Bounds for the signless Laplacian energy

Nair Abreu [a,1], Domingos M. Cardoso [b,2], Ivan Gutman [c,3], Enide A. Martins [b,2], María Robbiano [d,*,4]

[a] Universidade Federal do Rio de Janeiro, Brazil
[b] Universidade de Aveiro, Portugal
[c] University of Kragujevac, Serbia
[d] Universidad Católica del Norte, Chile

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ABSTRACT

The energy of a graph G is the sum of the absolute values of the eigenvalues of the adjacency matrix of G. The Laplacian (respectively, the signless Laplacian) energy of G is the sum of the absolute values of the differences between the eigenvalues of the Laplacian (respectively, signless Laplacian) matrix and the arithmetic mean of the vertex degrees of the graph. In this paper, among some results which relate these energies, we point out some bounds to them using the energy of the line graph of G. Most of these bounds are valid for both energies, Laplacian and signless Laplacian. However, we present two new upper bounds on the signless Laplacian which are not upper bounds for the Laplacian energy.

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1. Preliminaries

In this paper, we consider simple undirected graphs G, herein just called graphs, with vertex set $\mathcal{V}(G)$ and edge set $\mathcal{E}(G)$. We say that G is an (n, m)-graph if G has n vertices v_i, $1 \leqslant i \leqslant n$, and m edges $e_k = v_i v_j = v_j v_i$, $1 \leqslant k \leqslant m$, with $i, j \in \{1, \ldots, n\}$. The complement of a graph G is denoted

$\overline{G}$. The complete graph on n vertices, K_n, is such that each pair of vertices is connected by an edge. Then, the null graph is $\overline{K}_n$. The line graph $\mathcal{L}(G)$ of G is the graph whose vertex set is in one-to-one correspondence with the edge set of the graph G and where two vertices of $\mathcal{L}(G)$ are adjacent if and only if the corresponding edges in G have a vertex in common [15]. If G is a graph and H one of its induced subgraphs, then we denote by $G − H$ the graph obtained from G by deleting all the vertices in $\mathcal{V}(H)$ (and therefore, eliminating all the edges with at least one end-vertex in $\mathcal{V}(H)$).

The adjacency matrix $A(G)$ of G is a $(0, 1)$-matrix of order n such that $a_{ij} = 1$ if the vertex v_i is adjacent to the vertex v_j and 0 otherwise. Let $D(G)$ be the diagonal matrix of order n whose (i, i)-entry is the degree d_i of the vertex $v_i \in \mathcal{V}(G)$. Then the matrices $L(G) = D(G) − A(G)$ and $L^+(G) = D(G) + A(G)$ are the Laplacian and the signless Laplacian matrices, respectively. For details on their spectral properties see [3,4].

The eigenvalues $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ of the adjacency matrix $A(G)$ of the graph G are also called the eigenvalues of G. For details on spectral graph theory see [2]. The energy of the graph G is defined as

$$E(G) = \sum_{j=1}^{n} |\lambda_j(G)|.$$

Details on the properties of graph energy can be found in [9,10,13,25,11,17].

Let $\mu_1, \mu_2, \ldots, \mu_n$ and $\mu_1^+, \mu_2^+, \ldots, \mu_n^+$ be the eigenvalues of the matrices $L(G)$ and $L^+(G)$, respectively. Then the Laplacian energy of G is defined as [14],

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$$

and, in analogy to $LE(G)$, the signless Laplacian energy may be defined as

$$LE^+(G) = \sum_{i=1}^{n} \left| \mu_i^+ - \frac{2m}{n} \right|.$$

Details on the properties of Laplacian energy can be found in [14,24,25,23].

The concept of matrix energy, put forward by Nikiforov [20], represents a far-reaching generalization of graph energy. Let C be a real matrix of order $s \times t$, with singular values $s_1(C), s_2(C), \ldots, s_q(C)$. Its energy, $E(C)$, is defined as $s_1(C) + s_2(C) + \cdots + s_q(C)$, where $q \leqslant min\{s, t\}$. Therefore, if C is real and symmetric of order n, then $s_i(C), i = 1, \ldots, n$ are equal to the absolute values of their eigenvalues.

According to the above definitions,

$$E(G) = E(A(G)),$$

$$LE(G) = E\left(L(G) - \frac{2m}{n}\mathbb{I}_n\right),$$

and

$$LE^+(G) = E\left(L^+(G) - \frac{2m}{n}\mathbb{I}_n\right),$$

where $\mathbb{I}_n$ stands for the identity matrix of order n.

2. Energy and signless Laplacian energy

This section begins with the following two well known results which will be used along the paper.

Theorem 1 [6]**.** *Let $A, B \in \mathbb{R}^{n \times n}$ and let $C = A + B$. Then*

$$E(C) \leqslant E(A) + E(B). \tag{1}$$

Moreover, equality holds if and only if there exists an orthogonal matrix P such that PA and PB are both positive semidefinite matrices.

Consider the matrix $B = \left(A^T A\right)^{1/2}$ such that $B^T B = A^T A$ and denote $|A| \triangleq \left(A^T A\right)^{1/2}$. Then, we have the following version of the polar decomposition theorem.

Theorem 2 [18]. *If $A \in \mathbb{R}^{n \times n}$, then there exist positive semidefinite matrices X, $Y \in \mathbb{R}^{n \times n}$ and orthogonal matrices P, $Q \in \mathbb{R}^{n \times n}$, such that $A = PX = YQ$. Moreover, $X = |A|$, $Y = \left(AA^T\right)^{1/2}$ are unique matrices that satisfy these equalities. In addition, the matrices P and Q are uniquely determined if and only if A is nonsingular.*

The inequality $LE(G) \leqslant E(G) + \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right|$ was proven in [21] and the equality case was studied in [22]. In an analogous manner it can be shown that

$$LE^+(G) \leqslant E(G) + \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right|. \tag{2}$$

If the graph G is connected then the equality holds if and only if G is regular.

The Zagreb index of a graph G is defined as

$$Z_g(G) = \sum_{i=1}^{n} d_i^2. \tag{3}$$

So, using this invariant, we can obtain:

Theorem 3. *If G is a connected (n, m)-graph, then*

$$LE^+(G) \leqslant E(G) + \sqrt{nZ_g(G) - 4m^2}. \tag{4}$$

This inequality holds as equality if and only if G is regular.

Proof. By Cauchy–Schwarz inequality, we get

$$\sum_{j=1}^{n} |d_j - \frac{2m}{n}| \leqslant \sqrt{n \sum_{j=1}^{n} \left[d_j - \frac{2m}{n} \right]^2} = \sqrt{nZ_g(G) - 4m^2}.$$

The inequality holds as equality if and only if $\left| d_j - \frac{2m}{n} \right| = c$, $\forall j = 1, \ldots, n$. Thus, from inequality (2), the result follows. $\square$

Let G be an (n, m)-graph with edge set $\{e_1, e_2, \ldots, e_m\}$ and let $G(e_k)$, $k \in \{1, 2, \ldots, m\}$ be a spanning subgraph of G with only one edge e_k connecting the vertices v_i and v_j for some $i, j \in \{1, 2, \ldots, n\}$. Then, the signless Laplacian matrix of $G(e_k)$ is

$$L^+(G(e_k))_{rs} = \begin{cases} 1 \text{ if } r, s \in \{i, j\} \\ 0 \text{ otherwise.} \end{cases}$$

If $\alpha \in \mathbb{R}$, then

$$E(L^+(G(e_k)) - \alpha \, \mathbb{I}_n) = (n - 1) |\alpha| + |2 - \alpha|. \tag{5}$$

Therefore, for $k = 1, \ldots, m$, the signless Laplacian matrix of G can be expressed in terms of $L^+(G(e_k))$ as

$$L^+(G) = \sum_{k=1}^{m} L^+(G(e_k)). \tag{6}$$

In order to conclude that the upper bound for the Laplacian energy, obtained in [22, Theorem 7], is also valid for the signless Laplacian energy of an (n, m)-graph, we need the following:

Lemma 4 [22]. *Let $0 < a < 1$ and consider the matrices*

$$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}, \quad S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} a+1 & 0 \\ 0 & a-1 \end{bmatrix}.$$

Then, $A = SDS^{-1}$, $|A| = S|D|S^{-1}$ and $A = Q|A|$.

In the above Lemma, $|A|$ and $|D|$ stand for the matrix functions, pertaining to the ordinary function $f(x) = |x|$. Thus, in particular,

$$|D| = \begin{bmatrix} |a+1| & 0 \\ 0 & |a-1| \end{bmatrix}.$$

Using Lemma 4 and adapting the proof given in [22], it is straightforward to conclude the next upper bound on $LE^+(G)$.

Theorem 5. *Let G be an (n, m)-graph. Then*

$$LE^+(G) \leqslant 4m \left(1 - \frac{1}{n} \right). \tag{7}$$

The equality holds if and only if either G is a null graph (that is a graph with n vertices and without edges) or G is a graph with only one edge plus $n - 2$ isolated vertices.

3. Relations between Laplacian and signless Laplacian energy

It is well known that the spectra of $L(G)$ and $L^+(G)$ coincide if and only if the graph G is bipartite (see [7,8]). Then, of course, $LE(G) = LE^+(G)$. It is also elementary to see that if the graph G is regular, then $LE(G) = LE^+(G) = E(G)$. Therefore, the concept of signless Laplacian energy could be of interest only for non-bipartite, non-regular graphs, in which case $LE^+(G)$ would differ from $LE(G)$.

There exist non-bipartite non-regular graphs for which the inequality

$$LE^+(G) < LE(G) \tag{8}$$

holds, and other such graphs for which

$$LE^+(G) > LE(G) \tag{9}$$

is satisfied. The graphs G_1 and G_2, depicted in Fig. 1, are examples for the validity of (8) and (9), respectively. Whether there are non-bipartite, non-regular graphs satisfying

$$LE^+(G) = LE(G) \tag{10}$$

remains an open problem.

By using Theorem 1 we may prove the following result.

Theorem 6. *Let G be an (n, m)-graph. Then*

$$\left| LE^+(G) - LE(G) \right| \leqslant 2E(G). \tag{11}$$

The equality holds if G is the null graph.

Proof. From the equalities

$$L(G) - \frac{2m}{n}\mathbb{I}_n = D(G) - \frac{2m}{n}\mathbb{I}_n - A(G)$$
$$L^+(G) - \frac{2m}{n}\mathbb{I}_n = D(G) - \frac{2m}{n}\mathbb{I}_n + A(G)$$

it follows that

$$\left(L^+(G) - \frac{2m}{n}\mathbb{I}_n\right) - \left(L(G) - \frac{2m}{n}\mathbb{I}_n\right) = 2A(G).$$

Then,

$$L(G) - \frac{2m}{n}\mathbb{I}_n = -2A(G) + \left(L^+(G) - \frac{2m}{n}\mathbb{I}_n\right)$$

and

$$L^+(G) - \frac{2m}{n}\mathbb{I}_n = 2A(G) + \left(L(G) - \frac{2m}{n}\mathbb{I}_n\right).$$

By application of Theorem 1, we obtain

$$
\begin{aligned}
LE(G) &= E\left(L(G) - \frac{2m}{n}\mathbb{I}_n\right) \leqslant E\left(-2A(G)\right) + E\left(L^+(G) - \frac{2m}{n}\mathbb{I}_n\right) \\
&= 2E(G) + LE^+(G) \\
LE^+(G) &= E\left(L^+(G) - \frac{2m}{n}\mathbb{I}_n\right) \leqslant E(2A(G)) + E\left(L(G) - \frac{2m}{n}\mathbb{I}_n\right) \\
&= 2E(G) + LE(G)
\end{aligned}
$$

(12)

(13)

and thus the inequality in (11) follows. Finally, if G is the null graph, then it is immediate that the inequality (11) holds as equality. □

Theorem 7. *Let G be an (n, m)-graph, $A = A(G)$ its adjacency matrix and $\overline{D} = D(G) - \frac{2m}{n}I_n$. Then,*

$$\max\{2E(G), 2E(\overline{D})\} \leqslant LE^+(G) + LE(G) \leqslant 2E(G) + 2E(\overline{D}).$$

All the inequalities hold as equalities if and only if G is regular.

Notice that $E(\overline{D}) = \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right|$, cf. Eq. (2).

Proof. Consider the $2n \times 2n$ matrices $X = \begin{pmatrix} \overline{D} & A \\ A & \overline{D} \end{pmatrix}$ and $J = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{I}_n & \mathbb{I}_n \\ \mathbb{I}_n & -\mathbb{I}_n \end{pmatrix}$. It is not difficult to prove that $J^2 = \mathbb{I}_{2n}$ and

$$JXJ = \begin{pmatrix} \overline{D}+A & 0 \\ 0 & \overline{D}-A \end{pmatrix} = Y.$$

Therefore, $E(X) = E(Y) = LE^+(G) + LE(G)$. We observe that

$$X = \begin{pmatrix} \overline{D} & 0 \\ 0 & \overline{D} \end{pmatrix} + \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$$

and by applying Theorem 1, the right inequality is obtained. This result may also be obtained from Theorems 1 and 4 in [22]. According to [22], the right inequality holds as equality if and only if G is

regular. Regarding the left inequality, since the matrices X and $Z = \begin{pmatrix} \overline{D} & -A \\ -A & \overline{D} \end{pmatrix}$ are similar, by applying Theorem 1, we obtain

$$4E(\overline{D}) = E(X + Z) \leqslant E(X) + E(Z) = 2E(X) \Rightarrow 2E(\overline{D}) \leqslant E(X)$$

and

$$4E(A) = E(X - Z) \leqslant E(X) + E(Z) = 2E(X) \Rightarrow 2E(A) \leqslant E(X).$$

Thus, the left inequality follows. From Theorem 1, the left inequality holds as equality if and only if there exist orthogonal matrices P and Q such that $PX = M_1$, $PZ = M_2$, $QX = N_1$, $Q(-Z) = N_2$ are positive semidefinite matrices. From Theorem 2, $M_1 = |X| = |Z| = M_2$, and $N_1 = |X| = |Z| = N_2$, which implies $X = Z$ or $X = -Z$. In both cases G is a regular graph. □

As direct consequence of Theorems 6 and 7, we may conclude that if G is an (n, m)-graph, then

$$\frac{|LE^+(G) - LE(G)|}{2} \leqslant E(G) \leqslant \frac{LE^+(G) + LE(G)}{2}.$$

Based on the results obtained in Section 2 and also in [12], it seems that the Laplacian and signless Laplacian energies satisfy bounds of equal form. However, here we point out some differences between both graph invariants. In particular, we introduce some bounds for the signless Laplacian energy which, for certain kind of graphs, are not valid for the Laplacian energy.

Let G be an (n, m)-graph whose vertex degrees are $d_1, \ldots, d_n$ and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then, the following is known:

1. The line graph $\mathcal{L}(G)$ has m vertices and q edges, where

$$q = -m + \frac{1}{2}\sum_{i=1}^{n} d_i^2. \tag{14}$$

2. According to [16],

$$E(G) \leq \lambda_1 + \sqrt{n - 1}\sqrt{2m - \lambda_1^2}. \tag{15}$$

Furthermore, if G is an (n, m)-graph with $m \geq n$, using (14), (15) and Theorem 9-(b) and (c), we arrive at the inequalities

$$LE^+(G) \leq E(\mathcal{L}(G)) \leq \lambda_1(\mathcal{L}(G)) + \sqrt{m - 1}\sqrt{2q - \lambda_1^2(\mathcal{L}(G))}, \tag{16}$$

where $\lambda_1(\mathcal{L}(G))$ denotes the index of $\mathcal{L}(G)$.

In some cases, we can see that while the above upper bound is true for the signless Laplacian energy, it is not valid for the Laplacian energy. For instance, considering the graph G_1 depicted in Fig. 1, we obtain

$$5.123 = LE^+(G_1) \leq \lambda_1(\mathcal{L}(G_1)) + \sqrt{m - 1}\sqrt{2q - \lambda_1^2(\mathcal{L}(G_1))}$$
$$= 5.773 < 6 = LE(G_1).$$
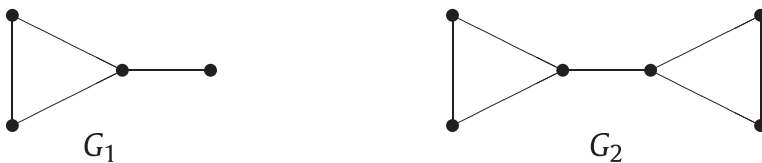

Fig. 1. The graphs G_1 and G_2 are such that $LE^+(G_1) < LE(G_1)$ and $LE^+(G_2) = 8.666 > 8.456 = LE(G_2)$.

Taking into account the definition of Zagreb index (3) and the equality (14), it is straightforward to conclude that

$$q = -m + \frac{1}{2} Z_g(G). \tag{17}$$

Let us consider an (n, m)-graph G. Then, it follows:

1. As proven in [1] and [19], the inequalities

$$2\sqrt{m} \leqslant E(G) \leqslant \sqrt{2mn} \tag{18}$$

hold and the upper bound is attained if and only if G is either the null graph or m copies of K_2.

2. As proven in [16], the inequality

$$E(G) \leqslant \frac{2m}{n} + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]} \tag{19}$$

holds and the upper bound is attained if and only if G is either the null graph, m copies of K_2, the complete graph K_{n-1} or a strongly regular connected graph with two non-trivial eigenvalues, both having the absolute value equal to $\sqrt{\left(2m - \left(\frac{2m}{n}\right)^2\right)/(n-1)}$.

Therefore, using the lower bound in (18) and the upper bound (19) (since, for every graph, the upper bound in (18) is worse than the (19)), it follows

$$\sqrt{2Z_g(G) - 4m} \leqslant E(\mathcal{L}(G)) \leqslant \Upsilon^+(G), \tag{20}$$

where

$$\Upsilon^+(G) = \frac{1}{m}\left[Z_g(G) - 2m + \sqrt{(m-1)(Z_g(G) - 2m)(m^2 - Z_g(G) + 2m)}\right].$$

Therefore, applying (20) and Theorem 9, beyond the inequality

$$\sqrt{2(Z_g(G) - 2m)} \leqslant LE^+(G) \tag{21}$$

we arrive at the inequalities

$$LE^+(G) \leqslant \begin{cases} \Upsilon^+(G), & \text{if } m \geqslant n \\ \Upsilon^+(G) + \frac{4m}{n}(n - m) & \text{if } m < n. \end{cases} \tag{22}$$

The upper bound (22) (for $m \geq n$) also shows us that $LE^+(G)$ and $LE(G)$ behave differently. In fact, using the same graph displayed in Fig. 1, we get

$$5.123 = LE^+(G)$$
$$\leqslant \frac{Z_g(G) - 2m + \sqrt{(m-1)(Z_g(G) - 2m)(m^2 - Z_g(G) + 2m)}}{m} = 5.854$$
$$< 6 = LE(G).$$

Now, let us compare $\Upsilon^+(G)$ with the lower bound in (21). Since,

$$Z_g(G) = \sum_{i=1}^{n} d_i^2 = \sum_{ij \in \mathcal{E}(G)} (d_i + d_j) \leq 2m\Delta(G) \Rightarrow Z_g(G) - 2m \leq 2m(\Delta(G) - 1),$$

where $\Delta(G)$ denotes the maximum degree of the vertices of G, and

$$4m^2 = \left(\sum_{i=1}^{n} d_i\right)^2 \leqslant n \sum_{i=1}^{n} d_i^2 \Rightarrow 2m\left(\frac{2m}{n} - 1\right) \leqslant Z_g(G) - 2m.$$

Denoting $\bar{d}(G) = \frac{2m}{n}$, it follows:

$$\Upsilon^+(G) = \frac{1}{m}\left[\sqrt{Z_g(G) - 2m} + \sqrt{(m-1)(m^2 - Z_g(G) + 2m)}\right]\sqrt{Z_g(G) - 2m}$$

$$\leqslant \frac{1}{m}\left[\sqrt{2m(\Delta(G) - 1)} + \sqrt{2m(m-1)\left(\frac{m}{2} - (\bar{d} - 1)\right)}\right]\sqrt{Z_g(G) - 2m}$$

$$= \left(\sqrt{\frac{\Delta(G) - 1}{m}} + \sqrt{\frac{(m-1)\left(\frac{m}{2} - (\bar{d}(G) - 1)\right)}{m}}\right)\sqrt{2(Z_g(G) - 2m)}$$

$$\leq \left(\sqrt{\frac{m-1}{m}} + \sqrt{\frac{(m-1)\left(\frac{m}{2} - (\bar{d}(G) - 1)\right)}{m}}\right)\sqrt{2(Z_g(G) - 2m)}$$

$$\leq \left(1 + \sqrt{\frac{m}{2} - (\bar{d}(G) - 1)}\right)\sqrt{2(Z_g(G) - 2m)}.$$

Therefore,

$$\sqrt{2(Z_g(G) - 2m)} \leqslant LE^+(G) \leq \Upsilon^+(G)$$

$$\leqslant \left(1 + \sqrt{\frac{m}{2} - (\bar{d}(G) - 1)}\right)\sqrt{2(Z_g(G) - 2m)}.$$

Finally, we conclude that the above inequalities give the range where $LE^+(G)$ may vary in function of the parameter $\sqrt{2(Z_g(G) - 2m)}$.

4. A bound for the Laplacian energy of trees

In this section, denoting the number of edges of a graph G by $m(G)$, we prove a proposition which is an application of the following results given in [5,12].

Theorem 8 [5]. *Let H be an induced subgraph of G. Then*

$$E(H) + E(G - H) \leqslant E(G).$$

Equality holds if and only if each edge of G is in H.

Theorem 9 [12]. *If G is a graph with $n \geqslant 2$ vertices and $m(G) = m \geqslant 1$, then the following statements hold:*

(a) *If $m < n$, then $LE^+(G) - \frac{4m}{n}(n - m) \leqslant E(\mathcal{L}(G)) < LE^+(G)$. The left-hand equality is attained if and only if G is the direct sum of m copies of the complete graph K_2.*

(b) If $m > n$, then $LE^+(G) < E(\mathcal{L}(G)) < LE^+(G) + 4(m - n)$.

(c) Finally, $LE^+(G) = E(\mathcal{L}(G))$ if and only if $m = n$.

Based on the above results, we may prove the following theorem.

Theorem 10. *Let T be a tree on n vertices and H a connected subgraph of T on k vertices. Then*

$$LE(H) + LE(T - H) < LE(T) + 4(1 + \delta),$$

where δ is the number of connected components of $T - H$.

Proof. Since the number of edges of T is $m = n - 1$ and the subgraph H is connected (therefore a subtree) of order k, it follows that the number of connected components of $T - H$ is

$$\delta = n - 1 - (k - 1) - m(T - H) = n - k - m(T - H).$$

Notice that δ also coincides with the number of edges between the vertices of T which are in and out of the subtree H. Hence, $m(T - H) = n - k - \delta$. Taking into account that T is bipartite, it follows that $LE^+(T) = LE(T)$ (cf. [7,8]). By using Theorem 9-(a), we obtain

$$LE(T) - \frac{4(n - 1)}{n} \leqslant E(\mathcal{L}(T)) < LE(T),$$

where the left-hand side inequality is attained as equality if and only if $T \cong K_2$.

It is immediate that, if H is a subtree of T, then $\mathcal{L}(H)$ is an induced subgraph of $\mathcal{L}(T)$. Then, by applying Theorem 8, we have

$$E(\mathcal{L}(H)) + E(\mathcal{L}(T) - \mathcal{L}(H)) \leqslant E(\mathcal{L}(T)). \tag{23}$$

Since $\mathcal{L}(G - H)$ is an induced subgraph of $\mathcal{L}(T) - \mathcal{L}(H)$, then

$$E(\mathcal{L}(G - H)) \leqslant E(\mathcal{L}(T) - \mathcal{L}(H))$$

and from (23), we arrive at the inequality

$$E(\mathcal{L}(H)) + E(\mathcal{L}(T - H)) \leqslant E(\mathcal{L}(T)).$$

Now, applying Theorem 9-(a) to the subtree H and to the subgraph $T - H$, it follows

$$LE(H) - \frac{4(k - 1)}{k} \leqslant E(\mathcal{L}(H)) < LE(H)$$

and

$$LE(T - H) - \frac{4(n - k - \delta)}{n - k}\delta \leqslant E(\mathcal{L}(T - H)) < LE(T - H).$$

By adding the above inequalities, we arrive at

$$LE(H) + LE(T - H) - 4\frac{(k - 1)(n - k) + (n - k - \delta)k\delta}{(n - k)k}$$
$$\leqslant E(\mathcal{L}(H)) + E(\mathcal{L}(T - H)) \leqslant E(\mathcal{L}(T)) < LE(T).$$

This implies,

$$LE(H) + LE(T - H) < LE(T) + 4\frac{(k-1)(n-k) + (n-k-\delta)k\delta}{(n-k)k}$$

$$= LE(T) + 4\left(\frac{k-1}{k} + \frac{n-k-\delta}{n-k}\delta\right)$$

$$< LE(T) + 4(1 + \delta)$$

since $(k-1)/k$ and $(n-k-\delta)/(n-k)$ are both less than 1. $\square$

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