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A new exact solution for pricing European options in a two-state regime-switching economy

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ABSTRACT

In this study, we derive a new exact solution for pricing European options in a two-state regime-switching economy. Two coupled Black–Scholes partial differential equations (PDEs) under the regime switching are solved using the Fourier Transform method. A key feature of the newly-derived solution is its simplicity in the form of a single integral with a real integrand, which leads to great computational efficiency in comparison with other closed-form solutions previously presented in the literature. Numerical examples are provided to demonstrate some interesting results obtained from our pricing formula.

1. Introduction

It is well known that the classical Black–Scholes model with constant volatility does not fully reflect the stochastic nature of financial markets. Consequently, there is a need for more realistic models that better reflect random market movements. One such formulation is a model with regime switching, in which the key parameters of an asset depend on the market mode (or "regime") that switches among a finite number of states. From an economic perspective, regime-switching behavior captures the changing preferences and beliefs of investors concerning asset prices as the state of a financial market changes. Since being introduced by Hamilton [1], there has been a growing body of empirical evidence suggesting that the distributions of asset returns in some cases are better described by a regime-switching process (see [2–7]).

Pricing financial derivatives with regime-switching models has been discussed in literature. Bollen [8] presented a lattice-based method for pricing both European-style and American-style derivatives. Like other lattice-based numerical approaches adopted to price financial derivatives without the assumption of regime switching, Bollen's approach is financially intuitive and easy to implement. However, for European-style derivatives under the assumption of only two economic states, most researchers have focused their attention on developing closed-form exact solutions. Naik [9] was the first to discuss pricing and hedging European-style options when the volatility of the risky asset is assumed to randomly jump between two states. He found an exact closed-form pricing formula in the form of a double integral for an arbitrary security with a given payoff function. Di Masi et al. [10] discussed mean–variance hedging for European options where the drift rate and volatility are driven by a regime-switching process. Herzel [11] argued that a closed-form solution for a European contingent claim can be found in terms of a "basis" option. But, he only wrote down the partial differential equation (PDE) that the option prices under a two-state regime-switching model must satisfy once the value of the "basis" option has been found, without actually solving the derived PDE. Guo [12] presented a closed-form formula for the arbitrage-free price of a European call option in a two-state economy. The result found by Guo [12] is more general than that found

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by Naik [9] as the drift rate, volatility and continuous dividend yield are all assumed to be dependent upon the economic state. Buffington and Elliot [13,14] showed how the governing PDEs are formed for European-style options and presented a closed-form solution for this problem through the derivation of the characteristic function of the occupation times in each state. However, if one adopts their formula, a Fourier inversion must be performed numerically. Fuh et al. [15] claimed that Guo [12] made an error, and they too presented a closed-form formula in a very similar form. Once the probability density function of the occupation time in each state is found explicitly, as done in [15], it is not surprising that the European-style option prices for a two-state regime-switching economy can simply be written in a closed form as a discounted expectation of the terminal payoff under the risk-neutral measure. More recently, Sepp and Skachkov [16] adopted a similar approach to the one used in this paper, finding a two-branch solution to the PDEs associated with two-state regime-switching for a European call option in the Laplace space. However, they did not perform the Laplace inversion analytically and resorted to the use of a robust numerical scheme for the calculation of option values.

Unfortunately, all existing formulae are written in the form of either a double integral, as a direct result of taking the discounted expectation, or a closed-form solution in a transform space such as the Laplace space; no one has managed to show that a closed-form solution can be written in the form of a single integral with real integrand. Reducing the final form of the closed-form solution from a double integral to a single integral comprised of elementary functions not only simplifies the appearance of the formula, but also enhances the computational efficiency if numerical values need to be produced. The contribution of this paper is to provide a new closed-form formula to value European options in a two-state regime-switching economy. This is achieved through an exact solution to the PDE system for a European put option found via the Fourier transform method. A key feature of our new formula is that we have successfully performed Fourier inverse transform analytically and thus produced a final pricing formula containing a single integral of a real-valued function. Therefore, in comparison with other approaches in the literature, our new formula displays advantages in computational efficiency and accuracy.

The rest of the paper is organized as follows. In Section 2, the asset price dynamics in a regime-switching economy are briefly described, followed by a detailed description of the newly found closed-form formula for the value of a European put option. In Section 3, numerical examples are given for the purpose of illustration, followed by the conclusions in Section 4. Any mathematical derivations that are not immediately needed in the main body of the paper, yet are important for readers who may be interested in the details of derivation, are left to the Appendices.

2. New solution

We model a European put option in a regime-switching economy where the drift rate and volatility are subject to random shifts between two states. The asset-price dynamics in a regime-switching economy have been described previously in [12–15]. However, for completeness, we start this section by briefly describing them as well.

The fluctuations of an asset are assumed to follow a stochastic process described by the stochastic differential equation

$$dS_t = \mu_{X_t} S_t dt + \sigma_{X_t} S_t dW_t \tag{1}$$

where X is a continuous-time Markov chain with a finite state space. The drift rate, μ_{X_t}, and the volatility rate, σ_{X_t}, of the asset are functions of X_t. W is the standard Wiener process and the processes X and W are assumed to be independent. For each state, the drift rate and the volatility rate are assumed to be given constants. Furthermore, it is assumed that the volatility rates are distinct (i.e. $\sigma_{X_s} \neq \sigma_{X_t}$ if $X_s \neq X_t$).

In this paper, we assume X is a two-state Markov chain which jumps between two states,

$$X_t = \begin{cases} 1, & \text{when the economy is in a state of growth} \\ 2, & \text{when the economy is in a state of recession.} \end{cases}$$

The transition between states occurs as a Poisson process, i.e.

$$P(t^*_{jk} > t) = e^{-\lambda_{jk} t}, \quad j, k = 1, 2,\ j \neq k$$

where λ_{jk} is the transition rate from state j to state k and t^*_{jk} is the time spent in state j before entering state k.

The market price of risk associated with a change in state is not uniquely determined since the market is incomplete. In this paper, we assume that the risk associated with a regime switch is diversifiable and therefore not priced. Naik [9] demonstrates that this assumption does not result in a loss of generality, since one only needs to adjust the rate parameters of the transition process to account for non-diversifiable risk. Under this assumption, a system of coupled Black–Scholes equations for the value of a European put option can be derived (cf. [13,14]), with the movement of the underlying asset being described by Eq. (1), as

$$\begin{cases} \dfrac{\partial V_1}{\partial t} + \dfrac{1}{2}\sigma_1^2 S^2 \dfrac{\partial^2 V_1}{\partial S^2} + rS\dfrac{\partial V_1}{\partial S} - rV_1 = \lambda_{12}(V_1 - V_2) \\ V_1(0, t) = Ee^{-r(T-t)} \\ \lim\limits_{S\to\infty} V_1(S, t) = 0 \\ V_1(S, T) = \max\{E - S, 0\} \end{cases} \tag{2}$$

$$\begin{cases} \dfrac{\partial V_2}{\partial t} + \dfrac{1}{2}\sigma_2^2 S^2 \dfrac{\partial^2 V_2}{\partial S^2} + rS\dfrac{\partial V_2}{\partial S} - rV_2 = \lambda_{21}(V_2 - V_1) \\ V_2(0,t) = Ee^{-r(T-t)} \\ \lim\limits_{S\to\infty} V_2(S,t) = 0 \\ V_2(S,T) = \max\{E - S, 0\} \end{cases} \tag{3}$$

where S is the value of the underlying asset, t is the current time, $V_j(S,t)$ $(j = 1, 2)$ is the option value when in state j of the economy, r is the risk-free interest rate (assumed to be constant), E is the strike price and T is the expiration time of the option.

We begin by introducing the following dimensionless variables

$$q_j(x,\tau_j) = \frac{e^x V_j(S,t)}{E}, \quad x = \ln\left(\frac{S}{E}\right), \ \tau_j = \frac{\sigma_j^2}{2}(T-t)$$

for $j = 1, 2$. Apart from the inclusion of the exponential factor in $q_j(x,\tau_j)$, the above change of variables for option valuation problems is a commonly adopted approach to normalize the PDE systems. The exponential factor is included to ensure that the $q_j(x,\tau_j)$ functions are integrable under the Fourier transform (cf. [17]). With the new dimensionless variables, Eqs. (2) and (3) become

$$\begin{cases} -\dfrac{\partial q_1}{\partial \tau_1} + \dfrac{\partial^2 q_1}{\partial x^2} + (\gamma_1 - 3)\dfrac{\partial q_1}{\partial x} - (2\gamma_1 + \beta_{12} - 2)q_1 = -\beta_{12}q_2 \\ \lim\limits_{x\to-\infty} q_1(x,\tau_1) = 0 \\ \lim\limits_{x\to\infty} q_1(x,\tau_1) = 0 \\ q_1(x,0) = (e^x - e^{2x})^+ \end{cases} \tag{4}$$

$$\begin{cases} -\dfrac{\partial q_2}{\partial \tau_2} + \dfrac{\partial^2 q_2}{\partial x^2} + (\gamma_2 - 3)\dfrac{\partial q_2}{\partial x} - (2\gamma_2 + \beta_{21} - 2)q_2 = -\beta_{21}q_1 \\ \lim\limits_{x\to-\infty} q_2(x,\tau_2) = 0 \\ \lim\limits_{x\to\infty} q_2(x,\tau_2) = 0 \\ q_2(x,0) = (e^x - e^{2x})^+ \end{cases} \tag{5}$$

where $\gamma_j \equiv \frac{2r}{\sigma_j^2}$ and $\beta_{jk} \equiv \frac{2\lambda_{jk}}{\sigma_j^2}$, for $j, k = 1, 2\ j \neq k$, which can be viewed as the interest rate and the rate of leaving each state, relative to the volatility from that state, respectively.

Upon performing the Fourier transform defined as

$$\mathscr{F}q_j(x,\tau_j) = \int_{-\infty}^{\infty} e^{-i\omega x} q_j(x,\tau_j)dx = \hat{q}_j(\omega,\tau_j),$$

where $i = \sqrt{-1}$, Eqs. (4) and (5) are transformed to two coupled ordinary differential equations (ODEs) in the Fourier space

$$\begin{cases} \left[\dfrac{d}{d\tau_1} + B_{12}(\omega)\right]\hat{q}_1(\omega,\tau_1) = \beta_{12}\hat{q}_2(\omega,\tau_1) \\ \hat{q}_1(x,0) = \hat{q}_0 \\ \left.\dfrac{d\hat{q}_1}{d\tau_1}\right|_{\tau_1=0} + B_{12}(\omega)\hat{q}_0 = \beta_{12}\hat{q}_0 \end{cases} \tag{6}$$

$$\begin{cases} \left[A\dfrac{d}{d\tau_1} + B_{21}(\omega)\right]\hat{q}_2(\omega,\tau_1) = \beta_{21}\hat{q}_1(\omega,\tau_1) \\ \hat{q}_2(x,0) = \hat{q}_0 \\ A\left.\dfrac{d\hat{q}_2}{d\tau_1}\right|_{\tau_1=0} + B_{21}(\omega)\hat{q}_0 = \beta_{21}\hat{q}_0 \end{cases} \tag{7}$$

where

$$\hat{q}_0 = \mathscr{F}(e^x - e^{2x})^+ = \int_{-\infty}^{\infty} e^{-i\omega x}(e^x - e^{2x})^+ dx = \frac{1}{(1-i\omega)(2-i\omega)} \tag{8}$$

and

$$A = \frac{\sigma_1^2}{\sigma_2^2}, \qquad B_{jk}(\omega) = \omega^2 - i\omega(\gamma_j - 3) + (2\gamma_j + \beta_{jk} - 2), \quad j,k = 1,2,\ j \neq k.$$

Solving the coupled first-order linear ODEs is a relatively straightforward exercise. The result is

$$\begin{aligned}\hat{q}_1(\omega,\tau_1) &= \frac{\hat{q}_0\{[\beta_{12}-m_2-B_{12}(\omega)]e^{m_1\tau_1}-[\beta_{12}-m_1-B_{12}(\omega)]e^{m_2\tau_1}\}}{m_1-m_2}\\ &= \frac{[\beta_{12}-m_2-B_{12}(\omega)]e^{m_1\tau_1}-[\beta_{12}-m_1-B_{12}(\omega)]e^{m_2\tau_1}}{(1-i\omega)(2-i\omega)(m_1-m_2)}\\ \hat{q}_2(\omega,\tau_2) &= \frac{\hat{q}_0\{[\beta_{21}-Am_2-B_{21}(\omega)]e^{Am_1\tau_2}-[\beta_{21}-Am_1-B_{21}(\omega)]e^{Am_2\tau_2}\}}{A(m_1-m_2)}\\ &= \frac{[\beta_{21}-Am_2-B_{21}(\omega)]e^{Am_1\tau_2}-[\beta_{21}-Am_1-B_{21}(\omega)]e^{Am_2\tau_2}}{A(1-i\omega)(2-i\omega)(m_1-m_2)}\end{aligned}$$

where

$$m_{1,2} = -\frac{B_{12}(\omega)}{2}-\frac{B_{21}(\omega)}{2A}\pm\frac{\sqrt{[AB_{12}(\omega)-B_{21}(\omega)]^2+4A\beta_{21}\beta_{12}}}{2A}.$$

Of course, to obtain the option price, one needs to perform the Fourier inversion

$$q_j(x,\tau_j) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\hat{q}_j(\omega,\tau_j)d\omega, \quad \text{for } j=1,2. \tag{9}$$

Quite often, one may leave the final solution in the form of Eq. (9) (e.g., [18]), as, in most cases, finding Fourier inversion analytically is a non-trivial task and the success is usually very much case dependent, apart from some tedious algebraic manipulations. Some may even argue that the above expression is already of closed form as there is only one explicit integral left to be calculated, similar to the calculation of the normal cumulative distribution function required in the evaluation of the Black–Scholes formula. However, it must be pointed out that a significant difference between the two is that the integrand in the latter is a well-defined elementary real-valued function whereas the integrand in Eq. (9) is not. To appropriately derive a pricing formula, one should endeavor to analytically perform the Fourier inverse transform, whenever possible, so that any integral left in the final form of the formula only has real-valued integrand, which can thus be easily evaluated numerically.

We have successfully performed an analytical inversion of the Fourier transform as shown in Appendix A. After the Fourier inversion is performed, our solution can be written as, in terms the original variables and parameters,

$$\begin{aligned}V_j(S,t) = {} & Ee^{-r(T-t)}+\frac{1}{4\pi\sqrt{2}}\sqrt{SE}e^{-\frac{1}{2}\left(r+\lambda_{21}+\lambda_{12}+\frac{\sigma_1^2+\sigma_2^2}{8}\right)(T-t)}\int_0^\infty \frac{(-1)^{j-1}2f_1(\rho)(\lambda_{21}+\lambda_{12})}{M(\rho)\left(\rho^4+\frac{1}{16}\right)(\sigma_1^2-\sigma_2^2)}\\ & \times\left\{e^{X_j(\rho)}\left[\left(2\rho^2-\frac{1}{2}\right)\sin(f_2(\rho)+\theta(\rho)-Y_j(\rho))-\left(2\rho^2+\frac{1}{2}\right)\cos(f_2(\rho)+\theta(\rho)-Y_j(\rho))\right]\right.\\ & \left.-e^{-X_j(\rho)}\left[\left(2\rho^2-\frac{1}{2}\right)\sin(f_2(\rho)+\theta(\rho)+Y_j(\rho))-\left(2\rho^2+\frac{1}{2}\right)\cos(f_2(\rho)+\theta(\rho)+Y_j(\rho))\right]\right\}\\ & +\frac{2f_1(\rho)}{M(\rho)}\left\{e^{X_j(\rho)}\left[\sin(f_2(\rho)+\theta(\rho)-Y_j(\rho))+\cos(f_2(\rho)+\theta(\rho)-Y_j(\rho))\right]\right.\\ & \left.-\ e^{-X_j(\rho)}\left[\sin(f_2(\rho)+\theta(\rho)+Y_j(\rho))+\cos(f_2(\rho)+\theta(\rho)+Y_j(\rho))\right]\right\}\\ & +\frac{f_1(\rho)}{\rho^4+\frac{1}{16}}\left\{e^{X_j(\rho)}\left[\left(2\rho^2-\frac{1}{2}\right)\sin[f_2(\rho)-Y_j(\rho)]-\left(2\rho^2+\frac{1}{2}\right)\cos[f_2(\rho)-Y_j(\rho)]\right]\right.\\ & \left.+e^{-X_j(\rho)}\left[\left(2\rho^2-\frac{1}{2}\right)\sin[f_2(\rho)+Y_j(\rho)]-\left(2\rho^2+\frac{1}{2}\right)\cos[f_2(\rho)+Y_j(\rho)]\right]\right\}d\rho\end{aligned} \tag{10}$$

for $j=1,2$, where

$$\tau_- = \frac{\sigma_1^2-\sigma_2^2}{4}(T-t), \qquad \alpha_- = \frac{2(\lambda_{12}-\lambda_{21})}{\sigma_1^2-\sigma_2^2}, \qquad \mu^2 = \frac{4\lambda_{12}\lambda_{21}}{(\sigma_1^2-\sigma_2^2)^2}$$

$$M(\rho) = \left\{\left[\left(\frac{1}{4}+\alpha_-\right)^2-\rho^4+\mu^2\right]^2+4\rho^4\left(\frac{1}{4}+\alpha_-\right)^2\right\}^{\frac{1}{4}}$$

$$\theta(\rho) = \frac{1}{2}\tan^{-1}\left[\frac{2\rho^2\left(\frac{1}{4}+\alpha_-\right)}{\left(\frac{1}{4}+\alpha_-\right)^2-\rho^4+\mu^2}\right]$$

$$X_j(\rho) = (-1)^{j-1}M(\rho)\tau_-\cos\theta(\rho), \qquad Y_j(\rho) = (-1)^{j-1}M(\rho)\tau_-\sin\theta(\rho)$$

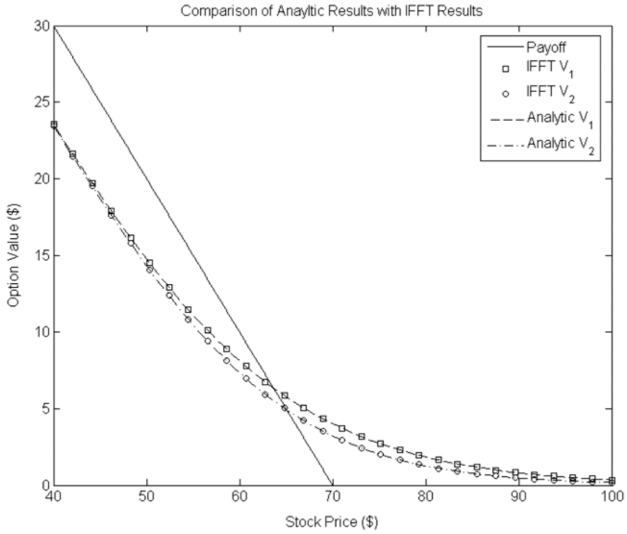

Fig. 1. Option values ($) vs. stock price ($) using Eq. (10) and using the IFFT.

and

$$f_1(\rho) = e^{-\frac{\rho}{\sqrt{2}}\left|\ln\left(\frac{S}{E}\right)+r(T-t)\right|}, \qquad f_2(\rho) = \frac{\rho^2}{4}(\sigma_1^2+\sigma_2^2)(T-t) - \frac{\rho}{\sqrt{2}}\left|\ln\left(\frac{S}{E}\right)+r(T-t)\right|.$$

One can observe that the newly found solution, Eq. (10), is in the form of a single integral with a real integrand comprised of elementary functions, whereas the existing formulae are all in the form of a double integral.

3. Numerical examples and discussion

In this section we provide some numerical examples to demonstrate that numerical values can be easily produced from our analytical solution. We also briefly discuss some implications of the regime switching model by using the new analytical solution.

Before a meaningful discussion is carried out, we first compare the results obtained using the newly-derived formula (10) with those obtained by numerically performing the inverse fast Fourier transform (IFFT) (cf. [19]). To a large extent, this helps verify that the tedious algebraic manipulations that we had carried out for the analytical inversion are correct.

For the purpose of illustration we take the following parameters for pricing a European put option in a two-state regime-switching economy: $E = \$70$, $r = 10\%$, $T - t = 1$ year, $\sigma_1 = 20\%$, $\sigma_2 = 30\%$, $\lambda_{21} = 1$ and $\lambda_{12} = 2$. In Fig. 1, numerical and analytical option values are plotted against the underlying stock price. The dashed line represents V_1, the option value in State 1 with volatility value at σ_1, while the dot–dashed line corresponds to V_2, the option value in State 2 with σ_2. One can clearly observe that the results from our exact solution match up perfectly with the numerical results for both V_1 and V_2.

It should be noted that the newly found solution, Eq. (10), is in the form of a single integral with a real integrand comprised of elementary functions. All existing solutions are in the form of a double integral, or are in a transform space and thus reliant on a numerical inversion. Of course, when the numerical values need to be computed from the newly found formula (10), a numerical quadrature scheme[2] is needed, as is the case when evaluating existing ones. A fundamental difference is that the calculation of a single integral is, in general, computationally more efficient than that of a double integral. Furthermore, since the integrand of the newly found formula is simpler than that of the existing formulae, the evaluation of this integrand within the numerical integration scheme has added further strength in terms of computational efficiency. As far as the existing solutions in the Laplace space are concerned, performing numerical Laplace inversions can be notoriously cumbersome (cf. [20]) and thus solutions in the form of a single integral with a real integrand are always preferred. For these reasons one can expect an improvement in computational efficiency from the newly found closed-form solution in comparison to the existing ones.

In Table 1, we compared our results with those published in [15] for a European call option in a two-state regime-switching economy, with $S = \$100$, $E = \$90$, $r = 10\%$, $\sigma_1 = 20\%$, $\sigma_2 = 30\%$, $\lambda_{21} = 1$ and $\lambda_{12} = 1$. The put-call parity is used to calculate the call option prices from our put formula. In order to demonstrate the fast convergence of the numerical computation of the single integral in Eq. (10), we displayed our results for different values of L, with L being the truncated upper limit that replaces the ∞ in the single integral in Eq. (10). As the values in the tables show, the computation of the

[2] The one adopted to produce the results presented in this paper is the adaptive Lobatto quadrature scheme provided in Matlab 7.0.

Table 1
Comparison of V_1 and V_2 in [15] and those obtained by using Eq. (10).

$T-t$ (years)	V_1 ($) in [15]	Our results V_1 ($)			V_2 ($) in [15]	Our results V_2 ($)		
		$L=10$	$L=100$	$L=1000$		$L=10$	$L=100$	$L=1000$
0.1	11.360	11.7962	11.3608	11.3608	10.993	11.5679	10.9932	10.9932
0.2	12.889	13.0262	12.8895	12.8895	12.164	12.5833	12.1647	12.1647
0.5	16.718	16.5813	16.7184	16.7184	15.614	15.6611	15.6144	15.6144
1.0	21.812	21.7785	21.8120	21.8120	20.721	20.6522	20.7216	20.7216
2.0	30.085	30.0892	30.0850	30.0850	29.287	29.2884	29.2877	29.2877
3.0	37.061	37.0618	37.0619	37.0619	36.476	36.4774	36.4766	36.4766

Table 2
Convergence test results with payoff $(S-E)^+ = \$(100-90) = \$10$.

L	10	100	1000
$V_i(100, T)$	\$10.56300639684868	\$10.00000714537026	\$9.99999999988907

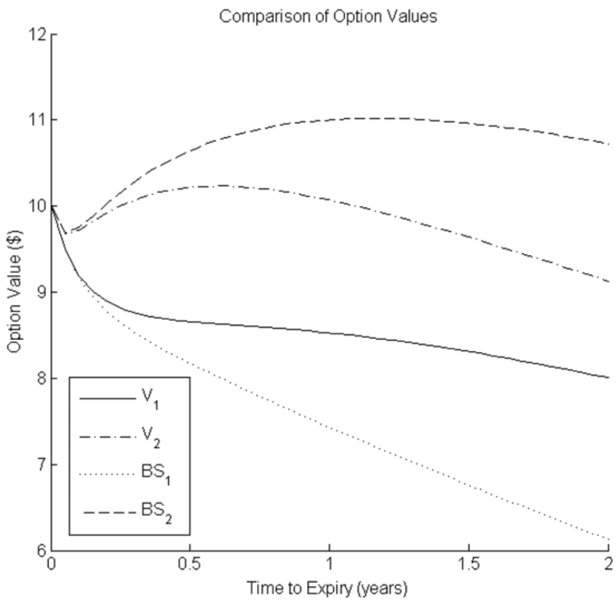

Fig. 2. Option value ($) vs. time to expiry (years).

single integral involved in our exact solution appears to converge very quickly; the results obtained with $L = 100$ appear to be accurate to 6 significant figures already. However, there are slight discrepancies between our solutions and those given in [15] at some time values, which seem to be due to the choices of truncation.

In general, as an option approaches expiry, the numerical quadrature becomes less accurate if L is too small. Away from expiration, the choice of $L = 10$ has resulted in relative errors in option values being less than 0.01%. Closer to expiration, a larger L is required to produce results with the same level of accuracy. This is due to the exponential factor in $f_1(\rho)$, which causes the integrand to vanish as $T-t$ increases. As we know the exact option value at expiration, we can analyze the convergence of our solution at $T-t=0$ by comparing the payoff to the analytic formulae for V_1 and V_2 in Eq. (10) (it is worth noting that the option prices obtained from Eq. (10) converge for $T-t=0$). The results of the comparison are provided in Table 2. The choice of a still relatively small value of $L = 100$ produces the option value at expiration with a relative error of the order of 10^{-6}. As $T-t=0$ is generally the worst case for convergence, we can conclude that setting $L = 100$ should produce convergent solutions for all $T-t \geq 0$. If we go beyond $L = 1000$, numerical results start to display large errors again, simply because the specified domain becomes too large for the effective part of the integrand to be properly resolved as pointed out by Zhu [21].

We now discuss some interesting implications of taking a simple two-state regime-switching model in pricing European options by comparing the option prices obtained using the newly-derived formula to those obtained using the classical Black–Scholes model. Depicted in Fig. 2 are four different option values as a function of time to expiration. V_1 and V_2 are the put option values obtained from the regime-switching model, while the curves with legends BS_1 and BS_2 are the option values calculated from the classical Black–Scholes model, corresponding to volatilities σ_1 and σ_2, respectively. The calculation was carried out with the parameter values: $S_0 = \$90$, $E = \$100$, $r = 10\%$, $\sigma_1 = 20\%$, $\sigma_2 = 30\%$ and $\lambda_{12} = \lambda_{21} = 1$. As can be seen from Fig. 2, the introduction of regime switching produces option values that lie in between the two classical Black–Scholes option values (i.e. $BS_1 < V_1 < V_2 < BS_2$) for non-zero rate parameters. This is not surprising at all, as option values are monotonically increasing functions of volatility. The difference between V_1 and BS_1 (V_2 and BS_2) represents the added (subtracted) value of having a certain non-zero probability that the underlying will spend time in a state of higher (lower) volatility. As time to expiration decreases, the expected amount of time spent in the other state of the economy decreases. This causes the added (subtracted) value due to regime switching to decrease. Of course, as time approaches expiration, all option values converge to the payoff, which is simply the difference between the strike price and the underlying in this case.

For volatility being higher than a certain value, clearly the put values are not monotonic functions of the time to expiry. Since the option price calculated with the regime-switching model is bounded above by its Black–Scholes counterpart with the volatility being set to σ_2 and below by its Black–Scholes counterpart with the volatility being set to σ_1, one can seek mathematical and financial explanations from these upper and lower bounds. While one can mathematically prove (cf. [22]), from the Black–Scholes formula, that this is true for a vanilla put option, it can be understood from a financial point of view as well. While the volatility contributes to the time value of a put option in a positively correlated manner, because the value of the option itself is a monotonically increasing function of volatility, just as that for the case of a call option, the discount effect, through the interest rate, acts against the time value of a put. In the extreme case, in the event that volatility is zero, the underlying asset's appreciation with time implies that the longer a put option is away from the expiry, the less the holder of the option would receive when exercising the option on the expiry date. Consequently, a relatively large interest rate, in comparison with the volatility level, would result in the put option value being a decreasing function of the time to expiry (this explains why the lower bound of BS_1 with a small volatility behaves as a decreasing function of the time to expiry in Fig. 2). On the other hand, when the volatility becomes larger, the time value of a put eventually becomes an increasing function of the time to expiry. Somewhere in between, with a right combination of the ratio of S/E (usually near the money) and the relative interest rate, $2r/\sigma^2$, (or $2r/\sigma_2^2$ for the regime-switching case), θ_τ may change signs[3] as displayed in the two top curves in Fig. 2.

4. Conclusions

In this paper, a new exact closed-form solution for European options in a two-state regime-switching economy is derived. The newly-obtained formula involves only the calculation of a single integral with a real integrand and can thus be very easily calculated, if numerical values are needed. Such a result is achieved through the analytic inversion of the Fourier transform.

An interesting extension of this paper would be to consider the case where the risk-free interest rate is state dependent. Extending the model such that $r_1 \neq r_2$ would yield similar PDEs under the restrictive assumption that the extra source of risk can be diversified (see [9] or [12]). However, if this assumption were not to be made, the pricing problem would lead to a non-trivial extension of the current model, which may further complicate the solution procedure. We are currently exploring this case and the results will be published in a forthcoming paper.

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Appendix A. Analytical Fourier inversion

The details for the Fourier inversion of $\hat{q}_1(\omega, \tau_1)$ are provided here, the result for $\hat{q}_2(\omega, \tau_2)$ will follow due to symmetry. Before trying to find its inverse Fourier transform, we rewrite $\hat{q}_1(\omega, \tau_1)$ in a more convenient form

$$\begin{aligned}\hat{q}_1(\omega, \tau_1) = \exp\left\{-\tau_+\left[\left(\omega + \frac{3i}{2}\right)^2 + \frac{1}{4} - i\omega\gamma_+ + 2\gamma_+ + \beta_+\right]\right\} \\ \times \left\{\left[\frac{\alpha_+}{-\left(\omega+\frac{3i}{2}\right)^2 - \frac{1}{4}} + 1\right]\frac{e^{g(\omega)\tau_-} - e^{-g(\omega)\tau_-}}{2g(\omega)} + \frac{e^{g(\omega)\tau_-} + e^{-g(\omega)\tau_-}}{2\left[-\left(\omega+\frac{3i}{2}\right)^2 - \frac{1}{4}\right]}\right\}\end{aligned} \tag{A.1}$$

[3] $\theta_\tau = \frac{\partial P}{\partial \tau}$ with P being the value of a put option and τ being the time to expiry.

where

$$\tau_{\pm} = \tau_1\left(\frac{A \pm 1}{2A}\right) = \frac{\sigma_1^2 \pm \sigma_2^2}{4}(T-t),$$
$$\alpha_{\pm} = \frac{A\beta_{12} \pm \beta_{21}}{A-1} = \frac{2(\lambda_{12} \pm \lambda_{21})}{\sigma_1^2 - \sigma_2^2},$$
$$\gamma_{+} = \frac{A\gamma_1 + \gamma_2}{A+1} = \frac{4r}{\sigma_1^2 + \sigma_2^2},$$
$$\beta_{+} = \frac{A\beta_{21} + \beta_{12}}{A+1} = \frac{2(\lambda_{12} + \lambda_{21})}{\sigma_1^2 + \sigma_2^2},$$
$$\mu^2 = \frac{4\beta_{21}\beta_{12}}{(A-1)^2} = \frac{4\lambda_{21}\lambda_{12}}{(\sigma_1^2 - \sigma_2^2)^2}$$

and

$$g(\omega) = \sqrt{\left[\left(\omega + \frac{3i}{2}\right)^2 + \frac{1}{4} + \alpha_{-}\right]^2 + \mu^2}.$$

Note that

$$\alpha_+^2 - \alpha_-^2 = \mu^2.$$

On initial inspection, $\hat{q}_1(\omega, \tau_1)$ appears to be a multi-valued function due to the presence of a square root function in $g(\omega)$. Despite its appearance, however, the function $\hat{q}_1(\omega, \tau_1)$ is analytic everywhere except at the rather obvious poles $\omega = -i$ and $\omega = -2i$ (see Appendix B for its proof). The terms involving $g(w)$ in $\hat{q}_1(\omega, \tau_1)$ are grouped in such a way that their contributions do not create the need for any branch cut.

$\hat{q}_1(\omega, \tau_1)$ as written in Eq. (A.1) provides a better mathematical description of the problem. The factor

$$-\frac{1}{\left(w + \frac{3i}{2}\right)^2 + \frac{1}{4}}$$

is the corresponding payoff function for a European put option in the Fourier space. Another factor that can be readily identified is

$$\exp\left\{-\tau_+\left[\left(\omega + \frac{3i}{2}\right)^2 + \frac{1}{4} - i\omega\gamma_+ + 2\gamma_+\right]\right\}$$

which, when combined with the payoff function, is the value of a European-style option in the Fourier space with the variance taken as the average of the variances from the two states (i.e. $\sigma_+^2 = (\sigma_1^2 + \sigma_2^2)/2$). The remaining component represents the effect of regime switching on an option with volatility σ_+, in the Fourier space.

In order to evaluate the complex integral in Eq. (9), two closed contours are constructed as shown in Fig. 3. By applying Cauchy's residue theorem on the contours, we are able to express the Fourier inversion integral in terms of the residue at $\omega = -i$ and a real-valued single integral.

In Fig. 3(a), *GA* is a straight line along the real axis; *AB* and *FG* are arc segments of a circle centered at $\omega = -\frac{3i}{2}$ with radius R eventually approaching infinity; and *BD* and *DF* are line segments connecting the end of the arcs to the center of the circle. In Fig. 3(b), *AB* is a straight line along the real axis; *BC* and *EA* are arc segments of a circle centered at $\omega = -\frac{3i}{2}$ with radius R eventually approaching infinity; and *CD* and *DE* are line segments connecting the end of the arcs to the center of the circle.

For $x + \gamma_+\tau_+ \geq 0$, the integrals on arcs *AB* and *FG* in Fig. 3(a) vanish as $R \to \infty$. Similarly, for $x + \gamma_+\tau_+ < 0$, the integrals on arcs *BC* and *EA* in Fig. 3(b) vanish as $R \to \infty$.

By Cauchy's residue theorem, for $x + \gamma_+\tau_+ \geq 0$,

$$\frac{1}{2\pi}\int_{GA} e^{i\omega x}\hat{q}_1(\omega, \tau_1)d\omega = \frac{1}{2\pi}\int_{FD} e^{i\omega x}\hat{q}_1(\omega, \tau_1)d\omega + \frac{1}{2\pi}\int_{DB} e^{i\omega x}\hat{q}_1(\omega, \tau_1)d\omega - i\,\mathrm{Res}_{\omega=-i}\{e^{i\omega x}\hat{q}_1(\omega, \tau_1)\} \tag{A.2}$$

and for $x + \gamma_+\tau_+ < 0$,

$$\frac{1}{2\pi}\int_{BA} e^{i\omega x}\hat{q}_1(\omega, \tau_1)d\omega = \frac{1}{2\pi}\int_{CD} e^{i\omega x}\hat{q}_1(\omega, \tau_1)d\omega + \frac{1}{2\pi}\int_{DE} e^{i\omega x}\hat{q}_1(\omega, \tau_1)d\omega - i\,\mathrm{Res}_{\omega=-i}\{e^{i\omega x}\hat{q}_1(\omega, \tau_1)\}. \tag{A.3}$$

As $R \to \infty$ the left-hand side of the above expressions converge to the Fourier inversion integral.

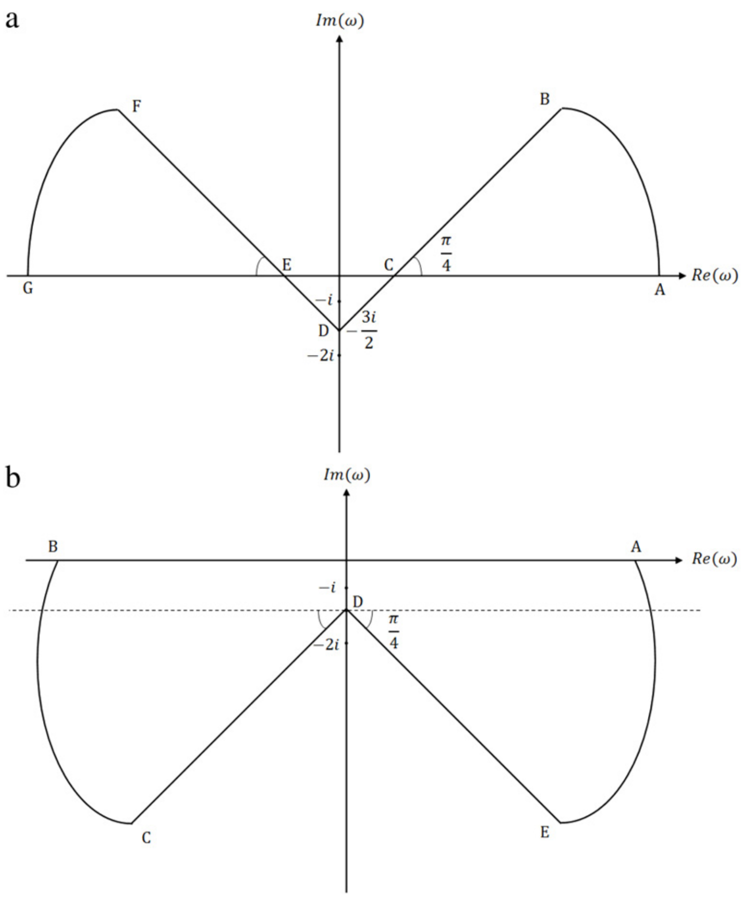

Fig. 3. Complex contours used for evaluating Eq. (9).

For both cases, the residue at $\omega = -i$ can be readily evaluated as

$$\mathrm{Res}_{\omega=-i}\left\{e^{i\omega x}\hat{q}_1(\omega,\tau_1)\right\}e = -\frac{e^{x-\gamma_+\tau_+}}{i}. \tag{A.4}$$

The task now remains to calculate the non-trivial integrals in Eqs. (A.2)–(A.3).

Consider the case $x+\gamma_+\tau_+ \geq 0$. On DB, let $\omega + \frac{3i}{2} = \rho e^{i\frac{\pi}{4}}$, $d\omega = e^{i\frac{\pi}{4}}d\rho = \frac{1}{\sqrt{2}}(1+i)d\rho$. Similarly, on FD, $\omega + \frac{3i}{2} = \rho e^{\frac{3\pi i}{4}}$ and $d\omega = e^{\frac{3\pi i}{4}}d\rho = \frac{1}{\sqrt{2}}(-1+i)d\rho$. Thus, combining the integrals on DB and FD results in the imaginary part of the integral on DB canceling that of the integral on FD such that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{FD} e^{i\omega x}\hat{q}_1(\omega,\tau_1)d\omega + \frac{1}{2\pi}\int_{DB} e^{i\omega x}\hat{q}_1(\omega,\tau_1)d\omega \\
&= \frac{1}{4\pi\sqrt{2}}\exp\left[\frac{3x-\gamma_+\tau_+}{2} - \frac{\tau_+}{4} - \beta_+\tau_+\right]\int_0^\infty \frac{f_1(\rho)\alpha_+}{M(\rho)\left(\rho^4+\frac{1}{16}\right)} \\
&\quad\times\left\{e^{X(\rho)}\left[\left(2\rho^2-\frac{1}{2}\right)\sin(f_2(\rho)+\theta(\rho)-Y(\rho)) - \left(2\rho^2+\frac{1}{2}\right)\cos(f_2(\rho)+\theta(\rho)-Y(\rho))\right]\right. \\
&\quad\left. -e^{-X(\rho)}\left[\left(2\rho^2-\frac{1}{2}\right)\sin(f_2(\rho)+\theta(\rho)+Y(\rho)) - \left(2\rho^2+\frac{1}{2}\right)\cos(f_2(\rho)+\theta(\rho)+Y(\rho))\right]\right\} \\
&\quad+\frac{f_1(\rho)}{M(\rho)}\left\{e^{X(\rho)}\left[\sin(f_2(\rho)+\theta(\rho)-Y(\rho))+\cos(f_2(\rho)+\theta(\rho)-Y(\rho))\right]\right. \\
&\quad\left. -e^{-X(\rho)}\left[\sin(f_2(\rho)+\theta(\rho)+Y(\rho))+\cos(f_2(\rho)+\theta(\rho)+Y(\rho))\right]\right\}
\end{aligned}$$

$$+\frac{2f_1(\rho)}{\rho^4+\frac{1}{16}}\left\{e^{X(\rho)}\left[\left(2\rho^2-\frac{1}{2}\right)\sin[f_2(\rho)-Y(\rho)]-\left(2\rho^2+\frac{1}{2}\right)\cos[f_2(\rho)-Y(\rho)]\right]\right.$$
$$\left.+e^{-X(\rho)}\left[\left(2\rho^2-\frac{1}{2}\right)\sin[f_2(\rho)+Y(\rho)]-\left(2\rho^2+\frac{1}{2}\right)\cos[f_2(\rho)+Y(\rho)]\right]\right\}d\rho \tag{A.5}$$

where

$$f_1(\rho)=\exp\left[-\frac{\rho}{\sqrt{2}}(x+\gamma_+\tau_+)\right],\qquad f_2(\rho)=\rho^2\tau_+-\frac{\rho}{\sqrt{2}}(x+\gamma_+\tau_+)$$

and

$$M(\rho)=\left\{\left[\left(\frac{1}{4}+\alpha_-\right)^2-\rho^4+\mu^2\right]^2+4\rho^4\left(\frac{1}{4}+\alpha_-\right)^2\right\}^{\frac{1}{4}}$$
$$\theta(\rho)=\frac{1}{2}\tan^{-1}\left[\frac{2\rho^2\left(\frac{1}{4}+\alpha_-\right)}{\left(\frac{1}{4}+\alpha_-\right)^2-\rho^4+\mu^2}\right]$$
$$X(\rho)=M(\rho)\tau_-\cos\theta(\rho),\qquad Y(\rho)=M(\rho)\tau_-\sin\theta(\rho).$$

The right-hand side of Eq. (A.5) is a real valued integral. Combining Eqs. (A.2)–(A.5) and reintroducing the original variables, we obtain the formula for the regime-switching option value

$$V_1(S,t)=Ee^{-r(T-t)}+\frac{1}{4\pi\sqrt{2}}\sqrt{SE}e^{-\frac{1}{2}\left(r+\lambda_{21}+\lambda_{12}+\frac{\sigma_1^2+\sigma_2^2}{8}\right)(T-t)}\int_0^\infty\frac{2f_1(\rho)(\lambda_{21}+\lambda_{12})}{M(\rho)\left(\rho^4+\frac{1}{16}\right)(\sigma_1^2-\sigma_2^2)}$$
$$\times\left\{e^{X(\rho)}\left[\left(2\rho^2-\frac{1}{2}\right)\sin(f_2(\rho)+\theta(\rho)-Y(\rho))-\left(2\rho^2+\frac{1}{2}\right)\cos(f_2(\rho)+\theta(\rho)-Y(\rho))\right]\right.$$
$$\left.-e^{-X(\rho)}\left[\left(2\rho^2-\frac{1}{2}\right)\sin(f_2(\rho)+\theta(\rho)+Y(\rho))-\left(2\rho^2+\frac{1}{2}\right)\cos(f_2(\rho)+\theta(\rho)+Y(\rho))\right]\right\}$$
$$+\frac{2f_1(\rho)}{M(\rho)}\left\{e^{X(\rho)}\left[\sin(f_2(\rho)+\theta(\rho)-Y(\rho))+\cos(f_2(\rho)+\theta(\rho)-Y(\rho))\right]\right.$$
$$\left.-e^{-X(\rho)}\left[\sin(f_2(\rho)+\theta(\rho)+Y(\rho))+\cos(f_2(\rho)+\theta(\rho)+Y(\rho))\right]\right\}$$
$$+\frac{f_1(\rho)}{\rho^4+\frac{1}{16}}\left\{e^{X(\rho)}\left[\left(2\rho^2-\frac{1}{2}\right)\sin[f_2(\rho)-Y(\rho)]-\left(2\rho^2+\frac{1}{2}\right)\cos[f_2(\rho)-Y(\rho)]\right]\right.$$
$$\left.+e^{-X(\rho)}\left[\left(2\rho^2-\frac{1}{2}\right)\sin[f_2(\rho)+Y(\rho)]-\left(2\rho^2+\frac{1}{2}\right)\cos[f_2(\rho)+Y(\rho)]\right]\right\}d\rho \tag{A.6}$$

valid for $\ln\left(\frac{S}{E}\right)+r(T-t)\geq 0$.

For the case when $\ln\left(\frac{S}{E}\right)+r(T-t)<0$ $(x+\gamma_+\tau_+<0)$, the integral in Eq. (A.3) can be evaluated in the same way as that of Eq. (A.2). We obtain the same result except that ρ is replaced with $-\rho$ in the integrand.

Redefining the functions

$$f_1(\rho)=e^{-\frac{\rho}{\sqrt{2}}\left|\ln\left(\frac{S}{E}\right)+r(T-t)\right|},$$
$$f_2(\rho)=\frac{\rho^2}{4}(\sigma_1^2+\sigma_2^2)(T-t)-\frac{\rho}{\sqrt{2}}\left|\ln\left(\frac{S}{E}\right)+r(T-t)\right|$$

we can generalize the option value formula in Eq. (A.6) for all $\ln\left(\frac{S}{E}\right)+r(T-t)$.

Appendix B. Proof of $\hat{q}_1(\omega,\tau_1)$ being analytic

We show that $\hat{q}_1(\omega,\tau_1)$ is analytic, apart from the rather obvious simple poles at $\omega=-i$ and $\omega=-2i$.

The multi-valued function $g(\omega)$ is the only component that would possibly create the need for a branch cut for $\hat{q}_1(\omega,\tau_1)$ as stated in the form of (A.1). Consider the conformal mapping

$$\Omega=\left[\left(\omega+\frac{3i}{2}\right)^2+\frac{1}{4}+\alpha_-\right]^2+\mu^2$$

with Ω being the argument of the square-root function in $g(\omega)$. In terms of Ω, the two terms that may create the need for a branch cut are

$$\frac{e^{\sqrt{\Omega}\tau_-} - e^{-\sqrt{\Omega}\tau_-}}{2\sqrt{\Omega}} \tag{B.1}$$

and

$$\frac{e^{\sqrt{\Omega}\tau_-} + e^{-\sqrt{\Omega}\tau_-}}{2}. \tag{B.2}$$

We are concerned that the presence of the multi-valued function, $\sqrt{\Omega}$, may cause $\hat{q}_1(\omega, \tau_1)$ to be multi-valued. We define the branch cut of $\sqrt{\Omega}$ along the negative real axis on the Ω-plane. As $\sqrt{\Omega}$ is analytic at all points on the Ω-plane, but discontinuous across the negative real axis, we need to prove that the expressions in (B.1) and (B.2) are continuous across this branch cut of the Ω-plane in order to prove that they are analytic.

First consider (B.1). Approaching the branch cut from above, i.e. let $\Omega = re^{i\pi}$,

$$\begin{aligned}\frac{e^{\sqrt{\Omega}\tau_-} - e^{-\sqrt{\Omega}\tau_-}}{2\sqrt{\Omega}} &= \frac{e^{\sqrt{r}e^{i\frac{\pi}{2}}\tau_-} - e^{-\sqrt{r}e^{i\frac{\pi}{2}}\tau_-}}{2\sqrt{r}e^{i\frac{\pi}{2}}} \\ &= \frac{e^{i\sqrt{r}\tau_-} - e^{-i\sqrt{r}\tau_-}}{2i\sqrt{r}}\end{aligned}$$

and approaching the branch cut from below, i.e. let $\Omega = re^{-i\pi}$,

$$\begin{aligned}\frac{e^{\sqrt{\Omega}\tau_-} - e^{-\sqrt{\Omega}\tau_-}}{2\sqrt{\Omega}} &= \frac{e^{\sqrt{r}e^{-i\frac{\pi}{2}}\tau_-} - e^{-\sqrt{r}e^{-i\frac{\pi}{2}}\tau_-}}{2\sqrt{r}e^{-i\frac{\pi}{2}}} \\ &= \frac{e^{i\sqrt{r}\tau_-} - e^{-i\sqrt{r}\tau_-}}{2i\sqrt{r}}\end{aligned}$$

yield the same result, proving that the expression in (B.1) is continuous across the branch cut of $g(\omega)$.

At the origin of the Ω-plane the denominator and numerator of (B.1) are zero. To observe the behavior of (B.1) at the origin we parameterize $\Omega = re^{i\theta}$ and find the limit

$$\lim_{r\to 0} \frac{e^{\sqrt{r}e^{i\frac{\theta}{2}}\tau_-} - e^{-\sqrt{r}e^{i\frac{\theta}{2}}\tau_-}}{2\sqrt{r}e^{i\frac{\theta}{2}}} = \tau_-.$$

Therefore, the singularity at the origin of the Ω-plane is a removable singularity.

Now consider the expression in (B.2). Approaching the branch cut from above, i.e. let $\Omega = re^{i\pi}$,

$$\begin{aligned}e^{\sqrt{\Omega}\tau_-} + e^{-\sqrt{\Omega}\tau_-} &= e^{\sqrt{r}e^{i\frac{\pi}{2}}\tau_-} + e^{-\sqrt{r}e^{i\frac{\pi}{2}}\tau_-} \\ &= e^{i\sqrt{r}\tau_-} + e^{-i\sqrt{r}\tau_-}\end{aligned}$$

and approaching the branch cut from below, i.e. let $\Omega = re^{-i\pi}$,

$$\begin{aligned}e^{\sqrt{\Omega}\tau_-} + e^{-\sqrt{\Omega}\tau_-} &= e^{\sqrt{r}e^{-i\frac{\pi}{2}}\tau_-} + e^{-\sqrt{r}e^{-i\frac{\pi}{2}}\tau_-} \\ &= e^{i\sqrt{r}\tau_-} + e^{-i\sqrt{r}\tau_-}\end{aligned}$$

yield the same result, proving that the expression in (B.2) is continuous across the branch cut.

Thus, we conclude that $\hat{q}_1(\omega, \tau_1)$ is analytic except at the simple poles $\omega = -i$ and $\omega = -2i$.

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