4 - 405a^3J_2^4d\\
&+ 540a^2bJ_1J_2^3d - 810a^2J_1^3J_2d^2 - 540a^2J_1^2J_2^2cd - 90ab^2J_1^2J_2^2d + 594abJ_1^4d^2\\
&+ 630abJ_1^3J_2cd - 495aJ_1^4c^2d - 54aJ_1^4c\hat{D}d - 27aJ_1^4\hat{D}cd + 162aJ_1^4\hat{D}^2d\\
&+ 270aJ_1^3J_2\hat{D}bd + 405aJ_1^2J_2^2\hat{D}ad + 9aJ_1J_2^4 - 120b^3J_1^3J_2d + 72b^2J_1^4cd\\
&+ 36b^2J_1^4\hat{D}d + 4bJ_1^4c^3 - 144bJ_1^4\hat{D}bd - 18bJ_1^4\hat{D}^2c - 540bJ_1^3J_2\hat{D}ad\\
&- 4bJ_1^2J_2^3 + 24J_1^4J_2d - 18J_1^4c^2\hat{D}b + 702J_1^4c\hat{D}ad - 36J_1^4c\hat{D}^2b\\
&+ 486J_1^4\hat{D}a\hat{D}d - 54J_1^4\hat{D}b\hat{D}c - 54J_1^4\hat{D}^3b - 3J_1^3J_2^2c = 0.
\end{aligned}
$$

Note that the first two equations of this system, $J_3 = -J_2^2/J_1$, $J_4 = J_2^3/J_1^2$, are equivalent to the invariant system (5.7)*. The other equations define relative invariants up to the fourth-order.*

The coefficient $\alpha(t)$ of the resulting linear equation is given by

$$\alpha = J_1/\big(54\varphi_p^3\big) \neq 0.$$

Furthermore, the linearizing contact transformation is obtained by solving the following compatible system of 11 *equations for four functions* $\varphi(x,y,p)$, $\psi(x,y,p)$, $g(x,y,p)$ *and* $k(x,y,p)$:

$$
\begin{aligned}
&J_1\hat{D}\varphi = \varphi_p J_2, \qquad 3J_1^2\varphi_y = \varphi_p\big(-3J_{1p}J_2 + 3J_{2p}J_1 - 3aJ_2^2 + 2bJ_1J_2 - J_1^2c\big),\\
&6J_1^2\varphi_p\varphi_{ppp} = 9\varphi_{pp}^2J_1^2 + \varphi_p^2\big(12J_{1p}aJ_2 - 12J_{2p}aJ_1 - 6a_pJ_1J_2 + 3b_pJ_1^2 + 3a^2J_2^2\\
&\qquad\qquad - 2abJ_1J_2 + 4aJ_1^2c - b^2J_1^2 - 3J_1^2\hat{D}a\big),\\
&\varphi_pJ_1\hat{D}g = \varphi_pg_pJ_2 - J_1k,\\
&3\varphi_p^2J_1^2g_y = 3\varphi_{pp}J_1^2k + \varphi_pJ_1k(3aJ_2 - bJ_1)\\
&\qquad\qquad + \varphi_p^2g_p\big(-3J_{1p}J_2 + 3J_{2p}J_1 - 3aJ_2^2 + 2bJ_1J_2 - J_1^2c\big),\\
&54\varphi_pg_{pp} = 54\varphi_{pp}g_p + 54ak - J_1\psi,\\
&\psi_p = \varphi_pg, \qquad J_1\hat{D}\psi = \varphi_pJ_2g,\\
&3J_1^2\psi_y = -3J_{1p}\varphi_pJ_2g + 3J_{2p}\varphi_pJ_1g + 3J_1^2k + \varphi_pg\big(-3aJ_2^2 + 2bJ_1J_2 - J_1^2c\big),\\
&3\varphi_pJ_1^2\hat{D}k = -3J_{1p}\varphi_pJ_2k + 3J_{2p}\varphi_pJ_1k + 3\varphi_{pp}J_1J_2k + \varphi_pJ_1k(bJ_2 - J_1c),\\
&108\varphi_pJ_1^4k_y = 54kJ_1(2\varphi_{pp}J_1 - a_p\varphi_pJ_2p)(J_{2p}J_1 - J_{1p}J_2)\\
&\qquad + 36\varphi_{pp}J_1^2k\big(-3aJ_2^2 + 2bJ_1J_2 - J_1^2c\big)\\
&\qquad + \varphi_pk\big[18a_pJ_1J_2p\big(3aJ_2^2 - 2bJ_1J_2 + J_1^2c\big) - 36b_pJ_{1p}J_1^2J_2p\\
&\qquad + 36b_pJ_{2p}J_1^3p + 3b_pJ_1^2\big(-9aJ_2^2p + 8bJ_1J_2p - 6J_1^2cp + 12J_1J_2\big)\\
&\qquad + 18J_{1pp}J_{1p}J_1J_2p - 18J_{1pp}J_{2p}J_1^2p\\
&\qquad + 6J_{1pp}J_1p\big(3aJ_2^2 - 2bJ_1J_2 + J_1^2c\big) - 24J_{1p}^3J_2p\\
&\qquad + 24J_{1p}^2J_{2p}J_1p + J_{1p}^2\big(-123aJ_2^2p + 22bJ_1J_2p - 8J_1^2cp + 72J_1J_2\big)\\
&\qquad - 24J_{1p}J_{1y}J_1^2p + 6J_{1p}J_{2p}J_1(30aJ_2p - bJ_1p - 12J_1)\\
&\qquad + 2J_{1p}\big(-63a^2J_2^3p + 45abJ_1J_2^2p - 39aJ_1^2J_2cp + 90aJ_1J_2^2\\
&\qquad + 4b^2J_1^2J_2p + bJ_1^3cp - 42bJ_1^2J_2 + 12J_1^3c + 27J_1^2J_2\hat{D}ap\big)\\
&\qquad + 6J_{1y}J_1^2(-3aJ_2p + bJ_1p + 12J_1) - 81J_{2p}^2aJ_1^2p\\
&\qquad + 6J_{2p}J_1\big(18a^2J_2^2p - 12abJ_1J_2p + 12aJ_1^2cp - 18aJ_1J_2\\
&\qquad - 2b^2J_1^2p + 6bJ_1^2 - 9J_1^2\hat{D}ap\big) + 27d_paJ_1^4p\\
&\qquad + 54a_yJ_1^3J_2p - 45b_yJ_1^4p + 18J_{1py}J_1^3p - 18c_paJ_1^3J_2p + 6c_pbJ_1^4p\\
&\qquad - 36c_pJ_1^4 + 9(\hat{D}b)_pJ_1^4p - 27a^3J_2^4p + 36a^2bJ_1J_2^3p - 54a^2J_1^3J_2dp\\
&\qquad - 36a^2J_1^2J_2^2cp + 108a^2J_1J_2^3 - 6ab^2J_1^2J_2^2p + 36abJ_1^4dp\\
&\qquad + 42abJ_1^3J_2cp - 108abJ_1^2J_2^2 - 33aJ_1^4c^2p + 9aJ_1^4\hat{D}cp - 108aJ_1^4d\\
&\qquad + 108aJ_1^3J_2c + 18aJ_1^3J_2\hat{D}bp + 27aJ_1^2J_2^2\hat{D}ap - 8b^3J_1^3J_2p\\
&\qquad + 6b^2J_1^4cp - 18bJ_1^4\hat{D}bp - 36bJ_1^3J_2\hat{D}ap + J_1^4J_2p + 54J_1^4c\hat{D}ap
\end{aligned}
$$

$$+\,36J_1^4\hat{D}b - 27J_1^4\hat{D}^2ap - 108J_1^3J_2\hat{D}a\big],$$
$$3\varphi_p J_1 k_p = 3\varphi_{pp} J_1 k + \varphi_p k(3aJ_2 - bJ_1).$$

5.4. Examples

Example 5.1. It is known [7] (see also [5, Section 8.3.3]) that the equations

$$\text{(C)}\quad y''' = \frac{3y'y''^2}{1+y'^2} \qquad \text{and} \qquad \text{(H)}\quad y''' = \frac{3y''^2}{2y'} \tag{5.9}$$

describing the families of circles and hyperbolas, respectively, are connected by a complex transformation, and that Eq. (5.9)(H) can be linearized to the equation $u''' = 0$ by a contact transformation (specifically, by the Legendre transformation). One can readily check that Eq. (5.9)(C) also satisfies the conditions (5.6), and hence can be reduced to $u''' = 0$ by a (real valued) contact transformation. The reckoning yields the following linearizing transformation:

$$\varphi = \frac{y(1+\sqrt{p^2+1})}{p} - x, \qquad \psi = -y\left(1 + \frac{1+\sqrt{p^2+1}}{p^2}\right),$$
$$g = -\frac{1+\sqrt{p^2+1}}{p}.$$

An alternative transformation is

$$\varphi = -\big(p + \sqrt{1+p^2}\big), \qquad \psi = (px - y)\big(p + \sqrt{1+p^2}\big),$$
$$g = y - x\big(p + \sqrt{1+p^2}\big).$$

Remark 5.1. It is stated in [10] that the contact transformation

$$t = -2xg(x, y, p), \qquad u = y + xp, \qquad u' = g(x, y, p),$$
$$\text{where } g^2 = -p, \tag{5.10}$$

maps the equation $u''' = 0$ to the equation for circles (5.9)(C). However, the transformation (5.10) relates the equation $u''' = 0$ with Eq. (5.9)(H) but not with (C).

Example 5.2. Consider again the equations of the form (2.6). One can readily verify that two of the conditions for linearization by a contact transformation are satisfied, namely $J_1 = 0$ and $J_2 = 0$. Equating to zero two other invariants, J_3 and J_4, we conclude that Eq. (2.6) can be mapped by a contact transformation to the equation $u''' = 0$ if and only if the following equations hold:

$$2(3B_2 - 3A_{1x} - A_0A_1) = 7(A_{0y} - A_{1x}), \qquad 9B_3 = 3A_{1y} + A_1^2,$$
$$3(A_{0y} - A_{1x})_y - A_1(A_{0y} - A_{1x}) = 0,$$
$$6(A_{0y} - A_{1x})_x + 2A_0(A_{0y} - A_{1x}) - 3\big(3B_1 - A_0^2 - 3A_{0x}\big)_y = 0,$$
$$9A_{0xx} + 18A_{0x}A_0 + 54B_{0y} - 27B_{1x} + 4A_0^3 - 18A_0B_1 + 18A_1B_0 = 0. \tag{5.11}$$

The last equation (5.11) yields $\Omega = 0$, where Ω is defined by (2.29). Invoking Theorem 2.1 and noting that Eqs. (2.22)–(2.23) imply the first four equations (5.11), we conclude that Eq. (2.6) is linearizable simultaneously by contact and point transformations if and only if its coefficients satisfy the equation $\Omega = 0$ and Eqs. (2.22)–(2.23).

For example, the equation

$$y''' + \frac{3}{y}y'y'' - 3y'' - \frac{3}{y}y'^2 + 2y' - y = 0$$

is linearizable by a point transformation, but it is not linearizable by a contact transformation.

On the other hand, the equation

$$y''' + yy'' + \frac{1}{54}\left(63p^2 + 24y^2p + y^4\right) = 0$$

can be linearized by a contact transformation, but cannot be linearized by a point transformation since $A_{0y} - A_{1x} \neq 0$.

Example 5.3. The equation

$$y''' + yy'' + \beta\left(1 - y'^2\right) = 0$$

widely used in hydrodynamics (it is called the Blasius equation when $\beta = 0$, the Hiemenz flow when $\beta = 1$, and also known as the Falkner–Skan equation [9]) is linearizable neither by point nor contact transformation.

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