Almost splitting sets in integral domains

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Abstract

Let $A$ be an integral domain, $S$ a saturated multiplicative subset of $A$, and $N(S) = \{0 \neq x \in A | (x, s)_s = A$ for all $s \in S\}$. Then $S$ is called an almost splitting set if for each $0 \neq d \in A$, there is an integer $n = n(d) \geq 1$ such that $d^n = st$ for some $s \in S$ and $t \in N(S)$. Let $B$ be an overring of $A$, $X$ an indeterminate over $B$, $R = A + XB[X]$, and $D = A + X^2B[X]$. In this paper, we study almost splitting sets and show that $D$ is an AGCD-domain if and only if $R$ is an AGCD-domain and $\text{char}(A) \neq 0$. As a corollary, we have that $D$ is an AGCD-domain if $A$ is an integrally closed AGCD-domain, $\text{char}(A) \neq 0$, and $B = AS$, where $S$ is an almost splitting set of $A$.

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1. Introduction

Let $D$ be an integral domain with quotient field $K$, $S$ a saturated multiplicative subset of $D$, and $N(S) = \{0 \neq x \in D | (x, s)_s = D$ for all $s \in S\}$. Then $S$ is called a splitting set if each $0 \neq d \in D$ may be written as $d = sa$ for some $s \in S$ and $a \in N(S)$. Following [5], we say that $S$ is a t-splitting set if for each $0 \neq d \in D$, $dD = (AB)_t$ for some integral ideals $A$ and $B$ of $D$, where $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. It is easy to see that a splitting set is a t-splitting set, but a t-splitting set need not be a splitting set (see Proposition 2.7). However, if $\text{Cl}(D) = 0$, then a t-splitting set $S$ of $D$ is a splitting set. For if $0 \neq d \in D$, then...
We prove in Section 3 that $d D_S \cap D$ is a $t$-invertible $t$-ideal of $D$ [5, Corollary 2.3; 25, Lemma 3.17]; hence $d D_S \cap D$ is principal. Thus $S$ is a splitting set [3, Theorem 2.2].

Now we have a very similar and interesting question. “What are the properties of a $t$-splitting set $S$ of $D$ when $Cl(D)$ is torsion?” Fortunately, by an argument similar to the one given in the proof of the case when $Cl(D) = 0$, we can show that for each $0 \neq d \in D$, there is an integer $n = n(d) \geq 1$ such that $d^n D_S \cap D$ is principal (see the proof of Corollary 2.4). (This is equivalent to the fact that $d^n = s t$ for some $s \in S$ and $t \in N(S)$; see Lemma 2.2.) This type of multiplicative sets was introduced by Dumitrescu et al. [19] to study when $t$-splitting set $S$ of $D$ when $Cl(D)$ is torsion when $S$ is a splitting set [3, Theorem 2.2].

Let $S$ be a $t$-splitting set of an integral domain $D$, and let $\mathcal{T} = (A_1 \cdots A_n) | A_i = (d_i D_S \cap D$ for some $0 \neq d_i \in D)$. Then $D_S = \cap \{D_P \mid P \in t\text{-Max}(D) \text{ and } P \cap S = \emptyset\}$, $D_{\mathcal{T}} = \cap \{D_P \mid P \in t\text{-Max}(D) \text{ and } P \cap S \neq \emptyset\}$, and $D = D_S \cap D_{\mathcal{T}}$, where $D_{\mathcal{T}} = \{x \in K \mid x C \subseteq D$ for some $C \in \mathcal{T}\}$ [5, Lemma 4.2 and Theorem 4.3]. A $t$-splitting set $S$ of $D$ is a $t$-complemented $t$-splitting set if $D_{\mathcal{T}} = D_T$ for some multiplicative subset $T$ of $D$, and the saturation of $T$ is called the $t$-complement of $S$. It is known, and easily proved, that if $S$ is a $t$-complemented $t$-splitting set, then $N(S)$ is the $t$-complement of $S$ and $N(S)$ is also a $t$-complemented $t$-splitting set with $t$-complement $N(N(S)) = S$, the saturation of $S$ in $D$ [5, p. 15]. A $t$-splitting set was introduced in [5] to show that $D(S) = D + X D_S[X]$ is a PVMD if and only if $D$ is a PVMD and $S$ is a $t$-splitting set of $D$. (Recall that $D$ is a Prüfer $v$-multiplication domain (PVMD) if every finite type $v$-ideal of $D$ is $t$-invertible.)

In Section 2, we study almost splitting sets. In particular, we show that an almost splitting set is a $t$-complemented $t$-splitting set and that if $S$ is an almost splitting set of $D$, then $Cl(D)$ is torsion if and only if $Cl(D_S)$ and $Cl(D_{N(S)})$ are both torsion. We also give an example of a $t$-complemented $t$-splitting set $S$ of $D$ which is not an almost splitting set such that both $Cl(D_S)$ and $Cl(D_{N(S)})$ are torsion, but $Cl(D)$ is not torsion. Let $B$ be an overring of an integral domain $A$, $X$ an indeterminate over $B$, $R = A + X B[X]$, and $D = A + X^2 B[X]$. We prove in Section 3 that $D$ is an AGCD-domain if and only if $R$ is an AGCD-domain and $char(A) \neq 0$. As a corollary, we have that $D$ is an AGCD-domain if $A$ is an integrally closed AGCD-domain, $char(A) \neq 0$, and $B = A_S$, where $S$ is an almost splitting set of $A$.

Throughout this paper, $D$ is an integral domain with quotient field $K$, $U(D)$ is the group of units of $D$, and $char(D)$ is the characteristic of $D$. An overring of $D$ means a ring between $D$ and $K$. As usual, for $f \in K[X]$, the content $A_f$ of $f$ is the fractional ideal of $D$ generated by the coefficients of $f$. Recall that for a nonzero fractional ideal $I$ of $D$, $I^{-1} = \{x \in K \mid x I \subseteq D\}$, $I_v = (I^{-1})^{-1}$, and $I_l = \cup \{(a_1, \ldots, a_n) \mid (0) \neq (a_1, \ldots, a_n) \subseteq I\}$. We say that $I$ is a divisorial ideal or $v$-ideal (resp., $t$-ideal) if $I = I_v$ (resp., $I = I_t$), while $I_v$ is a finite type $v$-ideal if $I_v = (a_1, \ldots, a_n)$ for some $(0) \neq (a_1, \ldots, a_n) \subseteq I$. Let $t\text{-Max}(D)$ be the set of ideals maximal among proper integral $t$-ideals of $D$. It is well known that (i) $t\text{-Max}(D) \neq \emptyset$ if $D$ is not a field, (ii) every ideal in $t\text{-Max}(D)$ is prime, (iii) $D = \cap_{P \in t\text{-Max}(D)} D_P$, and (iv) every
prime ideal minimal over a $t$-ideal is a $t$-ideal, in particular, every height-one prime ideal is a $t$-ideal.

A nonzero fractional ideal $I$ of $D$ is said to be $t$-invertible if $(II^{-1})_t = D$. It is well known that the set $T(D)$ of $t$-invertible fractional $t$-ideals of $D$ is an abelian group under the $t$-multiplication $I * J = (IJ)_t$. Let $\text{Prin}(D)$ be its subgroup of nonzero principal fractional ideals. We recall that as in [14,15], the $(t)$-class group of $D$ is the quotient group $\text{Cl}(D) = T(D)/\text{Prin}(D)$. If $D$ is a Krull domain, then $\text{Cl}(D)$ is just the divisor class group (see [21]). Many researchers have studied the class group of integral domains; for example, see [3,7–9,13,15,20,22].

Let $S$ be a multiplicative subset of $D$. Then the set $N(S) = \{0 \neq t \in D | (s,t)_v = D \text{ for all } s \in S\}$ is a saturated multiplicative subset of $D$ called the $m$-complement of $S$. It is clear that $S \cap N(S) \subseteq U(D)$ (equality holds if $S$ is saturated) and that $S$ is a splitting set if and only if $S$ is saturated and $SN(S) = D\setminus \{0\}$. The reader is referred to [2,6,10] for the $m$-complement of a multiplicative set, to [2,3,5,10,16,18] for splitting or $t$-splitting sets, and to [8,11,12] for integral domains of the form $A + X^2 B[X]$. Any undefined concepts or notation are standard as in [23,26].

2. Almost splitting sets

We begin this section with the following well-known results. The reader may consult [25] or Zafrullah’s survey article [28] for the $t$-operation.

**Lemma 2.1.** Let $I$ be a nonzero fractional ideal of an integral domain $D$, and let $S$ be a multiplicative subset of $D$.

1. If $I_t$ is of finite type, then $(ID_S)^{-1} = I^{-1}D_S$ and $(ID_S)_v = (I_vD_S)_v$. In particular, if $I$ is $t$-invertible, then $(ID_S)_v = I_vD_S$.
2. $(ID_S)_t = (I_tD_S)_t$ for any $I$.
3. $(ID)_t \cap D$ is a $t$-ideal of $D$.
4. $I$ is $t$-invertible if and only if $I_t$ is of finite type and $I$ is $t$-locally principal.
5. If $I$ is a $t$-ideal, then $I = \cap_{P \in \text{Max}(D)} ID_P$.
6. If $I = I_t \subseteq D$ and $(I, s)_t = D$ for all $s \in S$, then $ID_S \cap D = I$.

**Proof.** For (1) and (2), see [25, Lemma 3.4] or [28, Lemma 1.4]. Conditions (3)–(5) appear in [25, Corollary 2.7, Proposition 2.8(3), and Lemma 3.17]. (6) For $0 \neq x \in ID_S \cap D$, let $A = (I : x) = \{a \in D | ax \in I\}$. Then $I \subseteq A$ and $A \cap S \neq \emptyset$ because $x \in ID_S$, and so $A_t = D$. Note that $A_t = A$ since $I$ is a $t$-ideal [23, Exercise 1, p. 406]. Hence $A = D$, and thus $x \in I$. The reverse inclusion is clear. \qed

Let $S$ be a saturated multiplicative subset of an integral domain $D$. Recall that $S$ is a splitting set if and only if for each $0 \neq d \in D$, $dD_S \cap D$ is principal [3, Theorem 2.2] and that $S$ is a $t$-splitting set if and only if for each $0 \neq d \in D$, $dD_S \cap D$ is $t$-invertible [5, Corollary 2.3]. The following lemma is the almost splitting set analog which appears in [6, Proposition 2.7]. We recall it for easy reference of the reader.
Lemma 2.2. Let $S$ be a saturated multiplicative subset of an integral domain $D$. Then $S$ is an almost splitting set if and only if for each $0 \neq d \in D$, there is an integer $n = n(d) \geq 1$ such that $d^n D_S \cap D$ is principal.

Our first result shows that an almost splitting set is a $t$-complemented $t$-splitting set. Hence “splitting set $\Rightarrow$ almost splitting set $\Rightarrow$ $t$-complemented $t$-splitting set $\Rightarrow$ $t$-splitting set”. However, the converse implications do not hold; for example, see Proposition 2.7 and [5, p. 15].

Proposition 2.3. An almost splitting set is a $t$-complemented $t$-splitting set.

Proof. Let $S$ be an almost splitting set of an integral domain $D$, and let $N(S) = \{0 \neq x \in D | (x, s)_v = D$ for all $s \in S\}$. We first show that $S$ is a $t$-splitting set. By Anderson et al. [5, Corollary 2.3], we need only show that for each $0 \neq d \in D$, $d D_S \cap D$ is $t$-invertible. Let $A = d D_S \cap D$, and let $n = n(d) \geq 1$ be an integer such that $d^n = st$ for some $s \in S$ and $t \in N(S)$. Then $d^n D_S \cap D = st D_S \cap D = t D_S \cap D = t D$ by Lemma 2.1(6). If $d^n D_S \cap D = (A^n)_t$, then $(A^n)_t = t D$. So $(A^n)_t$, and hence $A$, is $t$-invertible. Thus it suffices to show that $d^n D_S \cap D = (A^n)_t$.

Since $A = d D_S \cap D$ and $d \in A$, we have $d D_S = AD_S$, and hence $d^n D_S = A^n D_S = (A^n D_S)_t \supseteq (A^n)_t D_S \supseteq A^n D_S$ by Lemma 2.1(2). So $d^n D_S = (A^n)_t D_S = A^n D_S$ and $(A^n)_t \subseteq d^n D_S \cap D$. For the reverse containment, let $x \in d^n D_S \cap D$ and $I = ((A^n)_t : x) = \{a \in D | xa = (A^n)_t\}$. Then $I$ is a $t$-ideal [23, Exercise 1, p. 406], and since $d^n D_S = (A^n)_t D_S$, we have $I \cap S = \emptyset$. Moreover, since $t \in d^n D_S \cap D \subseteq d D_S \cap D = A$, it follows that $t^n \in A^n \subseteq (A^n)_t \subseteq I$. Let $s \in I \cap S$. Then $D = (s, t^n)_v \subseteq I$, and thus $x \in x D = x(s, t^n)_v \subseteq x I \subseteq (A^n)_t$.

We next show that $S$ is $t$-complemented. Let $T = \{A_1 \cdots A_k | A_i = d_i D_S \cap D$ for some $0 \neq d_i \in D\}$; then $D_T = \cap \{D_P | P \cap S \neq \emptyset$ and $P \in \text{t-Max}(D)\}$ [5, Lemma 4.2 and Theorem 4.3]. We claim that $D_N(S) = D_T$. Clearly, $D_N(S) \subseteq D_T$ since $t D_S \cap D = D$ for all $t \in N(S)$ by Lemma 2.1(6). For the reverse containment, let $x \in D_T$. Then $x A_1 \cdots A_k \subseteq D$ for some $A_1 \cdots A_k \in T$. Since $A_i = d_i D_S \cap D$ for some $0 \neq d_i \in D$, there is an integer $m \geq 1$ such that $((A_1 \cdots A_k)^n)_t = a D$ for some $a \in N(S)$ (see the above paragraph). Hence $xa \in x((A_1 \cdots A_k)^n)_t \subseteq x(A_1 \cdots A_k)_t \subseteq D$, and thus $x \in D_N(S)$. □

Corollary 2.4. Let $D$ be an integral domain with $C_l(D)$ torsion, and let $S$ be a saturated multiplicative subset of $D$. Then $S$ is an almost splitting set if and only if $S$ is a $t$-splitting set.

Proof. Assume that $S$ is a $t$-splitting set, and let $0 \neq d \in D$. Then $d D = (AB)_t$ for some $t$-invertible integral ideals $A$ and $B$ of $D$ such that $A_t \cap S = s A_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. Since $C_l(D)$ is torsion, there is an integer $n \geq 1$ such that $(A^n)_t = a D$ for some $0 \neq a \in D$. Clearly $(a, s)_v = D$ for all $s \in S$; so $a D_S \cap D = a D$ by Lemma 2.1(6). Since $d D = (AB)_t$, it follows that $d^n D_S = ((A B^n)_t D_S)_t = ((A^n)_t D_S)_t$ (Lemma 2.1(2)). So $d^n D_S \supseteq (A^n)_t D_S \supseteq (A^n B^n)_t D_S \cap D = d^n D_S$, or $d^n D_S = (A^n)_t D_S$. Hence $d^n D_S \cap D = (A^n)_t D_S \cap D = a D$. Thus $S$ is an almost splitting set by Lemma 2.2. The converse always holds by Proposition 2.3. □
Remark 2.5. Let $D$ be an integral domain, $X$ an indeterminate over $D$, and $\emptyset \neq S \subseteq \{ f \in D[X] | (A_f)_v = D \}$ a saturated multiplicative subset of $D[X]$. In [16, Proposition 3.7], we showed that $S$ is a $t$-complemented $t$-splitting set. Note that $\text{Cl}(D) = \text{Cl}(D[X])$ if and only if $D$ is integrally closed [22, Theorem 3.6]. Thus if $D$ is an integrally closed domain with $\text{Cl}(D)$ torsion, then $S$ is an almost splitting set by Corollary 2.4.

Recall that an integral domain $D$ is a GCD-domain (resp., UMT-domain) if and only if $D \setminus \{0\}$ is a splitting set (resp., $t$-splitting set) in $D[X]$ [3, Example 4.7] (resp. [16, Corollary 2.9]). An integral domain $D$ is called a UMT-domain if every upper to zero in $D[X]$ is a maximal $t$-ideal. It is well known that if $D$ is an integrally closed UMT-domain if and only if $D$ is a PVMD [24, Proposition 3.2]. We next give the almost splitting set analog.

Proposition 2.6. Let $D$ be an integrally closed domain and $X$ an indeterminate over $D$. Then $D \setminus \{0\}$ is an almost splitting set in $D[X]$ if and only if $D$ is an AGCD-domain.

Proof. Recall that an integrally closed domain $D$ is an AGCD-domain if and only if $D$ is a PVMD with $\text{Cl}(D)$ torsion [27, Corollary 3.8 and Theorem 3.9].

$(\Rightarrow)$ Suppose that $D \setminus \{0\}$ is an almost splitting set in $D[X]$, and let $0 \neq f \in D[X]$. Then there is an integer $n = n(f) \geq 1$ such that $f^n = ag$ for some $0 \neq a \in D$ and $g \in D[X]$ with $(d, g)_v = D[X]$ for all $0 \neq d \in D$. Clearly, $(A_g)_v = D$; hence $(A_{f^n})_v = (A_f)_v = (A_{ag})_v = aD$ as $D$ is integrally closed [23, Proposition 34.8]. Thus $A_f$ is $t$-invertible, which implies that $D$ is a PVMD. Moreover, since $(A_{f^n})_v$ is principal, we can conclude that $\text{Cl}(D)$ is torsion.

$(\Leftarrow)$ Assume that $D$ is an AGCD-domain, and let $0 \neq f \in D[X]$. Then there is an integer $n = n(f) \geq 1$ such that $(A_{f^n})_v = aD$ for some $a \in D$; so $(A_{f^n})_v = aD$ [23, Proposition 34.8] because $D$ is integrally closed. Let $g = f^n/a$. Then $f^n = ag$ and $g \in D[X]$ with $(A_g)_v = D$; so $(d, g)_v = D[X]$ for all $0 \neq d \in D$ [24, Proposition 1.1]. Thus $D \setminus \{0\}$ is an almost splitting set. □

We next give an example of a $t$-complemented $t$-splitting set which is not an almost splitting set.

Proposition 2.7. Let $D$ be an integral domain, $X$ an indeterminate over $D$, and $S = \{ uX^n | u \in U(D) \}$ and $n = 0, 2, 3, \ldots \). Then:

1. $S$ is a saturated multiplicative subset of $D[X^2, X^3]$.
2. $S$ is a $t$-complemented $t$-splitting set of $D[X^2, X^3]$ and the $t$-complement of $S$ is $D[X^2, X^3] \setminus X^2D[X]$.
3. $S$ is an almost splitting set of $D[X^2, X^3]$ if and only if $\text{char} \ (D) \neq 0$.
4. $S$ is not a splitting set of $D[X^2, X^3]$.

Proof. Recall that $X^2D[X]$ is a height-one maximal $t$-ideal of $D[X^2, X^3]$ and if $Q$ is a maximal $t$-ideal of $D[X^2, X^3]$, then either $Q = X^2D[X]$ or $Q \cap S = \emptyset$ [8, Lemma 1]. Also, note that $D[X^2, X^3]_S = D[X, X^{-1}] = D[X]_S$.

(1) This is clear.
(2) We first show that $S$ is a $t$-splitting set. To do this, it suffices to show that for each $0 \neq f \in D[X^2, X^3]$, $fD[X] \cap D[X^2, X^3]$ is $t$-invertible [5, Corollary 2.3]. Let $I = fD[X] \cap D[X^2, X^3]$. Then $ID[X] \cap D[X^2, X^3]$ (note that $(a + Xg)(a - Xg) = a^2 - X^2g^2 \in D[X^2, X^3]$ for all $a \in D$ and $g \in D[X]$), and $I$ is a $t$-ideal of $D[X^2, X^3]$ by Lemma 2.1(3).

Let $Q$ be a maximal $t$-ideal of $D[X^2, X^3]$. If $Q = X^2D[X]$, then $ID[X^2, X^3]Q = D[X^2, X^3]Q$. Assume that $Q \neq X^2D[X]$. Then $Q \cap S \neq \emptyset$, and so $ID[X^2, X^3]Q = (ID[X^2, X^3]S)Q = fD[X^2, X^3]Q$. Thus $I$ is $t$-locally principal. Hence if $I$ is of finite type, then $I$ is $t$-invertible by Lemma 2.1(4). Let $g \in I \setminus X^2D[X]$. Then $fD[X^2, X^3]g \in (g, X^2f)_vD[X^2, X^3] \subseteq I = fD[X^2, X^3]$; so $(g, X^2f)_vD[X^2, X^3] = I = fD[X^2, X^3]$. Hence $I = (g, X^2f)_v$ for all maximal $t$-ideals $Q$ of $D[X^2, X^3]$, and thus $I = (g, X^2f)_v$ by Lemma 2.1(5).

We next show that $S$ is $t$-complemented. Let $Q$ be a maximal $t$-ideal of $D[X^2, X^3]$ such that $Q \cap S \neq \emptyset$. Then $Q = X^2D[X]$, and hence $\cap (DQ | Q \cap S \neq \emptyset$ and $Q \in t$-Max($D$)) = $D[X^2, X^3]Q$. Thus $S$ is $t$-complemented with $t$-complement $D[X^2, X^3]Q$.

(3) ($\Rightarrow$) Assume that $S$ is an almost splitting set, and let $f = X^2(1 + X)$. Then $f \in D[X^2, X^3]$, and since $S$ is an almost splitting set, there is an integer $n = n(f) \geq 1$ such that $f^nD[X] \subseteq D[X^2, X^3]$ for some $0 \neq g \in D[X^2, X^3]$ by Lemma 2.2. It is clear that $g(0) \neq 0$, $f^nD[X] = gD[X]$, and $f^n \in gD[X^2, X^3]$. So $f^n = uX^ng$ for some $u \in U(D)$ and integer $m \geq 0$, and hence $(1 + X)^n = ug$ because $g(0) \neq 0$. Note that $g \in D[X^2, X^3]$ and $(1 + X)^n = 1 + nX + [n(n + 1)/2]X^2 + \cdots + X^n$; so $nX = 0$. Thus $char(D) \neq 0$.

($\Leftarrow$) Assume that $char(D) = p \neq 0$, and let $0 \neq f = X^n g \in D[X^2, X^3]$, where $n \geq 0$ is an integer and $g \in D[X]$ with $g(0) \neq 0$. Then $g^p \in D[X^2, X^3]$ and $f^pD[X] = g^pD[X]$. If $h \in D[X]$ such that $g^p h \in D[X^2, X^3]$, then $h \in D[X^2, X^3]$ because $g^p(0) \neq 0$ and $g^p \in D[X^2, X^3]$. So $f^pD[X] \cap D[X^2, X^3] = g^pD[X] \cap D[X^2, X^3] = g^pD[X^2, X^3]$. Thus by Lemma 2.2, $S$ is an almost splitting set.

(4) Let $f = X^2(1 + X) \in D[X^2, X^3]$. Then $fD[X] \cap D[X^2, X^3]$ is not principal, and thus $S$ is not a splitting set [3, Theorem 2.2].

**Corollary 2.8** (cf. Anderson et al. [11, Theorem 2.5]). Let $D$ be an integral domain, $X$ an indeterminate over $D$, and $S = \{ux^n | u \in U(D)$ and $n = 0, 2, 3, \ldots \}$. Let $I$ be a nonzero integral ideal of $D[X]$ such that $ID[X] \subseteq D[X] = I$. Then $I$ is a $t$-ideal of $D[X]$ if and only if $I \cap D[X^2, X^3]$ is a $t$-ideal of $D[X^2, X^3]$.

**Proof.** Let $T = \{ux^n | u \in U(D)$ and $n = 0, 1, 2, \ldots \}$, and note that $D[X]T = D[X^2, X^3]$. ($\Rightarrow$) Assume that $I$ is a $t$-ideal of $D[X]$. Then $ID[X]T = ID[X] \subseteq D[X^2, X^3] [3, Corollary 3.5]$ since $T$ is a splitting set in $D[X]$ [3, Example 4.5], and hence $I \cap D[X^2, X^3] = (ID[X] \cap D[X^2, X^3])ID[X^2, X^3] = ID[X] \cap D[X^2, X^3]$ is a $t$-ideal of $D[X^2, X^3]$ by Lemma 2.1(3). ($\Leftarrow$) Assume that $I \cap D[X^2, X^3]$ is a $t$-ideal of $D[X^2, X^3]$, and let $J = I \cap D[X^2, X^3]$. Then $JD[X] = ID[X] \cap D[X^2, X^3] = I = JD[X] \cap D[X]$ since $S$ is a $t$-splitting set in $D[X^2, X^3]$ (Proposition 2.7(2)). Thus $I = ID[X] \cap D[X] = JD[X] \subseteq D[X]$ is a $t$-ideal of $D[X]$ (Lemma 2.1(3)).
It is well known that if $S$ is a splitting set of an integral domain $D$, then $\text{Cl}(D) = \text{Cl}(D_S) \oplus \text{Cl}(D_{N(S)})$ [3, Corollary 3.8]. This result cannot be generalized to a t-splittable set [5, Remark 4.13]. We next give an example which shows that [3, Corollary 3.8] cannot be extended to an almost splitting set.

**Example 2.9.** Let $D$ be an integral domain with quotient field $K$, $X$ an indeterminate over $D$, $S = \{uX^n | u \in U(D) \text{ and } n = 0, 2, 3, \ldots \}$, and $N(S) = \{ f \in D[X^2, X^3] | (f, uX^n)_v = D[X^2, X^3] \text{ for all } uX^n \in S \}$. Then $S$ is a t-splittable set iff there is an integer $n \geq 1$ such that $(S_1)_v = (S_1)_v \cap (N_1)_v$, for some $\emptyset \neq S_1 \subseteq S$ and $\emptyset \neq N_1 \subseteq N(S)$.

**Theorem 2.10.** Let $D$ be an integral domain, $S$ an almost splitting set of $D$, and $N(S) = \{ 0 \neq t \in D | (s, t)_v = D \text{ for all } s \in S \}$. Then $S$ is a t-splittable set iff $N(S)$ is torsion, and let $S$ be a t-splittable set.

(1) If $I$ is a t-invertible integral t-ideal of $D$, then there is an integer $n \geq 1$ such that $(I^n)_v = ((S_1)_v) = (S_1)_v \cap (N_1)_v$, for some $\emptyset \neq S_1 \subseteq S$ and $\emptyset \neq N_1 \subseteq N(S)$.

(2) $\text{Cl}(D)$ is torsion if and only if $\text{Cl}(D_S)$ and $\text{Cl}(D_{N(S)})$ are torsion.

**Proof.** (1) Let $I = (a_1, \ldots, a_k)_v$. Then there is an integer $n \geq 1$ such that $a^n_i = s_it_i$ for some $s_i \in S$ and $t_i \in N(S)$. Since $I$ is t-invertible, $(I^n)_v = (a^n_1, \ldots, a^n_k)_v = (s_it_1, \ldots, s_k t_k)_v$ [1, Lemma 3.3]. Let $Q$ be a maximal t-ideal of $D$. Then since $Q \cap S = \emptyset$ or $Q \cap N(S) = \emptyset$, we have $(s_1t_1, \ldots, s_k t_k)_Q = ((s_1, \ldots, s_k)(t_1, \ldots, t_k))_Q$. So by Lemma 2.1(1),

$((I^n)_v)_Q = (((s_1t_1, \ldots, s_k t_k)_v)_Q = (((s_1, \ldots, s_k)(t_1, \ldots, t_k))_Q \supseteq (((s_1, \ldots, s_k)(t_1, \ldots, t_k))_Q \supseteq (((s_1, \ldots, s_k t_k)_v)_Q = ((I^n)_v)_Q$.

Hence $(s_1t_1, \ldots, s_k t_k)_v = ((s_1, \ldots, s_k)(t_1, \ldots, t_k))_v$ for all maximal t-ideals $Q$ of $D$. Thus $(I^n)_v = ((s_1, \ldots, s_k)(t_1, \ldots, t_k))_v = (s_1, \ldots, s_k)_v \cap (t_1, \ldots, t_k)_v$ by Lemma 2.1(5) and the fact that $(s_1, \ldots, s_k) + (t_1, \ldots, t_k) = D$.

(2) $\Rightarrow$ Recall that almost splitting sets are t-splittable sets (Proposition 2.3); hence the map $\phi : \text{Cl}(D) \to \text{Cl}(D_S) \oplus \text{Cl}(D_{N(S)})$, given by $[I] \to ([ID_S], [ID_{N(S)})]$, is surjective [5, Remark 4.13]. Thus if $\text{Cl}(D)$ is torsion, then $\text{Cl}(D_S) \oplus \text{Cl}(D_{N(S)})$, and hence both $\text{Cl}(D_S)$ and $\text{Cl}(D_{N(S)})$, are torsion.

$\Leftarrow$ Assume that $\text{Cl}(D_S)$ and $\text{Cl}(D_{N(S)})$ are both torsion, and let $I$ be a t-invertible integral t-ideal of $D$. Then $ID_S$ and $ID_{N(S)}$ are t-invertible, and thus there exists an integer $n \geq 1$ such that $(ID_S)_v = ((I)^n)_v$, $ID = aD_S$ and $(ID_{N(S)})_v = ((I)^n)_v$, $D_{N(S)} = bD_S$ for some $a, b \in D$ (see Lemma 2.1(1) for the equalities). Since $(I^n)_v$ is a t-invertible t-ideal
and $S$ is an almost splitting set, by (1) we can choose another integer $m \geq 1$ such that $(I^{nm})_{t} = (((I^{n})_{t})^{m})_{t} = ((S_{1})_{t}(N_{1}))_{t}$, $a^{m} = s't$, and $b^{m} = st'$ for some $\emptyset \neq S_{1} \subseteq S$, $\emptyset \neq N_{1} \subseteq N(S)$, $s,s' \in S$, and $t, t' \in N(S)$. Also, since $I$ is $t$-invertible, by Lemma 2.1(1)

\[(N_{1})_{t}D_{S} = (I^{nm})_{t}D_{S} = ((I^{n})_{t}D_{S})^{m}_{t} = a^{m}D_{S} = tD_{S}
\]

and $(S_{1})_{t}D_{N(S)} = (I^{nm})_{t}D_{N(S)} = ((I^{n})_{t}D_{N(S)})^{m}_{t} = b^{m}D_{N(S)} = sD_{N(S)}$. Therefore, $(I^{nm})_{t} = ((S_{1})_{t}(N_{1}))_{t} = (S_{1})_{t} \cap (N_{1})_{t} = ((S_{1})_{t}D_{N(S)} \cap D) \cap ((N_{1})_{t}D_{S} \cap D) = (sD_{N(S)} \cap D) \cap (tD_{S} \cap D) = sD \cap tD = stD$ by Lemma 2.1(6). This means that $Cl(D)$ is torsion.

Let $S$ be a $t$-complemented $t$-splitting set of an integral domain $D$. As we noted in the proof of $(\Rightarrow)$ of Theorem 2.10(2), if $Cl(D)$ is torsion, then $Cl(D_{S})$ and $Cl(D_{N(S)})$ are both torsion (or see [5, Remark 4.13]). Our next example shows that the converse does not hold.

**Example 2.11.** Let the notation be as in Example 2.9. Assume that $D$ is an integrally closed domain with $Cl(D)$ torsion. Then $Cl(D[X^{2}, X^{3}_{N(S)}]) = 0$ and $Cl(D[X^{2}, X^{3}_{S}]) = Cl(D[X]) = Cl(D)$ is torsion [22, Theorem 3.6]. But since $Cl(D[X^{2}, X^{3}]) = Cl(D) \oplus K$ [8, Corollary 7], $Cl(D[X^{2}, X^{3}])$ is not torsion if and only if $char(K) = 0$, if and only if $S$ is not an almost splitting set (cf. Proposition 2.7(3) and Theorem 2.10(2)). For example, if $D = \mathbb{Z}$ is the ring of integers, then $Cl(\mathbb{Z}[X^{2}, X^{3}_{N(S)}]) = Cl(\mathbb{Z}[X^{2}, X^{3}_{S}]) = 0$ but $Cl(\mathbb{Z}[X^{2}, X^{3}]) = \mathbb{Q}$ is torsion-free, where $\mathbb{Q}$ is the additive group of rational numbers.

Let $D$ be an integral domain and $X^{1}(D)$ the set of height-one prime ideals of $D$. Then $D$ is called a weakly Krull domain if $D = \cap_{p \in X^{1}(D)}D_{p}$ and the intersection has finite character. Recall that $D$ is an almost weakly factorial domain (AWFD) if for each nonzero nonunit $x \in D$, some positive power of $x$ is a product primary elements. It is known that $D$ is an AWFD if and only if $D$ is a weakly Krull domain and $Cl(D)$ is torsion [4, Theorem 3.4]. For more on weakly Krull domains and AWFD’s, see [4,12].

**Corollary 2.12.** Let $S$ be an almost splitting set of an integral domain $D$ and $N(S) = \{0 \neq x \in D | (x,s)_{v} = D$ for all $s \in S\}$.

(1) $D$ is an AGCD-domain if and only if $D_{S}$ and $D_{N(S)}$ are AGCD-domains.

(2) $D$ is weakly Krull if and only if $D_{S}$ and $D_{N(S)}$ are weakly Krull.

(3) $D$ is an AWFD if and only if $D_{S}$ and $D_{N(S)}$ are AWFDs.

**Proof.** (1) Assume that both $D_{S}$ and $D_{N(S)}$ are AGCD-domains, and let $0 \neq a, b \in D$. Then as $S$ is an almost splitting set, there is an integer $n \geq 1$ such that $a^{n} = s_{1}t_{1}$ and $b^{n} = s_{2}t_{2}$ for some $s_{1} \in S$ and $t_{1} \in N(S)$. By assumption and [27, Lemma 3.6], there is another integer $m \geq 1$ such that $s_{1}^{m}D_{N(S)} \cap s_{2}^{m}D_{N(S)} = sD_{N(S)}$ and $t_{1}^{m}D_{S} \cap t_{2}^{m}D_{S} = tD_{S}$ for some $s, t \in D$. Recall that for any $0 \neq x, y, d \in D$, if $xD_{N(S)} \cap yD_{N(S)} = dD_{N(S)}$, then $x^{k}D_{N(S)} \cap y^{k}D_{N(S)} = d^{k}D_{N(S)}$ for all integers $k \geq 1$ [27, Lemma 3.6]. Thus as $S$ and $N(S)$ are almost splitting sets, we may assume that $s \in S$ and $t \in N(S)$. So $s_{1}^{m}D \cap s_{2}^{m}D = (s_{1}^{m}D_{N(S)} \cap D) \cap (s_{2}^{m}D_{N(S)} \cap D) = sD_{N(S)} \cap D = sD$ by Lemma 2.1(6). Similarly, $t_{1}^{m}D \cap t_{2}^{m}D = tD$. Thus $a^{nm}D \cap b^{nm}D = s_{1}^{nm}D \cap s_{2}^{nm}D = s_{1}^{m}D \cap s_{2}^{m}D \cap t_{1}^{m}D \cap t_{2}^{m}D = sD \cap tD = stD$.

The converse always holds for any multiplicative subset of $D$. The proof is analogous.
(2) It is well known that if $D$ is weakly Krull, then $D_N$ is also weakly Krull for any multiplicative subset $N$ of $D$. The converse follows directly from the fact that $D = D_S \cap D_{N(S)}$ [2, Proposition 1.1].

(3) This is an immediate consequence of (2) and Theorem 2.10(2) since $D$ is an AWFD if and only if $D$ is weakly Krull and $CI(D)$ is torsion [4, Theorem 3.4].

3. AGCD-domains of the form $A + X^2B[X]$

Let $B$ be an overring of an integral domain $A$, $X$ an indeterminate over $A$, and $R = A + XB[X]$. In [19, Theorem 3.1], the authors showed that $R$ is an integrally closed AGCD-domain if and only if $A$ is an integrally closed AGCD-domain and $B = A_S$, where $S$ is an almost splitting set in $A$. They also gave some examples of non-integrally closed AGCD-domains. For example, if $A$ is an integrally closed AGCD-domain of char $(A) = p \neq 0$ such that $A \neq A^p$, then $A[X^p, X^{p+1}, \ldots, X^{2p+1}]$ and $A^p + XA[X]$ are non-integrally closed AGCD-domains. The purpose of this section is to prove that the domain $A + X^2B[X]$ is an AGCD-domain if and only if $A + XB[X]$ is an AGCD-domain and char$(A) \neq 0$. Using this result and [19, Theorem 3.1], we can construct simple examples of non-integrally closed AGCD-domains.

Let $A \subseteq B$ be an extension of integral domains. Following [17], we say that $B$ is $t$-linked over $A$ if $I^{-1} = A$ for a nonzero finitely generated ideal $I$ of $A$ implies $(IB)^{-1} = B$; equivalently, if $P$ is a maximal $t$-ideal of $B$, then $(P \cap A)^t \subseteq A$. Recall that $A$ is of finite $t$-character if each nonzero nonunit element of $A$ belongs to only finitely many maximal $t$-ideals of $A$. Examples of integral domains of finite $t$-character include Krull domains, Mori domains, Noetherian domains, and one-dimensional semi-quasilocal domains.

Let $A \subseteq B$ be an extension of integral domains, $X$ an indeterminate over $A$, $R = A + XB[X]$, and $D = A + X^2B[X]$. In [12, Lemma 4.1], the authors proved that the map $Spec(R) \rightarrow Spec(D)$, given by $Q \mapsto Q \cap D$, is an order-preserving bijection. In particular, if $A = B$, then the bijection preserves $t$-ideals, i.e., $Q$ is a prime $t$-ideal of $R$ if and only if $Q \cap D$ is a prime $t$-ideal of $D$ [11, Theorem 2.5] (or see Corollary 2.8). We next show that this holds for maximal $t$-ideals when $B$ is an overring of $A$.

Lemma 3.1. Let $B$ be an overring of an integral domain $A$, $X$ an indeterminate over $A$, $R = A + XB[X]$, $D = A + X^2B[X]$, and $Q$ a nonzero prime ideal of $R$. Then $Q$ is a maximal $t$-ideal of $R$ if and only if $Q \cap D$ is a maximal $t$-ideal of $D$. In particular, $R$ is $t$-linked over $D$, and $R$ is of finite $t$-character if and only if $D$ is of finite $t$-character.

Proof. Recall that the map $Spec(R) \rightarrow Spec(D)$, given by $Q \mapsto Q \cap D$, is an order-preserving bijection [12, Lemma 4.1]. So we need only show that if $Q$ is a maximal $t$-ideal of $R$, then $Q \cap D$ is a $t$-ideal of $D$ and that if $Q \cap D$ is a maximal $t$-ideal of $D$, then $Q$ is a $t$-ideal of $R$. (This means that $Q$ is a maximal $t$-ideal of $R$ if and only if $Q \cap D$ is a maximal $t$-ideal of $D$.)

Let $K$ be the quotient field of $A$, $Q$ a nonzero prime ideal of $R$, $P = Q \cap D$, and $S = \{X^n | n = 0, 2, 3, \ldots\}$. Note that $Q \cap A = P \cap A; Q \cap S = 0 \Leftrightarrow P \cap S = 0; R_S = B[X, X^{-1}] = B[X]_S = D_S; \text{ and } PB[X]_S = QB[X]_S$. 


Case 1: $Q \cap A = (0) \iff P \cap A = (0)$. Note that $R_{A\setminus\{0\}} = K[X]$, $D_{A\setminus\{0\}} = K[X^2, X^3]$, and $\dim(K[X]) = \dim(K[X^2, X^3]) = 1$; so $ht Q = ht P = 1$. Thus $Q$ and $P$ are prime $t$-ideals of $R$ and $D$, respectively.

Case 2: $Q \cap A \neq (0)$ and $Q \cap S = \emptyset \iff P \cap A \neq (0)$ and $P \cap S = \emptyset$.

Assume that $(PB[X]_S)_I = (QB[X]_S)_I = B[X]_S$. Then there is a finitely generated subideal $I$ of $Q$ such that $(IB[X]_S)_v = B[X]_S$. Note that for any $0 \neq a \in Q \cap A$, $(I, a) \subseteq Q$, $(I, a)B[X]_S)_v = B[X]_S$, and $(I, a)$ is finitely generated. Replacing $I$ with $(I, a)$, we may assume that $I \cap A \neq (0)$. So $(R : I) \subseteq K[X]$. Since $R \subseteq B[X]_S$, it follows that $(R : I) \subseteq (B[X]_S : I) = B[X]_S$, and thus $(R : I) \subseteq B[X]_S \cap K[X] = B[X]$. Hence $XB[X] \subseteq (R : B[X]) \subseteq (R : (R : I)) = I_v$, and thus $X \subseteq I_v \subseteq Q_I$. Therefore, if $Q$ is a $t$-ideal, then $(PB[X]_S)_I = (QB[X]_S)_I \subseteq B[X]_S$. Similarly, if $P$ is a prime $t$-ideal, then $(QB[X]_S)_I = (PB[X]_S)_I \subseteq B[X]_S$.

Assume that $Q$ or $P$ is a maximal $t$-ideal. Then $(PB[X]_S)_I = (QB[X]_S)_I \subseteq B[X]_S$ by the above paragraph, and hence $PB[X]_S = (PB[X]_S)_I = (QB[X]_S)_I = Q[B[X]_S]$ (cf. Lemma 2.1(3)). Thus $Q = Q[B[X]_S] \cap R$ and $P = P[B[X]_S] \cap D$ are $t$-ideals by Lemma 2.1(3).

Case 3: $Q \cap A \neq (0)$ and $Q \cap S \neq \emptyset \iff P \cap A \neq (0)$ and $P \cap S \neq \emptyset$. It is clear that $Q = (Q \cap A) + XB[X]$ and $P = (Q \cap A) + X^2B[X]$. We first show that (1) if $I$ is a nonzero (integral) ideal of $R$ such that $I \cap A \neq (0)$ and $X^2 \in I_1$, then $I_v = (I_{-1}^{-1} \cap B)^{-1} \cap B + X^2B[X]$, where $I_0 = \{ f(0) | f \in I_1 \}$. Let $\omega \in I_1^{-1} \cap B$. Then $\omega \in B[X]$ because $I \cap A \neq (0)$ and $X^2 \in I$. Note that since $f \omega = f + 0$ for any $f \in I$, we have $f(0)\omega(0) = A$, and so $\omega(0) \in I_{-1}^{-1} \cap B$. Hence $I_{-1} = (I_{-1}^{-1} \cap B) + X^2B[X]$ since $X^2B[X] \subseteq I_{-1}^{-1}$. The same argument also shows that $I_v = (I_{-1}^{-1} \cap B)^{-1} \cap B + X^2B[X]$. Similarly, we can show that (2) if $J$ is a nonzero (integral) ideal of $D$ such that $J \cap A \neq (0)$ and $X^2 \in J$, then $J_v = (J_{-1}^{-1} \cap B)^{-1} \cap B + X^2B[X]$, where $J_0 = \{ g(0) | g \in J \}$.

Assume that $Q$ is a $t$-ideal of $R$, and let $J$ be a finitely generated subideal of $P$. Note that $X^2 \in P$ and $J \subseteq (J, a, X^2) \subseteq P$ for any $0 \neq a \in A \cap Q$. So replacing $J$ with $(J, a, X^2)$, we may assume that $J \cap A \neq (0)$ and $X^2 \in J$. By (1) and (2), $(J_R)_v = (J_{-1}^{-1} \cap B)^{-1} \cap B + X^2B[X]$. Since $Q$ is a $t$-ideal and $JR$ is a finitely generated subideal of $Q$, $(J_{-1}^{-1} \cap B)^{-1} \cap B \subseteq Q \cap A$, and thus $J_v \subseteq (Q \cap A) + X^2B[X] = P$, which implies that $P$ is a $t$-ideal.

We next assume that $P$ is a $t$-ideal, and let $I$ be a finitely generated subideal of $Q$. As in the above paragraph, we may assume that $I \cap A \neq (0)$ and $X^2 \in I$. Let $I_0 = \{ f(0) | f \in I \}$. Then $I_0 \neq (0)$ and $I_0$ is a finitely generated subideal of $Q \cap A$ because $I$ is finitely generated and $XB[X] \subseteq Q$. Note that $(I_0, X^2)D$ is a finitely generated subideal of $P$ such that $(g(0) | g \in (I_0, X^2)D) = I_0$, $(I_0, X^2)D \cap A \neq (0)$, and $X^2 \in (I_0, X^2)D$. So by (1) and (2), $(I_0, X^2)D_v = (I_{-1}^{-1} \cap B)^{-1} \cap B + X^2B[X]$ and $I_v = (I_{-1}^{-1} \cap B)^{-1} \cap B + X^2B[X]$. Since $P$ is a $t$-ideal, $(I_{-1}^{-1} \cap B)^{-1} \cap B \subseteq Q \cap A$, and thus $I_v \subseteq (Q \cap A) + XB[X] = Q$. This shows that $Q$ is a $t$-ideal.

Let $A \subseteq B$ be an extension of integral domains. Then $B$ is said to be a root extension of $A$ if for each $x \in B$, $x^n \in A$ for some integer $n \geq 1$. 
Lemma 3.2. Let $A \subseteq B$ be an extension of integral domains, $X$ an indeterminate over $B$, $R = A + XB[X]$, and $D = A + X^2B[X]$. Then $R$ is a root extension of $D$ if and only if $\text{char} (A) \neq 0$.

Proof. Assume that $R$ is a root extension of $D$. Then $(1 + X)^n \in D$ for some integer $n \geq 1$. Hence $nX = 0$, and thus $\text{char} (A) \neq 0$. Conversely, if $\text{char} (A) = p \neq 0$, then for any $f \in R$, $f^p \in D$. Thus $R$ is a root extension of $D$.

We next give the main result of this section. This result combined with [19, Theorem 3.1(a)] gives many examples of non-integrally closed AGCD-domains (see Corollary 3.5).

Theorem 3.3. Let $B$ be an overring of an integral domain $A$, $X$ an indeterminate over $A$, $R = A + XB[X]$, and $D = A + X^2B[X]$. Then $R$ is an AGCD-domain and $\text{char} (A) \neq 0$ if and only if $D$ is an AGCD-domain.

Proof. ($\Rightarrow$) Assume that $R$ is an AGCD-domain and $\text{char} (A) = p \neq 0$. We first note that (#) if $f \in D$ with $f(0) \neq 0$, then $fR \cap D = fD$. For if $g = a_0 + a_1X + \cdots + a_nX^n \in R$ such that $fg \in D$, then $f(0)a_1 = 0$ since $fg = f(0)a_0 + f(0)a_1X + X^2g_1$ for some $g_1 \in B[X]$; so $a_1 = 0$. Hence $g \in D$, and thus $fR \cap D = fD$.

Let $0 \neq f, g \in D$.

Case 1: $f(0) \neq 0$ and $g(0) \neq 0$. Since $R$ is an AGCD-domain, there is an integer $n = n(f,g) \geq 1$ such that $f^nR \cap g^nR = hR$ for some $h \in R$. Note that $f^n(0) \neq 0$, $g^n(0) \neq 0$, and $\text{char} (A) = p$; hence $h(0) \neq 0$ and $h \in D$. Thus $f^{np}D \cap g^{np}D = (f^{np}R \cap D) \cap (g^{np}R \cap D) = (f^{np}R \cap g^{np}R) \cap D = h^pR \cap D = h^pD$ by (#) and [27, Lemma 3.6].

Case 2: $f(0) \neq 0$ and $g = X^mg_1$, where $m \geq 2$ and $g_1 \in B[X]$ with $g_1(0) \neq 0$. Let $0 \neq s \in A$ such that $sg_1 \in R$ (note that $B$ is an overring of $A$). Replacing $f, g, g_1$, and $m$ with $(sf)^p$, $(sg)^p = X^{mp}(sg_1)^p$, and $mp$, respectively, we may assume that $g_1 \in D$. Thus by the proof of Case 1, $f^nR \cap g^n_1R = hR$ and $f^nD \cap g^n_1D = hD$ for some integer $n \geq 1$ and $h \in D$ with $h(0) \neq 0$.

Note that $R$ is an AGCD-domain and that for any integer $k \geq 1$, if $f^kR \cap g^R$ is principal, then $f^{nk}R \cap g^{nk}R$ is also principal [27, Lemma 3.6]. So we may assume that $f^nR \cap g^nR = bR$ for some $b \in R$. Since $X^{am}h \in f^nR \cap g^nR$, we have $X^{am}h = bc$ for some $c \in R$. Also, since $b \in f^nR \cap g^n_1R = hR$, we have $b = hd$ for some $d \in D$. Hence $X^{am}h = hdc$, and so $X^{am} = dc$. Finally, since $b \in g^nR$, we have $b = g^n = X^{am}g^n_1r$ for some $r \in R$, and so $h = g^n_1rc$. Hence $c \in U(A)$ as $h(0) \neq 0$, and thus $f^nR \cap g^nR = X^{am}hR$.

Let $g^n_1h_1 \in f^nD \cap g^nD$, where $h_1 \in D$. Then $g^n_1h_1 = X^{am}h_2x$ for some $x \in R$ by the above paragraph; so $(g^n_1)\alpha = h_2x$. Thus by (#), $x \in D$ because $g_1, h_1 \in D$ and $h(0) \neq 0$, and hence $f^nD \cap g^nD \subseteq X^{am}hD$. The reverse containment follows directly from the fact that $f^nD \cap g^nD = hD$ and $g = X^{m}g_1$. Therefore, $f^nD \cap g^nD = X^{am}hD$.

Case 3: $f = X^k f_1$ and $g = X^m g_1$, where $m \geq k \geq 2$, $f_1 \in B[X]$ with $f_1(0) \neq 0$, and $g_1 \in B[X] \cap 0 \neq 0$. As in the proof of Case 2, we may assume that $f_1, g_1 \in D$. If $k = m$, then there exists an integer $n \geq 1$ such that $f^n_1D \cap g^n_1D$ is principal by Case 1. Thus $f^n_1D \cap g^nD = (X^k f_1)^nD \cap (X^k g_1)^nD = X^{nk}(f^n_1D \cap g^n_1D)$ is principal. If $m > k$, then replacing $f$ and $g$ with $f^2$ and $g^2$, we may assume that $m - k \geq 2$; so $X^{m-k}g_1 \in D$. 


Thus by Case 2, \( f_1^n D \cap (X^{m-k}g_1)^n D = h D \) for some integer \( n \geq 1 \) and \( h \in D \). Hence 
\[
f^n D \cap \cap g^m D = (X^f f_j)^n D \cap (X^m g_1)^n D = X^{nk}(f_1^n D \cap (X^{m-k}g_1)^n D) = X^{nk}h D.
\]

(\( \Leftarrow \)) Assume that \( D \) is an AGCD-domain. Note that if \( \text{char}(A) \neq 0 \), then \( R \) is a \( t \)-linked root extension of \( D \) by Lemmas 3.1 and 3.2, and thus \( R \) is an AGCD-domain [19, Remark 4.1(b)]. So it suffices to show that \( \text{char}(A) \neq 0 \).

Let \( f = X^2(1 + X) \) and \( I = (f, 1 - X^2)_v \subseteq D \). We first prove that \( I \) is \( t \)-locally principal, and thus \( t \)-invertible by Lemma 2.1(4). Let \( Q \) be a maximal \( t \)-ideal of \( D \) and \( S = \{X^n | n = 0, 2, 3, \ldots \} \). If \( Q \cap S \neq \emptyset \), then \( I \not\subseteq Q \), and so \( ID_Q = D_Q \). Next assume that \( Q \cap S = \emptyset \). Note that \( f D_S \subseteq (f, 1 - X^2)_S \cap (f, 1 - X^2)_D = (f, 1 - X^2)_D = (f)_D \subseteq (f)_D \) by Lemma 2.1(1); so \( ID_Q = (ID_S)_Q = ((f)_D)_Q = f D_Q \).

Recall that an AGCD-domain has a torsion class group [1, Theorem 3.4]. So \( (P^n)_v = ((f, 1 - X^2)_v)^n = (f, 1 - X^2)_v = h D \) for some \( h \in D \) and integer \( n \geq 1 \). Thus \( h B[X, X^{-1}] = h D_S = (f, 1 - X^2)_v D_S = ((f, 1 - X^2)_v)_v = ((1 + X)_v D_S = (1 + X^m B[X, X^{-1}] \) (the third equality follows from Lemma 2.1(1) because \( (f, 1 - X^2)_v \) is \( t \)-invertible), and so \( h = u X^n (1 + X)^m \) for some \( u \in U(B) \) and integer \( m \). But since \( (1 - X^2)_v \in h D \), we have \( h(0) \neq 0 \), and so \( m = 0 \). Hence \( h = u(1 + X)^m = u + unX + \ldots + uX^n \), and thus \( n = 0 \) because \( h \in D \) and \( u \in U(B) \). This means that \( \text{char}(A) \neq 0 \). \( \Box \)

Let \( A \) be an integrally closed AGCD-domain with \( \text{char}(A) \neq 0 \). Then \( A[X] \) is an AGCD-domain. So by Theorem 3.3 and [8, Corollary 7], \( A[X^2, X^3] \) is a non-integrally closed AGCD-domain with \( Cl(A[X^2, X^3]) = Cl(A) \otimes K \), where \( K \), the quotient field of \( A \), is considered as an additive abelian group. It is interesting to note here that \( Cl(A[X^2, X^3]) \) is torsion.

**Corollary 3.4.** Let \( A \) be an integral domain, \( S \) a saturated multiplicative subset of \( A \), \( X \) an indeterminate over \( A \), \( R = A + XAS[X] \), and \( D = A + X^2AS[X] \). Then the following statements are equivalent:

1. \( D \) is an AGCD-domain.
2. \( R \) is an AGCD-domain and \( \text{char}(A) \neq 0 \).
3. \( A \) and \( A_S[X] \) are AGCD-domains, \( \text{char}(A) \neq 0 \), and \( S \) is an almost splitting of \( A \).
4. \( A \) is an AGCD-domain, \( A_S[X] \subseteq A'_S[X] \) is a root extension, and \( \text{char}(A) \neq 0 \), where \( A' \) is the integral closure of \( A \).

**Proof.** (1) \( \leftrightarrow \) (2): This is Theorem 3.3. (2) \( \leftrightarrow \) (3) \( \leftrightarrow \) (4): See [6, Theorem 3.10]. \( \Box \)

**Corollary 3.5.** Let \( B \) be an overring of an integral domain \( A \), \( X \) an indeterminate over \( A \), \( R = A + XB[X] \), and \( D = A + X^2B[X] \). Then the following statements are equivalent:

1. \( D \) is an AGCD-domain with integral closure \( R \).
2. \( R \) is an integrally closed AGCD-domain and \( \text{char}(A) \neq 0 \).
3. \( A \) is an integrally closed AGCD-domain, \( \text{char}(A) \neq 0 \), and \( B = A_S \), where \( S \) is an almost splitting set of \( A \).
Proof. (1) ⇔ (2): This follows directly from Theorem 3.3 because $R$ is integral over $D$.
(2) ⇔ (3): See [19, Theorem 3.1(b)]. □

In [19], the authors studied integrally closed AGCD-domain of finite $t$-character of the form $A + XB[X]$ and constructed non-integrally closed AGCD-domains of finite $t$-character using local algebraic techniques. The following corollary gives many simple examples of non-integrally closed AGCD-domains of finite $t$-character.

**Corollary 3.6.** Let $A$ be an AGCD-domain with $\text{char}(A) \neq 0$, $X$ an indeterminate over $A$, $S$ an almost splitting set of $A$, and $D = A + X^2A_S[X]$. Then $D$ is an AGCD-domain of finite $t$-character if $A$ is an integrally closed AGCD-domain of finite $t$-character and $S$ does not contain any infinite sequence of mutually $v$-coprime nonunit elements.

**Proof.** By Dumitrescu et al. [19, Theorem 3.1], $R = A + XA_S[X]$ is an integrally closed AGCD-domain of finite $t$-character. So $D$ is an AGCD-domain of finite $t$-character by Lemma 3.1 and Corollary 3.5. □

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**References**