A Sharp Inequality of Markov Type for Polynomials Associated with Laguerre Weight

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The best possible constant $A_n$ in an inequality of Markov type
\[
\left\| \frac{d}{dx}(e^{-x}p_n(x)) \right\|_{[0,\infty)} \leq A_n \left\| e^{-x}p_n(x) \right\|_{[0,\infty)},
\]
where $\| \cdot \|_{[0,\infty)}$ denotes the sup-norm on the half real line $[0, \infty)$ and $p_n$ is an arbitrary polynomial of degree at most $n$, is determined in terms of the weighted Chebyshev polynomials associated with the Laguerre weight $e^{-x}$ on $[0, \infty)$. © 2001 Elsevier Science

1. INTRODUCTION

Among the many theorems relating the derivative of a polynomial to the polynomial itself, two of the most well-known inequalities are Bernstein’s inequality
\[
|p'(x)| \leq \frac{n}{\sqrt{1-x^2}} \|p\|_{[-1,1]}, \quad -1 < x < 1, \quad p \in P_n,
\]
and Markov’s inequality
\[
\|p'\|_{[-1,1]} \leq n^2 \|p\|_{[-1,1]}, \quad p \in P_n'. \quad (1)
\]
Here $P_n$ and $P_n'$ denote the polynomials of degree $n$ and the polynomials of degree $n$ with real coefficients, respectively, and $\| \cdot \|_{[-1,1]}$ denotes the sup-norm on the interval $[-1, 1]$; that is, $\|p\| = \sup\{|p(x)|: -1 \leq x \leq 1\}$. Notice that the version of Bernstein’s inequality mentioned here is a pointwise

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inequality while Markov’s inequality is global. Even though Markov’s inequality is global, we find that Markov’s inequality gives us a better local estimate than that of Bernstein for \( x \) near \(-1\) or \(1\). But, when \( x \) is not near \(-1\) or \(1\), the estimate given by Bernstein’s result can be much better than Markov’s. In this paper, we will be concerned with inequalities similar to the latter, the so-called Markov-type inequalities.

In general, the Markov-type inequalities involve finding

\[
A_n := \sup \frac{\| (wp) \|}{\| wp \|}
\]

as a function of \( n \), where the supremum is taken over all \( p \) from a subset of either \( P_n \) or \( P_n' \), the weight \( w \) is fixed, and \( \cdot \) is a norm. The classical Markov inequality (1) is the case when the supremum is taken over \( P_n' \), \( w \equiv 1 \), \( \| \cdot \| = \| \cdot \|_{[-1, 1]} \), and \( A_n \) is determined as

\[
A_n = \frac{\| T_n \|}{\| T_n \|}
\]

where \( T_n(x) = \frac{1}{2^n} \cos(n \arccos x) \), the \( n \)th Chebyshev polynomial of the first kind.

Various problems arise from considering different norms, weights, and sets over which the supremum is taken. When sup-norm is used, there are only two other cases besides the classical Markov inequality where \( A_n \) is precisely determined. In these results \( A_n \) is expressed in terms of the weighted Chebyshev polynomials. The first of these cases, studied by Frappier [6], involved taking the supremum over polynomials with restrictions on the location of their zeros with \( w \equiv 1 \).

The second case, when \( w(x) = e^{-x^2} \), \( \mathbf{l} \cdot \| = \mathbf{l} \cdot \|_{(-\infty, \infty)} \) and the supremum is taken over \( P_n' \), was first considered by Mohapatra et al. [14] and later by Li et al. [10]. In [10] the \( A_n \) is determined in terms of \( T_n(\cdot, e^{-x^2}) \), the weighted Chebyshev polynomial of degree \( n \) associated with the Hermite weight \( e^{-x^2} \) on \(( -\infty, \infty ) \). Precisely, the following result is proved:

**Theorem A.** Let \( T_n = T_n(\cdot, e^{-x^2}) \) and \( \mathbf{l} \cdot \| = \mathbf{l} \cdot \|_{(-\infty, \infty)} \). Then

\[
\max_{p \in \mathcal{P}_n, p \neq 0} \frac{\| (e^{-x^2} p) \|}{\| e^{-x^2} p \|} = \frac{\| (e^{-x^2} T_n) \|}{\| e^{-x^2} T_n \|}
\]

Many results on estimates of \( A_n \) associated with Markov-type inequalities have been obtained (cf. [12, pp. 700–723] and the references therein).
One of the most recent of such results was a rational version of the Markov-type inequality with the sup-norm, namely when \( \| \cdot \| = \| \cdot \|_{[-1, 1]} \) and the weight \( w \equiv 1/q \), where \( q \) is a polynomial of degree \( n \) with distinct zeros outside of \([-1, 1]\). This case was first studied by Borwein et al. [4] and later by Min [13]. Although \( A_n \) is still unknown, estimates have been found in terms of the \( n \)th Chebyshev rational function that can be viewed as a weighted Chebyshev polynomial.

In this paper we will concentrate on the Laguerre weight \( e^{-x} \) with \( \| \cdot \| = \| \cdot \|_{[0, \infty)} \) and obtain \( A_n \) in terms of the weighted Chebyshev polynomials. We will adapt the approach of Markov (see, for example [1, 10, 19]) by considering first a pointwise version of the problem and viewing this new problem as that of minimizing some linear functional under appropriate constraints. Before we state our result we mention some related results for Markov-type inequalities involving Laguerre weight.

In 1964, Szegö [17] proved the following:

**Theorem B.** For \( p \in \mathcal{P}'_n \),

\[
\| p(x) e^{-x} \|_{[0, \infty)} \leq M_n \| p(x) e^{-x} \|_{[0, \infty)},
\]

where \( M_n \sim n \); i.e., \( M_n/n \) is bounded away from zero and infinity.

This result of Szegö implies an estimate of \( A_n \), viz. \( A_n = O(n) \). For \( L^2 \) norm, the corresponding sharp Markov-type inequality associated with the Laguerre weight is obtained by Turan [18]. This result was rediscovered by Shampine [15] who also obtained best constants for similar inequalities where second derivatives are used. Further results and references can be found in Dörfier’s paper [5].

Markov-type inequalities in \( L^p \) spaces associated with weights \( e^{-|x|^a} \), \( a > 1 \), have been considered in [7–9], but only estimates of \( A_n \) are obtained. We remark that even in the \( L^2 \) Markov inequality on \([-1, 1]\),

\[
\left( \int_{-1}^1 |p'(x)|^2 \, dx \right)^{1/2} \leq A_n \left( \int_{-1}^1 |p(x)|^2 \, dx \right)^{1/2}, \quad p \in \mathcal{P}_n,
\]

the extremal polynomials that reduce the inequality to an equality have not yet been found (see [2, 7]).

We now turn to our main result.

## 2. THE MAIN RESULT

Let \( w \) be a weight function. The weighted Chebyshev polynomial \( T_n(x; w) = x^n + \text{lower degree terms} \in \mathcal{P}_n \) can be characterized by \( w(x) T_n(x; w) \) having
an alternating set of \( n+1 \) points; see for example [11]. We will use \( \hat{T}_n \) to denote the normalized weighted Chebyshev polynomial; that is, \( \hat{T}_n = T_n / \| w T_n \| \).

The main result of this paper is the following:

**Theorem 1.** With \( w(x) = e^{-x} \), \( p_n \in \mathcal{P}_n \), and \( \| \cdot \| = \| \cdot \|_{[0, \infty)} \),

\[
A_n = \sup \{ (w p_n)' : \| w p_n \| = 1, p_n \in \mathcal{P}_n \} = \| (w \hat{T}_n)' (0) \|.
\]

**Remark.** Notice that the norm \( \| \cdot \| \) is always taken on weighted polynomials \( w p_n \) and \( (w p_n)' = w (p_n' - p_n) \). By a well-known result of Mhaskar and Saff (cf. [11]), for \( w(x) = e^{-x} \), we have

\[
\| w p_n \|_{[0, \infty)} \|_{[0, \infty)} = \| w p_n \|_{[0, \infty)} \|_{[0, \infty)} = \| (w p_n)' \|_{[0, \infty)}.
\]

Thus, we can use the compact intervals \([0, \pi n/2]\) to replace \([0, \infty)\) when dealing with the sup-norm.

The outline on the proof goes like this: We will first consider the following pointwise extremal problem: For \( y \geq 0 \),

\[
(P_y) \quad \begin{cases} \text{Maximize} & (w p_n)' (y) \\ \text{subject to} & p_n \in \mathcal{P}_n \text{ and } \| w p_n \| \leq 1. \end{cases}
\]

A standard compact argument shows that the extremal solutions of \((P_y)\) always exists. Let \( M_n (y) \) denote the maximum value obtained while solving the problem \((P_y)\). Then to find \( A_n \), we find the largest value of \( M_n (y) \) as a function of \( y \). It turns out that extremal solutions of \((P_y)\) for the maximum points \( y \) of \( M_n (y) \) are the weighted Chebyshev polynomials.

To prove Theorem 1, we need the following result of Shapiro [16]. Let \( P \) be a normed space with norm \( \| \cdot \| \) and let \( L \) be a linear functional on \( P \). Consider the more general problem

\[
(P_L) \quad \begin{cases} \text{Maximize} & L(p) \\ \text{subject to} & p \in P \text{ and } \| p \| \leq 1. \end{cases}
\]

**Lemma 2 (Representation Theorem).** Let \( C(S) \) be the set of real-valued continuous functions on a compact Hausdorff space \( S \). Let \( P \) be an \( n \)-dimensional linear subspace of \( C(S) \) over \( \mathbb{R} \). Let \( \| \cdot \| \) be the sup-norm taken on the set \( S \). Let \( L \neq 0 \) be a real-valued linear functional on \( P \). Then there
exist points \( \tau_1, \tau_2, \ldots, \tau_r \in S \) and nonzero real numbers \( \lambda_1, \lambda_2, \ldots, \lambda_r \), where \( 1 \leq r \leq n \) such that

\[
L(p) = \sum_{i=1}^{r} \lambda_i p(\tau_i), \quad p \in P,
\]

and

\[
\|L\| = \sum_{i=1}^{r} |\lambda_i|,
\]

where

\[
\|L\| := \sup\{|L(p)| : \|p\| \leq 1, p \in P\}.
\]

The proof of this result can be found in [3, 16].

Now, let us go back to the pointwise problem \((P_y)\) for \( w(x) = e^{-x} \). Letting

\[
P = wP^* := \{wP_\alpha : p_\alpha \in P^* \}
\]

and \( L(wp) = (wp)^{(y)}(y) \) and taking \( S = [0, n\pi/2] \) (in view of the Remark following Theorem 1) in the Representation Theorem, we have the following:

**Lemma 3.** The weighted polynomial \( wQ \in wP^* \) is an extremal element for \((P_y)\) if and only if there exist \( \lambda_i \neq 0 \) and \( \tau_j, j = 1, 2, \ldots, r \) for some \( r = r(y) \), \( 1 \leq r \leq n+1 \), with \( 0 \leq \tau_1 < \tau_2 < \cdots < \tau_r \leq n\pi/2 \) such that

\[
(wp_\alpha)^{(y)} = \sum_{i=1}^{r} \lambda_i (wp_\alpha)(\tau_i) \quad \text{for all} \quad p_\alpha \in P^*,
\]

\[
\text{sgn } \lambda_j = \text{sgn}(wQ_\alpha)(\tau_j), \quad \text{and}
\]

\[
|(wQ_\alpha)(\tau_j)| = \|wQ\| = 1, \quad j = 1, 2, \ldots, r.
\]

**Remarks.** (i) This result completely characterizes the extremal elements for \((P_y)\). (ii) The last condition in (4) suggests that Chebyshev polynomials will be of interest.

### 3. ADDITIONAL LEMMAS

Let \( w(x) = e^{-x} \). In this section, we shall obtain lemmas which will be used in the proof of our theorem.

Now let \( y \geq 0 \) and let \( w(x) Q_\alpha(x) := w(x) Q_\alpha(x, y) \) be an extremal element for \((P_y)\).
Our first lemma says something about the case where \( r = n + 1 \) in Lemma 3.

**Lemma 4.** If \( r = n + 1 \), then \( Q_n = \hat{T}_n \) or \( -\hat{T}_n \).

*Proof.* First note that since \((wQ_n)' = w(Q_n' - Q_n)\), \((wQ_n)'\) has at most \( n \) zeros. Now suppose there is at least one \( j \) such that \( wQ_n(\tau_j) = wQ_n(\tau_{j+1}) \). If this is the case, then \((wQ_n)'(x)\) has a zero in \((\tau_j, \tau_{j+1})\). But \((wQ_n)'(\tau_j) = 0\) for \( j = 2, 3, ..., n + 1 \). So \((wQ_n)'\) has at least \( n + 1 \) zeros, a contradiction. Hence, \( Q_n \) must have the maximal equioscillation property enjoyed by only \( \pm \hat{T}_n \).

The next result tells us more about the zeros of \( w(x)p(x) \) and its derivative when \( p \in \mathcal{P}_n' \).

**Lemma 5.** Let \( p_n \in \mathcal{P}_n' \) have \( n \) simple zeros in \([0, \infty)\). Then \((wp_n)'\) has exactly \( n \) zeros in \((0, \infty)\). Further, the zeros of \((wp_n)'\) in \((0, \infty)\) and the zeros of \( p_n \) interlace so that the largest zero of \( p_n \) is less than the largest zero of \( (wp)' \).

*Proof.* First note that \((wp_n)' = w(p_n' - p_n)\). Let \( q = p_n' - p_n \). It is obvious that \( q \) can have at most \( n \) zeros. We now show that \( q \) has \( n \) distinct zeros in \((0, \infty)\).

Let \( x_1 \) and \( x_2 \) be two consecutive zeros of \( p_n \). Then \( q(x_1) = p_n'(x_1) \) and \( q(x_2) = p_n'(x_2) \) have opposite signs as implied by the assumption that \( p_n \) has \( n \) simple zeros. Thus, \( q \) and therefore \((wp_n)'\) has a zero between every two consecutive zeros of \( p \). So, if \( x^* \) denotes the largest zero of \( p_n \), then \( q \) has at least \( n - 1 \) distinct zeros in \((0, x^*)\). Suppose, without loss of generality, that the leading coefficient of \( p_n \) is positive. Then, since the degree of \( p_n \) is one more than the degree of \( p_n' \), we see that \( q(x) < 0 \) for sufficiently large \( x \). But \( q(x^*) = p_n'(x^*) > 0 \). So \( q \) has one more zero in \((x^*, \infty)\). Thus, we have accounted for \( n \) distinct zeros of \( q \) in \((0, \infty)\). These zeros interlace with those of \( p_n \) with the largest zero of \( q \) being larger than the largest zero of \( p_n \).

**Lemma 6.** For problem \( (P_0) \), that is, when \( y = 0 \) in the problem \( (P_y) \), we have \( r = n + 1 \) in Lemma 3, and hence by Lemma 4, we have \( Q_n(x, 0) = \pm \hat{T}_n \).

*Proof.* Suppose \( y = 0 \) and \( r \leq n \). Choose \( \tau_{r+1}, \tau_{r+2}, ..., \tau_n \) each in \((0, \infty)\) so that \( \tau_i \neq \tau_j, i \neq j \). Let \( p_n(x) := \prod_{i=1}^{n+1} (x - \tau_i) \). By Lemma 5 we have that \((wp_n)'\) has exactly \( n \) simple zeros, all of which are in \((\tau_1, \infty)\). But by (4), we have that \( y \) is a zero of \((wp_n)'\). Since \( y = 0, y \notin (\tau_1, \infty) \). So \((wp_n)'\) has \( n + 1 \) distinct zeros, a contradiction.
4. PROOF OF THE MAIN RESULT

Recall that \( w(x) = e^{-x} \) and \( \| \cdot \| = \| \cdot \|_{(0, \infty)}. \)

First note that

\[
\sup \left\{ \| (wp) \|_{(0, a)} : \| wp \|_{(0, a)} \leq 1, \ p \in \mathcal{P}_n \right\} = \max \left\{ \left( \frac{\partial}{\partial x} w(x) \right) Q_n(x, y) \ : \ y \neq 0 \right\} = \| (wT_n') \|_{a}.
\]

for any \( a \in \mathbb{R} \). This follows from the fact that \( w(x-a) p(x-a) = w(x) \times e^{ap(x-a)} \in \mathcal{P}_n \). This translation invariance property of the problem will be crucial in our proof. We explore this property as follows.

Let \( y > 0 \). Let \( Q_n(x, y) \) be an extremal polynomial in \( x \) for the pointwise problem \( (P_y) \). Consider \( p_n(x) := w(y) Q_n(x + y, y) \). Note that

\[
\| w(x) p_n(x) \| = \| w(x + y) Q_n(x + y, y) \| \leq 1.
\]

Further, note that by Lemma 6

\[
\frac{\partial}{\partial x} w(x) Q_n(x, y) \bigg|_{x=y} = \frac{\partial}{\partial x} w(x) p_n(x) \bigg|_{x=0} \leq \frac{\partial}{\partial x} w(x) Q_n(x, 0) \bigg|_{x=0} = \| (wT_n') \|_{0}.
\]

Thus, we see that

\[
\sup \left\{ \| (wp_n) \| : \| wp_n \| \leq 1, \ p_n \in \mathcal{P}_n \right\} = \max \left\{ \left( \frac{\partial}{\partial x} w(x) Q_n(x, y) \right) \ : \ y \geq 0 \right\} = \| (wT_n') \|_{0}.
\]

On the other hand, note that for some \( y_0 > 0 \),

\[
\| (wT_n') \|_{0} = \| (wT_n') \|_{y_0} \leq \left| \frac{\partial}{\partial x} w(x) Q_n(x, y_0) \right|_{x=y_0} = \| (wT_n') \|_{y_0}.
\]

So we have

\[
\| (wT_n') \|_{0} = \| (wT_n') \|_{y_0}.
\]

This completes our proof.
REFERENCES


