Subgroups Which Are the Union of Three Conjugate Classes

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Subgroups which are the union of two conjugate classes were studied previously (M. Shahryari and M. A. Shahabi, Subgroups which are the union of two conjugate classes, Groups Conference, St. Andrews, Bath, 1997). In this article we will determine the structure of subgroups which are the union of three conjugate classes.

1. INTRODUCTION

Let $G$ be a finite group and let $H$ be a normal subgroup of $G$ which is a union of three conjugate classes of $G$, i.e.,

$$H = 1 \cup \text{Cl}_G(h) \cup \text{Cl}_G(g).$$

In this note we will determine the structure of $H$. First we give an example to show that the class of all groups $G$ with this property is too large for classification.

**Example 1.1.** Let $G$ be a finite group such that $S_3 \leq G$. Then it is easy to see that $S_3$ is a union of three conjugate classes of $G$.

In this article we prove that under the above assumption, $H$ is an elementary abelian $p$-group for some odd prime $p$, a metabelian $p$-group, or an extension of an elementary abelian group with a certain cyclic group. We can distinguish three cases.

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Case A. Let $h^{-1} \in \text{Cl}_{G}(g)$. In this case we will prove that $H$ is an elementary abelian $p$-group of odd order. Also we will prove that

$$|H| - \frac{1}{2} \mid \chi(1)$$

for any irreducible character $\chi$ of $G$ with the property $[\chi, 1] = 0$.

Case B. Let $h^{-1} \in \text{Cl}_{G}(g)$ and $\gcd(o(h), o(g)) \neq 1$. In this case, we show that $H$ is a metabelian $p$-group.

Case C. Let $h^{-1} \in \text{Cl}_{G}(g)$ and $\gcd(o(h), o(g)) = 1$. This is a more complicated case. We prove that $H$ is a Frobenius group with the elementary abelian kernel $H'$. Actually, we prove that $H$ is an extension of some elementary abelian group with a cyclic group of prime order, such as $S_{3}$, $D_{10}$, and many other groups.

2. GENERALITIES

Case A. Let $h^{-1} \in \text{Cl}_{G}(g)$. It is easy to see that $\text{Cl}_{G}(h)^{-1} \subseteq \text{Cl}_{G}(g)$, where

$$\text{Cl}_{G}(h)^{-1} = \{x^{-1} | x \in \text{Cl}_{G}(h)\}.$$  

Also we have $\text{Cl}_{G}(g)^{-1} \subseteq \text{Cl}_{G}(h)$, so $\text{Cl}_{G}(h)^{-1} = \text{Cl}_{G}(g)$.

**Proposition 1.** Let $H$ be as above. Then $H$ is an elementary abelian $p$-group of odd order. We have $H \leq G'$, except in the case $H = Z_{3}$. Also, for any $\chi \in \text{Irr}(G), [\chi, 1] = 0$ implies

$$|H| - \frac{1}{2} \mid \chi(1).$$

**Proof:** Since $|H| = 1 + |\text{Cl}_{G}(h)| + |\text{Cl}_{G}(g)| = 1 + 2|\text{Cl}_{G}(h)|$, $|H|$ is odd. We know that all nonidentity elements of $H$ have the same order. Let $p$ be a prime divisor of $|H|$. Then $H$ has an element of order $p$, so every element of $H$ has order $p$ or $1$. Hence $H$ is a $p$-group in which $x^{p} = 1$ for all $x \in H$.

Note that $H$ is a minimal normal subgroup of $G$, since if $1 \neq K \unlhd G$ and $K < H$, then we have

$$K = 1 \cup \text{Cl}_{G}(h) \quad \text{or} \quad K = 1 \cup \text{Cl}_{G}(g).$$

In any case, $K$ is not a subgroup of $G$, but we have $H' \unlhd G$ and $H' \neq H$, so $H' = 1$. 

Let \( \lambda \) be a linear character of \( G \). Then \( \lambda_H \) is a linear character of \( H \) and hence \([\lambda, 1_H] = 0\) or 1. Let \([\lambda, 1_H] = 0\). This implies that
\[
1 + |\text{Cl}_G(h)(\lambda(h) + \lambda(g)) = 0,
\]
so \( 1/|\text{Cl}_G(h)| \) is an algebraic integer. This implies that \(|\text{Cl}_G(h)| = 1\) and so \(|H| = 3\). Hence if \( H \neq Z_3 \), then \([\lambda, 1_H] = 1\) for any linear character \( \lambda \).

Hence \( H \leq G' \).

Let \( \chi \in \text{Irr}(G) \) and \([\chi, 1_H] = 0\). Then
\[
\chi(1) + |\text{Cl}_G(h)(\chi(h) + \chi(g)) = 0,
\]
so we have
\[
-\chi(1) = \frac{|H| - 1}{2}(\chi(h) + \chi(g)).
\]
But \( \chi(h) + \chi(g) \) is an algebraic integer, hence
\[
\frac{|H| - 1}{2} |\chi(1)|.
\]

This completes the proof.

Case B. Let \( h^{-1} \in \text{Cl}_G(h) \) and \( \gcd(o(h), o(g)) \neq 1 \). In this case, \( g^{-1} \in \text{Cl}_G(g) \). Without loss of generality we can assume that \( gh \in \text{Cl}_G(h) \).

This assumption implies that \( hg \in \text{Cl}_G(h) \).

**Proposition 2.** With the above assumptions, \( H \) is a metabelian \( p \)-group.

**Proof.** Let \( p \) be a prime divisor of \(|H|\). Then \( H \) has a \( p \)-element \( x \). Let \( q \) be another prime which divides \(|H|\). Then there is a \( y \in H \) such that \( o(y) = q \). If any of the \( x \) and \( y \) lies in \( \text{Cl}_G(h) \), then the other will be in \( \text{Cl}_G(g) \). Let \( o(h) = p \). Then \( o(g) = q \), which is a contradiction. Hence \( H \) is a \( p \)-group. Let \( H' \neq 1 \). We know that \( H' \trianglelefteq G \), so
\[
H' = 1 \cup \text{Cl}_G(h) \quad \text{or} \quad H' = 1 \cup \text{Cl}_G(g).
\]

Let \( H' = 1 \cup \text{Cl}_G(h) \). We have \( gh \in \text{Cl}_G(h) \); hence \( g = gh h^{-1} \in H' \), a contradiction. So \( H' = 1 \cup \text{Cl}_G(g) \). We prove that \( H' \) is an elementary abelian group. Note that for any \( x \in H' \), \( x^p = 1 \). Let
\[
C_1, C_2, \ldots, C_k
\]
be the set of all conjugate classes of \( H' \) such that \( C_1 = \{1\} \). The group \( G \) acts on this set by conjugation as well as on the set \( \text{Irr}(H') \). Also, we have
\[
\chi^*(C_i) = \chi(C_i^*),
\]
for any \( x \in G \), so by the Brauer’s theorem on character tables, the number of orbits of \( G \) in these actions is the same. But \( G \) has only two orbits on the set \( C_1, \ldots, C_k \). Hence \( \text{Irr}(H') \) has only two orbits under the action of \( G \), namely, \( (1_H) \) and \( \text{Irr}^*(H') = \text{Irr}(H') \setminus 1_H \), so all elements of \( \text{Irr}^*(H') \) have the same degree, say \( m \). But
\[
|H'| = \sum_{\lambda \in \text{Irr}^*(H')} \lambda(1)^2.
\]
This implies
\[
|H'| = 1 + (k - 1)m^2.
\]
Since \( m | |H'| \), so \( m | 1 \). This shows that \( m = 1 \), hence \( H' \) is abelian.

**Case C.** Suppose \( h^{-1} \in \text{Cl}_G(h) \) and \( \gcd(o(h), o(g)) = 1 \). Without loss of generality we can assume that \( gh \in \text{Cl}_G(h) \). As in the Case B, we obtain \( hg \in \text{Cl}_G(h) \). Using a series of lemmas, we determine the structure of \( H \).

**Lemma 3.** We have \( |H| = p^m q^n \) for some distinct primes \( p \) and \( q \), and some integers \( m \) and \( n \).

**Proof.** Let \( p \) and \( q \) be distinct primes such that \( p | o(h) \) and \( q | o(g) \). Using the Cauchy theorem we obtain \( o(h) = p \) and \( o(g) = q \). So every element of \( H \) has order 1, \( p \), or \( q \). Hence \( |H| = p^m q^n \) for some \( m \) and \( n \).

**Lemma 4.** \( Z(H) = 1 \).

**Proof.** Let \( x \in Z(H) \) and \( x \neq 1 \). We have \( x \in \text{Cl}_G(h) \) or \( x \in \text{Cl}_G(g) \). If \( x \in \text{Cl}_G(h) \), then \( o(x) = p \), so \( o(xg) = pq \), a contradiction. Similarly if we assume \( x \in \text{Cl}_G(g) \), then we get again a contradiction.

**Lemma 5.** We have \( H' = 1 \cup \text{Cl}_G(g) \) and \( H' \) is an elementary abelian \( q \)-group.

**Proof.** The proof is similar to the Case B.

**Lemma 6.** \( H \) is a Frobenius group with kernel \( H' \) and its complement is \( \mathbb{Z}_p \).

**Proof.** We know that \( \gcd(p^m, q^n) = 1 \) and every element of \( H \) is a root of \( x^{p^n} = 1 \) or \( x^{q^n} = 1 \). Also the set
\[
\{ x \in H \mid x^{p^n} = 1 \} = H'
\]
is a nontrivial proper normal subgroup of \( H \). So by [2], \( H \) is a Frobenius group with kernel \( H' \). Let \( K \) be its complement. Then \( H/H' \cong K \), so \( K \) is abelian. But every element of \( K \) has order 1 or \( p \). Hence \( K \) is an elementary abelian \( q \)-group. It is known that a Frobenius complement cannot contain any subgroup of type \( (p, p) \). This shows that \( |K| = p \).
COROLLARY 7. (i) $|H| = pq^n$ and either $p$ or $q$ is 2.
(ii) $p | q^n - 1$ and $p - 1 | q^n$.
(iii) $|\text{Cl}_G(g)| = q^n - 1$ and $|\text{Cl}_G(h)| = (p - 1)q^n$.

We summarize the results we obtained in the following theorem. Note that $V(n, q)$ is the $n$-dimensional vector space over the field of order $q$.

THEOREM 8. Suppose $h^{-1} \in \text{Cl}_G(h)$ and $\gcd(o(h), o(g)) = 1$. Then $h$ is a Frobenius group with elementary abelian kernel $H' = V(n, q)$ and complement $Z_p$.

3. EXAMPLES

Finally, in this section we give some examples to show that the results of Theorem 8 cannot be strengthened.

EXAMPLE 3.1. Let $q$ be an odd prime and let $n$ be an integer. Let $Z_2 = \langle y \rangle$ act nontrivially on $V(n, q)$. So there is a homomorphism $\phi: Z_2 \to \text{GL}_n(q)$. Then $\phi(y)$ is an involution in $\text{GL}_n(q)$, so it is diagonalizable, i.e., there is a basis of $V(n, q)$ over $Z$, say $x_1, \ldots, x_n$, such that the matrix representation of $\phi(y)$ with respect to this basis is of the form

$$
\begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix},
$$

where $\lambda_i = \pm 1$ for any $1 \leq i \leq n$. Let $H = V(n, q) \cdot Z_2$, the semidirect product. Then $H$ has a presentation of the form

$$H = \langle x_1, \ldots, x_n, y : x_1^q = \cdots = x_n^q = y^2 = 1, x_i x_j = x_j x_i, x_i^y = x_i^{\lambda_i} \rangle.$$

If there exists an $i$ such that $\lambda_i = +1$, then $x_i \in Z(H)$. In this case $H$ is not a Frobenius group, so we may assume $\lambda_i = -1$ for any $i$. Hence the presentation of $H$ reduces to the form

$$H = \langle x_1, \ldots, x_n, y : x_1^q = \cdots = x_n^q = y^2 = 1, x_i x_j = x_j x_i, x_i^y = x_i^{-1} \rangle.$$

Every element of $H$ has a unique expression of the form $x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} y^j$, such that $0 \leq r_i \leq q - 1$ and $j = 0, 1$. We prove that $\text{Aut}(H)$ has only
three orbits on \( H \), namely,

\[
X = \{ x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} \mid 0 \leq r_i \leq q - 1 \text{ and } \exists i: r_i \neq 0 \},

Y = \{ x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} y \mid 0 \leq r_i \leq q - 1 \}.
\]

We prove this in two steps:

**Step 1.** \( \text{Aut}(H) \) is transitive on \( X \). Let \( x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} \in X \). We must obtain an automorphism \( f: H \to H \) such that \( f(x_i) = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} \).

If \( r_1 \neq 0 \), define \( f(x_1) = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}, f(x_i) = x_i \) for \( 2 \leq i \leq n \), and \( f(y) = y \). Then we can see that the set

\[
\{ f(x_1), f(x_2), \ldots, f(x_n), f(y) \}
\]

generates \( H \) and satisfies the same relations as in the presentation of \( H \). So \( f \) can be extended to an automorphism of \( H \).

Let \( r_1 = 0 \) and \( j \) be a number such that \( r_j \neq 0 \). Define \( f(x_1) = x_2^{r_2} \cdots x_n^{r_n}, f(x_j) = x_1, f(x_i) = x_i \) for other \( i \), and \( f(y) = y \). Still the set

\[
\{ f(x_1), f(x_2), \ldots, f(x_n), f(y) \}
\]

generates \( H \) and satisfies the same relations as in the presentation of \( H \). So it can be extended to an automorphism of \( H \). This argument shows that \( \text{Aut}(H) \) is transitive on \( X \).

**Step 2.** \( \text{Aut}(H) \) is transitive on \( Y \). Let \( x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} y \in Y \). We must obtain an automorphism \( f: H \to H \) such that \( f(y) = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} y \). It is easy to see that the function \( f \), defined by \( f(x_i) = x_i \) for all \( i \), \( f(y) = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} y \), can be extended to an automorphism of \( H \).

Now consider the natural action of \( \text{Aut}(H) \) on \( H \). The group \( H \) has three orbits under this action. So if we define \( G = \text{Hol}(H) \), the holomorph of \( H \), then \( H \) is a normal subgroup of \( G \) containing just three conjugate classes of \( G \). Some special cases of this example are dihedral groups \( D_{2q} \), such as \( D_6 = S_3 \) and \( D_{10} \).

**Example 3.2.** In this example we suppose \( q = 2, n = 2, \) and \( p = 3 \). Let \( Z_3 = \langle y \rangle \) act on \( V(2, 2) \) in which the matrix representation of \( y \) with respect to the standard basis of \( V(2, 2) \) is

\[
R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Let \( x_1 = [1, 0]^t \) and \( x_2 = [0, 1]^t \). Then the semidirect product \( H = V(2, 2) \cdot Z_3 \) has a presentation of the form

\[
H = \langle x_1, x_2, y : x_1^2 = x_2^2 = y^3 = 1, x_1 x_2 = x_2 x_1, x_1 y = x_1, x_2 y = x_2 \rangle.
\]
We have $|H| = 12$ and every element of $H$ has a unique expression of the form $x_1^{r_1}x_2^{r_2}y^j$ such that $r_1 = 1, 2$ and $j = 0, 1, 2$. We prove that $\text{Aut}(H)$ has three orbits on $H$, namely,

$$\{1\},$$

$$X = \{x_1^{r_1}x_2^{r_2} | r_1 = 0, 1 \text{ and } \exists i: r_i \neq 0\},$$

$$Y = \{x_1^{r_1}x_2^{r_2}y^j | r_1 = 0, 1, j = 1, 2\}.$$

Let $x_1^{r_1}x_2^{r_2} \in X$. We find an automorphism $f: H \to H$ such that $f(x_1) = x_1^{r_1}x_2^{r_2}$. If $r_1 = 0$ and $r_2 = 1$, then the function

$$f(x_1) = x_2, \quad f(x_2) = x_1x_2, \quad f(y) = y$$

can be extended to an automorphism of $H$. If $r_1 = 1$ and $r_2 = 0$, then the function

$$f(x_1) = x_1x_2, \quad f(x_2) = x_1, \quad f(y) = y$$

can be extended to an automorphism of $H$. Hence $\text{Aut}(H)$ is transitive on $X$. Using a similar method it can be shown that $\text{Aut}(H)$ is transitive on $Y$. So if we define $G = \text{Hol}(H)$, then $H$ is a normal subgroup of $G$, containing just three conjugate classes of $G$.

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