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New generalized and improved (G'/G)-expansion method for nonlinear evolution equations in mathematical physics

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Abstract In this article, new extension of the generalized and improved (G'/G)-expansion method is proposed for constructing more general and a rich class of new exact traveling wave solutions of nonlinear evolution equations. To demonstrate the novelty and motivation of the proposed method, we implement it to the Korteweg-de Vries (KdV) equation. The new method is oriented toward the ease of utilize and capability of computer algebraic system and provides a more systematic, convenient handling of the solution process of nonlinear equations. Further, obtained solutions disclose a wider range of applicability for handling a large variety of nonlinear partial differential equations.

MATHEMATICS SUBJECT CLASSIFICATION: 35C07; 35C08; 35P99

1. Introduction

Nonlinear partial differential equations (PDEs) are widely used as models to express many important physical phenomena. The analytical solutions of nonlinear PDEs play an important role in nonlinear science and engineering. In order to understand better, as well as further applications in practical life, it is important to generate exact traveling wave solutions. In the recent past, a diverse group of scientists presented a variety of methods to construct analytical and numerical solutions. For instance, the Hirota's bilinear transformation method [1], the truncated Painleve expansion method [2], the Backlund transformation method [3], the Weirstrass elliptic function method [4], the inverse scattering method [5], the tanh-coth method [6,7], the Riccati equation method [8–10], the Jacobi elliptic function expansion method [11], the F-expansion method [12,13], the Exp-function method [14–17], the sine-cosine method [18] and others [19–21].

Recently, Wang et al. [22] presented one of the powerful methods and called the (G'/G)-expansion method for constructing traveling wave solutions of some nonlinear evolution equations (NLEEs). In this method, they implemented the second order linear ordinary differential equation (ODE)

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$G'' + \lambda G' + \mu G = 0$, where λ and μ are arbitrary constants. Afterward, many scientists applied the (G'/G)-expansion method to study various NLEEs for obtaining traveling wave solutions, can be found [23–27].

To express the applicability and effectiveness of the (G'/G)-expansion method, further research has been accomplished by a diverse group of researchers. For instance, Zhang et al. [28] introduced the improved (G'/G)-expansion method to establish many traveling wave solutions. In Ref. [22], Wang et al. presented $u(\xi) = \sum_{i=0}^{m} a_i (G'/G)^i$, where $a_m \neq 0$ as the traveling wave solutions. However, Zhang et al. [28] described the traveling wave solutions in the form $u(\xi) = \sum_{i=-m}^{m} a_i (G'/G)^i$, where a_{-m} or a_m may be zero, but both a_{-m} and a_m cannot be zero at the same time. Following Zhang et al. [28], many researchers implemented the improved (G'/G) method to construct traveling wave solutions of several nonlinear PDEs [29–32].

Very recently, Akbar et al. [33] introduced a powerful method for obtaining many new solutions of three nonlinear equations and called the generalized and improved (G'/G)-expansion method. In this method, they also used the second-order linear ODE as auxiliary equation, and the presentation of the solutions is: $u(\xi) = \sum_{n=-m}^{m} \frac{e_{-n}}{(d+(G'/G))^n}$, where either e_{-m} or e_m may be zero, but both e_{-m} and e_m cannot be zero simultaneously. After that, Naher et al. [34] investigated higher-dimensional nonlinear equation via this method for constructing new traveling wave solutions.

The importance of our present work is, in order to generate many new and more general exact traveling wave solutions, new extension of the generalized and improved (G'/G)-expansion method is proposed. For illustration and to show the advantages of the proposed method, the KdV equation has been investigated and constructed a rich class of new traveling wave solutions.

2. Description of new generalized and improved (G'/G)-expansion method

Let us consider a general nonlinear PDE:

$$S(u, u_t, u_x, u_{tt}, u_{xt} u_{xx}, \ldots) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function, S is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order partial derivatives and nonlinear terms are involved.

Step 1. We suppose that the combination of real variables x and t by a complex variable ξ

$$u(x,t) = u(\xi), \quad \xi = x \pm Vt, \tag{2}$$

where V is the speed of the traveling wave. Now using Eq. (2), Eq. (1) is converted into an ordinary differential equation for $u = u(\xi)$:

$$Q(u, u', u'', u''', \ldots) = 0, \tag{3}$$

where the superscripts indicate the ordinary derivatives with respect to ξ.

Step 2. According to possibility, Eq. (3) can be integrated term by term one or more times, yields constant(s) of integration. The integral constant may be zero, for simplicity.

Step 3. Suppose that the traveling wave solution of Eq. (3) can be expressed as follows:

$$u(\xi) = \sum_{g=-N}^{N} e_g (d+H)^g + \sum_{g=1}^{N} f_g (d+H)^{-g}, \tag{4}$$

where either e_{-N} or e_N or f_N may be zero, but these e_{-N}, e_N and f_N cannot be zero at a time, $e_g(g = 0, \pm 1, \pm 2, \ldots, \pm N)$, $f_g(g = 1, 2, \ldots, N)$ and d are arbitrary constants to be determined later and $H(\xi)$ is

$$H(\xi) = (G'/G), \tag{5}$$

where $G = G(\xi)$ satisfies the following nonlinear ordinary differential equation (ODE) [35]:

$$AGG'' - BGG' - C(G')^2 - EG^2 = 0, \tag{6}$$

where the primes denote derivatives with respect to ξ and A, B, C, E are real parameters.

Step 4. To determine the positive integer N, taking the homogeneous balance between the highest order nonlinear terms and the highest order derivatives appearing in Eq. (3).

Step 5. Substituting Eqs. (4) and (6) including Eq. (5) into Eq. (3) with the value of N obtained in Step 4, we obtain polynomials in $(d+H)^N (N = 0, \pm 1, \pm 2, \ldots)$ and $(d+H)^{-N} (N = 1, 2, 3, \ldots)$. Then, we collect each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for $e_g(g = 0, \pm 1, \pm 2, \ldots, \pm N)$, $f_g(g = 1, 2, \ldots, N)$, d and V.

Step 6. Suppose that the value of the constants e_g $(g = 0, \pm 1, \pm 2, \ldots, \pm N)$, $f_g(g = 1, 2, \ldots, N)$, d and V can be found by solving the algebraic equations which are obtained in step 5. Since the general solution of Eq. (6) is well known to us, substituting the values of e_g $(g = 0, \pm 1, \pm 2, \ldots, \pm N)$, f_g $(g = 1, 2, \ldots, N)$,d and V into Eq. (4), we can obtain more general type and new exact traveling wave solutions of the nonlinear partial differential Eq. (1). Following Naher and Abdullah [35], we have the following solutions of Eq. (5), using the general solutions of Eq. (6):

Family 1. When $B \neq 0$, $\Psi = A - C$ and $\Omega = B^2 + 4E(A - C) > 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)} \tag{7}$$

Family 2. When $B \neq 0$, $\Psi = A - C$ and $\Omega = B^2 + 4E(A - C) < 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\Psi} + \frac{\sqrt{-\Omega}}{2\Psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) + C_2 \cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{C_1 \cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) + C_2 \sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)} \tag{8}$$

Family 3. When $B \neq 0$, $\Psi = A - C$ and $\Omega = B^2 + 4E(A - C) = 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\Psi} + \frac{C_2}{C_1 + C_2 \xi} \tag{9}$$

Family 4. When $B = 0$, $\Psi = A - C$ and $\Delta = \Psi E > 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{\Delta}}{\Psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)} \quad (10)$$

Family 5. $B = 0$, $\Psi = A - C$ and $\Delta = \Psi E < 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{-\Delta}}{\Psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) + C_2 \cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{C_1 \cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) + C_2 \sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)} \quad (11)$$

3. Application of the method

In this section, we apply proposed method to study the KdV equation.

3.1. The KdV equation

Let us consider the KdV equation

$$u_t + uu_x + \delta u_{xxx} = 0. \quad (12)$$

Now, we use the wave transformation $\xi = x - Vt$ into the Eq. (12), which yields:

$$-Vu' + uu' + \delta u''' = 0. \quad (13)$$

Eq. (13) is integrable, therefore, integrating with respect ξ once yields:

$$K - Vu + \frac{1}{2}u^2 + \delta u'' = 0, \quad (14)$$

where K is an integral constant which is to be determined.

Taking homogeneous balance between u^2 and u'' in Eq. (14), we obtain $N = 2$.

Therefore, the solution of Eq. (14) is of the form:

$$u(\xi) = e_0 + e_1(d+H) + e_2(d+H)^2 + (e_{-1} + f_1)(d+H)^{-1} + (e_{-2} + f_2)(d+H)^{-2}, \quad (15)$$

where e_{-2}, e_{-1}, e_0, e_1, e_2, f_1, f_2 and d are constants to be determined.

Substituting Eq. (15) together with Eqs. (5) and (6) into Eq. (14), the left-hand side is converted into polynomials in $(d+H)^N (N = 0, \pm 1, \pm 2, \ldots)$ and $(d+H)^{-N} (N = 1, 2, 3, \ldots)$. We collect each coefficient of these resulted polynomials to zero, yields a set of simultaneous algebraic equations (for simplicity, which are not presented) for e_{-2}, e_{-1}, e_0, e_1, e_2, f_1, f_2, d, K and V. Solving these algebraic equations with the help of algebraic software Maple, we obtain following.

Case 1:

$$e_{-2} = -f_2, e_{-1} = -f_1, e_0 = e_0, e_1 = \frac{12\delta\Psi(B + 2d\Psi)}{A^2},$$
$$e_2 = \frac{-12\delta\Psi^2}{A^2},$$
$$V = \frac{(A^2 e_0 + \delta B^2) + 4\delta\Psi(3d(d\Psi + B) - 2E)}{A^2},$$
$$d = d, K = \frac{P_1 + P_2 + P_3}{2A^4}, \quad (16)$$

where $P_1 = e_0 A^2(e_0 A^2 + 2\delta B^2) + 8\delta e_0 A^2 \Psi \ (3d(B + d\Psi) - 2E)$

$$P_2 = 48\delta^2 d\Psi^2(3d^2(2\Psi B + d(A+C)^2) - 4(E(B + d\Psi) + 3dAC)),$$
$$P_3 = 2\delta^2(2d^2 AC(288AC(d^2 - 1) - 25dB\Psi) + 12\Psi(B^2(dB - 1) + \Psi(2E^2 + 7d^2 B^2))),$$

$\Psi = A - C$, e_0, f_1, f_2, A, B, C and E are free parameters.

Case 2:

$$e_{-2} = \frac{-12\delta(d^2\Psi(d^2\Psi + 2(dB - E)) + (dB - E)^2)}{A^2} - f_2, e_0 = e_0, e_1 = 0, e_2 = 0,$$
$$e_{-1} = \frac{12\delta(d\Psi(2d^2\Psi + 3dB - 2E) + B(dB - E))}{A^2} - f_1, d = d,$$
$$V = \frac{(A^2 e_0 + \delta B^2) + 4\delta\Psi(3d(B + d\Psi) - 2E)}{A^2},$$
$$K = \frac{P_1 + P_2 + P_3}{2A^4}, \quad (17)$$

where $P_1 = e_0 A^2(e_0 A^2 + 2\delta B^2) + 8\delta e_0 A^2 \Psi \ (3d(B + d\Psi) - 2E)$

$$P_2 = 48\delta^2 d\Psi^2(3d^2(2\Psi B + d(A+C)^2) - 4(E(B + d\Psi) + 3dAC)),$$
$$P_3 = 2\delta^2(2d^2 AC(288AC(d^2 - 1) - 25dB\Psi) + 12\Psi(B^2(dB - 1) + \Psi(2E^2 + 7d^2 B^2))),$$

$\Psi = A - C$, e_0, f_1, f_2, A, B, C and E are free parameters.

Case 3:

$$e_{-2} = \frac{-3\delta(8\Delta(2\Delta + B^2) + B^4)}{4A^2\Psi^2} - f_2, e_{-1} = -f_1, e_0 = e_0, e_1 = 0,$$
$$e_2 = \frac{-12\delta\Psi^2}{A^2}, d = \frac{-B}{2\Psi},$$
$$K = \frac{A^4 e_0^2 - 4\delta(4\Delta(A^2 e_0 + 6\delta(2\Delta + B^2)) + B^2(A^2 e_0 + 3\delta B^2)))}{2A^4},$$
$$V = \frac{(A^2 e_0 - 2\delta B^2) - 8\delta\Delta}{A^2}, \quad (18)$$

where $\Psi = A - C$, $\Delta = \Psi E$, e_0, f_1, f_2, A, B, C and E are free parameters.

Case 4:

$$e_{-2} = \frac{-3\delta(8\Delta(2\Delta + B^2) + B^4)}{4A^2\Psi^2} - f_2, e_{-1} = -f_1, e_0 = e_0, e_1 = 0,$$
$$e_2 = 0, d = \frac{-B}{2\Psi},$$
$$K = \frac{A^4 e_0^2 - 4\delta(4\Delta(A^2 e_0 + 6\delta(2\Delta + B^2)) + B^2(A^2 e_0 + 3\delta B^2)))}{2A^4},$$
$$V = \frac{(A^2 e_0 - 2\delta B^2) - 8\delta\Delta}{A^2}, \quad (19)$$

where $\Psi = A - C$, $\Delta = \Psi E$, e_0, f_1, f_2, A, B, C and E are free parameters.

For case 1, substituting Eq. (16) into Eq. (15), along with Eq. (7) and simplifying, yields following traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u1_1(\xi) = e_0 + \frac{3\delta}{A^2}\left(\left(B^2 - \Omega\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right) + 4d\Psi(B + d\Psi)\right),$$
$$u1_2(\xi) = e_0 + \frac{3\delta}{A^2}\left(\left(B^2 - \Omega\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right) + 4d\Psi(B + d\Psi)\right),$$

substituting Eq. (16) into Eq. (15), along with Eq. (8) and simplifying, our exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u1_3(\xi) = e_0 + \frac{3\delta}{A^2}\left(\left(B^2 + \Omega\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right) + 4d\Psi(B + d\Psi)\right),$$
$$u1_4(\xi) = e_0 + \frac{3\delta}{A^2}\left(\left(B^2 + \Omega\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right) + 4d\Psi(B + d\Psi)\right),$$

substituting Eq. (16) into Eq. (15), together with Eq. (9) and simplifying, our obtained solution becomes:

$$u1_5(\xi) = e_0 + \frac{3\delta}{A^2}\left(\left(B^2 - \left(\frac{2\Psi C_2}{C_1 + C_2\xi}\right)^2\right) + 4d\Psi(B + d\Psi)\right),$$

substituting Eq. (16) into Eq. (15), along with Eq. (10) and simplifying, we obtain following traveling wave solutions (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u1_6(\xi) = e_0 + \frac{12\delta}{A^2}\left(d\Psi(B + d\Psi) + \sqrt{\Delta}\left(B\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) - \sqrt{\Delta}\coth^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)\right),$$
$$u1_7(\xi) = e_0 + \frac{12\delta}{A^2}\left(d\Psi(B + d\Psi) + \sqrt{\Delta}\left(B\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) - \sqrt{\Delta}\tanh^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)\right),$$

substituting Eq. (16) into Eq. (15), together with Eq. (11) and simplifying, our obtained exact solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$) respectively:

$$u1_8(\xi) = e_0 + \frac{12\delta}{A^2}\left(d\Psi(B + d\Psi) + \sqrt{\Delta}\left(iB\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) + \sqrt{\Delta}\cot^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)\right),$$
$$u1_9(\xi) = e_0 + \frac{12\delta}{A^2}\left(d\Psi(B + d\Psi) - \sqrt{\Delta}\left(iB\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right) - \sqrt{\Delta}\tan^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)\right),$$

where $\xi = x - \frac{(A^2 e_0 + \delta B^2) + 4\delta\Psi(3d(d\Psi + B) - 2E)}{A^2} t$.

Similarly, for case 2, substituting Eq. (17) into Eq. (15), along with Eqs. (7)–(11) and simplifying, our traveling wave solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$, for 1st two solutions, again these conditions for $u2_3$ and $u2_4$, also same conditions could be applied for solutions $u2_6$ and $u2_7$, moreover, mentioned conditions are implemented to solutions $u2_8$ and $u2_9$) respectively:

$$u2_1(\xi) = e_0 + (e_{-1} + f_1)\left(d + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi}\coth\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)^{-1} + (e_{-2} + f_2)\left(d + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi}\coth\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)^{-2},$$
$$u2_2(\xi) = e_0 + (e_{-1} + f_1)\left(d + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi}\tanh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)^{-1} + (e_{-2} + f_2)\left(d + \frac{B}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi}\tanh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)^{-2},$$
$$u2_3(\xi) = e_0 + (e_{-1} + f_1)\left(d + \frac{B}{2\Psi} + \frac{\sqrt{-\Omega}}{2\Psi}\cot\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^{-1} + (e_{-2} + f_2)\left(d + \frac{B}{2\Psi} + \frac{\sqrt{-\Omega}}{2\Psi}\cot\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^{-2},$$
$$u2_4(\xi) = e_0 + (e_{-1} + f_1)\left(d + \frac{B}{2\Psi} - \frac{\sqrt{-\Omega}}{2\Psi}\tan\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^{-1} + (e_{-2} + f_2)\left(d + \frac{B}{2\Psi} - \frac{\sqrt{-\Omega}}{2\Psi}\tan\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^{-2},$$
$$u2_5(\xi) = e_0 + (e_{-1} + f_1)\left(d + \frac{B}{2\Psi} + \frac{C_2}{C_1 + C_2\xi}\right)^{-1} + (e_{-2} + f_2)\left(d + \frac{B}{2\Psi} + \frac{C_2}{C_1 + C_2\xi}\right)^{-2},$$
$$u2_6(\xi) = e_0 + (e_{-1} + f_1)\left(d + \frac{\sqrt{\Delta}}{\Psi}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^{-1} + (e_{-2} + f_2)\left(d + \frac{\sqrt{\Delta}}{\Psi}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^{-2},$$
$$u2_7(\xi) = e_0 + (e_{-1} + f_1)\left(d + \frac{\sqrt{\Delta}}{\Psi}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^{-1} + (e_{-2} + f_2)\left(d + \frac{\sqrt{\Delta}}{\Psi}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^{-2},$$
$$u2_8(\xi) = e_0 + (e_{-1} + f_1)\left(d + \frac{\sqrt{-\Delta}}{\Psi}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^{-1} + (e_{-2} + f_2)\left(d + \frac{\sqrt{-\Delta}}{\Psi}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^{-2},$$
$$u2_9(\xi) = e_0 + (e_{-1} + f_1)\left(d - \frac{\sqrt{-\Delta}}{\Psi}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^{-1} + (e_{-2} + f_2)\left(d - \frac{\sqrt{-\Delta}}{\Psi}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^{-2},$$

where $\xi = x - \frac{(A^2 e_0 + \delta B^2) + 4\delta\Psi(3d(d\Psi + B) - 2E)}{A^2} t$.

Similarly, for case 3, substituting Eq. (18) into Eq. (15), along with Eqs. (7)–(11) and simplifying, the analytical solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$, for 1st two solutions, again these conditions for $u3_3$ and $u3_4$, also same conditions could be applied for solutions $u3_6$ and $u3_7$, furthermore, mentioned conditions have been executed to solutions $u3_8$ and $u3_9$) respectively:

$$u3_1(\xi) = e_0 - \frac{3\delta\Omega}{A^2}\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right) + (e_{-2} + f_2)\frac{4\Psi^2}{\Omega}\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right),$$
$$u3_2(\xi) = e_0 - \frac{3\delta\Omega}{A^2}\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right) + \frac{4(e_{-2} + f_2)\Psi^2}{\Omega}\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right),$$
$$u3_3(\xi) = e_0 + \frac{3\delta\Omega}{A^2}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) - \frac{4(e_{-2} + f_2)\Psi^2}{\Omega}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right),$$
$$u3_4(\xi) = e_0 + \frac{3\delta\Omega}{A^2}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) - \frac{4(e_{-2} + f_2)\Psi^2}{\Omega}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right),$$
$$u3_5(\xi) = e_0 - \frac{12\delta\Psi^2}{A^2}\left(\frac{C_2}{C_1 + C_2\xi}\right)^2 + (e_{-2} + f_2)\left(\frac{C_2}{C_1 + C_2\xi}\right)^{-2},$$
$$u3_6(\xi) = e_0 - \frac{12\delta\Psi^2}{A^2}\left(\frac{-B}{2\Psi} + \frac{\sqrt{\Delta}}{\Psi}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2 + (e_{-2} + f_2)\left(\frac{-B}{2\Psi} + \frac{\sqrt{\Delta}}{\Psi}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^{-2},$$
$$u3_7(\xi) = e_0 - \frac{12\delta\Psi^2}{A^2}\left(\frac{-B}{2\Psi} + \frac{\sqrt{\Delta}}{\Psi}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2 + (e_{-2} + f_2)\left(\frac{-B}{2\Psi} + \frac{\sqrt{\Delta}}{\Psi}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^{-2},$$
$$u3_8(\xi) = e_0 - \frac{12\delta\Psi^2}{A^2}\left(\frac{-B}{2\Psi} + \frac{\sqrt{-\Delta}}{\Psi}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2 + (e_{-2} + f_2)\left(\frac{-B}{2\Psi} + \frac{\sqrt{-\Delta}}{\Psi}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^{-2},$$
$$u3_9(\xi) = e_0 - \frac{12\delta\Psi^2}{A^2}\left(\frac{-B}{2\Psi} - \frac{\sqrt{-\Delta}}{\Psi}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2 + (e_{-2} + f_2)\left(\frac{-B}{2\Psi} - \frac{\sqrt{-\Delta}}{\Psi}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^{-2},$$

where $\xi = x - \frac{(A^2 e_0 - 2\delta B^2) - 8\delta\Delta}{A^2} t$.

Similarly, for case 4, substituting Eq. (19) into Eq. (15), along with Eqs. (7)–(11) and simplifying, our solutions become (if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$, for 1st two solutions, and same conditions applied for $u4_3$ and $u4_4$, again these conditions are implemented for solutions $u4_6$ and $u4_7$, moreover, mentioned conditions are employed to solutions $u4_8$ and $u4_9$) respectively:

$u4_1(\xi) = e_0 - \frac{3\delta(8\Delta(2\Delta+B^2)+B^4)}{A^2\Omega}\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right),$

$u4_2(\xi) = e_0 - \frac{3\delta(8\Delta(2\Delta+B^2)+B^4)}{A^2\Omega}\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right),$

$u4_3(\xi) = e_0 + \frac{3\delta(8\Delta(2\Delta+B^2)+B^4)}{A^2\Omega}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right),$

$u4_4(\xi) = e_0 + \frac{3\delta(8\Delta(2\Delta+B^2)+B^4)}{A^2\Omega}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right),$

$u4_5(\xi) = e_0 - \frac{3\delta(8\Delta(2\Delta+B^2)+B^4)}{4A^2\Psi^2}\left(\frac{C_2}{C_1+C_2\xi}\right)^{-2},$

$u4_6(\xi) = e_0 - \frac{3\delta(8\Delta(2\Delta+B^2)+B^4)}{4A^2\Psi^2}\left(\frac{2\Psi\sinh\left(\left(\sqrt{\Delta}/\Psi\right)\xi\right)}{B\sinh\left(\left(\sqrt{\Delta}/\Psi\right)\xi\right)-2\sqrt{\Delta}\cosh\left(\left(\sqrt{\Delta}/\Psi\right)\xi\right)}\right)^2,$

$u4_7(\xi) = e_0 - \frac{3\delta(8\Delta(2\Delta+B^2)+B^4)}{4A^2\Psi^2}\left(\frac{2\Psi\cosh\left(\left(\sqrt{\Delta}/\Psi\right)\xi\right)}{B\cosh\left(\left(\sqrt{\Delta}/\Psi\right)\xi\right)-2\sqrt{\Delta}\sinh\left(\left(\sqrt{\Delta}/\Psi\right)\xi\right)}\right)^2,$

$u4_8(\xi) = e_0 - \frac{3\delta(8\Delta(2\Delta+B^2)+B^4)}{4A^2\Psi^2}\left(\frac{2\Psi\sin\left(\left(\sqrt{-\Delta}/\Psi\right)\xi\right)}{B\sin\left(\left(\sqrt{-\Delta}/\Psi\right)\xi\right)-2\sqrt{-\Delta}\cos\left(\left(\sqrt{-\Delta}/\Psi\right)\xi\right)}\right)^2,$

$u4_9(\xi) = e_0 - \frac{3\delta(8\Delta(2\Delta+B^2)+B^4)}{4A^2\Psi^2}\left(\frac{2\Psi\cos\left(\left(\sqrt{-\Delta}/\Psi\right)\xi\right)}{B\cos\left(\left(\sqrt{-\Delta}/\Psi\right)\xi\right)-2\sqrt{-\Delta}\sin\left(\left(\sqrt{-\Delta}/\Psi\right)\xi\right)}\right)^2,$

where $\xi = x - \frac{(A^2e_0-2\delta B^2)-8\delta\Delta}{A^2}t$.

4. Discussions

The advantages and validity of the proposed method over the generalized and improved (G'/G)-expansion method are explained in the following.

4.1. Advantages

The vital advantage of proposed method over the generalized and improved (G'/G)-expansion method is that this method provides new and more general type exact traveling wave solutions with many real parameters. The traveling wave solutions of NLEEs have its significant to reveal the internal mechanism of the physical phenomena.

4.2. Validity

Akbar et al. [33] used linear ordinary differential equation as auxiliary equation and the solutions presented in the form $u(\xi) = \sum_{n=-m}^{m}\frac{e_{-n}}{(d+(G'/G))^n}$, where either e_{-m} or e_m may be zero, but both e_{-m} and e_m cannot be zero simultaneously. On the contrary, we implement nonlinear ordinary differential equation with four real parameters in proposed method and presenting traveling wave solutions, $u(\xi) = \sum_{g=-N}^{N}e_g(d+H)^g + \sum_{g=1}^{N}f_g(d+H)^{-g}$, where $H(\xi) = (G'/G)$.

It is important to point out that some of our solutions are identical with already published results, if parameters taken particular values which validates our proposed method. Furthermore, In Ref. [33] Akbar et al. implemented the generalized and improved (G'/G)-expansion method to the celebrated KdV equation and obtained ten solutions. Despite the fact, we have generated thirty six solutions via the proposed method (solutions $u1_1$–$u4_9$).

5. Conclusions

In this article, new extension of the generalized and improved (G'/G)-expansion method has been first proposed. In order to demonstrate the effectiveness and advantages of the algorithm, we apply it to the KdV equation. Further, a rich class of solutions consisting of the hyperbolic functions, the trigonometric functions and the rational forms have been generated. Moreover, this study shows that, the proposed method can also be applied to deal with the higher-dimensional, higher-order and variable coefficients nonlinear evolution equations for obtaining various solutions.

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