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Cubic Vector Fields Symmetric with Respect to a Center*

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In this paper we study cubic vector fields which are symmetric with respect to a center. Our perspective is from the viewpoint of invariant algebraic curves of such systems. We give here a new proof of the integrability of symmetric systems with respect to a center by the method of Darboux, which uses invariant algebraic curves. The first integrals of the systems are all elementary and we give here their complete list. We next study the global geometry of such systems. We give the bifurcation diagram of the phase portraits of the vector fields. We show that although most bifurcations correspond to bifurcations of the algebraic invariant curves, unlike what happens in the quadratic case, the changes in the invariant algebraic curves do not completely determine the bifurcation diagram. We prove that there appear other bifurcations of saddle connections, whose equations are transcendental.

1. INTRODUCTION

In this paper we study cubic vector fields which are symmetric with respect to a weak focus, i.e., a singular point with pure imaginary eigenvalues. When we place the weak focus at the origin such a system can be written in the form

$$\begin{aligned} \dot{x} &= -y + \sum_{i+j=3} a_{ij} x^i y^j = -y + p_3(x, y) \\ \dot{y} &= x + \sum_{i+j=3} b_{ij} x^i y^j = x + q_3(x, y). \end{aligned} \tag{1.1}$$

In [M] Malkin found necessary and sufficient conditions for (1.1) to have a center (i.e., a singularity surrounded by closed integral curves) at the origin. In [LS] Lunkevich and Sibirsky proved the integrability of

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systems satisfying these conditions. To prove the integrability of the systems they made use of Sibirsky's algebraic invariant theory [S2].

Recently the method of Darboux was used to give a proof of the integrability of all quadratic systems with a center, [Sc1, Sc2]. Darboux's method has the advantage of being both very simple and at the same time unifying since this method can be applied for all systems regardless of the specific values in the parameter space. Furthermore the geometric content of the method: the invariant algebraic curves, throws light on the behaviour of the integral curves of the systems.

This paper has a double purpose. First we give a new proof of the integrability of systems (1.1) with a center at the origin by using the method of Darboux for all values of the parameters.

We next study the global geometry of cubic systems symmetric with respect to a center. To do this we use the conditions for a center as they appear in the paper of Sibirsky [S1]. They consist of three different cases. The first case is that of a Hamiltonian system. The second case is that of a symmetric system with respect to an axis: we say that the system has a center of type II. Because we only consider symmetric systems with respect to the origin, such systems necessarily have two symmetry axes. Finally the last case (center of type III) consists of systems having an invariant algebraic curve of degree 4 and one of degree 6. In this paper we limit ourselves to non-Hamiltonian vector fields.

Systems with centers of type II are indexed by parameters in $\mathbb{S}^3$. In the "generic case" they have two invariant conics, which suffice to construct a first integral. In opposition to what happens with quadratic systems the bifurcation diagram of the algebraic curves does not control completely the bifurcation diagram for the phase portraits of (1.1) with a center of type II, i.e., while bifurcations of singular points are given by bifurcations of the invariant algebraic curves, there exist bifurcations of saddle connections, which do not correspond to bifurcations of invariant algebraic curves. We give here a complete bifurcation diagram for the singular points and we show that there must exist surfaces of saddle connections. These surfaces are given by a transcendental equation in the parameter space. Each bifurcation of saddle connection can be viewed as a bifurcation of simultaneous tangency to the two coordinate axes for a trajectory of a linear system (Fig. 15). The exact position and shape of the surfaces of saddle connections is only conjectural, but we can prove the existence of some curves lying on them.

For centers of type III all trajectories are algebraic curves of degree $\leqslant 12$, among which we find one quartic and two sextic curves. Such systems are indexed by parameters in $\mathbb{S}^1$, but generically they all have the same phase portrait up to topological equivalence. The quartic and one of the sextic curves suffice to derive a first integral.

2. Basic Notions and Results: The Method of Darboux

In this section we recall briefly the notion of invariant algebraic curves and the way in which they are found. We show how they can be used to derive first integrals for a polynomial system. Let us start with a polynomial system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y). \end{aligned} \tag{2.1}$$

Definition 2.1. An invariant algebraic curve of a system (2.1) is a curve in the complex plane given by an equation $f(x, y) = 0$, with $f(x, y) \in \mathbb{C}[x, y]$ such that there exists $K(x, y) \in \mathbb{C}[x, y]$ satisfying

$$Df(x, y) = f(x, y)\, K(x, y), \tag{2.2}$$

where

$$Df = \dot{f} = f_x P + f_y Q, \tag{2.3}$$

and $K(x, y) \in \mathbb{C}[x, y]$. $K(x, y)$ is called the cofactor of the invariant algebraic curve $f = 0$.

Remarks 2.2. (1) If (2.1) is a polynomial system of degree n and $f(x, y)$ is a polynomial of degree m, then Df, the derivative of f along the integral curves of (2.1), is a polynomial in x and y of degree less than or equal to $(m+n-1)$ and $K(x, y)$ is a polynomial of degree less than or equal to $(n-1)$.

(2) We look for invariant algebraic curves

$$f(x, y) = \eta + \sum_{i+j=1}^{m} f_{ij} x^i y^j = \eta + f_2(x, y) + \cdots + f_m(x, y), \tag{2.4}$$

with $\eta = 0, 1$ and

$$K(x, y) = \sum_{i+j=0}^{n-1} k_{ij} x^i y^j. \tag{2.5}$$

We solve (2.2) by identifying corresponding monomials yields $f(x, y)$ and $K(x, y)$. The f_{ij}'s and k_{ij}'s are solutions of nonlinear equations.

THEOREM 2.3 [Da]. (1) *If the system* (2.1) *of degree n has more than* $N(n)=n(n+1)/2$ *irreducible invariant algebraic curves* $f_1(x, y)=0, ..., f_p(x, y)=0$, *then it has a first integral of the form*

$$F(x, y)=\prod_{i=1}^{p} f_i^{\alpha_i}, \tag{2.6}$$

with $\alpha_i \in \mathbb{C}$. *We call such an integral a Darboux first integral.*

(2) *If the system* (2.1) *has p invariant algebraic curves* $f_1(x, y)=0, ..., f_p(x, y)=0$, *with respective cofactors* $K_1(x, y), ..., K_p(x, y)$, *and if there exist* $\alpha_i \in \mathbb{C}$, $i=1, ..., p$, *not all zero such that*

$$\sum_{i=1}^{p} \alpha_i K_i(x, y)=0, \tag{2.7}$$

then the system has the Darboux first integral (2.6).

(3) *If the system* (2.1) *has p invariant algebraic curves* $f_1(x, y)=0, ..., f_p(x, y)=0$, *with respective cofactors* $K_1(x, y), ..., K_p(x, y)$, *and if there exist* $\alpha_i \in \mathbb{C}$, $i=1, ..., p$ *such that*

$$\operatorname{div}(P, Q)=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=\sum_{i=1}^{p} \alpha_i K_i(x, y), \tag{2.8}$$

then

$$R(x, y)=\prod_{i=1}^{p} f_i^{-\alpha_i} \tag{2.9}$$

is an integrating factor, i.e.,

$$\operatorname{div}(RP, RQ)=0. \tag{2.10}$$

In particular if $R(x, y)$ *exists on a simply connected domain U, then the vector field* (RP, RQ) *is Hamiltonian on U.*

Proof. (1) The space of polynomials of degree less than or equal to $(n-1)$ has the dimension $N(n)=n(n+1)/2$. Hence, for $p>N(n)$, the cofactors $K_i(x, y)$ are linearly dependent; i.e., there exist $\alpha_i \in \mathbb{C}$, $i=1, ..., p$, not all zero, such that (2.7) is satisfied. Then

$$DF=F\left(\sum_{i=1}^{p} \alpha_i K_i(x, y)\right)=0. \tag{2.11}$$

(2) It is easily checked that the previous method can work with less than $N(n)$ invariant algebraic curves, as long as there exist α_i, $i=1, ..., p$, not all zero satisfying (2.7). ∎

The method was further refined by Prelle and Singer [PS]. They proved that if a system has an elementary first integral, then it has invariant algebraic curves (cf. [Sc2]). Special cases were also studied by Christopher [C]. Since these will not be needed here we do not describe the results.

In this paper we study cubic systems symmetric with respect to a non-degenerate center. We suppose that the center is at the origin and that the system has the form (1.1). An algebraic characterization of centers for system (1.1) was first given by Malkin [M]. To simplify the conditions, Sibirsky applied first a rotation to obtain $a_{21}+b_{12}=0$ in (1.1) [SI]. He wrote such a system in the form

$$\begin{aligned} \dot{x} &= -y-(\omega+\theta-a)\,x^3-(\eta-3\mu)\,x^2y \\ &\quad -(3\omega-3\theta+2a-\xi)\,xy^2-(\mu-\nu)\,y^3 \\ \dot{y} &= x+(\mu+\nu)\,x^3+(3\omega+3\theta+2a)\,x^2y \\ &\quad +(\eta-3\mu)\,xy^2+(\omega-\theta-a)\,y^3. \end{aligned} \tag{2.12}$$

THEOREM 2.4 [M, Si]. *System* (1.1) *has a center at the origin if and only if one of the following conditions is satisfied*:

$$\text{(H)}\quad \xi=a=0 \tag{2.13}$$

$$\text{(II)}\quad \xi=\nu=\theta=0 \tag{2.14}$$

$$\text{(III)}\quad \xi=\nu=\omega=\eta=4(\mu^2+\theta^2)-a^2=0 \tag{2.15}$$

The condition (H) corresponds to Hamiltonian systems. We shall prove the integrability of the systems (II) and (III) by the method of Darboux. The necessity of the conditions comes from the calculations of the Lyapunov constants (see [Sh] for the definition and method), which must all vanish at a center:

THEOREM 2.5 [Si]. *System* (2.12) *has the following Lyapunov coefficients* V_i, *under the condition that* $V_j=0$ *for* $j<i$:

$$\begin{aligned} V_1 &= \xi \\ V_2 &= -a\nu \\ V_3 &= a\theta\omega \\ V_4 &= a^2\theta\eta \\ V_5 &= -a^2\theta[4(\mu^2+\theta^2)-a^2]. \end{aligned} \tag{2.16}$$

3. Centers of Type II

We consider systems (2.12) of type II (i.e., satisfying (2.14)). Such systems are symmetric with respect to two orthogonal lines. We can make a rotation so as to assume that the symmetry-axes are the coordinates axes. Hence we consider a system

$$\begin{aligned} \dot{x} &= -y + ax^2y + by^3 \\ \dot{y} &= x + cx^3 + dxy^2, \end{aligned} \tag{3.1}$$

with a, b, c, d not all zero.

Our study of these systems consists of several steps. First we use a reduction to a linear system to show integrability. This achieves the proof of the sufficiency of the Malkin's conditions. We then describe the invariant algebraic curves of the systems, which are significant in the derivation of the bifurcation diagram of the family of systems (3.1). Finally we derive the bifurcation diagram.

3.1. *Reduction to a Linear System*

This reduction was suggested to us by R. Kooij.

Proposition 3.1. *The change of variables*

$$(u, v) = (x^2, y^2), \tag{3.2}$$

together with the time-rescaling $T = 2xyt$ brings the system (3.1) *to the linear system*

$$\begin{aligned} \dot{u} &= -1 + au + bv \\ \dot{v} &= 1 + cu + dv. \end{aligned} \tag{3.3}$$

It is a diffeomorphism in the first open quadrant. Since the coordinate axes of (3.1) are symmetry axes the whole phase portrait can be recovered from the phase portrait of (3.3) in the first quadrant.

First integrals for system (3.3) will yield first integrals of system (3.1) via (3.2). Also the phase portrait of (3.1) can be recovered from integral curves of (3.3) in the first quadrant.

3.2. *First Integrals of a Linear System*

We consider here linear systems (2.1) with max(deg P, deg Q) = 1. Such systems will be called non-constant. We derive the first integral of any such non-constant linear system from its invariant algebraic curves. For this we reduce the linear system to a convenient normal form.

TABLE I

Normal form	First integral	Invariant curves
$\dot{x}=1$ $\dot{y}=x$	$F_1=y-x^2/2$ Hamiltonian system	$y=x^2/2+C, \quad C\in\mathbb{R}$
$\dot{x}=0$ $\dot{y}=x$	$F_2=x$ Hamiltonian system	$x=C, \quad C\in\mathbb{R}$ $x=0$ (line of sing. pts)
$\dot{x}=x$ $\dot{y}=a+bx+cy$ $c\neq 0, 1$	$F_3=x^{-c}[c(c-1)\,y+bcx+a(c-1)]$	$x=0$ $bcx+c(c-1)\,y+a(c-1)=0$
$\dot{x}=x$ $\dot{y}=a+bx$ $a\neq 0$	$F_4=x^a e^{bx-y}$	$x=0$
$\dot{x}=x$ $\dot{y}=bx$	$F_5=y-bx$	$x=0$ (line of sing. pts) $y=bx+C, \quad C\in\mathbb{R}$
$\dot{x}=x$ $\dot{y}=bx+y$ $b\neq 0$	$F_6=x^{-b}e^{y/x}$	$x=0$
$\dot{x}=x$ $\dot{y}=y$	$F_7=y/x$	$C_1x+C_2y=0, \quad C_1, C_2\in\mathbb{R}$

THEOREM 3.2. *The first integral of any non-constant linear system on* $\mathbb{C}^2$ *is given in Table* I.

Proof. Suppose first that the matrix of the linear system is nilpotent. Then we can bring it to its Jordan normal form. A translation in x transforms the system into one of the first two forms of Table I. The system is Hamiltonian, from which the integral follows.

If the matrix is not nilpotent, then a linear transformation brings it to a lower triangular matrix, with a nonzero coefficient in the upper left corner. Scaling, together with a translation in x, brings the system to the form

$$\begin{aligned}\dot{x}&=x\\ \dot{y}&=a+bx+cy,\end{aligned} \tag{3.4}$$

which always has the invariant line $x=0$. Due to this we have a Darboux first integral or an integrating factor. In case $c\neq 0, 1$, then the system has a second invariant line and a Darboux first integral given in Table I. In cases $c=0$, $a\neq 0$ and $c=1$, $b\neq 0$ integrating factors can be found as described in Section 2 and integration yields elementary first integrals. ∎

Remark 3.3. (1) We point out that all cases of Table I except the first can be derived from the generic Darboux integral F_3. Indeed

$$F_4 = \left(\lim_{c \to 0} x \left[1 + \frac{c}{a(c-1)} (bx + (c-1)\, y) \right]^{-1/c} \right)^a, \tag{3.5}$$

$$F_5 = \log(\lim_{a \to 0} F_4), \tag{3.6}$$

$$F'_6 = \lim_{c \to 1} \left(x \left[1 + \frac{c-1}{bc} \frac{cy+a}{x} \right]^{1/(1-c)} \right)^{-b}, \tag{3.7}$$

and F_6 is obtained from F'_6 by changing $y \mapsto y - a$,

$$F_7 = \log(\lim_{b \to 0} F_6). \tag{3.8}$$

To derive the second case we consider the system

$$\begin{aligned} \dot{x} &= dx \\ \dot{y} &= a + bx + cy, \end{aligned} \tag{3.9}$$

with the Darboux first integral

$$F'_3 = x^{-c} [bcx + c(c-d)\, y + a(c-d)]^d. \tag{3.10}$$

Then

$$F_2 = (\lim_{d \to 0} F'_3)^{-1/c}. \tag{3.11}$$

(2) Essentially two important phenomena occur with the invariant lines: one of them can disappear at infinity or they can coalesce. In both cases the first integral (F_4 or F_6) can be obtained as a limit process. Such phenomena have already been noted by Christopher [C].

3.3. *First Integrals of System* (3.1)

The existence of first integrals for system (3.1) is a consequence of the reduction of such systems to linear ones and of Theorem 3.2. Invariant lines of the linear system (3.3) correspond to invariant conics of the cubic system due to the change (3.2).

THEOREM 3.4. *For all values of the parameters the system* (3.1) *has the following first integrals* $F(x, y)$:

(1) *if* $(a-b+c-d)(ad-bc)\neq 0$, *and* $(d-a)^2+4bc>0$ *then*

$$F(x,y)=(f_{20}^{+}x^2+f_{02}^{+}y^2+1)^{(a+d+\sqrt{(a-d)^2+4bc})}$$
$$\times(f_{20}x^2+f_{02}y^2+1)^{(-a-d+\sqrt{(a-d)^2+4bc})}, \qquad (3.12)$$

with

$$f_{20}^{\pm}=\frac{2(ad-bc)-(a+c)(a+d)\pm(a+c)\sqrt{(a-d)^2+4bc}}{2(a-b+c-d)},$$
$$f_{02}^{\pm}=\frac{2(ad-bc)-(b+d)(a+d)\pm(b+d)\sqrt{(a-d)^2+4bc}}{2(a-b+c-d)}; \qquad (3.13)$$

(2) *if* $(a-b+c-d)(ad-bc)\neq 0$, *and* $\Delta=(d-a)^2+4bc<0$ *then*

$$F=\left[\frac{ad-bc}{a-b+c-d}\left(cx^4+(d-a)x^2y^2-by^4\right.\right.$$
$$\left.\left.+\frac{\alpha}{ad-bc}x^2+\frac{\beta}{ad-bc}y^2\right)+1\right]^{\sqrt{-\Delta}}$$
$$\times\exp\left[-2(a+d)\arctan\left(\frac{\sqrt{-\Delta}((a+c)x^2+(b+d)y^2)}{\alpha x^2+\beta y^2+2(a-b+c-d)}\right)\right], \qquad (3.14)$$

with

$$\alpha=2(ad-bc)-(a+c)(a+d)$$
$$\beta=2(ad-bc)-(b+d)(a+d); \qquad (3.15)$$

(3) *if* $a-b+c-d=0$ *and* $(b+c)(a-b)(a+c)\neq 0$, *then*

$$F(x,y)=(x^2+y^2)^{(b+c-a-d)/2(a+c)}\left[\frac{-c(a-b)}{b+c}x^2+\frac{b(a-b)}{b+c}y^2+1\right]; \qquad (3.16)$$

(4) *if* $(d-a)^2+4bc=0$ *and* $(a-b+c-d)(ad-bc)\neq 0$, *then*

$$F(x,y)=\left[\frac{-c(a+d)}{a+2c-d}x^2+\frac{a^2-d^2}{2(a+2c-d)}y^2+1\right]$$
$$\times\exp\left[\frac{4c(a+d)[(a+c)x^2+(b+d)y^2]}{(a+2c-d)[-2c(a+d)x^2+(a^2-d^2)y^2+2(a+2c-d)]}\right]; \qquad (3.17)$$

(5) *if* $a-b+c-d=b+c=0$ *and* $c(a+c)\neq 0$, *then*

$$F(x,y)=(x^2+y^2)\exp\left[-\frac{1+(a+c)\,y^2}{c(x^2+y^2)}\right]; \tag{3.18}$$

(6) *if* $ad-bc=0$ *and* $(a-b+c-d)(a+c)(a+d)\neq 0$ *then*

$$F(x,y)=e^{(a+d)(cx^2-ay^2)/(a+c)}\left[-\frac{(a+c)(a+d)}{a-b+c-d}x^2 -\frac{(b+d)(a+d)}{a-b+c-d}y^2+1\right]; \tag{3.19}$$

(7) *if* $ad-bc=0$ *and* $(a-b+c-d)(a+d)(b+d)\neq 0$ *then*

$$F(x,y)=e^{(a+d)(dx^2-by^2)/(b+d)}\left[-\frac{(a+c)(a+d)}{a-b+c-d}x^2 -\frac{(b+d)(a+d)}{a-b+c-d}y^2+1\right]; \tag{3.20}$$

(8) *if* $a-b=c-d=0$ *and* $b+c\neq 0$, *then*

$$F(x,y)=e^{cx^2-ay^2}(x^2+y^2); \tag{3.21}$$

(9) *if* $a+d=0$, *the system is Hamiltonian with*

$$F(x,y)=\tfrac{1}{2}(x^2+y^2)+\tfrac{1}{4}(cx^4-2ax^2y^2-by^4); \tag{3.22}$$

(10) *if* $a-d=b=c=0$, *then*

$$F(x,y)=\frac{-1+ax^2}{x^2+y^2}; \tag{3.23}$$

(11) *finally, if* $a-b+c-d=0$ *and* $a+c=b+d=0$, *then*

$$F(x,y)=x^2+y^2. \tag{3.24}$$

Proof. Long but straightforward calculations show that the functions are indeed first integrals. We explain in the remark below how these integrals were obtained.

First let us check that the list of conditions covers all possible parameter values $(a, b, c, d)\in\mathbb{S}^3$.

When $(a-b+c-d)(ad-bc)((d-a)^2+4bc)\neq 0$ we are in case (1) or (2) depending of the sign of Δ. In case (2) (3.12) is a first integral but it takes complex values. If we raise (3.12) to the power i ($i^2=-1$) then we get the first integral (3.14) which takes real values.

If $(a-d)^2+4bc=0$ and $(ad-bc)(a-b+c-d)\neq 0$ we are in case (4).

If $ad-bc=0$ and $[(a-d)^2+4bc](a-b+c-d)\neq 0$ we have $a+d\neq 0$ (otherwise $(a-d)^2+4bc=0$) and at least one of $a+c\neq 0$ or $b+d\neq 0$, yielding case (6) or (7).

If $a-b+c-d=0$ and $[(a-d)^2+4bc](ad-bc)\neq 0$, then necessarily $b+c\neq 0$ (otherwise we have $(a-d)^2+4bc=0$) and $(a-b)(a+c)\neq 0$ (otherwise we have $ad-bc=0$), yielding case (3).

If $(a-d)^2+4bc=ad-bc=0$ then $a+d=0$ and the system is Hamiltonian, yielding case (9).

If $(a-d)^2+4bc=a-b+c-d=0$, then necessarily $b+c=0$. Integrals are given by (5), (10) or (11) depending whether $a+c$ and c vanish or not.

Finally if $ad-bc=a-b+c-d=0$ then necessarily $(a-b)(a+c)=0$ and integrals are given in (8) or (11). ∎

How to Obtain the Integrals of (3.1). What we learn from Table 1 of Theorem 3.2 is the following: a non-Hamiltonian linear system always has an invariant line. Generically it has a second invariant line, yielding a Darboux integral. If one of the invariant line disappears at infinity then we know we must look for a first integral of the form $f\exp(g)$, with f and g polynomials of degree 1 in x and y. If the two invariant lines coallesce then we must look for a first integral of the form $f\exp(g/f)$, $f=0$ being the equation of the invariant line and g being a polynomial of degree 1 in x and y.

When we transform (3.1) into the linear system (3.3) via (3.2) we see that invariant lines of system (3.3) are in (1.1) correspondence with invariant symmetric conics $f_{20}x^2+f_{02}y^2+C=0$. These conics can be reducible or irreducible.

Hence, in practice we calculate directly the invariant algebraic curves of system (3.1). These are determined below. The condition $ad-bc=0$ corresponds to one of the conics passing at infinity (cases (5), (6), (7)). In this case we look for an integral of the form $f\exp(g)$, with $f=0$ an invariant conic and $g=Ax^2+By^2+C$. Similarly the condition $(d-a)^2+4bc=0$ corresponds to the coalescence of the two invariant conics (cases (3), (4)) and the form of the integral is $f\exp(g/f)$, with $f=0$ an invariant conic and $g=Ax^2+By^2+C$.

Remark 3.5. From Theorem 3.2 it is clear that condition $a-b+c-d=0$ has no special meaning as far as invariant lines of the linear system are concerned. In the cubic system it corresponds to the particular case of a reducible conic passing through the origin.

3.4. *Invariant Algebraic Curves of the Family of Systems* (3.1)

In this paragraph we study the low order invariant algebraic curves of the systems (3.1). These curves are helpful in the study of the family of systems (3.1). They also yield directly the integrals listed in Theorem 3.4.

Remark 3.6. The system (3.1) is symmetric under

$$(x, y, t, a, b, c, d) \mapsto (y, x, -t, -d, -c, -b, -a) \tag{3.25}$$

PROPOSITION 3.7. *Invariant lines for system* (3.1) *occur precisely in the following cases*

(1) $b=0$, $a \neq 0$, *in which case we have the two invariant lines which are components of* $ax^2-1=0$.

(2) $c=0$, $d \neq 0$, *in which case we have the two invariant lines, components of* $dy^2+1=0$.

(3) $a-b+c-d=0$ *in which case we have the two invariant lines, components of* $x^2+y^2=0$.

(4) $a+2b+c=b+2c+d=0$, $b, c \neq 0$, *in which case we have the four invariant lines*

$$\pm\sqrt{-c}\,x \pm \sqrt{b}\,y+1=0. \tag{3.26}$$

Some of these cases may occur simultaneously.

Proof. We first look for lines not passing through the origin. We take $f(x, y)=Ax+By+1$ and $K(x, y)$ as in (2.5), with $n=3$. Identifying linear and quadratic terms of the equation $Df=fK$ gives us $k_{00}=0$, $k_{10}=B$, $k_{01}=-A$, $k_{20}=-AB=-k_{02}$, $k_{11}=A^2-B^2$. Identification of the third order terms gives us the equations

$$\begin{aligned} B(A^2+c)&=0\\ A(-A^2+2B^2+a)&=0\\ B(-2A^2+B^2+d)&=0\\ A(B^2-b)&=0, \end{aligned} \tag{3.27}$$

which have the solutions

$$\begin{gathered} A=B=0\\ B=b=A^2-a=0\\ A=c=B^2+d=0\\ a+2b+c=b+2c+d=A^2+c=B^2-b=0. \end{gathered} \tag{3.28}$$

The case of lines passing through the origin is done similarly. ∎

We now look for invariant irreducible conics. We limit ourselves to conics symmetric with respect to the coordinate axes, since this will be sufficient to find the first integrals (non-symmetric conics could occur by pairs or 4-tuples). An irreducible symmetric invariant conic has an equation

$$f(x, y) = 1 + f_{20}x^2 + f_{02}y^2 = 0, \tag{3.29}$$

with $f_{20}f_{02} \neq 0$ and $f_{20}, f_{02} \in \mathbb{C}$.

We want to see under what conditions on the coefficients of the system we have such invariant conics and how many of them exist. For this we consider the equation $Df = fK$, with f and K as in (3.29) and (2.5). The calculations yield $k_{00} = k_{10} = k_{01} = 0 = k_{20} = k_{02} = 0$ and $k_{11} = 2(f_{02} - f_{20})$, giving

$$K(x, y) = 2(f_{02} - f_{20})\, xy. \tag{3.30}$$

Identifying the coefficients of x^3y and xy^3 we obtain the equations:

$$f_{20}a + f_{02}c = f_{20}(f_{02} - f_{20}), \tag{3.31}$$

$$f_{20}b + f_{02}d = f_{02}(f_{02} - f_{20}). \tag{3.32}$$

The equations (3.31) and (3.32) in the variables f_{20} and f_{02} define two conics whose projective completions have generically four intersection points, possibly counted with multiplicities (by Bezout's theorem). One of these points is (0, 0) in the affine plane (f_{20}, f_{02}), yielding no irreducible conic (3.29). The equations (3.31) and (3.32) have exactly one point of intersection at infinity, the point (1, 1, 0), corresponding to the line $f_{20} = f_{02}$. So there are at most two other intersection points of (3.31) and (3.32) in the affine plane (f_{20}, f_{02}).

From the form of their quadratic part, the two conics can have a common component only when (3.31) and (3.32) are both reducible. These cases are studied in Proposition 3.8. The remaining cases are studied in Proposition 3.9 and Theorem 3.10.

PROPOSITION 3.8. *The equations* (3.31) *and* (3.32) *are both reducible if and only if* $c(a + c) = 0 = b(b + d)$, *in which case system* (3.1) *has an irreducible conic if and only if we are in one of the following situation*

(1) $b = c = a - d = 0 \neq a$, *in which case the conic is anyone of the family*

$$1 + \alpha x^2 + (\alpha + a)\, y^2 = 0, \qquad \alpha(\alpha + a) \neq 0, \quad \alpha \in \mathbb{R}. \tag{3.33}$$

The integral in this case is rational.

(2) $c=b+d=0$ *and* a, d, $d-a\neq 0$, *in which case we have a unique irreducible invariant conic*

$$1+(d-a)\,x^2+dy^2=0. \tag{3.34}$$

(3) $a=c=b+d=0\neq b$, *in which case we have an infinite family of conics*

$$\alpha(x^2+y^2)+1=0, \qquad \alpha\neq 0. \tag{3.35}$$

In this case we have two symmetric lines of singular points $by^2-1=0$.

(4) $b=a+c=0$ *and* a, d, $d-a\neq 0$, *in which case we have a unique irreducible invariant conic*

$$1-ax^2+(d-a)\,y^2=0. \tag{3.36}$$

(5) $b=d=a+c=0\neq a$, *in which case we have an infinite family of conics* (3.35). *In this case we have two symmetric lines of singular points* $ax^2-1=0$.

(6) $a+c=b+d=0$ *and* a, $b\neq 0$, *in which case we have an infinite family of conics* (3.35) *and we have a conic of singular points* $1=ax^2+by^2$.

Proof. The equations (3.31) and (3.32) define a 4-parameter family of complex curves in the plane (f_{20}, f_{02}), the parameters being (a, b, c, d). Applying the criterion for reducibility of a conic in homogeneous coordinates in terms of its associate 3×3-determinant we have that the equation (3.31) (resp. (3.32)) is a reducible conic if and only if $c(a+c)=0$ (resp. $b(b+d)=0$). Let us assume that the conics are reducible. Then the parameters must satisfy one of the equations

$$\begin{aligned} &\text{(i)} \quad b=c=0\\ &\text{(ii)} \quad c=0=b+d\\ &\text{(iii)} \quad b=0=a+c\\ &\text{(iv)} \quad a+c=b+d=0. \end{aligned} \tag{3.37}$$

Let us consider these cases one-by-one

(i) $b=c=0$. In homogeneous coordinates (X, Y, Z), the two conics (3.31) and (3.32) are

$$\begin{aligned} X(aZ-Y+X)&=0\\ Y(dZ-Y+X)&=0. \end{aligned} \tag{3.38}$$

Only the solution $X-Y=-aZ=-dZ$ can produce irreducible conics (3.29). We have two possibilities: $Z=0=X-Y$ is a point at infinity of the affine plane (f_{20}, f_{02}). The second possibility is $a=d$ and $X-Y+aZ=0$, leading to family (3.33).

(ii) $c=0=b+d$. In projective coordinates the two conics are

$$\begin{aligned} X(aZ-Y+X)&=0 \\ (Y-X)(dZ-Y)&=0. \end{aligned} \tag{3.39}$$

Intersection points with $X=0$ yield no irreducible conic for (3.1). In case $Y-X=0=X-Y+aZ$, then $a=0$ or $Z=0$. In the latter we get the point at infinity, while in the former we have an infinite number of invariant conics $f_{20}(x^2+y^2)+1=0$ and the integral is rational, yielding case (3).

Finally, when $dZ-Y=aZ-Y+X=0$, then $Y=dZ$ and $X=(d-a)Z$. We have a unique irreducible conic $1+(d-a)x^2+dy^2=0$, for a, $d(d-a)\neq 0$, yielding case (2).

(iii) $b=0=a+c$. This case reduces to the previous one using symmetry (3.25) and yields cases (4) and (5).

(iv) $a+c=b+d=0$. The conclusion follows from the form of (3.1) in this case. ∎

PROPOSITION 3.9. *Assume that* $c(a+c)\neq 0$ *or* $b(b+d)\neq 0$. *Then*

(1) *the system* (3.1) *has at most two irreducible symmetric invariant conics* (3.29);

(2) *under the additional condition* $a-b+c-d=0$, *the system has two invariant lines* $(x^2+y^2=0)$ *and one irreducible conic*

$$-\frac{c(a-b)}{b+c}x^2+\frac{b(a-b)}{b+c}y^2+1=0, \tag{3.40}$$

precisely when $bc(a-b)(b+c)\neq 0$.

Proof. (1) It follows from the explanations before Proposition 3.8.

(2) Let us suppose for instance that $c(a+c)\neq 0$ (the other case follows using symmetry (3.25) and $a-b+c-d=0$. It is clear that $f_{20}=c$ is not a solution of (3.32) since $a+c=0$. We solve (3.31) for f_{02} and replace in (3.32). Since $f_{20}f_{02}\neq 0$ we get that f_{20} is solution of:

$$(b+c)(a+c)f_{20}+c(a-b)(a+c)=0, \tag{3.41}$$

which has a non-zero solution precisely when $(a-b)(b+c)\neq 0$. ∎

THEOREM 3.10. *We consider system* (3.1) *under condition* $a-b+c-d\neq 0$.

(1) *A necessary and sufficient condition for the system to have exactly two symmetric invariant irreducible conics* (3.29) *is that* $bc(ad-bc)((a-d)^2+4bc)\neq 0$. *The coefficients* f_{20} *and* f_{02} *are given by*

$$f_{20}=\frac{2(ad-bc)-(a+c)(a+d)\pm(a+c)\sqrt{(a-d)^2+4bc}}{2(a-b+c-d)} \tag{3.42}$$

$$f_{02}=\frac{2(ad-bc)-(b+d)(a+d)\pm(b+d)\sqrt{(a-d)^2+4bc}}{2(a-b+c-d)}. \tag{3.43}$$

A first conic, $F^+=0$, *corresponds to the two plus signs and a second one,* $F^-=0$, *to the two minus signs.*

(2) *If* $b=0$ *and* $cd(a-d)\neq 0$ *the system has one invariant irreducible conic*

$$-\frac{cd}{a+c-d}x^2+\frac{d(a-d)}{a+c-d}y^2+1=0. \tag{3.44}$$

(3) *If* $c=0$ *and* $ab(a-d)\neq 0$ *the system has one invariant irreducible conic*

$$\frac{a(d-a)}{a-b-d}x^2-\frac{ab}{a-b-d}y^2+1=0. \tag{3.45}$$

(4) *If* $ad-bc=0$ *and* $(a+c)(b+d)(a+d)\neq 0$ *the system has one invariant irreducible conic*

$$1-\frac{(a+c)(a+d)}{a-b+c-d}x^2-\frac{(b+d)(a+d)}{a-b+c-d}y^2=0. \tag{3.46}$$

(5) *If* $(a-d)^2+4bc=0$ *and* $bc(a+d)\neq 0$ *the system has one invariant irreducible conic*

$$1-\frac{c(a+d)}{a+2c-d}x^2+\frac{(a-d)(a+d)}{2(a+2c-d)}y^2=0. \tag{3.47}$$

Proof. We consider the equation $Df=fK$, with f and K given in (3.29) and (3.30), and we end with Eqs. (3.31) and (3.32). The first (second) equation is linear in f_{02} (f_{20}). In case $f_{20}\neq c$ and $f_{02}\neq -b$ we can solve (3.31) (resp. (3.32)) for f_{02} (resp. f_{20}) and replace in (3.32) (resp. (3.31)). Since $f_{20} f_{02} \neq 0$ we get that f_{20} and f_{02} are solutions of:

$$\begin{aligned}(a-b+c-d)\, f_{20}^2 &+ ((a+c)(a+d) \\ &-2(ad-bc))\, f_{20} + c(ad-bc) = 0, \end{aligned} \tag{3.48}$$

$$\begin{aligned}(a-b+c-d)\, f_{02}^2 &+ ((b+d)(a+d) \\ &-2(ad-bc))\, f_{02} - b(ad-bc) = 0. \end{aligned} \tag{3.49}$$

(3.48) and (3.49) have distinct solutions precisely when $(d-a)^2 + 4bc \neq 0$.

We now consider the cases $f_{20} = c$ and $f_{02} = -b$. If $f_{20} = c \neq 0$, then (3.31) gives us $a + c = 0$. Equation (3.32) becomes

$$f_{02}^2 - f_{02}(c+d) - cb = 0, \tag{3.50}$$

yielding solutions

$$f_{02} = \frac{c + d \pm \sqrt{(c+d)^2 + 4bc}}{2}. \tag{3.51}$$

These coincide with (3.43). In this case the two conics are tangent. The case $f_{02} = -b$ is done similarly. The two conics are irreducible precisely when (3.48) and (3.49) have no zero solutions, i.e., $bc(ad-bc) \neq 0$. The points (2)–(5) follow from straightforward verification. ∎

3.5. *Phase Portraits of the Systems* (3.1) *and Their Bifurcation Diagrams*

We begin by studying the nature of the singular points.

PROPOSITION 3.11. *System* (3.1) *has the following singular points*

(1) *the origin which is a center;*

(2) $Q^{\pm} = (0, \pm\sqrt{1/b})$ *for* $b > 0$, *with eigenvalues* $\lambda_{1,2} = \pm\sqrt{2(b+d)/b}$. *The points are saddles (resp. centers) for* $b + d > 0$ *(resp.* $b + d < 0$*). They are nilpotent for* $b + d = 0$, *of focus type if* $(a-d)^2 + 4bc < 0$, *elliptic type if resp.* $(a-d)^2 + 4bc > 0$, $ad - bc > 0$ *and saddle type if* $(a-d)^2 + 4bc > 0$, $ad - bc < 0$ *(see* [D] *for nilpotent points);*

(3) $Q_{\pm} = (\pm\sqrt{-1/c}, 0)$ *for* $c < 0$, *with eigenvalues* $\lambda_{1,2} = \pm\sqrt{2(a+c)/c}$. *They are saddles for* $a + c < 0$ *and centers for* $a + c > 0$. *They are nilpotent for* $a + c = 0$, *of focus type if* $(a-d)^2 + 4bc < 0$, *elliptic type if resp.* $(a-d)^2 + 4bc > 0$, $ad - bc > 0$ *and saddle type if* $(a-d)^2 + 4bc > 0$, $ad - bc < 0$;

(4) *the four points* $P_{\pm\pm} = (\pm\sqrt{(b+d)/(ad-bc)}, \pm\sqrt{-(a+c)/(ad-bc)})$ *for* $(b+d)(ad-bc) > 0$ *and* $(a+c)(ad-bc) < 0$, *whose eigenvalues are real and distinct for* $(a-d)^2 + 4bc > 0$, *in which case they are saddles if and only if* $ad - bc < 0$ *and nodes (alternately stable and unstable) if and only if*

$ad-bc>0$. If $(a-d)^2+4bc<0$ they are foci (alternately stable and unstable) or centers when $a+d=0$;

(5) *two pairs of singular points at infinity for $bc>0$, four pairs of singular points at infinity for $bc<0$, $b(d-a)>0$ and $(d-a)^2+4bc>0$, no singular point at infinity if $(d-a)^2+4bc<0$ or $((d-a)^2+4bc>0$, $bc<0$, and $b(d-a)<0)$. The points undergo bifurcations at $b=0$ (birth of two singular points from the y-direction in a nilpotent bifurcation involving a finite point), $c=0$ (similar in the x-direction), saddle-node bifurcation along the equator when $(d-a)^2+4bc=0$, and transcritical bifurcation between infinite points and finite points when $ad-bc=0$.*

Proof. (2) The Jacobian matrix at the points $(0, \pm\sqrt{1/b})$ is given by

$$J=\begin{pmatrix} 0 & 2 \\ 1+d/b & 0 \end{pmatrix}. \tag{3.52}$$

To study the type of the singular point when $b+d=0$, we localize the system at for instance $(0, \sqrt{1/b})$, by letting $y=Y+\sqrt{1/b}$. System (3.1) becomes

$$\begin{aligned} \dot{x} &= 2Y+\frac{a}{\sqrt{b}}x^2+(ax^2Y+bY^3+3\sqrt{b}\,Y^2) \\ \dot{Y} &= -2\sqrt{b}\,xY+cx^3+(-bxY^2). \end{aligned} \tag{3.53}$$

We next make the following weighted blow-up (as in [BM]):

$$\begin{aligned} x &= \varepsilon \\ Y &= \varepsilon^2 Z. \end{aligned} \tag{3.54}$$

After dividing by ε, the local system (3.53) becomes

$$\begin{aligned} \dot{\varepsilon} &= \frac{a}{\sqrt{b}}\varepsilon+2\varepsilon Z+o(\varepsilon) \\ \dot{Z} &= c-\frac{2}{\sqrt{b}}(a+b)\,Z-4Z^2+O(\varepsilon). \end{aligned} \tag{3.55}$$

It is easily checked that the condition for $\dot{Z}=0$ to have roots is precisely $(a-d)^2+4bc\geqslant 0$, yielding that the center case occurs when $(d-a)^2+4bc<0$. To decide between elliptic and saddle cases it is enough to check if the point splits into two nodes and a saddle or two saddles and a center.

(3) It is the same as (2) using the symmetry (3.25).

(4) The Jacobian matrix at these points is

$$J=\begin{pmatrix} 2axy & 2by^2 \\ 2cx^2 & 2dxy \end{pmatrix}. \tag{3.56}$$

The characteristic equation is $\lambda^2-(\operatorname{Tr} J)\,\lambda+\det J=0$ with discriminant $\operatorname{disc} J=-4(a+c)(b+d)[(a-d)^2+4bc)]/(ad-bc)^2$. Under the hypothesis of the existence of $P_{\pm\pm}$, $\operatorname{disc} J\geqslant 0$ if and only if $(d-a)^2+4bc\geqslant 0$.

(5) Let us first suppose $bc\neq 0$. To find the singular points at infinity we use the coordinates $(u, z)=(y/x,\ 1/x)$. After multiplication by z^2 the system becomes

$$\begin{aligned} \dot u &= c+(d-a)\,u^2-bu^4+z^2+u^2z^2 \\ \dot z &= -(au+bu^3)\,z+uz^3. \end{aligned} \tag{3.57}$$

Singular points at infinity are given by $z=0$ and the equation

$$-bu^4+(d-a)\,u^2+c=0, \tag{3.58}$$

yielding two singular points at infinity when $bc>0$ (case I) and four singular points if $bc<0$, $b(d-a)>0$ and $\Delta>0$ (case II). The Jacobian matrix at the singular points is given by

$$J=\begin{pmatrix} -4bu^3+2(d-a)\,u & 0 \\ 0 & -bu^3-au \end{pmatrix}=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{3.59}$$

Let us call u_1 (resp. $u_1\geqslant u_0$) the positive roots of (3.58) in case I (resp. case II). Using the symmetry (3.25) we can suppose $b>0$. Then necessarily $\lambda_1(u_1)<0$ and $\lambda_1(u_0)>0$. We also have:

$$u_i\lambda_2=-du_i^2-c=\frac{-d(d-a)-2bc\mp d\sqrt{\Delta}}{2b}. \tag{3.60}$$

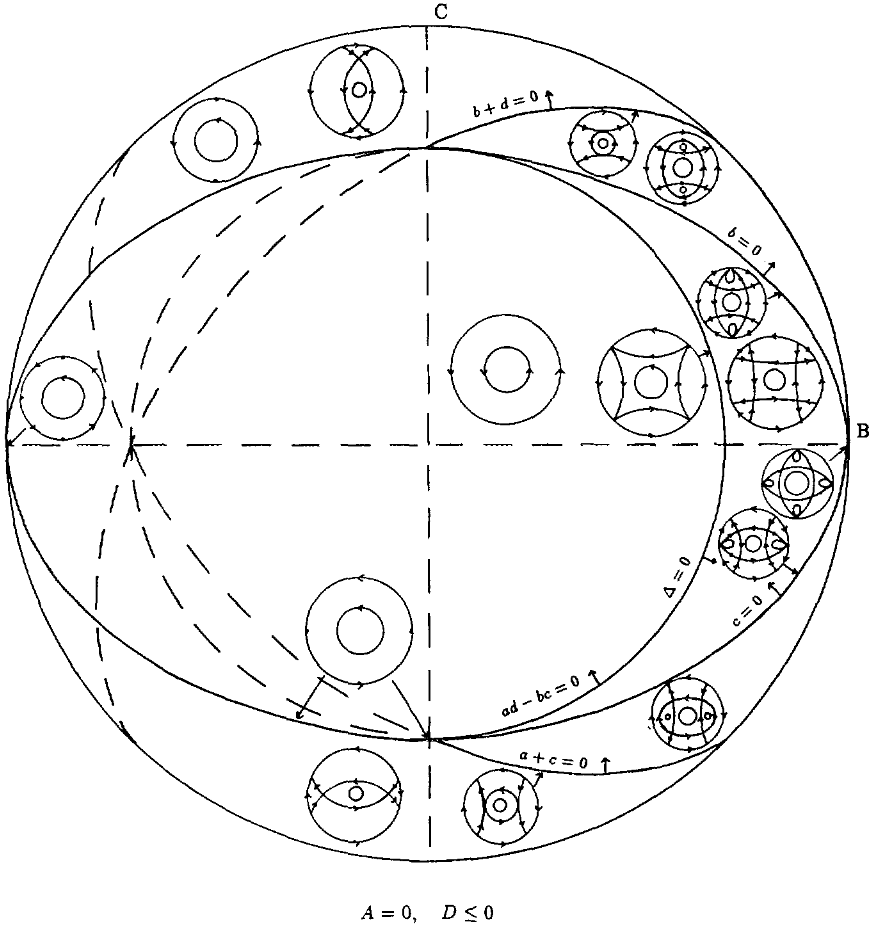
In case II, when $a+d\neq 0$, $\lambda_2(u_1)$ and $\lambda_2(u_0)$ necessarily have the same sign near $\Delta=0$ and their sign is the sign of $-d(d-a)-2bc$. One of the $\lambda_2(u_i)$ changes sign when $ad-bc$ changes sign. This occurs when the singular points $(\pm u_i, 0)$ coincide with the points $P_{\pm\pm}$. In case I then $\lambda_2(u_1)$ has the sign of $-d(d-a)-2bc-d\sqrt{\Delta}$. The case $b=0$ and/or $c=0$ can be studied easily: there are singular points at infinity on one or the two axes and their type is determined by blow-up. ∎

PROPOSITION 3.12. *The conics have real coefficients f_{20} and f_{02} precisely when $\Delta=(a-d)^2+4bc\geqslant 0$. In case $\Delta<0$ the real part of the conics is* *reduced to four real points, the points* $P_{\pm\pm}$ *of Proposition* 3.11, *when these exist. Otherwise the conics have no real points. When* $\Delta \geqslant 0$ *the following cases occur:*

(1) *The conic* $F_+ = 0$ *(resp.* $F_- = 0$*) is a hyperbola and passes through the singular points at infinity given by* $\cot^2 \theta = (a - d + \sqrt{\Delta})/2c$ *(resp.* $\cot^2 \theta = (a - d - \sqrt{\Delta})/2c$*) precisely when these points exist.*

(2) *The two conics* $F_+ = 0$ *and* $F_- = 0$ *intersect at the singular points* $P_{\pm\pm}$ *when these exist.*

(3) *If* $bc > 0$ *then the two conics are either an ellipse and a hyperbola, or a hyperbola and the empty conic (i.e.,* $f_{20}, f_{02} > 0$ *in* (3.29)).



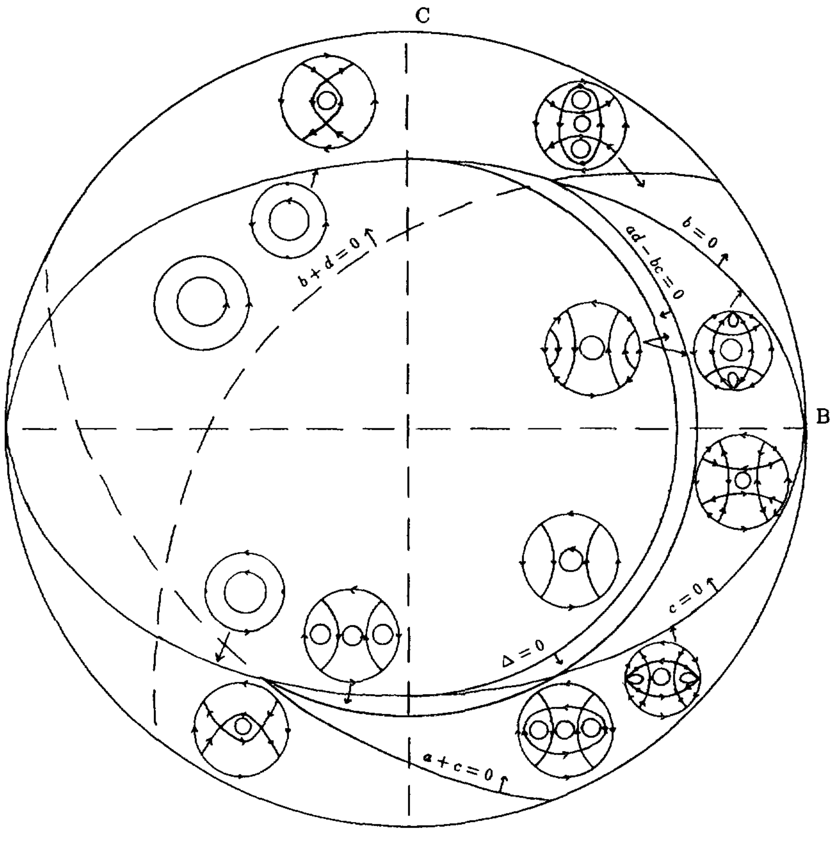
$A = 0, \quad D \leq 0$

FIGURE 1

(4) *If* $bc < 0$ *the two conics can be*

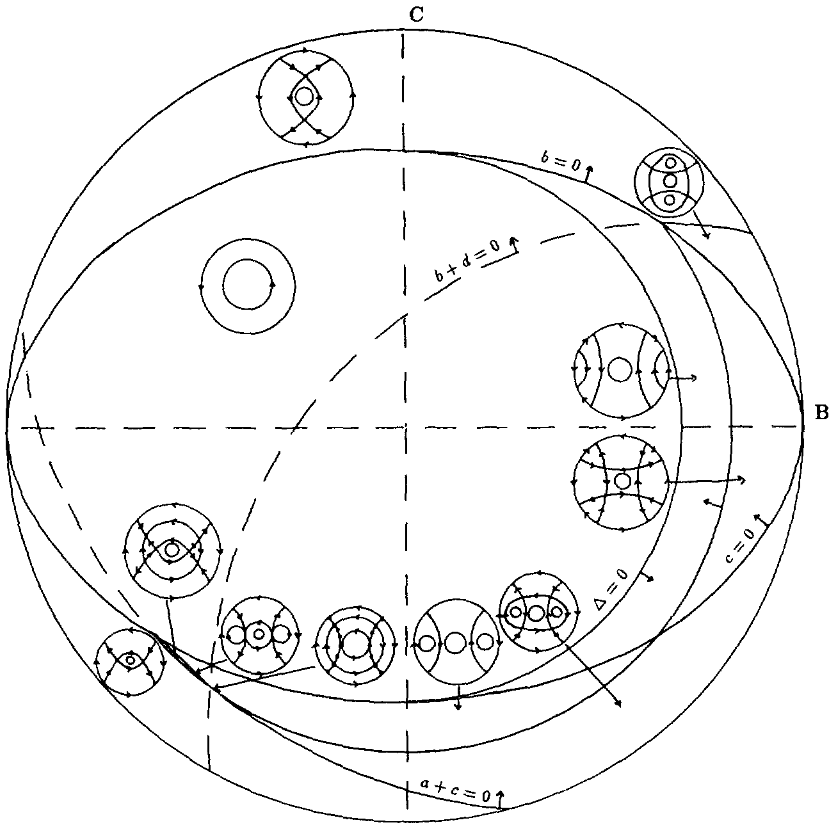
- *two ellipses,*
- *an ellipse and an empty conic,*
- *two empty conics,*
- *two hyperbolas with same direction,*
- *two hyperbolas with opposite directions.*

PROPOSITION 3.13. *The system* (3.1) *has no limit cycle.*



$0 < A < \frac{1}{2}, \quad D \leq 0$

FIGURE 2



$\frac{1}{2} < A < \frac{1}{\sqrt{2}}, \quad D \leq 0$

FIGURE 3

Proof. The divergence of the system is $\operatorname{div} = 2(a+d)xy$. When $a+d=0$ the system is Hamiltonian and has no limit cycle. When $a+d \neq 0$ the only periodic solutions cut the coordinate axes. Since the coordinate axes are symmetry axes the periodic solutions cannot be limit cycles. ▌

We now begin to construct the bifurcation diagrams of the system (3.1). To do this we first note that system (3.1) is symmetric under (3.25). The system is Hamiltonian for $a+d=0$. The symmetry allows us to consider only the case $a+d \geqslant 0$. We change to parameters

$$A = a+d \qquad B = a-d \qquad C = b+c \qquad D = b-c. \tag{3.61}$$

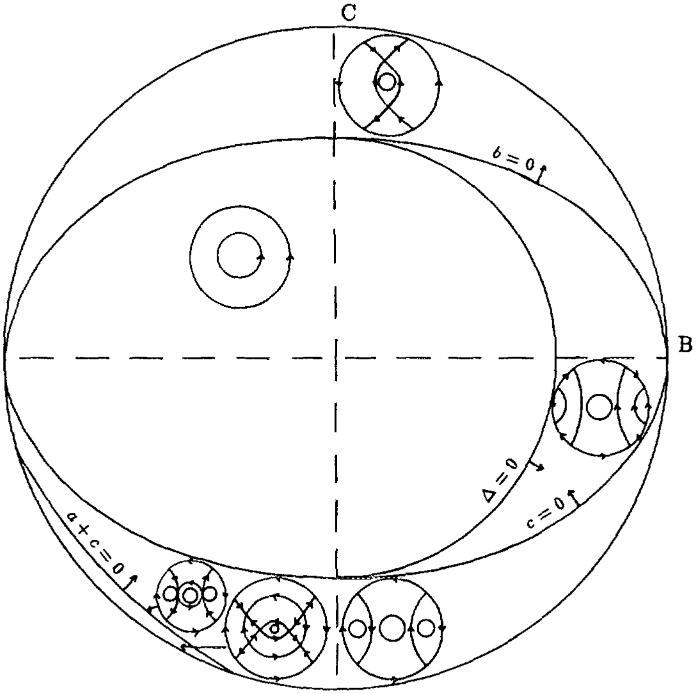
System (3.1) becomes

$$\dot{x} = -y + \frac{A+B}{2}x^2y + \frac{C+D}{2}y^3$$
$$\dot{y} = x + \frac{C-D}{2}x^3 + \frac{A-B}{2}xy^2. \tag{3.62}$$

and we rescale parameters so that $(A, B, C, D) \in \mathbb{S}^3$. Since $a+d \geqslant 0$, then $A \geqslant 0$. To draw the bifurcation diagram we express

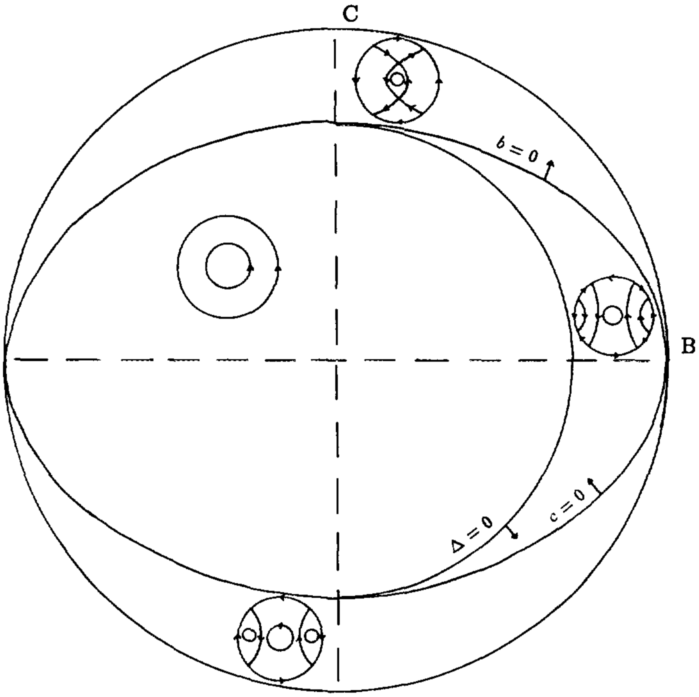
$$D = \pm\sqrt{1-A^2-B^2-C^2}. \tag{3.63}$$

The parameter space consists of two half 3-balls $A^2+B^2+C^2<1$, $A \geqslant 0$, (one for $D>0$, one for $D<0$) linked on a half 2-sphere $A^2+B^2+C^2=1$, $A \geqslant 0$. To draw these we cut them by planes $A = cst$.



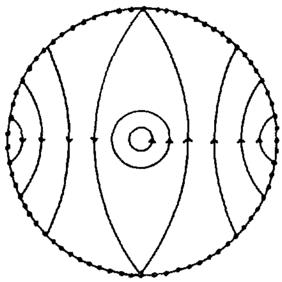
$\frac{1}{\sqrt{2}} < A < \sqrt{\frac{2}{3}}, \quad D \leq 0$

FIGURE 4



$A > \sqrt{\frac{2}{3}}, \quad D \leq 0$

FIGURE 5



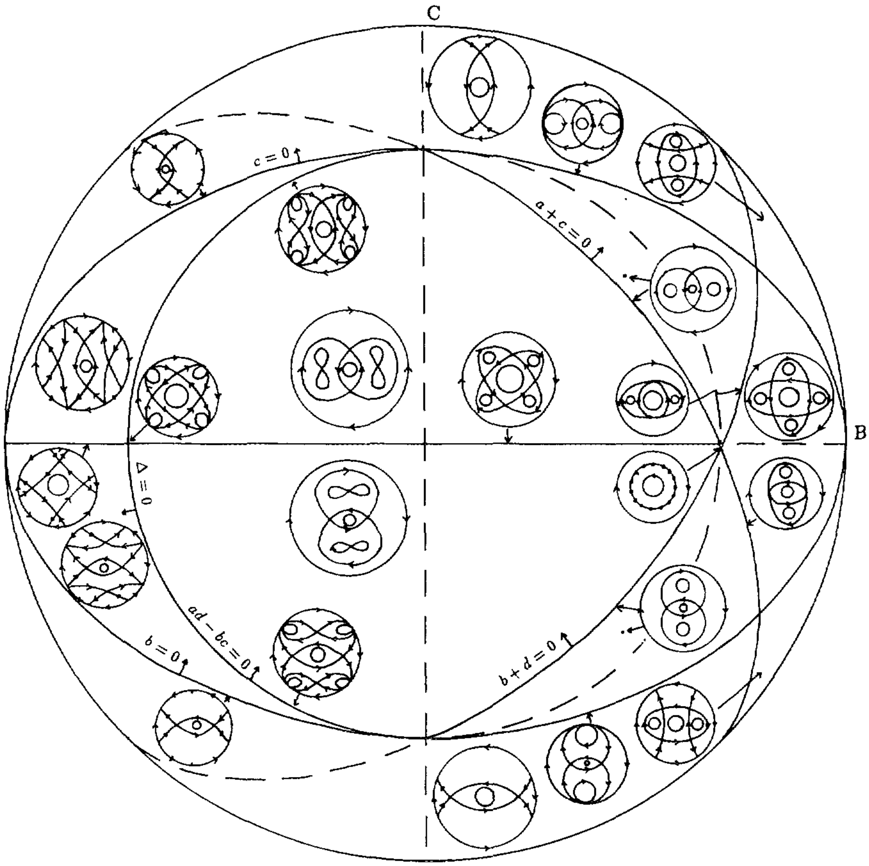
$A = 1, \quad D = 0$

FIGURE 6

In this parameter space we first consider the bifurcation surfaces of the singular points.

Proposition 3.14. *The following are bifurcation surfaces of the singular points, which correspond to bifurcations of the invariant conics*:

(1) $b=0$, *corresponding to a bifurcation of the singular points* $(0, \pm\sqrt{1/b})$ *as they pass through infinity, while one of the conics becomes reducible*;



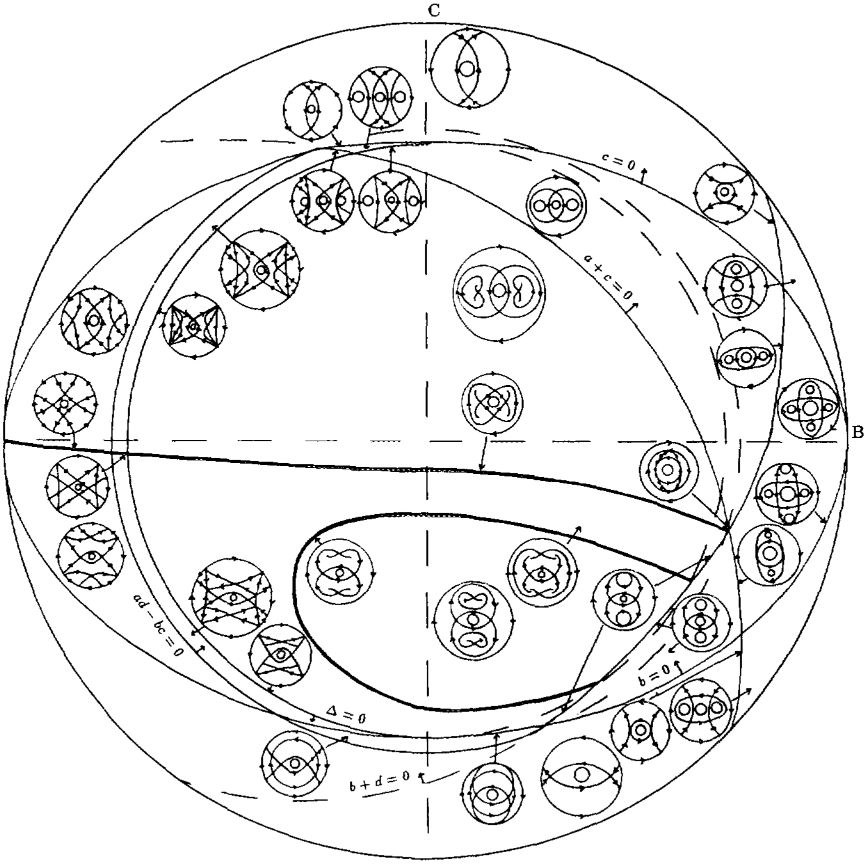
$A=0, \quad D \geq 0$

FIGURE 7

(2) $c = 0$, *corresponding to a bifurcation of the singular points* $(\pm\sqrt{-1/c}, 0)$ *as they pass through infinity, while one of the conics becomes reducible*;

(3) $ad - bc = 0$, *corresponding to a bifurcation of the four singular points* $P_{\pm\pm}$ *as they disappear at infinity. Meanwhile one of the conics becomes the double line at infinity*;

(4) $(a-d)^2 + 4bc = 0$, *corresponding to a bifurcation of the singular points at infinity. Meanwhile the two conics coincide. In the region where the four singular points* $P_{\pm\pm}$ *exist, the real parts of the conics pass from two intersecting conics to four isolated points.*

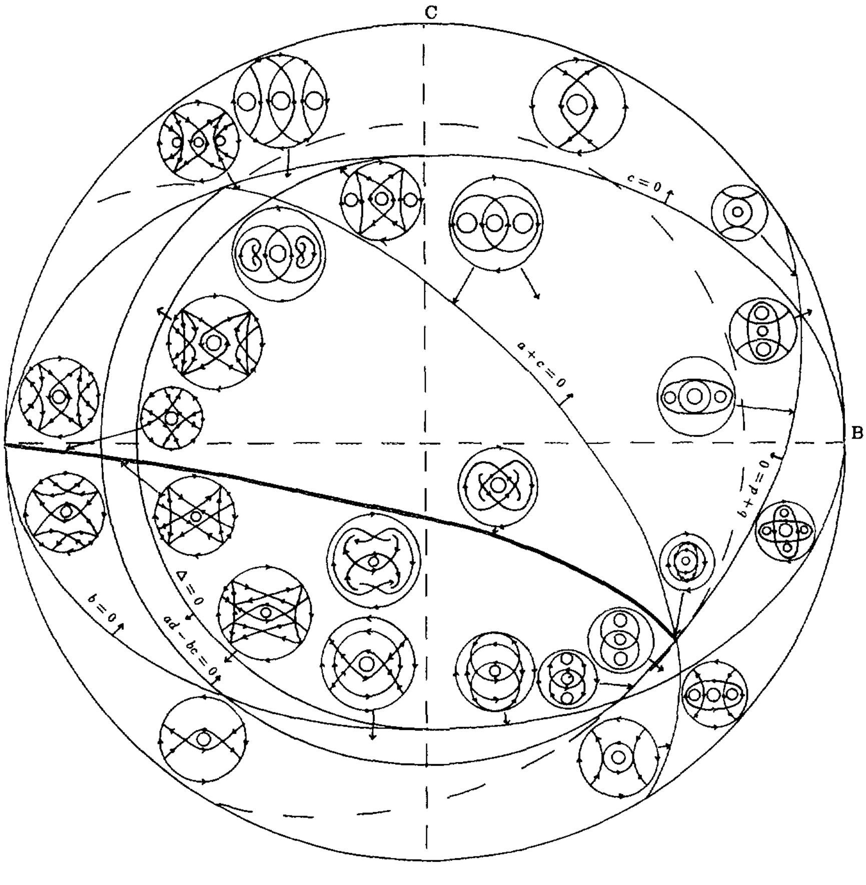


$0 < A < \sqrt{\frac{7-\sqrt{32}}{17}}, \quad D \geq 0$

FIGURE 8

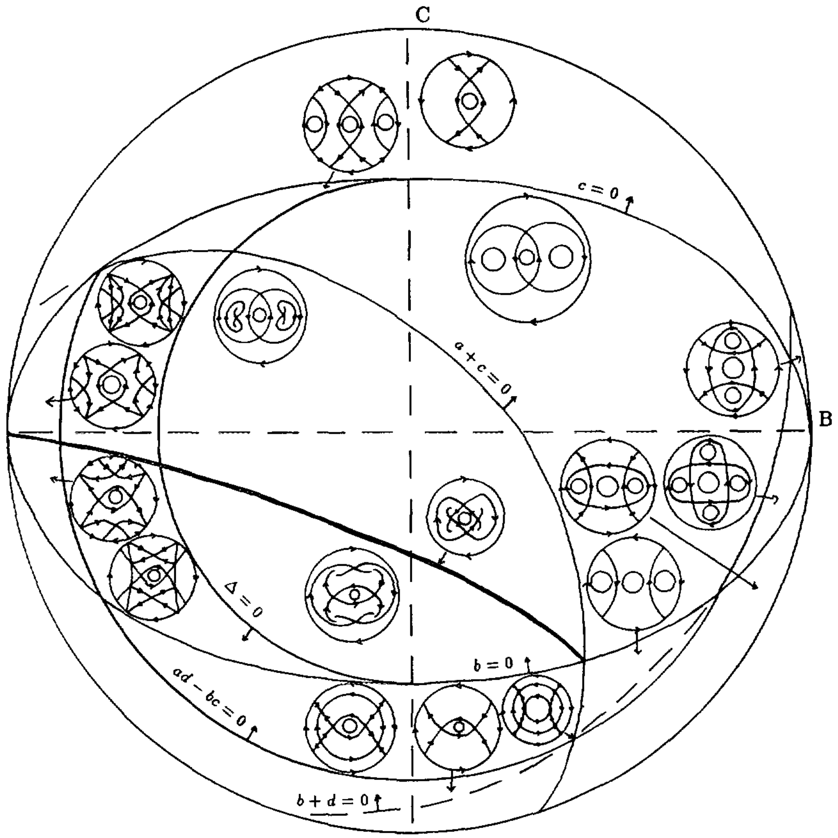
(5) $a+c=0$, *corresponding to the coalescence of the four singular points* $P_{\pm\pm}$ *with the points* $(\pm\sqrt{-1/c}, 0)$ *in a nilpotent bifurcation of saddle, elliptic or focus type. Meanwhile the two conics are tangent at the points* $(\pm\sqrt{-1/c}, 0)$;

(6) $b+d=0$, *corresponding to the coalescence of the four singular points* $P_{\pm\pm}$ *with the points* $(0, \pm\sqrt{1/b})$ *in a nilpotent bifurcation of saddle, elliptic or focus type. Meanwhile the two conics are tangent at the points* $(0, \pm\sqrt{1/b})$;



$\sqrt{\frac{7-\sqrt{32}}{17}} < A < \frac{1}{2}, \quad D \geq 0$

FIGURE 9



$\frac{1}{2} < A < \frac{1}{\sqrt{2}}, \quad D \geq 0$

FIGURE 10

(7) *$a + d = 0$, corresponding to $A = 0$, in which case the foci $P_{\pm\pm}$ become centers as the system is Hamiltonian.*

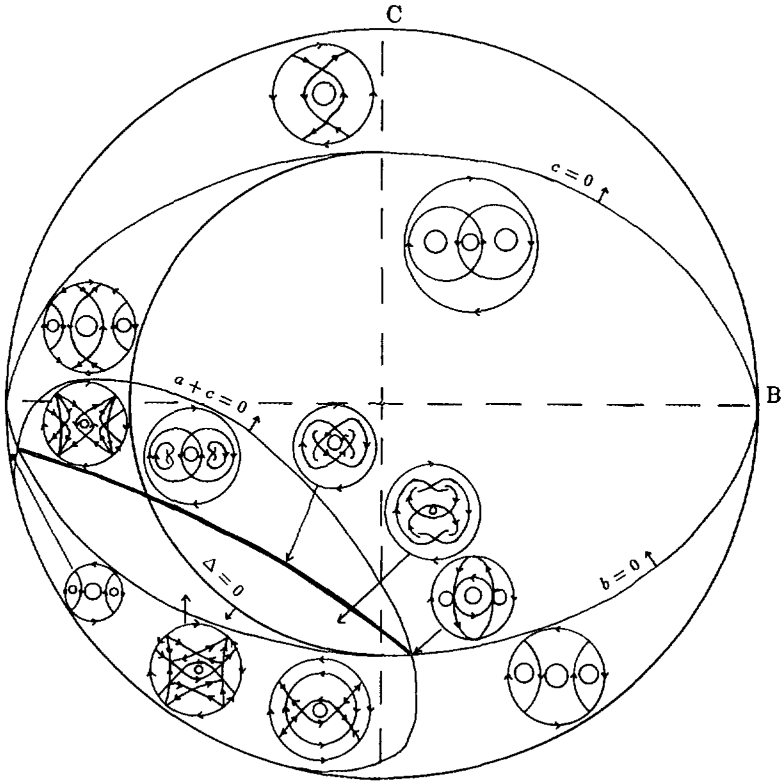
They appear in Figures 1–14.

Proof. The equations of the bifurcations surfaces in (A, B, C, D)-space are given by

(1) $b = 0$, i.e.,

$$2C^2 + B^2 = 1 - A^2, \tag{3.64}$$

with $C \leqslant 0$ $(C \geqslant 0)$ if $D \geqslant 0$ $(D \leqslant 0)$.



$\frac{1}{\sqrt{2}} < A < \sqrt{\frac{2}{3}}, \quad D \geq 0$

FIGURE 11

(2) $c = 0$, i.e.,

$$2C^2 + B^2 = 1 - A^2, \tag{3.65}$$

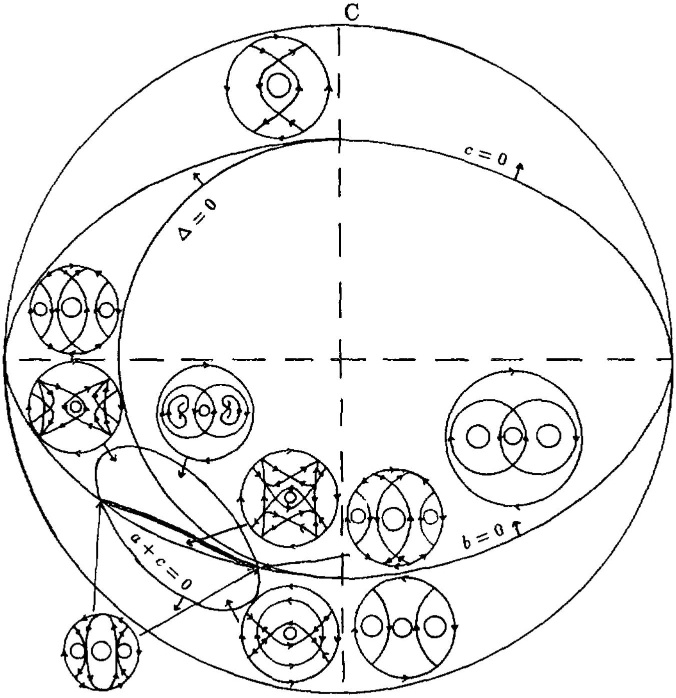
with $C \geqslant 0$ ($C \leqslant 0$) if $D \geqslant 0$ ($D \leqslant 0$).

(3) $ad - bc = 0$, i.e.,

$$1 - 2B^2 - 2C^2 = 0. \tag{3.66}$$

(4) $(a - d)^2 + 4bc = 0$, i.e.,

$$2B^2 + 2C^2 = 1 - A^2. \tag{3.67}$$



$\sqrt{\frac{2}{3}} < A < \sqrt{\frac{7+\sqrt{32}}{17}}, \quad D \geq 0$

FIGURE 12

(5) $a + c = 0$, i.e.,

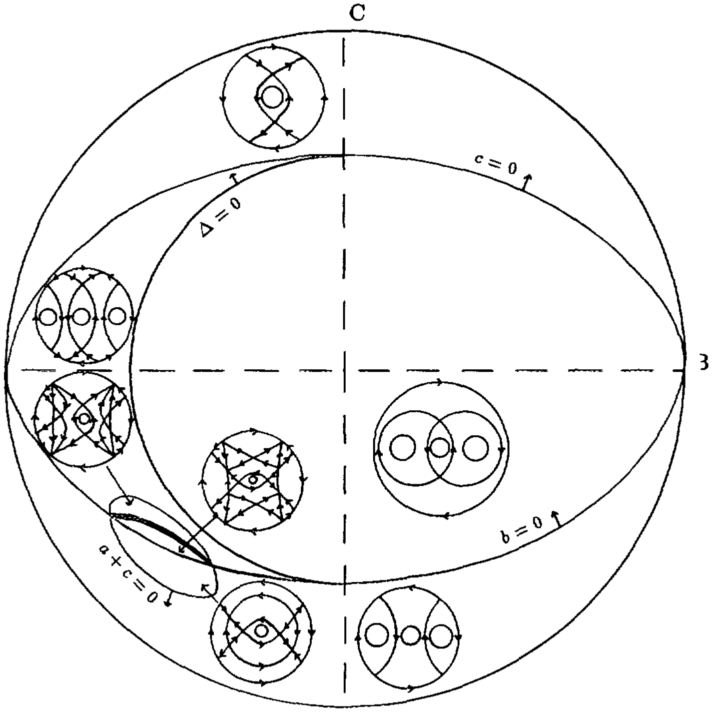
$$2B^2 + 2C^2 + 2BC + 2AB + 2AC = 1 - 2A^2, \tag{3.68}$$

with $A + B + C \geqslant 0$ ($A + B + C \leqslant 0$) if $D \geqslant 0$ ($D \leqslant 0$). This curve can be written

$$6\left(\frac{B+C}{2} + \frac{A}{3}\right)^2 + 2\left(\frac{B-C}{2}\right)^2 = 1 - \frac{4}{3}A^2. \tag{3.69}$$

(6) $b + d = 0$, i.e.,

$$2B^2 + 2C^2 - 2BC - 2AB + 2AC = 1 - 2A^2 \tag{3.70}$$



$\sqrt{\frac{7+\sqrt{32}}{17}} < A < \frac{\sqrt{3}}{2}, \quad D \geq 0$

FIGURE 13

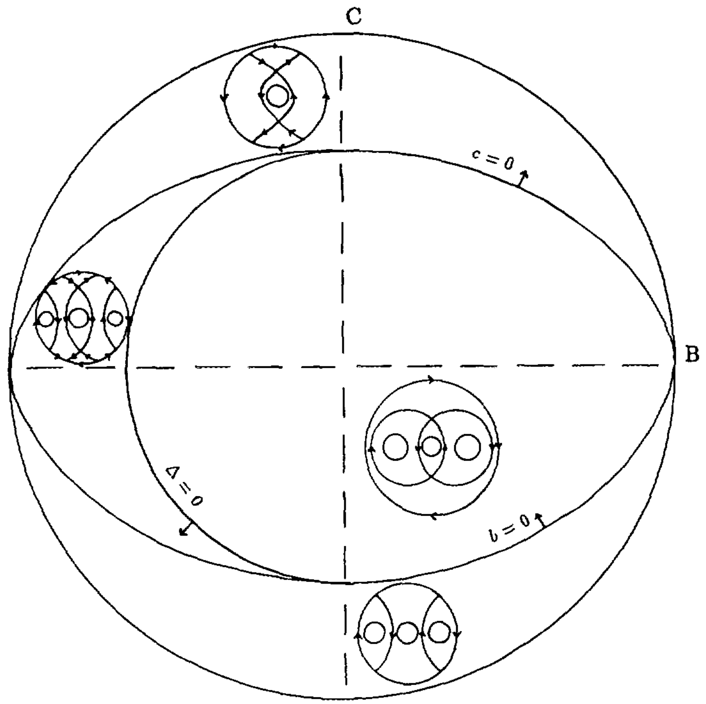
with $A-B+C \leqslant 0$ ($A-B+C \geqslant 0$) if $D \geqslant 0$ ($D \leqslant 0$). This curve can be written

$$2\left(\frac{B+C}{2}\right)^2 + 6\left(\frac{B-C}{2} - \frac{A}{3}\right)^2 = 1 - \frac{4}{3}A^2. \tag{3.71}$$

(7) $a+d=0$, i.e., $A=0$. ∎

Remark 3.15. The surface $a-b+c-d=0$ is a bifurcation surface for the conics only (as one of the conics coincides with the two lines $x^2+y^2=0$), corresponding to no bifurcation of the singular points in the phase portraits.

We consider now the existence of saddle connections surfaces on which occurs a saddle connection between $Q_+ = (\sqrt{-1/c}, 0)$ and $Q^+ = (0,$



$A > \frac{\sqrt{3}}{2}, \quad D \geq 0$

FIGURE 14

$\sqrt{1/b}$). This can be an "inner" or an "outer" saddle connection. Using the symmetries each saddle connection in the first quadrant yields four saddle connections between the points $Q_{\pm}$ and $Q^{\pm}$, forming either a lozenge (inner case) containing the origin or a clover (outer case) containing the origin together with the four singular points $P_{\pm\pm}$. The precise equations of the saddle connections are transcendental functions:

PROPOSITION 3.16. (1) *The equation of the bifurcation surface corresponding to saddle connections between the points* $Q_{\pm}$ *and* $Q^{\pm}$ *is given by*:

$$\left|\frac{a+c}{b+d}\right|^{2\sqrt{\Delta}} \left|\frac{c}{b}\right|^{-a-d-\sqrt{\Delta}} \left|\frac{2c+a-d-\sqrt{\Delta}}{a-2b-d+\sqrt{\Delta}}\right|^{2(a+d)} = 1, \tag{3.72}$$

when $(a-b+c-d)(ad-bc)\neq 0$ *and* $\Delta=(d-a)^2+4bc>0$, *and by*

$$-\frac{b(a+c)^2}{c(b+d)^2}\exp\left[\frac{2(a+d)}{\sqrt{-\Delta}}\arctan\frac{\sqrt{-\Delta}}{a-d}\right]=1, \tag{3.73}$$

when $(a-b+c-d)(ad-bc)\neq 0$ *and* $\Delta<0$.

Furthermore (3.72) *and* (3.73) *remain valid when* $b+c=0$ *and we let* $a+d\to 0$.

(2) *Points* $a+c=b=0$ *are limit points of the saddle bifurcation surface when* $c<0$, $d>0$.

(3) *Points* $a+c=b+d=0$ *are limit points of the saddle connection bifurcation surface when* $c<0$, $b>0$.

(4) *In case* $(a-d)^2+4bc=0$, *and* $(a+d)(a-b+c-d)\neq 0$ *the equation is given by*

$$-\frac{2c(b+d)}{(a-d)(a+c)}\exp\left[\frac{a+d}{a-d}\right]=1. \tag{3.74}$$

There is at least one point where this equation is satisfied in Figs. 8–11. *A branch of the curve of saddle connection tends to* $(a-d)^2+4bc=0$ *when* $a+d\to 0$.

(5) *In case* $ad-bc=0$ *the equation is given by*

$$1=-\frac{d}{a}\exp\left[\frac{(a+d)(d-c)}{c(b+d)}\right]=-\frac{d}{a}\exp\left[\frac{(a+d)(b-a)}{b(a+c)}\right]. \tag{3.75}$$

There is at least one point where the equation is satisfied in Figs. 8–11. *A branch of the curve of saddle connection tends to* $ad-bc=0$ *when* $a+d\to 0$.

(6) *Points on the surface of saddle connection are given by* $a+2b+c=b+2c+d=0$, *which are the points where the system has four invariant lines. They correspond to* $A+3C=B+D=0$ *in the figures. The point is located at* $C=0$ *and* $B=-1/\sqrt{2}$ *when* $A=0$. *When* A *increases it appears on all figures with* $D\geqslant 0$ *until* $A=\sqrt{3}/2$ *where it coincides with the point* $B=C=-1/(2\sqrt{3})$. *At this point* $b=0$, *i.e., two saddle points vanish at infinity.*

Proof. (1) There is a saddle connection if $F(\pm\sqrt{-1/c},0)=F(0,\pm\sqrt{1/b})$, where F is the first integral (3.12) or (3.14).

In case $\Delta=(a-d)^2+4bc>0$ the result follows from

$$F(\pm\sqrt{-1/c},0)=\frac{1}{E}\left|\frac{a+c}{c}\right|^{2\sqrt{\Delta}}|2c+a-d-\sqrt{\Delta}|^{a+d+\sqrt{\Delta}}$$
$$\times|2c+a-d+\sqrt{\Delta}|^{a-d+\sqrt{\Delta}}, \tag{3.76}$$

and

$$F(0, \pm\sqrt{1/b}) = \frac{1}{E}\left|\frac{b+d}{b}\right|^{2\sqrt{\Delta}} |a-2b-d+\sqrt{\Delta}|^{a+d+\sqrt{\Delta}}$$

$$\times |a-2b-d-\sqrt{\Delta}|^{-a-d+\sqrt{\Delta}}, \tag{3.77}$$

with $E=[2(a-b+c-d)]^{2\sqrt{\Delta}}$.

If we let $b+c=0$, then the right hand side of (3.72) becomes

$$\left|\frac{a+c}{c-d}\right|^{2\sqrt{(a-d)^2-4c^2}} \left|\frac{a+2c-d-\sqrt{(a-d)^2-4c^2}}{a+2c-d+\sqrt{(a-d)^2-4c^2}}\right|^{2(a+d)}, \tag{3.78}$$

which tends to 1 as $a+d \to 0$.

Similarly, when $\Delta < 0$

$$F(0, \pm\sqrt{1/b}) = \left(-\frac{(b+d)^2}{c(a-b+c-d)}\right)^{\sqrt{\Delta}}$$

$$\times \exp\left[-2(a+d)\arctan\frac{\sqrt{-\Delta}}{a-2b-d}\right], \tag{3.79}$$

$$F(\pm\sqrt{-1/c}, 0) = \left(\frac{(a+c)^2}{c(a-b+c-d)}\right)^{\sqrt{\Delta}}$$

$$\times \exp\left[2(a+d)\arctan\frac{\sqrt{-\Delta}}{a+2c-d}\right], \tag{3.80}$$

yielding a connection when (3.73) is satisfied. As in the previous case, if we let $b+c=0$, then (3.73) is satisfied at the limit when $a+d \to 0$.

(2) When $a+c=0=b$, we have the two invariant lines $cx^2+1=0$, which pass through the singular points $(\pm\sqrt{-1/c}, 0)$. The condition $d>0$ ensures that for small b the points $(0, \pm\sqrt{1/b})$ are saddles.

(3) When $a+c=b+d=0$ we have a conic of singular points $cx^2 - by^2+1=0$ passing through the singular points $(\pm\sqrt{-1/c}, 0)$ and $(0, \pm\sqrt{1/b})$.

(4) When $(a-d)^2+4bc=0$ we use the first integral (3.17). We have

$$F(\pm\sqrt{-1/c}, 0) = \frac{2(a+c)}{a+2c-d}\exp\left[-\frac{a+d}{a+2c-d}\right], \tag{3.81}$$

and

$$F(0, \pm\sqrt{1/b}) = -\frac{2(b+d)}{a-2b-d}\exp\left[\frac{2c(a+d)}{(a-d)(a+2c-d)}\right], \tag{3.82}$$

yielding a connection precisely when

$$-\frac{2c(b+d)}{(a-d)(a+c)}\exp\left[\frac{a+d}{a-d}\right] = 1. \tag{3.83}$$

If we let $b \to 0$ in the region $a-d<0$, then $a-d \to 0$ and the left hand side of (3.83) goes to zero. Similarly, if we let $a+c \to 0$ ($a-d$ is negative in the interesting region) then the left hand side goes to $+\infty$ (resp. $-\infty$) if we are in the region $a+c<0$ (resp. $a+c>0$). By the intermediate value theorem there appears a point of saddle connection in Figs. 7–11 corresponding to $0 \leqslant A \leqslant \sqrt{2/3}$.

Finally if we let $a+d \to 0$ then (3.83) is satisfied at the limit.

(5) When $ad-bc=0$, we use the first integral (3.20). We have

$$F(0, \pm\sqrt{1/b}) = -\frac{b+d}{a-b}\exp\left[-\frac{a+d}{b+d}\right], \tag{3.84}$$

and

$$F(\pm\sqrt{-1/c}, 0) = \frac{a+c}{c-d}\exp\left[-\frac{d(a+d)}{c(b+d)}\right], \tag{3.85}$$

yielding a connection precisely when

$$-\frac{d}{a}\exp\left[\frac{(a+d)(d-c)}{c(b+d)}\right] = 1. \tag{3.86}$$

If $b \to 0$ (resp. $c \to 0$) then $a \to 0$ and the left hand side of (3.86) tends to $+\infty$ (resp. 0), since b, $d>0$ and a, $c<0$ in the interesting region.

As in the previous case if we let $a+d \to 0$ then (3.86) is satisfied at the limit. ∎

Remark 3.17. Although we know the precise equation of the saddle connection surfaces, because these are given by transcendental functions, it is not easy to draw their sections with the planes $A = Cst$. In our drawings of the bifurcation diagrams (Figs. 8–13) the saddle connection curves are drawn as boldface curves and their position and shape is only conjectural.

We now gather all the informations obtained in the following theorem:

THEOREM 3.18. *The bifurcation diagram of singular points of* (3.1) *appears in Figs.* 1–14. *In the figures* $\Delta = (a-d)^2 + 4bc$. *The arrow near the name of a bifurcation curve* $f = 0$ *in the figures corresponds to the direction in which the quantity* f *increases.*

The bifurcation diagram of phase portraits of (3.1) *is complete except in the regions where both pairs of singular points* $Q_{\pm} = (\sqrt{-1/c}, 0)$ *and* $Q^{\pm} = (0, \sqrt{1/b})$ *exist and are saddle points, i.e.,* $b > 0$, $c < 0$, $b + d > 0$, $a + c < 0$. *In these regions the phase portaits in the first quadrant are known up to the relative position of the separatrices of* Q_+ *and* Q^+. *The two symmetry axes yield the phase portraits in the two other quadrants. More precisely*

(i) *The phase portraits are known for* $A = a + d = 0$ (*the Hamiltonian case*). *For each point on the plane* $A = 0$ *and not on a bifurcation line we can by continuity deduce the phase portraits for points in a small neighborhood.*

(ii) *The phase portraits are known on the interesection of the bifurcation surfaces* $a + c = 0$, $b + d = 0$, $b = 0$ *and* $c = 0$ *with the planes* $A = Cst$.

(iii) *The phase portraits are known everywhere on* $\Delta = 0$, *except for the position of the inner separatrices.*

(iv) *The phase portraits are known everywhere on* $ad - bc = 0$, *except for the position of the inner separatrices.*

(v) *On* $a + 2b + c = b + 2c + d = 0$ *the system has inner saddle connections given by the four invariant lines studied in Proposition* 3.7. *A limit point of this set is the point* $a + c = b = 0$ *on* $A = \sqrt{3}/2$.

(vi) *There exist points of inner saddle connections on* $\Delta = 0$ *and on* $ad - bc = 0$.

Proof. To derive the bifurcation diagram we start with the bifurcation diagram of the Hamiltonian system $A = 0$ (Figs. 1 and 7). In this case the phase portraits are easy to draw since we know the types of the singular points. Also, in the bifurcation diagram for $D \geqslant 0$ there appears a bifurcation curve of additional saddle connections ($b + c = 0 = C$): on this curve for $b + d > 0$ we have simultaneous inner and outer saddle connections when $\Delta > 0$ and only inner connections for $\Delta < 0$. Possible termination points (i.e., points of the adherence) for outer saddle connections are given by $\Delta = 0$.

The bifurcation diagram in the case $A > 0$, $D \leqslant 0$ (Figs. 2–6) is then easily derived from Propositions 3.11, 3.12, 3.13, since the phase portrait depends uniquely of the type of the singular points.

In the case $D \geqslant 0$, Proposition 3.11, 3.13 and the bifurcation diagram for $A = 0$ force the bifurcation diagram for small $A > 0$ far from the saddle connection and the curve $(a-d)^2 + 4bc = 0$.

The different bifurcation values of A are the following

(1) $A=\sqrt{(7-\sqrt{32})/17}$: this corresponds to $b+d=0$ becoming tangent to $(a-d)^2+4bc=0$ when $D \geqslant 0$. Meanwhile the curve of exterior saddle connection vanishes at the tangency point.

(2) $A=1/2$: this corresponds to $b+d=0$ (resp. $a+c=0$) becoming tangent to $ad-bc=0$ for $D \geqslant 0$ (resp. $D \leqslant 0$).

(3) $A=\sqrt{1/2}$: this corresponds to the disappearance of the curve $ad-bc=0$ through $D=0$.

(4) $A=\sqrt{2/3}$: this corresponds to the tangency of $a+c=0$ with $D=0$. For $A>\sqrt{2/3}$ the curve $a+c=0$ lies completely in $D>0$.

(5) $A=\sqrt{(7+\sqrt{32})/17}$: this corresponds to the tangency of $a+c=0$ with $(a-d)^2+4bc=0$ for $D>0$.

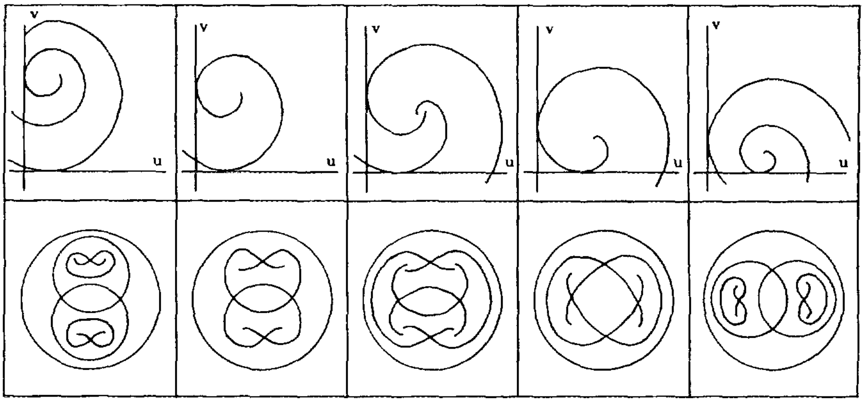
(6) $A=\sqrt{3/2}$: this corresponds to the curve $a+c=0$ shrinking to a point. This point is a termination point of $a+2b+c=b+2c+d=0$ where the system has four invariant lines yielding an inner saddle connection.

On $\Delta=0$ in the region $b>0$, $c<0$ the system has an invariant hyperbola passing through the singular points $P_{\pm\pm}$. Hence there are trajectories connecting these points to saddle points at infinity. From this it follows that there can be no outer saddle connections. The phase portrait then is known up to the relative position of the inner separatrices. The phase portraits on $a+c=0$ (resp. $b+d=0$) depend only on the type of the nilpotent points $Q_{\pm}$ (resp. $Q^{\pm}$). ∎

It seems natural that the curves of inner and outer saddle connections which coincide for $A=0$ would split from each other for $A>0$. The outer saddle connection coincides for $A=0$ with the curve $b+c=0$ in the region $(a-d)^2+4bc<0$ together with the portion of the circle $(a-d)^2+4bc=0$ in region B, $C<0$ (in this case the connection occurs at infinity). The saddle connection curves are drawn in Figs. 8–13 in the simplest possible way so as to glue together the different phase portraits which are forced by the case $A=0$. Note that they correspond to no bifurcation of the invariant algebraic curves. Hence the following conjecture is natural:

CONJECTURE 3.19. *The Figures* 1–14 *represent the complete bifurcation diagram of* (3.1).

Remark 3.20. The change of variables (3.2) brings (3.1) to the linear system (3.3) which we study in the first quadrant. Tangent points of trajectories with the coordinate axes correspond to singular points of (3.1). Hence we have a saddle connection between the singular points $Q_{\pm}$, and $Q^{\pm}$, precisely when the linear system (3.3) has a trajectory in the first quadrant tangent to both the coordinate axes. The correspondence between



Trajectories of the linear system (3-81) and corresponding trajectories of (3.1)

FIGURE 15

trajectories of the linear system (3.3) and trajectories of (3.1) is illustrated in Fig. 15.

3.6. *A Sufficient Condition for a Center*

The following theorem gives a sufficient condition for a system to have a center at the origin and is a partial converse to the existence of invariant algebraic curves for the stratum in the center space.

THEOREM 3.21. *If the system* (1.1) *has two distinct invariant conics* (*not necessarily irreducible*) *which are symmetric with respect to the two coordinate axes, then the origin is a center. Either the system is symmetric with respect to the coordinate axes or* $F(x, y) = x^2 + y^2$ *is a first integral for the system.*

Proof. By hypothesis we may assume that the equation of the conics is of the form

$$f_{20}x^2 + f_{02}y^2 + \delta = 0, \qquad \delta = 0, 1. \tag{3.87}$$

Let us start with conics not passing through the origin and assume each invariant conic has an equation (3.87) with $\delta = 1$. Then equation $Df = fK$ as in (2.2) yields $k_{00} = 0 = k_{10} = k_{01} = 0$. Identification of the quadratic terms gives $k_{20} = k_{02} = 0$ and $k_{11} = 2(f_{02} - f_{20})$. Identification of the x^4- and y^4-terms yields $a_{30} = b_{03} = 0$. Identification of the x^2y^2-terms gives

$$a_{12}f_{20} + b_{21}f_{02} = 0. \tag{3.88}$$

If this equation has two linearly independent solutions (f_{20}, f_{02}) and (f'_{20}, f'_{02}) then $a_{12} = b_{21} = 0$ and we have a system symmetric with respect to the axes, and hence a center at the origin.

If the two solutions are linearly dependent, then the identification of the x^3y- and xy^3-terms yields

$$\begin{aligned} 2f_{20}(f_{02} - f_{20}) &= a_{21} f_{20} + b_{30} f_{02} \\ 2f_{02}(f_{02} - f_{20}) &= a_{03} f_{20} + b_{12} f_{02}. \end{aligned} \tag{3.89}$$

Since the left hand sides are quadratic and the right hand sides are linear in f_{20} and f_{02}, then the only possibility to have two linearly dependent solutions is that both sides are zero, i.e., $f_{20} = f_{02}$ and $a_{21} + b_{30} = a_{03} + b_{12} = 0$. Together with the previous conditions $a_{30} = b_{03} = a_{12} + b_{21} = 0$, this is the condition for $F(x, y) = x^2 + y^2$ to be a first integral.

We can check that an invariant conic $g_{20}x^2 + g_{02}y^2 = 0$ passing through the origin necessarily satisfies $g_{20} = g_{02}$. It exists under the conditions

$$a_{30} - a_{12} - b_{21} + b_{03} = a_{03} - a_{21} + b_{12} - b_{30} = 0. \tag{3.90}$$

Let us suppose we have another invariant conic. Necessarily it does not pass through the origin and has the form (3.87). By the calculation above we have $a_{30} = b_{03} = 0$. By (3.90) we have $a_{12} + b_{21} = 0$. By (3.88) this yields either $a_{12} = b_{21} = 0$, in which case the system is symmetric with respect to the two axes or $f_{20} = f_{02}$. From (3.89) the latter gives $a_{21} + b_{30} = a_{03} + b_{12} = 0$. Together with the previous condition $a_{30} = b_{03} = a_{12} + b_{21} = 0$, this is the condition for $F(x, y) = x^2 + y^2$ to be a first integral. ∎

4. Centers of Type III

We study here a system

$$\begin{aligned} \dot{x} &= -y - (\theta - a)\, x^3 + 3\mu x^2 y - (2a - 3\theta)\, xy^2 - \mu y^3 \\ \dot{y} &= x + \mu x^3 + (3\theta + 2a)\, x^2 y - 3\mu xy^2 - (\theta + a)\, y^3, \end{aligned} \tag{4.1}$$

with

$$4(\theta^2 + \mu^2) = a^2. \tag{4.2}$$

The parameter space is a 2-cone in $\mathbb{R}^3$ and the bifurcation diagram has a conic structure. Hence, intersecting the cone with a sphere, we see that we must study (4.1) on two circles, for example $a = \pm 1$. The system has the following symmetries

$$(x, y, a) \mapsto (-y, x, -a)$$
$$(x, y, \theta, t) \mapsto (y, x, -\theta, -t). \tag{4.3}$$

Hence it is enough to study what happens in the case $a > 0$, $\theta \geq 0$, i.e., with parameters on $\mathbb{P}^1$.

THEOREM 4.1. *All the systems (4.1) have one invariant quartic and two invariant sextic curves. These curves all pass through two real singular points. They all have branches tangent to each other and to the equator at a pair of singular points at infinity. The curves appear in Fig. 16.*

Proof. An invariant quartic curve is given by $F_1 = 0$ with

$$F_1(x, y) = 1 - 2\mu x^2 + 2axy - 2\mu y^2 + \frac{(a+2\theta)^2}{4} x^4 - 2\mu(a+2\theta)\, x^3 y$$
$$+ 6\mu^2 x^2 y^2 - 2\mu(a - 2\theta)\, xy^3 + \frac{(a-2\theta)^2}{4} y^4, \tag{4.4}$$

and satisfies $DF_1 = K_1 F_1$ with $K_1(x, y) = 2a(x^2 - y^2)$.

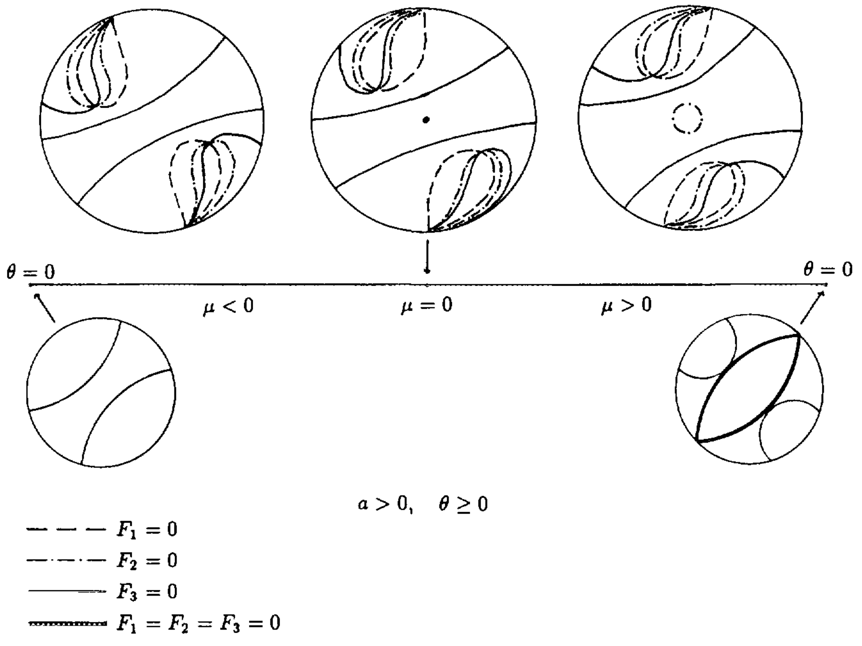


FIGURE 16

When looking for an invariant sextic curve $F(x, y) = 1 + \sum_{i+j=2}^{6} F_{ij} x^i y^j = 0$, we obtain as possible solutions for f_{11}: $f_{11} = 3a$, $f_{11} = 6a$, yielding for $\mu \neq 0$ two sextic curves $F_2 = 0$ and $F_3 = 0$ for which we have $K_2(x, y) = 3a(x^2 - y^2)$ and $K_3(x, y) = 6a(x^2 - y^2)$. The sextic curves $F_2(x, y)$ and $F_3(x, y)$ are given by:

$$\begin{aligned} F_2(x, y) = {} & 1 - \frac{3a^2}{4\mu} x^2 + 3axy - \frac{3a^2}{4\mu} y^2 + \frac{3a(a+2\theta)}{4} x^4 - \frac{3a(a+2\theta)(a-\theta)}{2\mu} x^3 y \\ & + \frac{9a^2}{2} x^2 y^2 - \frac{3a(a-2\theta)(a+\theta)}{2\mu} x y^3 + \frac{3a(a-2\theta)}{4} y^4 - \frac{a(a+2\theta)^3}{16\mu} x^6 \\ & + \frac{3a(a+2\theta)^2}{4} x^5 y - \frac{15a\mu(a+2\theta)}{4} x^4 y^2 + 10a\mu^2 x^3 y^3 \\ & - \frac{15a\mu(a-2\theta)}{4} x^2 y^4 + \frac{3a(a-2\theta)^2}{4} x y^5 - \frac{a(a-2\theta)^3}{16\mu} y^6, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} F_3(x, y) = {} & 1 + 6axy - \frac{3a(a-4\theta)}{4} x^4 - 6a\mu x^3 y + \frac{15a^2}{2} x^2 y^2 \\ & - 6a\mu x y^3 - \frac{3a(a+4\theta)}{4} y^4 + \frac{a\mu(a+2\theta)}{2} x^6 + 3a\theta(a+2\theta)\, x^5 y \\ & - \frac{3a\mu(3a+10\theta)}{2} x^4 y^2 + 4a(a^2 - 5\theta^2)\, x^3 y^3 - \frac{3a\mu(3a-10\theta)}{2} x^2 y^4 \\ & - 3a\theta(a-2\theta)\, x y^5 + \frac{a\mu(a-2\theta)}{2} y^6. \end{aligned} \tag{4.6}$$

The formula for F_2 is valid for $\mu \neq 0$, while the formulas for F_1 and F_3 are valid without restriction. Hence it is more natural to take as the first sextic curve $H_2(x, y) = \mu F_2(x, y) = 0$. This formula makes sense for all values of μ. In case $\mu = 0$, we can scale $(a, \theta) = (2, 1)$ (because of (2.2) and (2.3)). The sextic curve passes through the origin and reduces to

$$H_{20}(x, y) = -3x^2 - 3y^2 - 12x^3 y - 8x^6 = 0. \tag{4.7}$$

A first integral can be taken as

$$F(x, y) = F_1(x, y)^3\, F_3(x, y)^{-1}. \tag{4.8}$$

Among other choices we could choose $F_1(x, y)^3 H_2(x, y)^{-2}$ or $H_2(x, y)^2 F_3(x, y)^{-1}$.

For μ, $\theta \neq 0$, Sibirsky introduced the following change of coordinates

$$\begin{aligned} u &= cx - 2\mu y \\ v &= bx - 2\mu y, \end{aligned} \tag{4.9}$$

with

$$b = a - 2\theta > 0, \qquad c = a + 2\theta > 0, \tag{4.10}$$

which simplifies the expression for the invariant curves and helps visualizing them geometrically. Letting $b = 4\mu^2/c$, we obtain algebraic curves involving only the two parameters c and μ: The curves become $F_i = 0$ with:

$$F_1(u, v) = 1 - \frac{uv}{2\mu} + \frac{u^4}{4c^2}, \tag{4.11}$$

$$F_2(u, v) = 1 - \frac{3(c^2 + 4\mu^2)}{64c^2\mu^3}\left(4\mu^2u^2 + c^2v^2 - 2\mu u^3 v + \frac{2\mu^2u^6}{3c^2}\right), \tag{4.12}$$

$$\begin{aligned} F_3(u, v) = 1 &+ \frac{3(c^2 + 4\mu^2)}{2\mu(c^2 - 4\mu^2)^2}(u - v)(4\mu^2 u - c^2 v) + \frac{3(c^2 + 4\mu^2)}{(c^2 - 4\mu^2)^2} \\ &\times \left[\frac{c^2 + 4\mu^2}{16c^2}u^4 - u^3 v + \frac{5(c^2 + 4\mu^2)}{32\mu^2}u^2v^2 - \frac{3c^2(c^2 + 4\mu^2)}{256\mu^4}v^4\right] \\ &+ \frac{c^2 + 4\mu^2}{16\mu(c^2 - 4\mu^2)^2}u^3 \\ &\times \left[\frac{16\mu^2}{c^2}u^3 - \frac{3(c^2 + 4\mu^2)}{c^2}u^2 v + \frac{c^2 + 4\mu^2}{4\mu^2}v^3\right]. \end{aligned} \tag{4.13}$$

To draw $F_2 = 0$ we write it as

$$v^2 - \frac{2\mu}{c^2}u^3 v + \left(\frac{2\mu^2}{3c^4}u^6 + \frac{4\mu^2}{c^2}u^2 - \frac{64\mu^3}{3(c^2 + 4\mu^2)}\right) = 0. \tag{4.14}$$

The equation in v has discriminant

$$\Delta = \frac{4\mu^2}{3c^4}\left[u^6 - 12c^2u^2 + \frac{64\mu c^4}{4\mu^2 + c^2}\right] = \frac{4\mu^2}{3c^4}D. \tag{4.15}$$

The discriminant of the cubic polynomial D in $U = u^2$ being positive, and the sum of the roots of $D(U)$ being zero, D has one (resp. two) positive root(s) for $\mu < 0$ (resp. $\mu > 0$), yielding for $F_2 = 0$ a curve with two (resp.

three) connected components. We note that $F_2(u, 0)=0$ has no (resp. two) solution(s) for $\mu<0$ (resp. $\mu>0$). To look for the intersection points of $F_1=0$ and $F_2=0$ we first scale $c=1$. Solving $F_1=0$ in v and replacing in $F_2=0$ yields

$$0=u^8(1+4\mu^2)-24(1+4\mu^2)\,u^4+256\mu u^2-48(1+4\mu^2)=P(u^2), \tag{4.16}$$

The discriminant of $P(U)$ is

$$\mathrm{Disc}(P)=-452984832(4\mu^2-1)^4\,(4\mu^2+1)^4, \tag{4.17}$$

which is zero only for $\theta=0$. Hence the only bifurcations of roots of P occur for $\theta=0$. Checking for the two different values $(a, \mu, \theta)=(10, \pm 4, 3)$ we see that in all cases we get for P a negative root, a positive root and a pair of complex roots, yielding two intersection points of $F_1=0$ and $F_2=0$, which are symmetric with respect to the origin. In the particular case $\theta=0$, P has a triple positive (negative) root for $a=2\mu$ $(a=-2\mu)$.

$F_1=0$ and $F_2=0$ both have the same unique point at infinity $u=0$. $F_3=0$ has four points at infinity, among which the point $u=0$. These four points coincide with the four singular points of (4.1) at infinity, the equation of which is given by

$$(cx-2\mu y)(2\mu x^3+3cx^2y-6\mu xy^2-cy^3)=0. \tag{4.18}$$

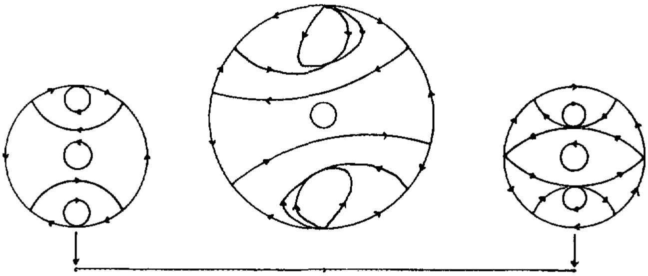
$F_3=0$ passes through the two finite intersection points of $F_1=0$ and $F_2=0$. To further investigate the form of $F_3=0$ let us notice that F_3 is a quartic polynomial $Q(v)$ in v. Looking to the sign changes in the coefficients we can note that $Q(v)$ has between one and three positive (resp. negative) roots and exactly one negative root for $u>0$ (resp. $u<0$) when $\mu>0$. When $\mu<0$ the sign of roots is reversed. Moreover the discriminant of Q is given by

$$x\Delta=-\frac{27}{268435456}\,\frac{(1+4\mu^2)^7}{\mu^{12}(1-4\mu^2)^{10}}\,P(u^2), \tag{4.19}$$

where $P(u^2)$ is given in (4.16). Hence $Q=0$ has two double points which are the intersection points of the three invariant curves $F_i=0$. This gives a total of six branches for $F_3=0$ as shown in Fig. 16.

We must now study what happens for $\theta=0$ and for $\mu=0$.

For $\theta=0$ the system is symmetric and we can use the results of Section 4. We must distinguish the two cases $a=\pm 2\mu$ with $a>0$. Using a rescaling it is enough to consider the two systems $(a, \mu)=(2, \pm 1)$. In order to work



$a > 0, \quad \theta \geq 0$

FIGURE 17

with the notation of Section 4 it is better to change to coordinates $(X, Y) = (x + y, -x + y)$, i.e., to consider the system

$$\dot{X} = -Y - \frac{5a + 6\mu}{4} X^2 Y + \frac{a + 2\mu}{4} Y^3$$
$$\dot{Y} = X + \frac{a - 2\mu}{4} X^3 - \frac{5a - 6\mu}{4} XY^2. \tag{4.20}$$

In the particular case $(a, \mu) = (2, 1)$ this gives the system

$$\dot{X} = -Y - 4X^2 Y + Y^3$$
$$\dot{Y} = X - XY^2, \tag{4.21}$$

which has the two invariant lines $Y = \pm 1$, tangent to the invariant hyperbola $3X^2 - Y^2 + 1 = 0$ (see Fig. 17). The invariant curves $F_i = 0$ reduce to

$$F_i(x, y) = (-x + y + 1)^{i+1} (x - y + 1)^{i+1} = 0, \qquad i = 1, 2, \tag{4.22}$$

and

$$F_3(x, y) = (x - y - 1)^2 (x - y + 1)^2 (1 + 2x^2 + 8xy + 2y^2) = 0. \tag{4.23}$$

The case $(a, \mu) = (2, -1)$ corresponds to the system

$$\dot{X} = -Y - X^2 Y$$
$$\dot{Y} = X + X^3 - 4XY^2, \tag{4.24}$$

with the invariant curves $F_i=0$ reduce to

$$F_i(x, y)=[1+(x+y)^2]^{i+1}=0, \qquad i=1, 2, \tag{4.25}$$

and

$$F_3(x, y)=[1+(x+y)^2]^2(1-2x^2+8xy-2y^2)=0. \tag{4.26}$$

For $\mu=0$ we scale $(a, \theta)=(2, 1)$ and study directly the system

$$\begin{aligned}\dot{x}&=-y+x^3-xy^2\\ \dot{y}&=x+7x^2y-3y^3,\end{aligned} \tag{4.27}$$

for which we have the invariant curves:

$$F_1(x, y)=1+4xy+4x^4=0, \tag{4.28}$$

and

$$F_3(x, y)=1+12xy+3x^4+30x^2y^2-9y^4+24x^5y-8x^3y^3=0. \tag{4.29}$$

These curves have the same form as in the general case μ, $\theta\neq 0$ and intersect in two points P_1 and P_2. The equation

$$H_{20}(x, y)=-3x^2-3y^2-12x^3y-8x^6=0 \tag{4.30}$$

corresponds to the particular case of a curve $H_2(x, y)$, when one branch reduces to a point. The two other branches pass through P_1 and P_2 respectively. $\mu=0$ corresponds to a bifurcation for the curve $H_2(x, y)=0$, but to no bifurcation of the phase portrait of (4.1). ∎

PROPOSITION 4.2. *The system* (4.1) *has four singular points at infinity, one repelling node and three saddles for* $\theta>0$. *For* $\theta=0$ *there are two saddles and a non-elementary singular point.*

Proof. To study singular points at infinity we change to coordinates $(x, y)=(\cos\phi/r, \sin\phi/r)$. After multiplying by r^2 system (4.1) becomes

$$\begin{aligned}\dot{r}&=[(\theta-a)\cos^4\phi-4\mu\cos^3\phi\sin\phi-6\theta\cos^2\phi\sin^2\phi+4\mu\cos\phi\sin^3\phi\\ &\quad+(\theta+a)\sin^4\phi]\,r+o(r)=A(\phi)\,r+o(r)\\ \dot{\phi}&=[\mu\cos^4\phi+(a+4\theta)\cos^3\phi\sin\phi-6\mu\cos^2\phi\sin^2\phi\\ &\quad+(a-4\theta)\cos\phi\sin^3\phi+\mu\sin^4\phi]+o(r)=B(\phi)+o(r).\end{aligned} \tag{4.31}$$

Singular points at infinity are solutions of $B(\phi)=0$.

We first treat the case $\mu=0$. Hence we can suppose $(a, \theta)=(2, 1)$, and we need only study the system

$$\begin{aligned} \dot{x} &= -y+x^3-xy^2 \\ \dot{y} &= x+7x^2y-3y^3, \end{aligned} \tag{4.32}$$

which has a repelling node on the y-axis and three saddles on the lines $y=0$ and $3x^2-y^2=0$.

We next consider the case $\mu \neq 0$. Let $\tau=\cos\phi/\sin\phi$. The singular points satisfy

$$[(a+2\theta)\,\tau-2\mu][(a-2\theta)\,\tau^3+6\mu\tau^2-3(a-2\theta)\,\tau-2\mu]=0. \tag{4.33}$$

The type of the singular points comes from the sign of the eigenvalues of the matrix J:

$$J=\begin{pmatrix} A(\phi) & 0 \\ * & B'(\phi) \end{pmatrix}. \tag{4.34}$$

At the singular point $\tau_4=2\mu/(a+2\theta)$ we have

$$J=\sin^4\theta \begin{pmatrix} \dfrac{12a^2\theta}{(a+2\theta)^2} & 0 \\ * & \dfrac{8a^2\theta}{(a+2\theta)^2} \end{pmatrix}, \tag{4.35}$$

yielding that the point is a repelling node. Let

$$D(\tau)=(a-2\theta)\,\tau^3+6\mu\tau^2-3(a-2\theta)\,\tau-2\mu. \tag{4.36}$$

The discriminant of D is easily checked to be positive (equal to $432a^2/(a-2\theta)^2$) so D always has three real roots. One can also see that B has two positive and two negative roots and it is easily checked that $D(\tau_0)=0$ precisely when $\theta=0$. When $\theta=0$ the system is symmetric and the type of the singular points is known. We limit ourselves here to the case $\theta \neq 0$. We let $\tau_1<\tau_2<\tau_3$ be the three roots of $D(\tau)$ and $\tau_4=2\mu/(a+2\theta)$. From the sign of $B'(\tau_4)$ which is the same as the sign of θ we can find the sign of the $B'(\tau_i)$ using the fact that the four roots of $B(\tau)$ are simple. We must study the sign of $A(\tau)$ under the condition $D(\tau)=0$ with

$$A(\tau)=(\theta-a)\,\tau^4-4\mu\tau^3-6\theta\tau^2+4\mu\tau+\theta+a. \tag{4.37}$$

We have:

$$(a-2\theta)^2 A(\tau)-(\theta-a)(a-2\theta)\,\tau D(\tau)-2\mu(a+\theta)\,D(\tau)$$
$$=2a(a-2\theta)[-3(a+\theta)\,\tau^2+4\mu\tau+a+\theta]=2a(a-2\theta)\,E(\tau). \quad (4.38)$$

We are interested to the behaviour of $D(\tau)$ when $E(\tau)=0$.

$$9(a+\theta)^2\,D(\tau)+3(a+\theta)(a-2\theta)\,\tau E(\tau)+2\mu(11a+5\theta)\,E(\tau)$$
$$=2(a-2\theta)[-\tau(a-2\theta)(a-\theta)+2\mu(a+\theta)]=F(\tau). \quad (4.39)$$

Hence $D(\tau)|_{E(\tau)=0}$ has the sign of $F(\tau)$. Let $\tau_0=2\mu(a+\theta)/((a-2\theta)(a-\theta))$ the root of $F(\tau)$. Then $E(\tau_0)=-18a\theta(a+\theta)^2/((a-2\theta)(a-\theta)^2)<0$. We must study the two cases μ positive or negative. We show the details for the case $\mu>0$. The other case can be done in the same way. In this case τ_0 is located to the right of the roots of $E(\tau)$, since $\tau_0>0$ and $E(\tau_0)<0$. Hence $F(\tau)|_{E(\tau)=0}>0$ from which $D(\tau)|_{E(\tau)=0}>0$ follows. Moreover $D(\tau_4)=-16a\theta\mu/(a+2\theta)^2<0$. Hence $\tau_1<\tau_2<0<\tau_4<\tau_3$. Since $A(\tau_i)$ has the sign of $E(\tau_i)$ we get $A(\tau_1)<0$, $A(\tau_2)>0$, $A(\tau_3)>0$, $A(\tau_4)>0$. From $B'(\tau_4)>0$ we can conclude that $B'(\tau_1)>0$, $B'(\tau_2)<0$, $B'(\tau_3)<0$, yielding three saddles in τ_1, τ_2, τ_3 and a node in τ_4. ∎

PROPOSITION 4.3. *System (4.1) has three finite singular points: the origin and two attracting nodes for $\theta\neq 0$. When $\theta=0$ and $\mu<0$, the origin is the unique singular point, while when $\theta=0$ and $\mu>0$ the system has three singular points, the origin and a pair of nilpotent elliptic points.*

Proof. Let us first notice that the system has no singular points on the x-axis since $a-\theta>0$. Singular points satisfy $x\dot{x}+y\dot{y}=0$. Letting $k=x/y$, this gives that k is solution of

$$f(k)=(a-\theta)\,k^4+4\mu k^3+6\theta k^2-4\mu k-(a+\theta)=0. \quad (4.40)$$

We first show that $f(k)=0$ has at most two real solutions for $\theta\neq 0$ by showing that $f'(k)=0$ has at most one real solution. Indeed the discriminant of the monic polynmial corresponding to $f'(k)$ is $\Delta=-27a^2\theta/(a-\theta)^3<0$. Since $f(0)<0$ this gives that $f(k)$ has exactly two real roots. Hence we have a possibility of two pairs of symmetric singular points (i.e., with same Poincaré index), the sum of the indices of which must be equal to 2 by the Poincaré index theorem. We show below that the system localized at these singular points has an invertible linear part, yielding points of index ± 1. The only possibility is then to have a unique pair of singular points of index one: these points are the intersection points of the invariant curves $F_i=0$, from which they cannot be foci. Hence we are left with two nodes. The divergence at the nodes is given by $\text{div}=5a(x^2-y^2)$.

Since $f(\pm 1) = 4\theta > 0$, the singular points are such that $|k| < 1$, i.e., the divergence is negative there.

To show that the jacobian matrix at the singular points is invertible we look at the system in polar coordinates $(x, y) = (r\cos\phi, r\sin\phi)$

$$\begin{aligned} \dot{r} &= \sin^4\phi\, f(k)\, r^3 \\ \dot{\phi} &= 1 + \sin^4\phi\, g(k)\, r^2, \end{aligned} \tag{4.41}$$

with $k = \cot\phi$, f given in (4.40) and

$$g(k) = \mu k^4 + (a + 4\theta)\, k^3 - 6\mu k^2 + (a - 4\theta)\, k + \mu. \tag{4.42}$$

Singular points are given by $f(k) = 0$ and $r^2 = 1/(\sin^4\phi g(k))$. The Jacobian matrix at these points is given by

$$J = \sin^4\phi \begin{pmatrix} 3r^2 f(k) & r^3 h(k) \\ 2rg(k) & r^2 g'(k) \end{pmatrix}, \tag{4.43}$$

with $\sin^4\phi\, h(\cot\phi) = (1/4)\, d/d\phi \sin^4\phi\, f(\cot\phi)$. The Jacobian at a point $f(k) = 0$ is nonzero if $g(k)\, h(k) \neq 0$. This is satisfied as soon as $\theta \neq 0$, since

$$\text{resultant}(f(k), g(k)) = -48a^6\theta^2, \tag{4.44}$$

and

$$\text{resultant}(f(k), h(k)) = -432a^6\theta^2. \tag{4.45}$$

In case $\theta = 0$, modulo the change of variables $(X, Y) = (x + y, -x + y)$ we must consider the two systems (4.21) and (4.24). In the first case we have the pair of nilpotent elliptic points located at $Y = \pm 1$, while in the second the origin is the unique singular point. ∎

THEOREM 4.4. *The bifurcation diagrams of the phase portraits of the systems* (4.1) *appear in Fig.* 17.

Proof. It follows from the previous propositions and Theorem 4.1. ∎

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