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Non-semisimple Hopf algebras of dimension p^2

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Abstract

Let H be a finite-dimensional Hopf algebra with antipode S of dimension pq over an algebraically closed field of characteristic 0, where $p \leqslant q$ are odd primes. If H is not semisimple, then the order of S^4 is p, and $\mathrm{Tr}(S^{2p})$ is an integer divisible by p^2. In particular, if $\dim H = p^2$, we prove that H is isomorphic to a Taft algebra. This completes the classification for the Hopf algebras of dimension p^2.

0. Introduction

Let p be a prime number and k an algebraically closed field of characteristic 0. If H is a semisimple Hopf algebra of dimension p^2, then H is isomorphic to a group algebra [9], namely $k[\mathbb{Z}_{p^2}]$ or $k[\mathbb{Z}_p \times \mathbb{Z}_p]$. For the non-semisimple case, the only known non-semisimple Hopf algebras of dimension p^2 are the Taft algebras [19] (cf. [11, 5]). The question whether the Taft algebras are the only non-semisimple Hopf algebras of dimension p^2 has remained open. In fact, the question was asked by Susan Montgomery in several international conferences. It was proved in [1, Theorem A] that if both H and H^* have non-trivial group-like elements or the order of the antipode is $2p$, then H is isomorphic to a Taft algebra provided $\dim H = p^2$. Here we prove that any non-semisimple Hopf algebra over

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k of dimension p^2 is isomorphic to a Taft algebra. Hence, the Hopf algebras over k of dimension p^2 can be completely classified (Theorem 5.5).

If $p \leqslant q$ and are odd primes, whether there is a non-semisimple Hopf algebra of dimension pq other than the Taft algebras is still in question. Nevertheless, we prove that for any non-semisimple Hopf algebra H of dimension pq, the order of S^4 is p, where S is the antipode of H. Moreover, $\mathrm{Tr}(S^{2p})$ is an integer divisible by p^2 (Theorem 5.4). The uniqueness of Taft algebras is a consequence of this result.

The article is organized as follows: In Section 1, we recall some notation, general theorems and some useful statements. In Section 2, we introduce the notion of the *index* of a Hopf algebra and we compute the index of the Taft algebras. In Section 3, we consider the common eigenspaces of S^2 and $r(g)$ where S and $r(g)$ are the antipode and the right multiplication by the distinguished group-like element g of the Hopf algebra H. We derive some arithmetic properties of the dimensions of these eigenspaces for the Hopf algebras of odd index. We further exploit the arithmetic properties of these numbers for Hopf algebras of odd prime index in Section 4. Using these arithmetic properties, we finally prove our main theorems in Section 5.

1. Notation and preliminaries

Throughout this paper k is an algebraically closed field of characteristic 0 and H is a finite-dimensional Hopf algebra over k with antipode S. Its comultiplication and counit are, respectively, denoted by Δ and ε. We will use Sweedler's notation [18]:

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}.$$

A non-zero element $a \in H$ is called group-like if $\Delta(a) = a \otimes a$. For the details of elementary aspects for finite-dimensional Hopf algebras, readers are referred to Refs. [10,18].

The set of all group-like elements $G(H)$ of H is a linearly independent set, and it forms a group under the multiplication of H. The divisibility of $\dim H$ by $|G(H)|$ is an immediate consequence of the following generalization of Lagrange's theorem, due to Nichols and Zoeller:

Theorem 1.1 [12]. *Let H be a finite-dimensional Hopf algebra over a field and B a Hopf subalgebra of H. Then H is a free B-module. In particular,* $\dim B$ *divides* $\dim H$.

The order of the antipode is of fundamental importance to the semisimplicity of H. We recall some important results on the antipode S of finite-dimensional Hopf algebras H.

Theorem 1.2 [6,7]. *Let H be a finite-dimensional Hopf algebra with antipode S over a field of characteristic* 0. *Then the following statements are equivalent:*

(i) *H is semisimple.*
(ii) *H^* is semisimple.*
(iii) $\mathrm{Tr}(S^2) \neq 0$.
(iv) $S^2 = \mathrm{id}_H$.

Let $\lambda \in H^*$ be a non-zero right integral of H^* and let $\Lambda \in H$ be a non-zero left integral of H. There exists $\alpha \in \mathrm{Alg}(H,k) = G(H^*)$, independent of the choice of Λ, such that $\Lambda a = \alpha(a)\Lambda$ for $a \in H$. Likewise, there is a group-like element $g \in H$, independent of the choice of λ, such that $\beta\lambda = \beta(g)\lambda$ for $\beta \in H^*$. We call g the distinguished group-like element of H and α the distinguished group-like element of H^*. Then we have a formula for S^4 in terms of α and g [13]:

$$S^4(a) = g\bigl(\alpha \rightharpoonup a \leftharpoonup \alpha^{-1}\bigr)g^{-1} \quad \text{for } a \in H, \tag{1.1}$$

where $\rightharpoonup$ and $\leftharpoonup$ denote the natural actions of the Hopf algebra H^* on H described by

$$\beta \rightharpoonup a = \sum a_{(1)}\beta(a_{(2)}) \quad \text{and} \quad a \leftharpoonup \beta = \sum \beta(a_{(1)})a_{(2)}$$

for $\beta \in H^*$ and $a \in H$. If λ and Λ are normalized, there are formulae for the trace of any linear endomorphism on H.

Theorem 1.3 [15, Theorem 1]. *Let H be a finite-dimensional Hopf algebra with antipode S over the field k. Suppose that λ is a right integral of H^*, and that Λ is a left integral of H such that $\lambda(\Lambda) = 1$. Then for any $f \in \mathrm{End}_k(H)$,*

$$\begin{aligned}\mathrm{Tr}(f) &= \sum \lambda\bigl(S(\Lambda_{(2)})f(\Lambda_{(1)})\bigr) = \sum \lambda\bigl((S \circ f)(\Lambda_{(2)})\Lambda_{(1)}\bigr)\\ &= \sum \lambda\bigl((f \circ S)(\Lambda_{(2)})\Lambda_{(1)}\bigr).\end{aligned}$$

We shall also need the following lemma of linear algebra:

Lemma 1.4 [1, Lemma 2.6]. *Let T be an operator on a finite-dimensional vector space V over the field k. Let p be a prime number and let $\omega \in k$ be a primitive pth root of unity.*

(i) *If $\mathrm{Tr}(T) = 0$ and $T^p = \mathrm{id}_V$, then $\dim V_i$ is constant where V_i is the eigenspaces of T associated with the eigenvalue ω^i. In particular, $p \mid \dim V$.*
(ii) *If $\mathrm{Tr}(T) = 0$ and $T^{2p} = \mathrm{id}_V$, then*

$$\mathrm{Tr}\bigl(T^p\bigr) = pd$$

for some integer d.

2. Index of a Hopf algebra

The distinguished group-like element g defines a coalgebra automorphism $r(g)$ on H as follows:

$$r(g)(a) = ag \quad \text{for } a \in H.$$

Since S^2 is an algebra automorphism on H and $S^2(g) = g$,

$$S^2 \circ r(g) = r(g) \circ S^2.$$

Moreover, both S^2 and $r(g)$ are of finite order. Therefore, S^2 and $r(g)$ generate a finite abelian subgroup of $\mathrm{Aut}_k(H)$. We will simply call the exponent of the subgroup generated by S^4 and $r(g)$ the *index* of H. It is easy to see that the index of H is also the smallest positive integer n such that

$$S^{4n} = \mathrm{id}_H \quad \text{and} \quad g^n = 1.$$

Obviously, $o(g) \mid n$ and $o(S^4) \mid n$, where $o(g)$ and $o(S^4)$ are the orders of g and S^4, respectively. By Eq. (1.1),

$$n \mid \mathrm{lcm}\big(o(g), o(\alpha)\big). \tag{2.1}$$

Example 2.1.

(i) If both H and H^* are unimodular, then $S^4 = \mathrm{id}_H$ by (1.1). Therefore, the index of H is 1. In particular, if H is semisimple, the index of H is 1.

(ii) Let $\xi \in k$ be a primitive nth root of unity. The Taft algebra [19] $T(\xi)$ over k is generated by x and a, as a k-algebra, subject to the relations

$$a^n = 1, \qquad ax = \xi xa, \qquad x^n = 0.$$

The Hopf algebra structure is given by

$$\Delta(a) = a \otimes a, \qquad S(a) = a^{-1}, \qquad \varepsilon(a) = 1,$$
$$\Delta(x) = x \otimes a + 1 \otimes x, \qquad S(x) = -xa^{-1}, \qquad \varepsilon(x) = 0.$$

It is known that $\{x^i a^j \mid 0 \leqslant i, j \leqslant n-1\}$ forms a basis for $T(\xi)$. In particular, $\dim T(\xi) = n^2$. The linear functional λ defined by

$$\lambda\big(x^i a^j\big) = \delta_{i,n-1}\delta_{j,0}$$

is a right integral of $T(\xi)^*$. One can easily see that a is the distinguished group-like element of $T(\xi)$. Moreover, $S^4(x) = \xi^2 x$ and $S^4(a) = a$. Since the order of a is n and $S^{4n} = \mathrm{id}_H$, the index of $T(\xi)$ is n.

Remark 2.2.

(i) If the index of the Hopf algebra H is greater than 1, then H is not semisimple by Example 2.1(i).

(ii) If $\dim H$ is odd, it follows from Theorem 1.1 that the order of the distinguished group-like element g of H and the order of the distinguished group-like element α of H^* are both odd. Hence, by the formula (2.1), the order of S^4 is also odd. Therefore, the index of H is odd.

3. Eigenspace decompositions for Hopf algebras of odd index

In this section, we will only consider those Hopf algebras H of odd index $n > 1$. Since $r(g)^n = S^{4n} = \mathrm{id}_H$, and S^2 and $r(g)$ are commuting operators on H, $r(g)$ and S^2 are simultaneously diagonalizable. Let $\omega \in k$ be a primitive nth root of unity. Then any eigenvalue of S^2 is of the form $(-1)^a\omega^i$ and the eigenvalues of $r(g)$ are of the form ω^j. Define

$$H^\omega_{a,i,j} = \left\{u \in H \;\middle|\; S^2(u) = (-1)^a\omega^i u,\ ug = \omega^j u\right\}$$
$$\text{for any } (a, i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_n.$$

We will simply write $\mathcal{K}_n$ for the group $\mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_n$, write $H^\omega_{(a,i,j)}$ for $H^\omega_{a,i,j}$ and $\boldsymbol{a}$ for (a, i, j). Since S^2 and $r(g)$ are simultaneously diagonalizable, we have the decomposition

$$H = \bigoplus_{\boldsymbol{a}\in\mathcal{K}_n} H^\omega_{\boldsymbol{a}}. \tag{3.1}$$

Note that $H^\omega_{\boldsymbol{a}}$ could be zero.

Since the distinguished group-like element α of H^* is an algebra map and $g^n = 1$, we have $\alpha(g)^n = 1$. Hence, $\alpha(g)$ is a nth root of unity, and so $\alpha(g) = \omega^x$ for some integer x. Using the eigenspace decomposition of H in (3.1), the diagonalization of the left integral Λ of H admits an interesting form.

Lemma 3.1. *Let H be a finite-dimensional Hopf algebra with antipode S of odd index $n > 1$ over the field k. Let g and α be the distinguished group-like elements of H and H^*, respectively. Suppose that Λ is a left integral of H, $\alpha(g) = \omega^x$, and $\boldsymbol{x} = (0, -x, x) \in \mathcal{K}_n$. Then*

$$\Delta(\Lambda) = \sum_{\boldsymbol{a}\in\mathcal{K}_n} \Big(\sum u_{\boldsymbol{a}} \otimes v_{-\boldsymbol{a}+\boldsymbol{x}}\Big) \tag{3.2}$$

where $\sum u_{\boldsymbol{a}} \otimes v_{-\boldsymbol{a}+\boldsymbol{x}} \in H^\omega_{\boldsymbol{a}} \otimes H^\omega_{-\boldsymbol{a}+\boldsymbol{x}}$.

Proof. Note that

$$H \otimes H = \bigoplus_{\boldsymbol{a},\boldsymbol{b}\in\mathcal{K}_n} H^\omega_{\boldsymbol{a}} \otimes H^\omega_{\boldsymbol{b}}.$$

In particular, we can write

$$\Delta(\Lambda) = \sum_{\boldsymbol{a},\boldsymbol{b}\in\mathcal{K}_n} \Bigl(\sum u_{\boldsymbol{a}} \otimes v_{\boldsymbol{b}}\Bigr)$$

where $\sum u_{\boldsymbol{a}} \otimes v_{\boldsymbol{b}} \in H^{\omega}_{\boldsymbol{a}} \otimes H^{\omega}_{\boldsymbol{b}}$. By [16, Proposition 3(d)],

$$S^2(\Lambda) = \alpha\bigl(g^{-1}\bigr)\Lambda = \omega^{-x}\Lambda.$$

Since S^2 is a coalgebra automorphism on H, we have

$$\begin{aligned}
\Delta(\Lambda) &= \sum_{(a,i,j),(b,s,t)\in\mathcal{K}_n} \Bigl(\sum u_{a,i,j} \otimes v_{b,s,t}\Bigr) \\
&= \sum_{(a,i,j),(b,s,t)\in\mathcal{K}_n} \omega^x S^2 \otimes S^2\Bigl(\sum u_{a,i,j} \otimes v_{b,s,t}\Bigr) \\
&= \sum_{(a,i,j),(b,s,t)\in\mathcal{K}_n} (-1)^{a+b}\omega^{x+i+s}\Bigl(\sum u_{a,i,j} \otimes v_{b,s,t}\Bigr). \qquad (3.3)
\end{aligned}$$

Since g is group-like and $\Lambda g = \alpha(g)\Lambda = \omega^x\Lambda$, we have

$$\begin{aligned}
\Delta(\Lambda) &= \sum_{(a,i,j),(b,s,t)\in\mathcal{K}_n} \Bigl(\sum u_{a,i,j} \otimes v_{b,s,t}\Bigr) \\
&= \sum_{(a,i,j),(b,s,t)\in\mathcal{K}_n} \omega^{-x} r(g) \otimes r(g)\Bigl(\sum u_{a,i,j} \otimes v_{b,s,t}\Bigr) \\
&= \sum_{(a,i,j),(b,s,t)\in\mathcal{K}_n} \omega^{-x+j+t}\Bigl(\sum u_{a,i,j} \otimes v_{b,s,t}\Bigr). \qquad (3.4)
\end{aligned}$$

Thus, if $\sum u_{a,i,j} \otimes v_{b,s,t} \neq 0$, by Eqs. (3.3) and (3.4),

$$1 = (-1)^{a+b}\omega^{x+i+s} \quad \text{and} \quad 1 = \omega^{-x+j+t},$$

or equivalently,

$$(b,s,t) = (a,-i,-j) + (0,-x,x) = -(a,i,j) + \boldsymbol{x}.$$

Thus,

$$\Delta(\Lambda) = \sum_{\boldsymbol{a}\in\mathcal{K}_n} \Bigl(\sum u_{\boldsymbol{a}} \otimes v_{-\boldsymbol{a}+\boldsymbol{x}}\Bigr). \quad \square \qquad (3.5)$$

In the sequel, we will call the expression in Eq. (3.2) the *normal form* of $\Delta(\Lambda)$ associated with ω. We will simply write $u_{\boldsymbol{a}} \otimes v_{-\boldsymbol{a}+\boldsymbol{x}}$ for the sum $\sum u_{\boldsymbol{a}} \otimes v_{-\boldsymbol{a}+\boldsymbol{x}}$ in the normal form of $\Delta(\Lambda)$.

The eigenspace decomposition $H = \bigoplus_{\boldsymbol{a}\in\mathcal{K}_n} H^{\omega}_{\boldsymbol{a}}$ is associated with a unique family of projections $E^{\omega}_{\boldsymbol{a}}$ ($\boldsymbol{a} \in \mathcal{K}_n$) from H onto $H^{\omega}_{\boldsymbol{a}}$ such that

(1) $E^\omega_{\boldsymbol{a}} \circ E^\omega_{\boldsymbol{b}} = 0$ for $\boldsymbol{a} \neq \boldsymbol{b}$, and
(2) $\sum_{\boldsymbol{a}\in\mathcal{K}_n} E^\omega_{\boldsymbol{a}} = \mathrm{id}_H$.

In particular, $\dim H^\omega_{\boldsymbol{a}} = \mathrm{Tr}(E^\omega_{\boldsymbol{a}})$ for all $\boldsymbol{a} \in \mathcal{K}_n$. By Lemma 3.1,

$$\Delta(\Lambda) = \sum_{\boldsymbol{a}\in\mathcal{K}_n} \big(E^\omega_{\boldsymbol{a}} \otimes E^\omega_{-\boldsymbol{a}+\boldsymbol{x}}\big)\Delta(\Lambda)$$

and hence $(E^\omega_{\boldsymbol{a}} \otimes E^\omega_{-\boldsymbol{a}+\boldsymbol{x}})\Delta(\Lambda)$ is identical to $\sum u_{\boldsymbol{a}} \otimes v_{-\boldsymbol{a}+\boldsymbol{x}}$ in the normal form (3.2) of $\Delta(\Lambda)$. Using the trace formulae in Theorem 1.3, we obtained the following lemma:

Lemma 3.2. *Let H be a finite-dimensional Hopf algebra with antipode S of odd index $n > 1$ over the field k, and let $\omega \in k$ be a primitive nth root of unity. Suppose that Λ is a left integral of H and that λ be a right integral of H^* such that $\lambda(\Lambda) = 1$. Then*

$$\dim H^\omega_{\boldsymbol{a}} = \lambda\big(S(v_{-\boldsymbol{a}+\boldsymbol{x}})u_{\boldsymbol{a}}\big) \tag{3.6}$$

for all $\boldsymbol{a} \in \mathcal{K}_n$, where $\sum_{\boldsymbol{a}\in\mathcal{K}_n} u_{\boldsymbol{a}} \otimes v_{-\boldsymbol{a}+\boldsymbol{x}}$ is the normal form of $\Delta(\Lambda)$ associated with ω.

Proof. Using the normal form of $\Delta(\Lambda)$ associated with ω and Theorem 1.3, for any $\boldsymbol{b} \in \mathcal{K}_n$,

$$\begin{aligned}\dim H^\omega_{\boldsymbol{a}} = \mathrm{Tr}\big(E^\omega_{\boldsymbol{a}}\big) &= \sum_{\boldsymbol{b}\in\mathcal{K}_n} \lambda\big(S(v_{-\boldsymbol{b}+\boldsymbol{x}})E^\omega_{\boldsymbol{a}}(u_{\boldsymbol{b}})\big) \\ &= \sum_{\boldsymbol{b}\in\mathcal{K}_n} \delta_{\boldsymbol{a},\boldsymbol{b}}\lambda\big(S(v_{-\boldsymbol{b}+\boldsymbol{x}})u_{\boldsymbol{a}}\big) = \lambda\big(S(v_{-\boldsymbol{a}+\boldsymbol{x}})u_{\boldsymbol{a}}\big). \quad \square\end{aligned}$$

The family of elements $S(v_{-\boldsymbol{a}+\boldsymbol{x}})u_{\boldsymbol{a}}$ appearing in (3.6) are in $H^\omega_{0,-x,*}$. Moreover, if H is non-semisimple, they satisfy a system of equations.

Lemma 3.3. *Let H be a finite-dimensional Hopf algebra with antipode S of odd index $n > 1$ over the field k, and let $\omega \in k$ be a primitive nth root of unity. Then*

$$\sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_n} (-1)^a \omega^{-i} \dim H^\omega_{a,i,j} = 0 \quad \textit{for } j \in \mathbb{Z}_n.$$

Proof. Let Λ be a left integral of H and let λ be a right integral of H^* such that $\lambda(\Lambda) = 1$. Since H is not semisimple, by [16, Theorem 4],

$$\sum S^3(\Lambda_2)\Lambda_1 = 0.$$

Hence for any integer e,

$$\sum S^3(\Lambda_2)\Lambda_1 g^e = 0,$$

where g is the distinguished group-like element of H. Let

$$h'_{\boldsymbol{a}} = S^3(v_{-\boldsymbol{a}+\boldsymbol{x}})u_{\boldsymbol{a}} \quad \text{for all } \boldsymbol{a} \in \mathcal{K}_n$$

where $\sum_{\boldsymbol{a}\in\mathcal{K}_n} u_{\boldsymbol{a}} \otimes v_{-\boldsymbol{a}+\boldsymbol{x}}$ is the normal form of $\Delta(\Lambda)$ associated with ω. Then

$$0 = \sum S^3(\Lambda_2)\Lambda_1 g^e = \sum_{(a,i,j)\in\mathcal{K}_n} h'_{a,i,j} g^e = \sum_{j\in\mathbb{Z}_n} \omega^{ej} \sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_n} h'_{a,i,j}$$

for $e = 0,\ldots,n-1$. Since $1,\omega,\ldots,\omega^{n-1}$ are distinct elements in k, the Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \cdots & \omega^{(n-1)^2} \end{bmatrix}$$

is invertible. Therefore,

$$\sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_n} h'_{a,i,j} = 0 \tag{3.7}$$

for $j \in \mathbb{Z}_n$. Notice that

$$S^3(v_{a,-i-x,-j+x}) = (-1)^a \omega^{-i-x} S(v_{a,-i-x,-j+x}).$$

Therefore, $h'_{a,i,j} = (-1)^a \omega^{-i-x} S(v_{-a,-i-x,-j+x})u_{a,i,j}$ for any $(a,i,j) \in \mathcal{K}_n$. Then Eq. (3.7) becomes

$$\sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_n} (-1)^a \omega^{-i} S(v_{-a,-i-x,-j+x})u_{a,i,j} = 0$$

for $j \in \mathbb{Z}_n$. Applying λ to the equation, we have

$$\sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_n} (-1)^a \omega^{-i} \lambda\big(S(v_{-a,-i-x,-j+x})u_{a,i,j}\big) = 0$$

for all $j \in \mathbb{Z}_n$. Then, the result follows from Lemma 3.2. □

Lemma 3.4. *Let H be a finite-dimensional unimodular Hopf algebra of odd index $n > 1$ over the field k, and let $\omega \in k$ be a primitive nth root of unity. Then*

$$\sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_n} (-1)^a \omega^{-i} \dim H^{\omega}_{a,i,l-2i} = 0$$

for $l \in \mathbb{Z}_n$.

Proof. Let α and g be the distinguished group-like elements of H^* and H, respectively. Since H is unimodular, $\alpha = \varepsilon$ and hence $\alpha(g) = 1 = \omega^0$. Let Λ

be a left integral of H and let λ be a right integral of H^* such that $\lambda(\Lambda) = 1$. It follows from Lemma 3.1 that the normal form of $\Delta(\Lambda)$ associated with ω is

$$\sum_{\boldsymbol{a}\in\mathcal{K}_n} u_{\boldsymbol{a}} \otimes v_{-\boldsymbol{a}}. \tag{3.8}$$

Since H is not semisimple,

$$0 = \varepsilon(\Lambda)1 = \sum \Lambda_1 S(\Lambda_2).$$

Thus, we have

$$0 = \sum_{\boldsymbol{a}\in\mathcal{K}_n} u_{\boldsymbol{a}} S(v_{-\boldsymbol{a}}). \tag{3.9}$$

Note that, by Eq. (1.1) and the unimodularity of H,

$$g^e a = S^{4e}(a) g^e$$

for any integer e and $a \in H$. Let $h_{\boldsymbol{a}} = u_{\boldsymbol{a}} S(v_{-\boldsymbol{a}})$ for $\boldsymbol{a} \in \mathcal{K}_n$. Then,

$$\begin{aligned} g^e h_{a,i,j} &= g^e u_{a,i,j} S(v_{a,-i,-j}) = \omega^{e(2i+j)} u_{a,i,j} S(v_{a,-i,-j}) \\ &= \omega^{e(2i+j)} h_{a,i,j}. \end{aligned} \tag{3.10}$$

By multiplying g^e on the left of Eq. (3.9), we have

$$0 = \sum_{(a,i,j)\in\mathcal{K}_n} \omega^{e(2i+j)} h_{a,i,j} = \sum_{l\in\mathbb{Z}_n} \omega^{el} \sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_n} h_{a,i,l-2i}. \tag{3.11}$$

By the same argument used in the proof of Lemma 3.3,

$$\sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_n} h_{a,i,l-2i} = 0 \tag{3.12}$$

for $l \in \mathbb{Z}_n$. Notice that, by [16, Theorem 3(a)],

$$\begin{aligned} \lambda\big(h_{a,i,j}\big) &= \lambda\big(u_{a,i,j} S(v_{a,-i,-j})\big) = \lambda\big(S^3(v_{a,-i,-j}) u_{a,i,j}\big) \\ &= (-1)^a \omega^{-i} \lambda\big(S(v_{a,-i,-j}) u_{a,i,j}\big). \end{aligned} \tag{3.13}$$

By Lemma 3.2 and Eq. (3.8),

$$\lambda\big(h_{a,i,j}\big) = (-1)^a \omega^{-i} \dim H^{\omega}_{a,i,j}.$$

Hence, we have

$$0 = \sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_n} \lambda\big(h_{a,i,l-2i}\big) = \sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_n} (-1)^a \omega^{-i} \dim H^{\omega}_{a,i,l-2i}$$

for $l \in \mathbb{Z}_n$. $\square$

4. Arithmetic properties of Hopf algebras with odd prime index

In this section, we will study the arithmetic properties for the Hopf algebras of odd prime index p. Let $\omega \in k$ be a primitive pth root of unity. The Taft algebra $T(\omega)$ [19] is then a Hopf algebra of this type by Example 2.1(ii). The quantum double of $T(\omega)$ is a unimodular Hopf algebra of index p (cf. [5]).

Lemma 4.1. *Let H be a finite-dimensional Hopf algebra of index p over the field k, where p is an odd prime. Let $\omega \in k$ be a primitive pth root of unity. Then for each $j \in \mathbb{Z}_p$ there exists an integer d_j such that*

$$\dim H^{\omega}_{0,i,j} - \dim H^{\omega}_{1,i,j} = d_j$$

for all $i \in \mathbb{Z}_p$.

Proof. By Lemma 3.3 we have

$$\sum_{i\in\mathbb{Z}_p} \omega^{-i}\left(\dim H^{\omega}_{0,i,j} - \dim H^{\omega}_{1,i,j}\right) = 0$$

for all $j \in \mathbb{Z}_p$. In particular, ω^{-1} is a root of the integral polynomial

$$f_j(x) = \sum_{i=0}^{p-1}\left(\dim H^{\omega}_{0,i,j} - \dim H^{\omega}_{1,i,j}\right)x^i.$$

Hence, $f_j(x) = d_j\Phi_p(x)$ for some $d_j \in \mathbb{Q}$, where $\Phi_p(x) = 1 + x + \cdots + x^{p-1}$ is the irreducible polynomial of ω^{-1} over $\mathbb{Q}$. Therefore,

$$\dim H^{\omega}_{0,i,j} - \dim H^{\omega}_{1,i,j} = d_j.$$

Since $\dim H^{\omega}_{0,i,j} - \dim H^{\omega}_{1,i,j}$ is an integer, and so is d_j. □

Lemma 4.2. *Let H be a finite-dimensional Hopf algebra of index p over the field k, where p is an odd prime. If H^* is not unimodular, then $p \mid \dim H$ and*

$$\sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_p} \dim H^{\omega}_{a,i,j} = \frac{\dim H}{p} \quad \text{for all } j \in \mathbb{Z}_p.$$

Proof. Since H^* is not unimodular, the distinguished group-like element g of H is not equal 1_H. Then, $\mathrm{Tr}(r(g)) = 0$ (cf. [8, Proposition 2.4(d)]). Moreover, $r(g)^p = \mathrm{id}_H$. Hence, by Lemma 1.4, $p \mid \dim H$ and the eigenspace of $r(g)$ associated with the eigenvalue ω^j is of dimension $\frac{\dim H}{p}$ for any $j \in \mathbb{Z}_p$. Note that

$$\bigoplus_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_p} H^{\omega}_{a,i,j}$$

is the eigenspace of $r(g)$ associated with ω^j. Therefore,

$$\frac{\dim H}{p} = \dim\Bigl(\bigoplus_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_p} H^{\omega}_{a,i,j}\Bigr) = \sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_p} \dim H^{\omega}_{a,i,j}. \quad \square$$

Lemma 4.3. *Let H be a finite-dimensional Hopf algebra of index p over the field k, where p is an odd prime. If H^* is not unimodular and H is unimodular, then*:

(i) *There is an integer d such that*

$$\dim H^{\omega}_{0,i,j} - \dim H^{\omega}_{1,i,j} = d \quad \textit{for any } i, j \in \mathbb{Z}_p.$$

(ii) $\mathrm{Tr}(S^{2p}) = p^2 d$.

Proof. (i) By Lemma 3.4, for any $l \in \mathbb{Z}_p$,

$$\sum_{i\in\mathbb{Z}_p} \bigl(\dim H^{\omega}_{0,i,l-2i} - \dim H^{\omega}_{1,i,l-2i}\bigr)\omega^{-i} = 0.$$

Since ω^{-1} is also a primitive pth root of unity in k, there exists an integer c_l such that

$$\dim H^{\omega}_{0,i,l-2i} - \dim H^{\omega}_{1,i,l-2i} = c_l \tag{4.1}$$

for $i \in \mathbb{Z}_p$. By Lemma 4.1, for any $i, l \in \mathbb{Z}_p$,

$$c_l = \dim H^{\omega}_{0,i,l-2i} - \dim H^{\omega}_{1,i,l-2i} = d_{l-2i}. \tag{4.2}$$

Since 2 and p are relative prime, $l, l-2, \ldots, l-2(p-1)$ is a complete set of representatives of $\mathbb{Z}_p$. Therefore,

$$d_j = c_l = d \quad \text{for any } j, l \in \mathbb{Z}_p.$$

(ii) Since p is odd,

$$\mathrm{Tr}\bigl(S^{2p}\bigr) = \sum_{i,j\in\mathbb{Z}_p} \dim H^{\omega}_{0,i,j} - \dim H^{\omega}_{1,i,j} = \sum_{i,j\in\mathbb{Z}_p} d = p^2 d. \quad \square$$

5. Hopf algebras of dimension *pq*

In this section, we will consider the Hopf algebras H of dimension pq where both $p \leqslant q$ are odd primes. In particular, we prove that if H is not semisimple and $\dim H = p^2$, then H is isomorphic to a Taft algebra. It was proved in [9, Theorem 2] that semisimple Hopf algebras of dimension p^2 are group algebras. Therefore, any Hopf algebra over k of dimension p^2 is either a group algebra or a Taft algebra. We begin the section with the following lemma.

Lemma 5.1. *Let p, q be two distinct prime numbers. Then there is no finite-dimensional Hopf algebra H of dimension pq over the field k such that $|G(H)| = p$ and $|G(H^*)| = q$.*

Proof. Suppose there is a Hopf algebra H of dimension pq such that $|G(H)| = p$ and $|G(H^*)| = q$. Let $g \in G(H)$ and $\alpha \in G(H^*)$ such that $o(g) = p$ and $o(\alpha) = q$. Note that $\beta^\ell(a) = \beta(a^\ell) = (\beta(a))^\ell$ for all $\beta \in G(H^*), a \in G(H)$, and $\ell \in \mathbb{Z}$. Therefore,

$$\alpha(g)^p = \alpha\big(g^p\big) = \alpha(1) = 1$$

and

$$1 = \varepsilon(g) = \alpha^q(g) = \alpha(g)^q.$$

Hence, $o(\alpha(g)) = 1$ and so $\alpha(g) = 1$. Let π be the Hopf algebra map which is the composite of $H \cong H^{**} \to k[G(H^*)]^*$, where the latter is the transpose of the inclusion $k[G(H^*)] \subseteq H^*$. Then the image of $B = k[G(H)]$ under π is one-dimensional. Thus $B^+ \subseteq \ker\pi$. It follows from [17, Theorem 2.4 (2a)] that $\dim H/B^+H = q$. Thus,

$$\dim B^+H = pq - q = \dim\ker\pi$$

and hence,

$$B^+H = \ker\pi.$$

Therefore, H/B^+H is isomorphic to $k[G(H^*)]^*$ as Hopf algebras. In particular, H/B^+H is semisimple. Let Λ be a non-zero left integral of H and Λ' a non-zero right integral of $k[G(H)]$. Since $\operatorname{char} k = 0$, $\varepsilon(\Lambda') \neq 0$ and hence, $\Lambda'\Lambda = \varepsilon(\Lambda')\Lambda \neq 0$. Therefore, $\Lambda \notin B^+H$ and so $\Lambda + B^+H$ is a non-zero left integral of H/B^+H. Since H/B^+H is semisimple, $\varepsilon(\Lambda) = \varepsilon(\Lambda + B^+H) \neq 0$. Hence, H is semisimple. By [3], H is trivial and so $|G(H)| = pq$ or $|G(H^*)| = pq$, a contradiction. □

Proposition 5.2. *Let H be a non-semisimple Hopf algebra of dimension pq with antipode S over the field k, where $p \leqslant q$ are odd primes. Then*

(i) *the order of S^4 is p, and*
(ii) *H is of index p.*

Proof. (i) Since H is not semisimple and $\dim H$ is odd, by [8, Theorem 2.1] or [1, Lemma 2.5], $S^4 \neq \mathrm{id}_H$ and H, H^* cannot be both unimodular. Let g be the distinguished group-like element of H, and α the distinguished group-like element of H^*. Then, $o(\alpha) < pq$ and $o(g) < pq$, for otherwise, H would be a group algebra, or the dual of a group algebra, both of which are semisimple. By Lemma 5.1,

$$\operatorname{lcm}\big(o(g), o(\alpha)\big) = p \text{ or } q. \tag{5.1}$$

By Eqs. (1.1) and (5.1), the order of S^4 is either p or q. If $p = q$, the order of S^4 and the index of H are obviously equal to p. We now assume $q > p$. Consider the following cases:

Case (a). H^* is not unimodular. In this case, $o(g) = p$ or q. Suppose that the order of S^4 is q. By Eq. (1.1), $q \mid \operatorname{lcm}(o(g), o(\alpha))$. Therefore, $\operatorname{lcm}(o(g), o(\alpha)) = q$ and hence $o(g) = q$. Thus, the index of H is also q. Let $\omega \in k$ be a qth primitive root of unity. By Lemma 4.1, for each $j \in \mathbb{Z}_q$ there is an integer d_j such that

$$\dim H^{\omega}_{0,i,j} - \dim H^{\omega}_{1,i,j} = d_j \quad \text{for all } i \in \mathbb{Z}_q. \tag{5.2}$$

Let $X_{i,j} = \min(\dim H^{\omega}_{0,i,j}, \dim H^{\omega}_{1,i,j})$. Then,

$$\dim H^{\omega}_{0,i,j} + \dim H^{\omega}_{1,i,j} = 2X_{i,j} + |d_j|$$

and so

$$\sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_q} \dim H^{\omega}_{a,i,j} = \sum_{i\in\mathbb{Z}_q} 2X_{i,j} + q|d_j| \tag{5.3}$$

for each $j \in \mathbb{Z}_q$. It follows from Lemma 4.2 that

$$\sum_{i\in\mathbb{Z}_q} 2X_{i,j} + q|d_j| = p. \tag{5.4}$$

Since p odd, by (5.4), $|d_j|$ must be odd. However, the left hand side of (5.4) is then strictly greater than p, a contradiction! Therefore, $o(S^4) = p$.

Case (b). H^* is unimodular. Then $H^{**} \cong H$ is not unimodular. By Theorem 1.2, H^* is not semisimple and $\dim H^* = pq$. It follows from Case (a) that the order of S^{*4} is p. Since $o(S^4) = o(S^{*4})$. Therefore, $o(S^4) = p$.

(ii) Let n be the index of H. Then, by (2.1), $n \mid \operatorname{lcm}(o(g), o(\alpha))$ and $o(S^4) \mid n$. Since $o(S^4) = p$ and $\operatorname{lcm}(o(g), o(\alpha)) = p$ or q, we have $n = p$. □

Lemma 5.3. *Let H be a Hopf algebra of dimension pq with antipode S over the field k, where p, q are odd primes. If $|G(H)| = |G(H^*)| = p$, then* $\operatorname{Tr}(S^{2p}) = p^2 d$ *for some integer d.*

Proof. Let $g \in G(H)$ and $\alpha \in G(H^*)$ such that both the orders of g and α are equal to p. Let B be the group algebra $k[g]$, and let $I = k[\alpha]^{\perp}$. Since $k[\alpha]$ is a Hopf subalgebra of H^*, I is then a Hopf ideal of H, and

$$H/I \cong k[\alpha]^* \cong B \tag{5.5}$$

as Hopf algebras. Let $\pi' : H \to B$ be the composition

$$H \xrightarrow{\eta} H/I \xrightarrow{\cong} B$$

where η is the natural surjection. Then H is a right B-comodule algebra, and

$$R = H^{\mathrm{co}\,B} = \{h \in H \mid (\mathrm{id}_H \otimes \pi')\Delta(h) = h \otimes 1_B\}$$

is a subalgebra of H. By [17, Theorem 2.2], H is the B-cross product

$$H \cong R \#_\sigma B. \tag{5.6}$$

In particular, $\dim R = q$. Let $\gamma : B \to H$ be the inclusion map. Then $\pi'\gamma : B \to B$ is a non-trivial Hopf algebra map. Otherwise $g \in R$ and hence $B \subseteq R$. It follows from (5.6) and Theorem 1.1 that R is a free B-module. Therefore, $p \mid q$. Hence $p = q$ and $B = R$. It follows from [2, Theorem 2.6(2)] that H is semisimple, a contradiction. Therefore, $\pi'\gamma$ is non-trivial. Since B is a group algebra of dimension p, $\pi'\gamma$ is actually an isomorphism. Let $\pi = (\pi'\gamma)^{-1}\pi'$. Then, $\pi : H \to B$ is a surjective Hopf algebra map and $\pi\gamma = \mathrm{id}_B$. Therefore, H is isomorphic to the biproduct $R \times B$ as Hopf algebras (cf. [14]). It was shown in [1, Section 4] that R is invariant under S^2. Moreover, in the identification $H \cong R \otimes B$ given by multiplication, one has

$$S^2 = T \otimes \mathrm{id}_B \tag{5.7}$$

for some linear endomorphism T on R. Since H is not semisimple, $\mathrm{Tr}(S^2) = 0$. By Eq. (5.7), $\mathrm{Tr}(S^2) = \mathrm{Tr}(T)p$. Therefore, $\mathrm{Tr}(T) = 0$. Moreover, $T^{2p} = \mathrm{id}_R$ as $S^{4p} = \mathrm{id}_H$ by Eq. (1.1). Hence, by Lemma 1.4, $\mathrm{Tr}(T^p) = pd$ for some integer d. Since $S^{2p} = T^p \otimes \mathrm{id}_B$, we have

$$\mathrm{Tr}(S^{2p}) = \mathrm{Tr}(T^p)\mathrm{Tr}(\mathrm{id}_B) = p^2 d. \quad \square$$

Theorem 5.4. *Let H be a non-semisimple Hopf algebra of dimension pq with antipode S over the field k, where $p \leqslant q$ are odd primes. Then* $\mathrm{Tr}(S^{2p}) = p^2 d$ *for some odd integer d.*

Proof. By Proposition 5.2, $S^{4p} = \mathrm{id}_H$. Let

$$H_\pm = \{h \in H \mid S^{2p}(h) = \pm h\}.$$

Then,

$$\dim H_+ - \dim H_- = \mathrm{Tr}(S^{2p}) \quad \text{and} \quad \dim H_+ + \dim H_- = pq.$$

Since pq is odd, $\mathrm{Tr}(S^{2p})$ is also an odd integer. Thus, if $\mathrm{Tr}(S^{2p}) = p^2 d$, then d must be an odd integer. Therefore, it suffices to show that $\mathrm{Tr}(S^{2p}) = p^2 d$ for some integer d. Since H is not semisimple, by Theorem 1.2, H^* is also not semisimple. By Proposition 5.2, the indexes of H and H^* are both p. Since $\dim H$ is odd, by [8, Theorem 2.2], not both of H and H^* are unimodular. We then have the following three cases:

(i) If H is unimodular and H^* is not unimodular, the result follows from Lemma 4.3.

(ii) If H is not unimodular and H^* is unimodular, by Lemma 4.3, $\mathrm{Tr}(S^{*2p}) = p^2 d$ for some odd integer d. The result follows from $\mathrm{Tr}(S^{*2p}) = \mathrm{Tr}(S^{2p})$.
(iii) If both H and H^* are not unimodular, by Lemma 5.1 and Proposition 5.2, the orders of the distinguished group-like elements of H and H^* are both equal to p. Thus, by Lemma 5.3, $\mathrm{Tr}(S^{2p}) = p^2 d$. □

As a consequence of the above theorem, we prove that any Hopf algebra of dimension p^2 is either a group algebra or a Taft algebra (see Example 2.1(ii)).

Theorem 5.5. *Let H be a Hopf algebra of dimension p^2 over the field k, where p is any prime number. Then H is isomorphic to one of the following Hopf algebras*:

(a) $k[\mathbb{Z}_{p^2}]$;
(b) $k[\mathbb{Z}_p \times \mathbb{Z}_p]$;
(c) $T(\omega), \omega \in k$ *a primitive pth of unity.*

Proof. If H is semisimple, it follows from [9, Theorem 2] that H isomorphic to $k[\mathbb{Z}_{p^2}]$ or $k[\mathbb{Z}_p \times \mathbb{Z}_p]$. It is also shown in [4] that if H is a non-semisimple Hopf algebra of dimension 4, then H isomorphic to the Taft algebra $T(-1)$. We may now assume H is not semisimple and p is odd. Let S be the antipode of H. By Proposition 5.2, $S^{4p} = \mathrm{id}_H$ and so S^{2p} is diagonalizable and the possible eigenvalues of S^{2p} are ± 1. Suppose $S^{2p} \neq \mathrm{id}_H$. Then, $\mathrm{Tr}(S^{2p})$ is an integer such that

$$-p^2 \leqslant \mathrm{Tr}\big(S^{2p}\big) < p^2.$$

By Theorem 5.4,

$$\mathrm{Tr}\big(S^{2p}\big) = p^2 d$$

for some odd integer d. Therefore, $\mathrm{Tr}(S^{2p}) = -p^2$ and hence $S^{2p} = -\mathrm{id}_H$. However, this is not possible since $S^{2p}(1_H) = 1_H$. Therefore, $S^{2p} = \mathrm{id}_H$. Then, S^2 has order 1 or p; thus S^2 has order p since H is assumed to be non-semisimple. It follows from [1, Theorem A(ii)] that H is isomorphic to a Taft algebra of dimension p^2. Hence, $H \cong T(\omega)$ for some primitive pth root of unity, $\omega \in k$. □

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