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Dense minimal asymptotic bases of order two ☆

Miroslawa Jańczak, Tomasz Schoen *

Department of Discrete Mathematics, Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, Poland

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ABSTRACT

We call a set A of positive integers an asymptotic basis of order h if every sufficiently large integer n can be written as a sum of h elements of A. If no proper subset of A is an asymptotic basis of order h, then A is a minimal asymptotic basis of that order. Erdős and Nathanson showed that for every $h \geqslant 2$ there exists a minimal asymptotic basis A of order h with $\mathrm{d}(A) = 1/h$, where $\mathrm{d}(A)$ denotes the density of A. Erdős and Nathanson asked whether it is possible to strengthen their result by deciding on the existence of a minimal asymptotic bases of order $h \geqslant 2$ such that $A(k) = k/h + O(1)$. Moreover, they asked if there exists a minimal asymptotic basis with $\limsup(a_{i+1} - a_i) = 3$. In this paper we answer these questions in the affirmative by constructing a minimal asymptotic basis A of order 2 fulfilling a very restrictive condition

$$\frac{1}{2}k \leqslant A(k) \leqslant \frac{1}{2}k + 1.$$

1. Introduction

A set A of positive integers is an asymptotic basis of order h if every sufficiently large integer n can be written as a sum of h elements of A. If A is an asymptotic basis of order h and no proper subset of A has this property then A is a minimal asymptotic basis of that order. Hence, for every element a of a minimal asymptotic basis A of order h there are infinitely many positive integers n, each of whose representations as a sum of h elements of A includes a as a summand.

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* Corresponding author.
E-mail addresses: mjanczak@amu.edu.pl (M. Jańczak), schoen@amu.edu.pl (T. Schoen).

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Put $A(k) = |\{a \in A\colon\ 1 \leqslant a \leqslant k\}|$. The lower density of A is defined by

$$\underline{\mathrm{d}}(A) = \liminf_{k\to\infty} \frac{A(k)}{k}.$$

If the limit $\lim_{k\to\infty} A(k)/k$ exists, then it is called the density of A and denoted by $\mathrm{d}(A)$. Nathanson and Sárközy [2] proved that if A is a minimal asymptotic basis of order h then $\underline{\mathrm{d}}(A) \leqslant 1/h$. Erdős and Nathanson [1] showed that for all $\alpha \in (0, 1/(2h-2))$ there exists a minimal asymptotic basis A of order h with $\mathrm{d}(A) = \alpha$. They also proved that for every $h \geqslant 2$ there exists a minimal asymptotic basis A of order h with $\mathrm{d}(A) = 1/h$. Erdős and Nathanson [1] asked whether it is possible to strengthen their result by deciding on the existence of a minimal asymptotic bases of order $h \geqslant 2$ such that

$$A(k) = k/h + O(1). \tag{1}$$

It is easy to see that there are no minimal asymptotic bases $A = (a_i)_{i=1}^{\infty}$ of order 2 with $\limsup(a_{i+1} - a_i) = 2$. The set constructed in [1] satisfies $\limsup(a_{i+1} - a_i) = 4$. Thus, Erdős and Nathanson asked if there exists a minimal asymptotic basis with

$$\limsup(a_{i+1} - a_i) = 3. \tag{2}$$

The object of this paper is to prove construct a minimal asymptotic basis A of order 2 fulfilling a very restrictive condition

$$\frac{1}{2}k \leqslant A(k) \leqslant \frac{1}{2}k + 1.$$

Clearly, this basis satisfies (2) and (1) in the case $h = 2$. As we also remark at the end of the note our example is, in a way, best possible.

2. A construction of a dense minimal basis

The main result of this note can be stated as follows.

Theorem 1. *There exists a minimal asymptotic basis A of order 2 such that for every $k \in \mathbb{N}$*

$$\frac{1}{2}k \leqslant A(k) \leqslant \frac{1}{2}k + 1. \tag{3}$$

Proof. Let $(t_n)_{n\geqslant 0}$ be the sequence such that $t_0 = 7$ and

$$t_{n+1} = 3t_n + 4,$$

so $t_n = 3^{n+2} - 2$. We shall construct inductively a sequence of sets $(A_n)_{n\geqslant 0}$, $A_n = \{a_1, \dots, a_{m_n}\}$ such that $m_n \geqslant n$ and for $n \geqslant 0$

$$A_n \subseteq [t_n], \quad t_n \in A_n, \tag{4}$$

$$\{6, \dots, t_n\} \subseteq A_n + A_n, \tag{5}$$

$$\frac{1}{2}k \leqslant A_n(k) \leqslant \frac{1}{2}k + 1 \quad \text{for every } k \in [t_n], \tag{6}$$

$$A_n(t_n) = \frac{1}{2}t_n + \frac{1}{2}, \tag{7}$$

and for $n \geqslant 0$, $t_n - 2$ has a unique representation in $A_n + A_n$ and this representation is of the form $t_n - 2 = b_n + a$, where $a \in A_n$ and $\{b_n\}_{n\in\mathbb{N}}$ is defined as follows

$$\begin{aligned} &b_1 = a_1, \\ &b_2 = a_1, \qquad b_3 = a_2, \\ &b_4 = a_1, \qquad b_5 = a_2, \qquad b_6 = a_3, \\ &\ldots. \end{aligned} \tag{8}$$

Observe that $b_n = a_i$ with $i \leqslant m_{n-1}$, so in the nth step the element b_n has been already defined and $b_n \leqslant t_{n-1}$.

We start with $A_0 = \{1, 2, 5, 7\}$, $t_0 = 7$. Clearly the conditions (4)–(7) are satisfied. Assume now that for some $n \geqslant 0$, A_n has been already defined and it satisfies (4)–(7).

We put

$$A'_{n+1} = A_n \cup C_n \cup D_n \cup \{t_{n+1} - 2, t_{n+1}\},$$

where

$$C_n = \{t_n + 1, t_n + 3, \ldots, 2t_n\},$$
$$D_n = \left\{d \in [2t_n + 1, t_{n+1} - 3]\colon\ t_{n+1} - 2 - d \notin A_n\right\}.$$

We will show now that

$$[6, t_{n+1}] \setminus \{t_{n+1} - 2\} \subseteq A'_{n+1} + A'_{n+1} \tag{9}$$

and

$$t_{n+1} - 2 \notin A'_{n+1} + A'_{n+1}. \tag{10}$$

By (5) we have

$$[6, t_n] \subseteq A_n + A_n \subseteq A'_{n+1} + A'_{n+1}.$$

Since $1, 2, t_n \in A_n$ it follows that $t_n + 1 \in A'_{n+1} + A'_{n+1}$ and for all $l \in \{1, 3, \ldots, t_n\}$

$$(t_n + l) + 1, (t_n + l) + 2 \in C_n + A_n \subseteq A'_{n+1} + A'_{n+1},$$

so $[t_n + 1, 2t_n + 2] \subseteq A'_{n+1} + A'_{n+1}$, and

$$t_n + (t_n + l), (t_n + 1) + (t_n + l) \in A'_{n+1} + A'_{n+1},$$

so $[2t_n + 1, 3t_n + 1] \subseteq A'_{n+1} + A'_{n+1}$. Notice that $t_{n+1} - 1 = 1 + (t_{n+1} - 2)$ and $t_{n+1} = 2 + (t_{n+1} - 2)$, so $t_{n+1} - 1$, $t_{n+1} \in A'_{n+1} + A'_{n+1}$. Observe that

$$t_{n+1} - 2 \notin A'_{n+1} + A'_{n+1}.$$

Indeed, by the definition of D_n it follows that $t_{n+1} - 2 \notin A_n + D_n$ and $t_{n+1} - 2 \in C_n + C_n$ is impossible because $t_{n+1} - 2$ is an odd number while all elements of $C_n + C_n$ are even.

We will prove that the counting function of the set A'_{n+1} satisfies (6) and (7). By $A'_{n+1} \cap [t_n] = A_n$, we have

$$A'_{n+1}(k) = A_n(k),$$

so A'_{n+1} fulfills (6) for all $k \in [1, t_n]$ by the inductional assumption. Since $t_n + l \in A'_{n+1}$ and $t_n + l + 1 \notin A'_{n+1}$ for odd $l \in \{1, \dots, t_n\}$ and (7), so

$$A'_{n+1}(t_n + l) = \frac{1}{2}(t_n + l) + 1$$

and

$$A'_{n+1}(t_n + l + 1) = \frac{1}{2}(t_n + l) + \frac{1}{2}.$$

Let $l \in [1, t_n + 1]$, so that $2t_n + 1 \leqslant t_{n+1} - 2 - l \leqslant 3t_n + 1 = t_{n+1} - 3$. We have $A'_{n+1}(l) = \frac{1}{2}l + C$, where $0 \leqslant C \leqslant 1$, therefore

$$\begin{aligned} A'_{n+1}(t_{n+1} - 2 - l) &= A'_{n+1}(2t_n) + D_n(t_{n+1} - 2 - l) \\ &= t_n + 1 + \bigl(t_n + 1 - (l - 1) - \bigl(A'_{n+1}(t_n + 1) - A'_{n+1}(l - 1)\bigr)\bigr) \\ &= t_n + 1 + \left(t_n + 2 - l - \left(\frac{1}{2}(t_n + 1) + 1 - \frac{1}{2}(l - 1) - C\right)\right) \\ &= \frac{1}{2}(t_{n+1} - 2 - l) + C. \end{aligned}$$

Note also that

$$\begin{aligned} A'_{n+1}(t_{n+1} - 2) &= t_n + 1 + A'_{n+1}(2t_n) - A'_{n+1}(t_n + 1) + 1 \\ &= 2t_n + 3 - \frac{1}{2}(t_n + 1) - 1 \\ &= \frac{1}{2}(t_{n+1} - 2) + \frac{1}{2}. \end{aligned}$$

Finally, since $t_{n+1} - 1 \notin A'_{n+1}, t_{n+1} \in A'_{n+1}$ we have

$$A'_{n+1}(t_{n+1} - 1) = \frac{1}{2}(t_{n+1} - 1),$$

$$A'_{n+1}(t_{n+1}) = \frac{1}{2}t_{n+1} + \frac{1}{2}.$$

Thus, A'_{n+1} satisfies (6) and (7).

Observe that $t_{n+1} - 2 - b_{n+1} \notin A'_{n+1}$. Now we can define the set A_{n+1} and prove that the number $t_{n+1} - 2$ has unique representation in $A_{n+1} + A_{n+1}$ in the form $t_{n+1} - 2 = b_{n+1} + a$, for some $a \in A_{n+1}$. Put

$$A_{n+1} = A'_{n+1} \cup \{t_{n+1} - 2 - b_{n+1}\} \setminus \{r_{n+1}\},$$

where b_{n+1} is an element of the sequence (8) and $r_{n+1} \in A'_{n+1}$ is chosen as follows. If $A'_{n+1}(t_{n+1}-2-b_{n+1}-1) = \frac{1}{2}(t_{n+1}-2-b_{n+1}-1)+1$, then $t_{n+1}-2-b_{n+1}-1 \in A'_{n+1}$ and we define

$$r_{n+1} = t_{n+1} - 2 - b_{n+1} - 1. \tag{11}$$

However, if $A'_{n+1}(t_{n+1}-2-b_{n+1}-1) = \frac{1}{2}(t_{n+1}-2-b_{n+1}-1)+\frac{1}{2}$, then we put

$$r_{n+1} = t_{n+1} - 2 - b_{n+1} + 1. \tag{12}$$

Notice, that $A'_{n+1}(t_{n+1}-2-b_{n+1}-1) = \frac{1}{2}(t_{n+1}-2-b_{n+1}-1)$ is not possible because we would have $A'_{n+1}(t_{n+1}-2-b_{n+1}) = \frac{1}{2}(t_{n+1}-2-b_{n+1})-\frac{1}{2}$, which contradicts (6).

Observe that

$$[6, t_{n+1}] \subseteq A_{n+1} + A_{n+1},$$

because every number from $[t_n, t_{n+1}-3]$ is a sum of two elements of A'_{n+1} less or equal $2t_n$, and $b_{n+1} \leqslant t_n$, so $r_{n+1} > 2t_n$. Moreover

$$t_{n+1} - 2 = b_{n+1} + (t_{n+1} - 2 - b_{n+1}),$$

$$t_{n+1} - 1 = 2t_n + (t_n + 3),$$

$$t_{n+1} = (t_{n+1} - 2) + 2.$$

To finish the proof we have to show that the counting function of the set A_{n+1} satisfies (6) and (7). If r_{n+1} is defined by (11), then $A'_{n+1}(t_{n+1}-2-b_{n+1}-1) = \frac{1}{2}(t_{n+1}-2-b_{n+1}-1)+1$, whence

$$A_{n+1}(r_{n+1}) = A'_{n+1}(r_{n+1}) - 1 = \frac{1}{2}r_{n+1}$$

and

$$\frac{1}{2}k \leqslant A_{n+1}(k) = A'_{n+1}(k) \leqslant \frac{1}{2}k + 1,$$

for every $k < r_{n+1}$ or $k \geqslant r_{n+1}+1$. In particular

$$A_{n+1}(t_{n+1}) = \frac{1}{2}t_{n+1} + \frac{1}{2}.$$

In the second case (12) we have $A'_{n+1}(t_{n+1}-2-b_{n+1}-1) = \frac{1}{2}(t_{n+1}-2-b_{n+1}-1)+\frac{1}{2}$. Then for each $k \leqslant r_{n+1}-2$ or $k \geqslant r_{n+1}$ one has

$$\frac{1}{2}k \leqslant A_{n+1}(k) = A'_{n+1}(k) \leqslant \frac{1}{2}k + 1.$$

Furthermore, by (12)

$$A_{n+1}(t_{n+1}-2-b_{n+1}) = A'_{n+1}(t_{n+1}-2-b_{n+1}) + 1 = \frac{1}{2}(t_{n+1}-2-b_{n+1}) + 1.$$

Now, let us set

$$A = \bigcup_{n=1}^{\infty} A_n.$$

Clearly, by (5) A is an asymptotic basis of $\mathbb{N}$. Moreover, by $A_{n+1} \cap [t_n] = A_n$ for every $n \geqslant 0$, and (6), A satisfies (3). To see that A is a minimal basis, observe that every element $a \in A$ occurs in the sequence $\{b_n\}_{n\in\mathbb{N}}$ infinitely many times. Therefore, there are infinitely many n's such that the number $t_n - 2$ has unique representation in $A + A$, which includes a as a summand. This completes the proof of Theorem 1. □

Finally, let us remark that the lower bound for $A(k)$ in (3) cannot be increased, i.e. no minimal basis A satisfies

$$\frac{1}{2}k + \frac{1}{2} \leqslant A(k) \leqslant \frac{1}{2}k + C$$

for every sufficiently large $k \in \mathbb{N}$, where C is a positive constant. Indeed, let m be a positive integer such that

$$A(m) = \frac{m}{2} + C$$

and let $a \in A$, $a > m$. Since A is a minimal basis, so there are infinitely many numbers $u > a + m$ such that for some $b \in A$, $a + b$ is the unique representation of u in $A + A$. Since u has a unique representation and $a, b > m$, the sets $u - (A \cap [m])$ and $A \cap [u - m, u - 1]$ are disjoint, so

$$\left|A \cap [u - m, u - 1]\right| \leqslant m - A(m) = \frac{m}{2} - C.$$

Thus

$$A(u - 1) \leqslant A(u - m - 1) + \left|A \cap [u - m, u - 1]\right| \leqslant \frac{u - 1}{2}.$$

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