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Generalized eikonal knots and new integrable dynamical systems

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Abstract

A new class of non-linear $O(3)$ models is introduced. It is shown that these systems lead to integrable submodels if an additional integrability condition (the generalized eikonal equation) is imposed. In the case of particular members of the family of the models the exact solutions describing toroidal solitons with a nontrivial value of the Hopf index are obtained. Moreover, the generalized eikonal equation is analyzed in detail. Topological solutions describing torus knots are presented. Multi-knot configurations are found as well.

1. Introduction

It has been recently shown that the complex eikonal equation

$$\partial_\mu u \partial^\mu u = 0 \tag{1}$$

plays a prominent role in nonlinear field theories in higher dimensions. Such models, widely applied in many physical contexts, are, in general, not integrable. As a consequence the spectrum of nontrivial (e.g., topological) solutions is scarcely known. Fortunately, the eikonal equation allows us to define integrable subsectors for such models with not-empty set of the topological solutions [1–6].

The nonlinear $O(3)$ models in two and three space dimensions can serve as the best example. These models, original based on an unit, three component vector field, can be reformulated in terms of a complex field u by means of the standard stereographic projection

$$\vec{n} = \frac{1}{1+|u|^2}\big(u+u^*, -i\big(u-u^*\big), |u|^2-1\big). \tag{2}$$

As the static, finite energy configurations are nothing else but maps from the compactified R^2 or R^3 on S^2, they can be classified by the pertinent topological invariant: the winding number or the Hopf index Q_H, respectively. As a result solutions describing topological vortices or knots can be obtained. The first model possessing the simplest knot soliton, i.e., the hopfion, is referred as the Nicole model [7]. Moreover, it has been proved that this solution fulfills additionally an integrability condition, i.e., the eikonal equation. The idea

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that hopfions can be quite easily found in integrable subsectors has enabled us to construct a family of the generalized Nicole models where hopfions with higher topological charges have been reported [8]. Strictly speaking, all these hopfions are identical to the eikonal knots [10,11] being solutions of the complex eikonal equation in toroidal coordinates[1]

$$x = \frac{\tilde{a}}{q} \sinh\eta \cos\phi,$$
$$y = \frac{\tilde{a}}{q} \sinh\eta \sin\phi,$$
$$z = \frac{\tilde{a}}{q} \sin\xi, \tag{3}$$

where $q = \cosh\xi - \cos\phi$ and $\tilde{a}$ is a scale constant. In fact it has been observed by Adam [10] that the following function

$$u(\eta,\xi,\phi) = f_0(\eta) e^{i(m\xi + k\phi)}, \tag{4}$$

where

$$f_0(\eta) = A \sinh^{\pm|k|}\eta \times \frac{(|m|\cosh\eta + \sqrt{k^2 + m^2\sinh^2\eta})^{\pm|m|}}{(|k|\cosh\eta + \sqrt{k^2 + m^2\sinh^2\eta})^{\pm|k|}}, \tag{5}$$

solves (1) and carries nonvanishing topological charge

$$Q_H = \pm|mk|. \tag{6}$$

Here m, k are integer numbers and A is a complex constant. Using the group of the target symmetries [9] one can construct even more general solutions

$$\tilde{u} = F(u), \tag{7}$$

where F is an arbitrary holomorphic function [10,11].

It should be emphasized that the eikonal knots are not only solutions of the toy models but can also find a physical application as approximate solutions of the Faddeev–Niemi effective model [13] for the low energy gluodynamics [14]. They provide an analytical framework in which qualitative (shape and topology) as well as quantitative (energy) features of the Faddeev–Niemi hopfions can be captured [11,15].

[1] The eikonal equation admits also other topological objects, e.g., vortices, braided strings or hedgehogs [12].

The fact that the complex eikonal equation has appeared to be very helpful in deriving hopfions in the Nicole-type models might indicate that in the case of other nonintegrable $O(3)$ models a similar condition might be introduced.

The main aim of the present Letter is to define a generalization of the eikonal equation which will enable us to derive new integrable submodels, for which new analytical toroidal solitons might be found.

2. Generalized eikonal knots

The original eikonal equation appears in the context of the nonlinear $O(3)$ model by the following vector quantity

$$K_\mu^{(1)} = \partial_\mu u. \tag{8}$$

In order to guarantee the existence of an integrable submodel this object must obey two conditions [3],

$$K_\mu^{(i)} \partial^\mu u = 0 \tag{9}$$

and

$$\mathrm{Im}\big(K_\mu^{(i)} \partial^\mu u^*\big) = 0, \tag{10}$$

where $i = 1, 2, \dots$ (see below). It is straightforward to notice that Eq. (9) gives the eikonal equation. Using $K_\mu^{(1)}$ one can construct a dynamical (but nonintegrable) model, i.e., the Nicole model [7]

$$L = \frac{1}{(1+|u|^2)^3} \big(K_\mu^{(1)} \partial^\mu u^*\big)^{3/2}, \tag{11}$$

where the power 3/2 is chosen to omit the Derick scaling argument for the nonexistence of the topological solitons. Now, Eq. (9) is an integrability condition for the model (for details see [7,8]).

This procedure, that is defining a quantity $K_\mu^{(i)}$ which fulfills (9), (10) and proposing a scale invariant Lagrangian built of it, can be repeated in the more complicated cases. Namely, Aratyn, Ferreira and Zimerman [3] have introduced

$$K_\mu^{(2)} = (\partial_\nu u)^2 \partial_\mu u^* + \alpha\big(\partial_\nu u \partial^\nu u^*\big)\partial_\mu u. \tag{12}$$

Now, two cases are possible. If $\alpha \neq -1$ then the integrability condition (9) leads to trivial solutions. Otherwise, for $\alpha = -1$ the condition is always, identically satisfied. Thus, any model based on $K_\mu^{(2)}$ with

$\alpha = -1$ is integrable. It is in contradiction to the Nicole-type models where Eq. (9) determines the integrable (essentially restricted) submodel. The pertinent Lagrangian reads

$$L = \frac{1}{(1+|u|^2)^3}\left(K_\mu^{(2)}\partial^\mu u^*\right)^{3/4}. \tag{13}$$

It has been checked that this model possesses infinitely many toroidal solitons with arbitrary Hopf charge [3] (for some generalization see [16]).

In our work we would like to focus on the next possible form of K_μ. We assume it as follows,

$$K_\mu^{(3)} = \alpha\left(\partial_\nu u\partial^\nu u^*\right)^2\partial_\mu u + \beta(\partial_\nu u)^2\left(\partial_\nu u\partial^\nu u^*\right)\partial_\mu u^* + \gamma(\partial_\nu u)^2\left(\partial_\nu u^*\right)^2\partial_\mu u, \tag{14}$$

where α, β, γ are real constants. The second condition (10) is immediately fulfilled whereas the first one (9) leads to an interesting formula. Indeed, inserting (14) into (9) we get

$$(\partial_\nu u)^2\left[\gamma(\partial_\nu u)^2\left(\partial_\nu u^*\right)^2 + (\alpha+\beta)\left(\partial_\nu u\partial^\nu u^*\right)^2\right] = 0. \tag{15}$$

It is not an identity unless $\alpha = -\beta$ and $\gamma = 0$. However, in this situation $K_\mu^{(3)}$ is proportional to $K_\mu^{(2)}$ and our problem can be reduced to the previously discussed model. Thus, from now we assume that $\alpha \neq -\beta$ and $\gamma \neq 0$ (for simplicity we assume that $\gamma = 1$). Then condition (15) can be rewritten in two parts

$$(\partial u)^2 = 0 \tag{16}$$

or

$$(\partial_\nu u)^2\left(\partial_\nu u^*\right)^2 + (\alpha+\beta)\left(\partial_\nu u\partial^\nu u^*\right)^2 = 0. \tag{17}$$

The first possibility is just the eikonal equation and does not lead to new integrable submodels. However, the second equation (17) (we called it generalized eikonal equation) provides a new integrability condition for models based on $K_\mu^{(3)}$.

Let us now solve the generalized eikonal equation. Due to the fact that we are mainly interested in obtaining of knotted configurations with a nontrivial value of the Hopf index we introduce the toroidal coordinates (3) and assume the following ansatz:

$$u(\eta,\xi,\phi) = f(\eta)e^{i(m\xi+n\phi)}. \tag{18}$$

Then formula (17) gives

$$\left[f'^2 - \left(m^2 + \frac{n^2}{\sinh^2\eta}\right)f^2\right]^2 + (\alpha+\beta)\left[f'^2 + \left(m^2 + \frac{n^2}{\sinh^2\eta}\right)f^2\right]^2 = 0, \tag{19}$$

where the prime denotes differentiation with respect to the η variable and f is a real shape function yet to be determined. The last equation can be rewritten as

$$f'^4 - 2\frac{(\alpha+\beta-1)}{(\alpha+\beta+1)}f'^2f^2\left(m^2+\frac{n^2}{\sinh^2\eta}\right) + f^4\left(m^2+\frac{n^2}{\sinh^2\eta}\right)^2 = 0 \tag{20}$$

and possesses the following roots

$$f'^2 = a^2(\alpha,\beta)\left(m^2+\frac{n^2}{\sinh^2\eta}\right)f^2, \tag{21}$$

where

$$a^2(\alpha,\beta) = \frac{(1-\alpha-\beta)}{(1+\alpha+\beta)} \pm \sqrt{\frac{(1-\alpha-\beta)^2}{(1+\alpha+\beta)^2} - 1}. \tag{22}$$

Eq. (21) admits a real solution only if $a^2 \geqslant 0$. Hence, we have a restriction for the parameters α and β in $K_\mu^{(3)}$,

$$-1 < \alpha + \beta \leqslant 0. \tag{23}$$

Finally we are able to solve (21). One can find that

$$f^{(\pm)} = A\sinh^{\pm a|k|}\eta \times \frac{(|m|\cosh\eta + \sqrt{k^2+m^2\sinh^2\eta})^{\pm a|m|}}{(|k|\cosh\eta + \sqrt{k^2+m^2\sinh^2\eta})^{\pm a|k|}}. \tag{24}$$

In other words, we have obtained a solution of the generalized eikonal equation

$$u(\eta,\xi,\phi) = A\sinh^{\pm a|k|}\eta \times \frac{(|m|\cosh\eta + \sqrt{k^2+m^2\sinh^2\eta})^{\pm a|m|}}{(|k|\cosh\eta + \sqrt{k^2+m^2\sinh^2\eta})^{\pm a|k|}} \times e^{i(m\xi+n\phi)}. \tag{25}$$

As the shape function smoothly interpolates from 0 to ∞ (or from ∞ to 0) this configuration carries the Hopf topological charge $Q_H = \pm|mk|$. It is clearly visible that the difference between the standard eikonal knots and the generalized solutions (25) emerges from the modified form of the shape function. The original shape function f_0 in Eq. (4) is replaced by f_0^a giving, depending on the value of the model parameters α and β, squeezed or stretched configurations.

The generalized eikonal equation has similar symmetries as its standard counterpart. As a consequence other solutions may be obtained by a simple algebraic transformation

$$\tilde{u} = F(u), \tag{26}$$

where F, exactly as for the eikonal equation, is a holomorphic function. Due to that, assuming that F is a polynomial, we are able to construct multi-knot configurations. Namely,

$$\begin{aligned} u(\eta, \xi, \phi) &= \sum_{j=1}^{N} A_j \sinh^{\pm a|k_j|} \eta \\ &\quad \times \frac{(|m_j| \cosh \eta + \sqrt{k_j^2 + m_j^2 \sinh^2 \eta})^{\pm a|m_j|}}{(|k_j| \cosh \eta + \sqrt{k_j^2 + m_j^2 \sinh^2 \eta})^{\pm a|k_j|}} \\ &\quad \times e^{i(m_j \xi + n_j \phi)} + c_0, \end{aligned} \tag{27}$$

where A_j and c_0 are complex constants and the integer parameters must obey the relation

$$\frac{m_j}{k_j} = \text{const}. \tag{28}$$

Analogously as in the standard case, such a multi-knot solution possesses the following topological charge (for configurations with '–' sign) [11]

$$Q_H = -\max\{m_j k_j,\ j = 1, \ldots, N\}. \tag{29}$$

Because of the fact that topological properties of the generalized and standard eikonal knots do not differ drastically, the detailed geometrical analysis of these solutions can be found in [11].

3. Integrable subsystems

Let us now introduce a new class of models based on the previously defined quantity $K_\mu^{(3)}$ for which integrable subsystems can be determined by the generalized eikonal equation. Moreover, we will show that some members of the family possess generalized eikonal knots as solutions of the dynamical equations of motion.

The pertinent model reads

$$\mathcal{S} = \int d^4x\, G\big(|u|\big)\big(K_\mu^{(3)} \partial^\mu u^*\big)^{1/2}, \tag{30}$$

where G is an arbitrary function of $|u|$ (sometimes called 'dielectric function').

Then the field equation is

$$\begin{aligned} &\partial_\mu \big[G\big(|u|\big)\big(K_\mu^{(3)} \partial^\mu u^*\big)^{-1/2} \mathcal{L}^\mu\big] \\ &\quad - \partial_{u^*} G\big(|u|\big)\big(K_\mu^{(3)} \partial^\mu u^*\big)^{1/2} = 0, \end{aligned} \tag{31}$$

where

$$\begin{aligned} \mathcal{L}_\mu = {}& (1+\beta)(\partial u)^2 \big[2\big(\partial u \partial u^*\big)\partial_\mu u^* + \big(\partial u^*\big)^2 \partial_\mu u\big] \\ & + 3\alpha \big(\partial u \partial u^*\big)^2 \partial_\mu u. \end{aligned} \tag{32}$$

One can notice that

$$\mathcal{L}_\mu \partial^\mu u^* = 3K_\mu^{(3)} \partial^\mu u^* \tag{33}$$

and

$$\begin{aligned} \mathcal{L}_\mu \partial^\mu u = {}& (\partial u)^2 \big[\big(2(1+\beta) + 3\alpha\big)\big(\partial u \partial u^*\big)^2 \\ & + (1+\beta)(\partial u)^2 \big(\partial u^*\big)^2\big]. \end{aligned} \tag{34}$$

In order to obtain an integrable submodel of (30) we impose the additional (integrability) condition [3]

$$\mathcal{L}_\mu \partial^\mu u = 0, \tag{35}$$

which leads to the generalized eikonal equation. Then, the remaining equation of motion takes the following form

$$\partial_\mu \big[G^{1/3}\big(|u|\big)\big(K_\mu^{(3)} \partial^\mu u^*\big)^{-1/2} \mathcal{L}^\mu\big] = 0. \tag{36}$$

In can be rewritten in the more compact form

$$\partial_\mu \mathcal{K}^\mu = 0, \tag{37}$$

where

$$\mathcal{K}^\mu = G^{1/3}\big(|u|\big)\big(K_\mu^{(3)} \partial^\mu u^*\big)^{-1/2} \mathcal{L}^\mu. \tag{38}$$

These both equations: dynamical equation (36) and the condition (35) define the integrable submodel. Here integrability is understood as the existence of an infinite family of the conserved current. In fact, using the results of [3] one can show that such currents are given by the expression

$$J_\mu = \mathcal{K}_\mu \frac{\partial H}{\partial u} - \mathcal{K}^*_\mu \frac{\partial H}{\partial u^*}, \tag{39}$$

where H is an arbitrary function of u and u^*.

Now, we will prove that the existence of these currents can result in the appearance of soliton solutions with a nontrivial value of the Hopf index. At the beginning we specify a particular form of the 'dielectric function' in the Lagrangian which allows us to obtain toroidal solitons in the exact form. We take

$$G_{m,a}(|u|) = \left(\frac{|u|^{\frac{1-am}{am}}}{1+|u|^{\frac{2}{am}}} \right)^3. \tag{40}$$

Taking into account ansatz (18) we get

$$G_{m,a}(f) = \left(\frac{f^{\frac{1-am}{am}}}{1+f^{\frac{2}{am}}} \right)^3. \tag{41}$$

Then the equation of motion (36) reads

$$\partial_\eta \left[G^{1/3} I_0 c_+ f' \right] - G^{1/3} I_0 c_- f \left(m^2 + \frac{k^2}{\sinh^2\eta} \right) + G^{1/3} I_0 c_+ f' \frac{\cosh\eta}{\sinh\eta} = 0, \tag{42}$$

where the following abbreviations have been made

$$I_0 = \left[\left((1+\beta)\omega_-^2 + \alpha\omega_+^2 \right)\omega_+ \right]^{-1/2}, \tag{43}$$

$$c_\pm = \pm 2(1+\beta)\omega_-\omega_+ + (1+\beta)\omega_-^2 + 3\alpha\omega_+^2 \tag{44}$$

and

$$\omega_\pm = f'^2 \pm \left(m^2 + \frac{k^2}{\sinh^2\eta} \right) f^2. \tag{45}$$

Of course, one should keep in mind that solutions of (42) must obey the generalized eikonal equation as well. Hence

$$\omega_\pm = f'^2 \left(1 \pm \frac{1}{a^2} \right) \equiv \lambda_\pm f'^2, \tag{46}$$

where

$$a^2 = \frac{-3\alpha-\beta-1}{3(\alpha+\beta+1)} \pm \sqrt{\left(\frac{3\alpha+\beta+1}{3(\alpha+\beta+1)} \right)^2 - 1}. \tag{47}$$

Then

$$c_\pm = \left[\pm 2(1+\beta)\lambda_-\lambda_+ + (1+\beta)\lambda_-^2 + 3\alpha\lambda_+^2 \right] f'^4 \equiv \sigma_\pm f'^4 \tag{48}$$

and

$$I_0 = \text{const} \times f'^{-3}. \tag{49}$$

Inserting (44)–(49) into (42) one obtains

$$\partial_\eta \ln\left(G^{1/3} f'^2 \right) - \frac{\sigma_-}{\sigma_+} \frac{f}{f'} \left(m^2 + \frac{k^2}{\sinh^2\eta} \right) + \partial_\eta \ln\sinh\eta = 0, \tag{50}$$

or equivalently

$$\partial_\eta \ln\left(G^{1/3} f'^2 \right) - \frac{\sigma_-}{a^2\sigma_+} \partial_\eta \ln f + \partial_\eta \ln\sinh\eta = 0. \tag{51}$$

Moreover, one can calculate that

$$\frac{\sigma_-}{\sigma_+} = a^2. \tag{52}$$

Thus finally, Eq. (51) can be integrated and we obtain

$$G^{1/3} f'^2 f^{-1} = \frac{\text{const}}{\sinh\eta}. \tag{53}$$

It can be checked that this equation is solved by the following function

$$f = \left(\frac{1}{\sinh\eta} \right)^{am}. \tag{54}$$

Since it also satisfies the generalized eikonal equation and corresponds via (18) to the configuration with the nontrivial topological charge $Q_H = -m^2$, we have proved that the spectrum of soliton solutions of the integrable submodel is not empty.

In order to complete the investigation of the generalized eikonal hopfions we calculate their total energy. It can be performed in the case of the hopfions for which the pertinent Lagrangian has been established. In other words, we do it for solutions (54), i.e., for knots with $m = k$.

Then, the energy of the static configurations reads

$$E = \int d^3x\, G(|u|) \left(\vec{K}^{(3)} \cdot \vec{\nabla} u^* \right)^{1/2}. \tag{55}$$

Taking into account that all solitons obey also the generalized eikonal equation we obtain that

$$E = (2\pi)^2 \sqrt{\lambda_+ \left[(1+\beta)\lambda_-^2 + \alpha\lambda_+ \right]} \times \int_0^\infty d\eta \sinh\eta \left(\frac{f^{\frac{1-am}{am}}}{1+f^{\frac{2}{am}}} \right)^3 f'^3, \tag{56}$$

or equivalently

$$E = (2\pi)^2 \sqrt{\lambda_+\big[(1+\beta)\lambda_-^2 + \alpha\lambda_+\big]}\, a^3 m^3 \times \int_0^\infty d\eta \, \frac{\cosh^3\eta}{\sinh^2\eta} \left(\frac{f^{\frac{1}{am}}}{1+f^{\frac{2}{am}}} \right)^3. \tag{57}$$

Finally, after inserting (54) into this formula we find

$$E = 2\pi^2 \sqrt{\lambda_+\big[(1+\beta)\lambda_-^2 + \alpha\lambda_+\big]}\, a^3 m^3. \tag{58}$$

As solutions (54) possess the Hopf index $Q_H = -m^2$, the energy depends on the topological charge in a rather nontrivial manner, i.e.,

$$E = 2\pi^2 \sqrt{\lambda_+\big[(1+\beta)\lambda_-^2 + \alpha\lambda_+\big]}\, a^3 |Q_H|^{3/2}. \tag{59}$$

One can notice that it resembles the energy-charge relation in the modified Nicole models [8]. This is not surprising since the Nicole-type models can be achieved from the dynamical systems introduced here in the limit $a = 1$.

Such overlinear dependence found for many (generalized) eikonal hopfions is rather puzzling and unexpected if we compare it with the famous Vakulenko–Kapitansky energy-charge inequality for the Faddeev–Niemi model [17], where $E \geqslant c|Q_H|^{3/4}$. Moreover, it has been proved that this sublinear dependence is valid also for the soliton solutions of the Aratyn–Ferreira–Zimerman model [3].

Of course, it must to be emphasized that the overlinear behavior is still only a conjecture. It is due to the fact that each soliton (generalized eikonal hopfion) has been derived in a different model. Thus, it is unknown whether for a fixed model (i.e., fixed values of the parameters in (32) and (40)) the topological solutions obey this relation. At this stage it would be hazardous to claim that all introduced models must lead to an overlinear dependence.

4. Further models

The construction introduced and analyzed in the previous section can be easily adapted to the more complicated (with a higher number of derivatives) $K_\mu^{(i)}$.

For example, let us focus on the next possibility and take

$$K_\mu^{(4)} = \alpha(\partial u)^2 \big(\partial u^*\big)^2 \big(\partial u \partial u^*\big) \partial_\mu u + \beta \big(\partial u \partial u^*\big)^3 \partial_\mu u + \gamma (\partial u)^2 \big(\partial u \partial u^*\big)^2 \partial_\mu u^* + \delta (\partial u)^4 \big(\partial u^*\big)^2 \partial_\mu u^*, \tag{60}$$

where $\alpha, \beta, \gamma, \delta$ are real constants.

The scale invariant action based on this quantity is given as follows

$$S = \int d^4x \, \tilde{G}\big(|u|\big) \big(K_\mu^{(4)} \partial^\mu u^*\big)^{3/8}, \tag{61}$$

where $\tilde{G}$ is any function of $|u|$. The corresponding integrable submodel can be found, in an analogous way as for (30), by imposing two conditions (9) and (10). Since (10) is always fulfilled let us turn to the first formula. One can show that, depending on the parameters of the model, this equation leads to two cases. If $\alpha = -\delta$ and $\beta = -\gamma$ then the condition is identically satisfied. It is identical to the Aratyn–Ferreira–Zimerman model and no new solutions can be obtained. Otherwise, we get an equation which is solved by generalized eikonal knots. Thus, the integrable subsystem of (61), though it exists, does not provide any new hopfions.

This feature appears to be quite general and one can observe it in the more complicated examples.

5. Conclusions

Let us briefly summarize the obtained results. A generalization of the standard complex eikonal equation has been proposed. This equation possesses various topologically nontrivial solutions. In particular, knotted configurations carrying arbitrary value of the Hopf charge have been explicitly derived. They appear to be deformed (squeezed or stretched) standard eikonal knots. Moreover, using the symmetry of the generalized eikonal equation, it is possible to construct multi-knot solutions (linked knots).

It has also been shown that this equation enables us to define a new class of integrable models, where integrability is understood as the existence of an infinite family of conserved currents. Then, the generalized eikonal equation is just the integrability condition. Additionally, we have proved that the integrability may lead to the appearance of soliton solutions. In the case of particular members of the family of models analysed here, we have found that the spectrum of solutions is not empty but consists of the generalized eikonal knots. Such Hopf solitons, i.e., hopfions, have finite energy which depends on the topological charge in a very nontrivial way.

There are several directions in which one could continue the present work. First of all, one should check whether, after fixing the model parameters in (32), (40), the energy of other hopfions is also proportional to $|Q_H|^{3/2}$. The validity of this conjecture (which could be tested at least by some numerical methods) might be very important in understanding of the interaction of Hopf solitons. Such a overlinear dependence might suggest that, in these models, hopfions with higher topological charges would decay into a collection of the simplest solitons, each with the unit Hopf index. It would be in contradiction to the standard clustering phenomena observed in the Faddeev–Niemi model, where, due to the sublinear behavior, a separated multi-soliton configuration tends to form a clustered, really knotted state [18,19].

Secondly, the generalized eikonal knots might be helpful in the construction of new approximated, analytical solutions of the Faddeev–Niemi hopfions (known only in the numeric form [20–22]), which could provide better accuracy in the approximation of the energy.

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