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Right-angled Artin groups are commensurable with right-angled Coxeter groups

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Abstract

For each right-angled Artin group there is a right-angled Coxeter group which contains it as a subgroup of finite index. A corollary is that right-angled Artin groups are linear. © 2000 Elsevier Science B.V. All rights reserved.

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0. Introduction

Recently, there has been a great deal of work in geometric group theory on right-angled Artin groups (otherwise known as "graph groups"). A key feature of any right-angled Artin group is that it acts cocompactly and isometrically on a CAT(0) cubical complex [4]. For example, this feature is heavily exploited by Bestvina and Brady [1] in the construction of their beautiful examples of groups which are type (FP) but are not finitely presented. Somewhat earlier, a similar result had been proved for right-angled Coxeter groups in [7]: such a Coxeter group acts cocompactly and isometrically on a CAT(0) cubical complex (see [5,6]). In this paper we show that given a right-angled Artin group A we can find a right-angled Coxeter group W' so that the corresponding cubical complexes are identical.

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1. The groups

Let Γ be a finite simplicial graph with vertex set I. Associated to Γ there are two groups — a *right-angled Coxeter group* W $(=W(\Gamma))$ and a *right-angled Artin group* A $(=A(\Gamma))$. For each $i \in I$ introduce symbols s_i and g_i and let $S = \{s_i\}_{i\in I}$ and $G = \{g_i\}_{i\in I}$. The group W is defined by a presentation with set of generators S and with relations, $s_i^2 = 1$ for all $i \in I$ and $(s_i s_j)^2 = 1$ whenever the vertices i and j span an edge of Γ. The group A is given by the presentation with generating set G and with relations $[g_i, g_j] = 1$ whenever i and j span an edge of Γ.

We define two other graphs Γ' and Γ'', as follows. The vertex set of Γ'' is $I \times \{0,1\}$. Two vertices $(i,1)$ and $(j,1)$ in $I \times 1$ are connected by an edge in Γ'' if and only if the corresponding vertices i and j span an edge in Γ. Any two distinct vertices in $I \times 0$ are connected by an edge. Finally, vertices $(i,0)$ and $(j,1)$ are connected by an edge if and only if $i \neq j$. The vertex set of Γ' is $I \times \{-1,1\}$. The subsets $I \times (-1)$ and $I \times 1$ both span copies of Γ. A vertex $(i,-1)$ in $I \times (-1)$ is connected to $(j,1)$ in $I \times 1$ if and only if $i \neq j$ and the vertices i and j span an edge of Γ.

Let $W' = W(\Gamma')$ and $W'' = W(\Gamma'')$ be the right-angled Coxeter groups associated to Γ' and Γ'', respectively. It will cause no confusion to denote the generators of W' corresponding to $I \times 1$, as well as the generators of W'' corresponding to $I \times 1$, by same set of symbols $S = \{s_i\}_{i\in I}$. (In both cases the subgroup $\langle S\rangle$ generated by S is isomorphic to the original Coxeter group W.) The remaining generators of W' corresponding to $I \times (-1)$ will be denoted by $T = \{t_i\}_{i\in I}$. The remaining generators in W'' corresponding to $I \times 0$ will be denoted by $R = \{r_i\}_{i\in I}$.

Let $(\mathbb{Z}/2)^I$ denote the direct sum of I copies of a cyclic group of order 2 and let $\{\bar{r}_i\}_{i\in I}$ be the standard set of generators.

Define homomorphisms $\varphi: W'' \to (\mathbb{Z}/2)^I$ and $\theta: W'' \to (\mathbb{Z}/2)^I$ by the formulas:

$$\varphi(r_i) = \bar{r}_i, \quad \varphi(s_i) = 1, \tag{1}$$

$$\theta(r_i) = \theta(s_i) = \bar{r}_i \tag{2}$$

for all $i \in I$.

We also have homomorphisms $\alpha: W' \to \ker\varphi \subset W''$ and $\beta: A \to \ker\theta \subset W''$ given by the formulas:

$$\alpha(s_i) = s_i, \quad \alpha(t_i) = r_i s_i r_i \tag{3}$$

and

$$\beta(g_i) = r_i s_i. \tag{4}$$

Theorem. *The homomorphisms* $\alpha: W' \to \ker\varphi$ *and* $\beta: A \to \ker\theta$ *are isomorphisms.*

Thus, A and W' are both subgroups of index 2^I in W''.

Remark. The homomorphisms φ and θ both take the subgroup $\langle R\rangle$ isomorphically onto $(\mathbb{Z}/2)^I$; moreover, the intersections $\langle R\rangle \cap \ker\varphi$ and $\langle R\rangle \cap \ker\theta$ are both trivial.

This gives two decompositions of W'' as a semi-direct product: $W'' = W' \rtimes (\mathbb{Z}/2)^I$ and $W'' = A \rtimes (\mathbb{Z}/2)^I$.

Corollary. *A is a subgroup of a $GL_N(\mathbb{R})$*

Proof. W'' is. See [3, Chapter 5, Section 4].

Remark. One can probably prove this corollary directly by writing down an explicit linear representation. However, we think that a proof of injectivity will necessarily use some geometric input. Thus, our use of commensurability is quite appropriate.

2. The complexes

Associated to any right-angled Coxeter group or any right-angled Artin group there is a natural contractible cubical cell complex on which the group acts properly and cocompactly. In the case of Coxeter groups, these complexes are described in [5,6] or [7]. In the case of Artin groups a description can be found in [4–6], [1] or [2]. We will recall these descriptions below. The main idea in the proof of our theorem is that the complex associated to the Coxeter group W' is identical to the complex associated to the Artin group A. Moveover, the group W'' acts on this complex and one can see geometrically that W' and A are both subgroups of index 2^I.

First consider the special case where I is a singleton, $I = \{i\}$. Then $W = \mathbb{Z}/2$ and $A = \mathbb{Z}$. The groups W' and W'' are both isomorphic to the infinite dihedral group $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$ ($W' = \langle t_i, s_i \rangle$, $W'' = \langle r_i, s_i \rangle$). The groups W', W'' and A can all be represented as transformation groups of $\mathbb{R}$ as follows. Let t_i, r_i and s_i denote the reflections across the points, -1, 0 and 1, respectively. Then W' and W'' are reflection groups on $\mathbb{R}$ with fundamental chambers $[-1, 1]$ and $[0, 1]$, respectively. The group $A = \mathbb{Z}$ acts by $r_i s_i$ which is translation by 2. Hence, $[0, 2]$ is a fundamental domain for A and the orbit space $\mathbb{R}/A$ is the circle formed by identifying the endpoints of $[0, 2]$.

Next suppose that I is an arbitrary finite set and that Γ is the complete graph on I. This yields a product of I copies of the situation in the previous paragraph: $W = (\mathbb{Z}/2)^I$, $A = \mathbb{Z}^I$ and $W' \cong (D_\infty)^I \cong W''$. These groups act on $\mathbb{R}^I$ by letting t_i, r_i and s_i denote the reflections across the hyperplanes $x_i = -1$, $x_i = 0$, and $x_i = 1$, respectively. (Here $(x_i)_{i \in I}$ are coordinates on $\mathbb{R}^I$.) The groups W' and W'' are reflection groups with respective fundamental chambers the cubes $[-1, 1]^I$ and $[0, 1]^I$. The generators $g_i = r_i s_i$ of A are translations by $2e_i$, where $(e_i)_{i \in I}$ is the standard basis. Thus, $\mathbb{R}^I/A$ is the torus T^I formed by identifying opposite faces of the cube $[0, 2]^I$. The theorem of the previous section is clearly true in this case. As we shall see below, in the general case, the cubical complexes which we are interested in have quotient spaces which are subcomplexes of either $[-1, 1]^I$, $[0, 1]^I$, or T^I.

Before describing these complexes, we first recall a general method of constructing actions of Coxeter groups as reflection groups. Let X be a topological space. A *mirror* *structure* over I on X is a family $\mathscr{M}=(X_i)_{i\in I}$ of closed subspaces of X indexed by I. Let Γ be a graph on I and W the associated right-angled Coxeter group.

Given a mirror structure $\mathscr{M}$ on X, define an equivalence relation $\sim$ on $W\times X$ by: $(w,x)\sim(w',x')$ if and only if $x=x'$ and $w^{-1}w'$ belongs to the subgroup $\langle S(x)\rangle$ generated by $S(x)=\{s_i \mid x\in X_i\}$. Set

$$\mathscr{U}(W,X,\mathscr{M})=(W\times X)/\sim .$$

There is a natural W-action on $\mathscr{U}(W,X,\mathscr{M})$ and X is the orbit space. If Y is any space with W-action and $f:X\to Y$ is any map such that $f(X_i)$ is contained in the fixed point set of s_i, for each $i\in I$, then f extends to a W-equivariant map $\tilde{f}:\mathscr{U}(W,X,\mathscr{M})\to Y$ by the formula $[w,x]\to wf(x)$. (Here $[w,x]$ denotes the image of (w,x) in $\mathscr{U}(W,X,\mathscr{M})$.) In particular, if W acts as a reflection group on Y and X is a fundamental chamber and X_i is defined to be the intersection of X with the fixed set of s_i, then the map $\mathscr{U}(W,X,\mathscr{M})\to Y$ induced by the inclusion $X\to Y$ is a homeomorphism. (Actually, we can take this to be the definition of "W is a reflection group on Y".)

Next we define a simplicial complex L $(=L(\Gamma))$ and a poset $\mathscr{S}$ $(=\mathscr{S}(\Gamma))$. L is the flag complex determined by Γ. In other words, L is the simplicial complex with vertex set I such that a nonempty subset J of I spans a simplex if and only if the subgraph Γ_J spanned by J is the complete graph on J (i.e., L is obtained from Γ by "filling in" all possible simplices of dimension greater than 1). $\mathscr{S}$ is the set of all such subsets J of I together with the empty set. It is partially ordered by inclusion.

Similarly, let $\mathscr{S}'$ and $\mathscr{S}''$ be the posets associated to Γ' and Γ'', respectively. Suppose $J'=\{(j_1,\varepsilon_1),\dots,(j_k,\varepsilon_k)\}$ is a subset of $I\times\{-1,1\}$. From the definition of Γ', we see that $J'\in\mathscr{S}'$ if and only if $j_1,\dots,j_k$ are distinct elements of I and $\{j_1,\dots,j_k\}\in\mathscr{S}$. In other words, $J'\in\mathscr{S}'$ if and only if it is the graph of a function $\varepsilon:J\to\{\pm1\}$ for some subset J of I such that $J\in\mathscr{S}$. An arbitrary subset J'' of $I\times\{0,1\}$ can be decomposed as $J''=J^0\times 0\cup J^1\times 1$, where for $\varepsilon=0,1$, J^ε is defined by $J^\varepsilon=\{j\in I \mid (j,\varepsilon)\in J''\}$. It follows from the definition of Γ'', that $J''\in\mathscr{S}''$ if and only if $J^1\in\mathscr{S}$ and $J^0\cap J^1=\emptyset$.

Next we define a cubical subcomplexes K of $[0,1]^I$ and K' of $[-1,1]^I$. (The cubical structure on $[-1,1]^I$ is defined by subdividing it into 2^I unit cubes.)

Define

$$K=[0,1]^I\cap\bigcup_{J\in\mathscr{S}}\mathbb{R}^J, \tag{5}$$

$$K'=[-1,1]^I\cap\bigcup_{J\in\mathscr{S}}\mathbb{R}^J, \tag{6}$$

where $\mathbb{R}^J$ denotes the linear subspace of $\mathbb{R}^I$ defined by the equations $x_i=0,\ i\in I-J$.

Next we define certain subcomplexes of K and K' indexed by $\mathscr{S}$ and $\mathscr{S}'$, respectively. We will also define subcomplexes of K indexed by $\mathscr{S}''$ and when we are interested in this system of subcomplexes we will write K'' instead of K. Given a subset J of I, set

$$K_J=K\cap\{x_j=1\}_{j\in J}, \tag{7}$$

where $\{x_j = 1\}_{j\in J}$ denotes the affine subspace of $\mathbb{R}^I$ defined by the equations within the brackets. Clearly, K_J is nonempty if and only if $J \in \mathscr{S}$. Given a subset J' of $I \times \{-1,1\}$ defined by the graph of a function $J \to \{-1,1\}$, define

$$K'_{J'} = K' \cap \{x_j = \varepsilon_j\}_{(j,\varepsilon_j)\in J'}. \tag{8}$$

This time $K'_{J'}$ is nonempty if and only if $J' \in \mathscr{S}'$. Finally, for each subset $J'' = J^0 \times 0 \cup J^1 \times 1$ of $I \times \{0,1\}$ set

$$K''_{J''} = K \cap \mathbb{R}^{J^0} \cap \{x_j = 1\}_{j\in J^1}. \tag{9}$$

We have that $K''_{J''}$ is nonempty if and only if $J^0 \cap J^1 = \emptyset$ and $J^1 \in \mathscr{S}$, that is to say, if and only if $J'' \in \mathscr{S}''$.

For each $i \in I$, set $K_i = K_{\{i\}}$. This defines a mirror structure $\mathscr{M} = (K_i)_{i\in I}$ over I on K. Similarly, we define a mirror structure $\mathscr{M}'$ over $I \times \{-1,1\}$ on K' and a mirror structure $\mathscr{M}''$ over $I \times \{0,1\}$ on K''. Set

$$\mathscr{U} = \mathscr{U}(W, K, \mathscr{M}),$$
$$\mathscr{U}' = \mathscr{U}(W', K', \mathscr{M}'),$$
$$\mathscr{U}'' = \mathscr{U}(W'', K'', \mathscr{M}'').$$

The cubical complex $\mathscr{U}$ is the natural contractible cell complex on which the Coxeter group W acts with compact quotient. (This follows from the description in [5] and the fact that a simplicial subdivision of K can be identified with the geometric realization of the poset $\mathscr{S}$ in such a fashion that K_J is the geometric realization of the subposet $\mathscr{S}_{\geq J}$.) The cubical complex K' is isomorphic to the cubical complex $K(\Gamma')$ defined by the graph Γ' (although, by definition, $K(\Gamma')$ is a subcomplex of $[0,1]^{I\times\{-1,1\}}$, it is isomorphic to K'). It follows that $\mathscr{U}'$ is also the natural contractible complex for W'. Since $K'' \not\cong K(\Gamma'')$, $\mathscr{U}''$ is not, *a priori*, contractible. However, by the following lemma, it is.

Lemma 1. *$\mathscr{U}''$ is contractible.*

Proof. By Corollary 10.3 in [5], we need to show that $K''_{J''}$ is acyclic for each $J'' \in \mathscr{S}''$ and that K'' is simply connected. Equation (9) can be rewritten as

$$K''_{J''} = K_{J^1} \cap K(\Gamma_{I-J^0}) = K(\Gamma_{I-J^0})_{J^1}. \tag{10}$$

In other words, $K''_{J''}$ is the J^1-face of the complex $K(\Gamma_{I-J^0})$ corresponding to the subgraph $K(\Gamma_{I-J^0})$. But for any graph $\hat{\Gamma}$ the complex $K(\hat{\Gamma})$ and each of its nonempty faces are contractible (they are cones).

We are now in position to prove the first half of the theorem.

Lemma 2. *The homomorphism $\alpha : W' \to \ker\varphi$ defined by* (3) *is an isomorphism. Moreover, there is a natural α-equivariant homeomorphism $\mathscr{U}' \to \mathscr{U}''$.*

Proof. For each $i \in I$, let $\bar{t}_i = \alpha(t_i) = r_i s_i r_i \in W''$ and let $\bar{T}$ denote the subset $\{\bar{t}_i\}_{i \in I}$ of W''. The subgroup $\langle S \cup \bar{T} \rangle$ of W'' is clearly in the kernel of φ. The group $(\mathbb{Z}/2)^I$ ($\cong \langle R \rangle$) acts as a reflection group on $[-1,1]^I$ with fundamental chamber $[0,1]^I$. The subcomplex K' is $(\mathbb{Z}/2)^I$-stable and $K' \cap [0,1]^I = K''$. Thus, K'' is the fundamental chamber for the $(\mathbb{Z}/2)^I$-action on K'. It follows that we can identify K' with a union of copies of K'' in $\mathscr{U}''$ as

$$K' = \langle R \rangle K''$$

where the right-hand side denotes the union of chambers gK'', $g \in \langle R \rangle$. The elements of $S \cup \bar{T}$ act as reflections on $\mathscr{U}''$ and one sees that $K' = \langle R \rangle K''$ is precisely the intersection of the corresponding half-spaces. Since the only other walls of $\mathscr{U}''$ which can intersect K' are those indexed by R, it follows that K' is a fundamental domain for the $\langle S \cup \bar{T} \rangle$-action on $\mathscr{U}''$. Since K' is the union of 2^I copies of K'', it follows that $\langle S \cup \bar{T} \rangle$ is of index 2^I in W''. Since this is also the index of $\ker \varphi$, we have that $\langle S \cup \bar{T} \rangle = \ker \varphi$, and hence, that α maps W' onto $\ker \varphi$. It follows from the general theory of groups generated by reflections that $\langle S \cup \bar{T} \rangle$ is a Coxeter group and that $S \cup \bar{T}$ is a fundamental set of generators for it. (The proof is the same as that of Théorème 1, p. 74 in [3] or Proposition 4.3 in [5].) Since $s_i \bar{t}_i = s_i r_i s_i r_i = (s_i r_i)^2$ we see that $s_i \bar{t}_i$ has infinite order. Also, for $i \neq j$, $\bar{t}_i \bar{t}_j = r_i s_i r_i r_j s_j r_j = (r_i r_j)(s_i s_j)(r_i r_j)$ so $\bar{t}_i \bar{t}_j$ has the same order as does $s_i s_j$. It follows that the product of any two elements in $S \cup \bar{T}$ have the same order as the corresponding elements of $S \cup T$. Hence, $\alpha : W' \to \langle S \cup \bar{T} \rangle$ is an isomorphism of Coxeter groups.

To prove the last sentence of the lemma, note that the natural inclusion $K' \hookrightarrow \mathscr{U}''$ induces a W'-equivariant homeomorphism $\mathscr{U}' \to \mathscr{U}''$. □

As is shown in [4], an Eilenberg–MacLane space for the Artin group A is the subcomplex $\mathscr{A}$ of T^I is given by

$$\mathscr{A} = \bigcup_{J \in \mathscr{S}} T^J,$$

where T^J is the torus formed by identifying opposite faces of $[0,2]^J$. Let $\{\bar{r}_i\}_{i \in I}$ denote the standard set of generators for $(\mathbb{Z}/2)^I$. Let $\bar{r}_i$ act on T^I by reflection across the hyperplane $x_i = 1$. Its fixed set consists of two parallel copies of $T^{I-\{i\}}$ corresponding to the hyperplanes $x_i = 0$ and $x_i = 1$. This defines an action of $(\mathbb{Z}/2)^I$ as a reflection group on T^I with fundamental chamber $[0,1]^I$. The subcomplex $\mathscr{A}$ is $(\mathbb{Z}/2)^I$-stable. A fundamental chamber for its action on $\mathscr{A}$ is K ($=K''$). Set $\hat{K} = K$. (We introduce a new symbol since we are going to define a new mirror structure on K.) The $(\mathbb{Z}/2)^I$-action gives a mirror structure $\hat{\mathscr{M}} = (\hat{K}_i)_{i \in I}$ which is closely related to the mirror structure $\mathscr{M}'' = (K''_{(i,\varepsilon)})_{(i,\varepsilon) \in I \times \{0,1\}}$. In fact,

$$\hat{K}_i = K''_{(i,0)} \cup K''_{(i,1)}.$$

Set

$$\hat{\mathscr{U}} = \mathscr{U}((\mathbb{Z}/2)^I, \hat{K}, \hat{\mathscr{M}}).$$

The fact that $(\mathbb{Z}/2)^I$ is a reflection group on $\mathscr{A}$ gives an identification of $\hat{\mathscr{U}}$ with $\mathscr{A}$. On the other hand, the inclusion $K'' \hookrightarrow \hat{\mathscr{U}}$ and the homomorphism $\theta : W'' \to (\mathbb{Z}/2)^I$ give a θ-equivariant map $\mathscr{U}'' \to \hat{\mathscr{U}}$ which is obviously a covering projection. Thus,

$$\mathscr{A} \cong \hat{\mathscr{U}} \cong \mathscr{U}''/\ker\theta.$$

Lifting this homeomorphism to the universal covers, we obtain a β-equivariant homeomorphism $\tilde{\mathscr{A}} \to \mathscr{U}''$, where $\tilde{\mathscr{A}}$ is the universal cover of $\mathscr{A}$ and $\beta : A \to \ker\theta$ is the homomorphism defined by (4). As an immediate corollary, we have the following lemma which gives the other half of the theorem.

Lemma 3. *The homomorphism $\beta : A \to \ker\theta$ is an isomorphism and the cubical complexes $\mathscr{U}'$, $\mathscr{U}''$ and $\tilde{\mathscr{A}}$ can be naturally identified with each other.*

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