On Hochschild Cohomology of Preprojective Algebras, I

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We study the Hochschild cohomology of a finite-dimensional preprojective algebra \( L \); this is periodic by a result of Schofield. In particular, for \( L \) of type \( A_n \), we obtain the dimensions and explicit characterizations and bases for all Hochschild cohomology groups.

Preprojective algebras were introduced by Gelfand and Ponomarev [GP] (and implicitly in the work of Riedtmann [R]) to study the representation theory of a finite quiver without oriented cycles. Gelfand and Ponomarev have shown that the preprojective algebra decomposes as the direct sum of all preprojective representations of the corresponding quiver, each occurring with multiplicity one, thus providing a method for constructing many indecomposable representations simultaneously. Subsequently preprojective algebras have played a role in classification problems of algebras of finite type and they are studied in [DR1, 2]. They have since occurred in other diverse topics and, for example, have been studied by Kronheimer [K] to deal with problems in differential geometry and by Lusztig [L] to construct canonical bases for quantum groups.
To any (undirected) graph $\Delta$ there is a preprojective algebra $\mathcal{P}(\Delta)$; for the definition see Section 1. This algebra is finite-dimensional if and only if $\Delta$ is a disjoint union of Dynkin diagrams. In [DR2], Dlab and Ringel observed that for type $A_n$ all modules of $\mathcal{P}(\Delta)$ are periodic. Subsequently it was proved by Schofield [S], by constructing an explicit minimal projective resolution, that for Dynkin diagrams as above the algebra $\mathcal{P}(\Delta)$ has a periodic resolution as a bimodule, of period dividing 6.

Here we use this projective resolution to study the Hochschild cohomology of $\Lambda$. The first part contains some general results on self-injective algebras and on finite-dimensional preprojective algebras. Moreover we calculate for type $A_n$ the dimensions of each cohomology group and give an explicit characterization. In particular we show that for $i \geq 1$, the dimension of $HH^i(\Lambda)$ is $(n - 1)/2$ for $n$ odd and $n/2$ for $n$ even. Moreover the dimension of $HH^0(\Lambda)$ is $(n + 1)/2$ for $n$ odd and $n/2$ for $n$ even. We summarize the results at the end. In the second part we will determine the ring structure of $HH^*(\Lambda)$ given by the Yoneda product, for type $A_n$, which is the same as the ring structure defined by the cup product ([GS], [G]). We assume throughout that $n \geq 2$.

For basic facts we refer to [ARS], [B].

1. PRELIMINARIES

1.1. We assume that $K$ is a field. Recall that a quiver is a directed graph which has vertices and arrows. An arrow $a$ starts at vertex $\mathbf{i}_a$ and it ends at vertex $\mathbf{t}_a$. Suppose $\mathcal{E}$ is a quiver which has a fixed point free involution $a \to \overline{a}$ on the arrows such that $\mathbf{i}_a = \mathbf{t}_a$ and $\mathbf{t}_a = \mathbf{i}_a$. Then the preprojective algebra associated to this quiver is $K\mathcal{E}/I$ where $I$ is the ideal generated by the relations

$$\sum_{a, \mathbf{i}_a = v} a\overline{a} = 0, \quad v \text{ a vertex of } \mathcal{E}. $$

The associated graph $\Delta$ of the preprojective algebra is obtained by replacing each pair $[a, \overline{a}]$ by an undirected edge. The preprojective algebra is then completely determined by this graph $\Delta$ and is denoted by $\mathcal{P}(\Delta)$. Here we study algebras where $\Delta$ is Dynkin, of the form $A, D, E$. It is known then that $\mathcal{P}(\Delta)$ is finite-dimensional; this is due to [R], [GP] (for a discussion see [DR2]). Our definition follows Schofield [S]; some authors use relations involving signs, but in our case (more generally for $\Delta$ a tree), they define the same algebras.

Preprojective algebras of Dynkin type are known to be self-injective, and for $\Delta = A_n$ we shall use an explicit Nakayama automorphism as described
in [S]. In this case, $\Delta$ has a unique graph automorphism of order two; this is the Nakayama permutation $\pi$.

1.2. We fix $\Delta$ and let $L = \mathcal{S}(\Delta)$. It is proved in [S] that the $L-L$ bimodule $L$ has a periodic minimal projective resolution of periodicity at most 6. We shall describe this resolution. Define projective $L-L$ bimodules

$$P_0 = P_2 = \bigoplus_i \Lambda(e_i \otimes e_i)\Lambda, \quad P_1 = \bigoplus_\alpha \Lambda(e_{\alpha} \otimes e_{\alpha})\Lambda$$

and define homomorphisms $\delta: P_1 \to P_0$ and $R: P_2 \to P_1$ by

$$\delta(e_{\alpha} \otimes e_{\alpha}) = \alpha \otimes e_{\alpha} - e_{\alpha} \otimes \alpha =: \lambda_{\alpha}$$

$$R(e_i \otimes e_i) = \sum_{i\alpha = \alpha} e_{\alpha} \otimes \bar{\alpha} + \alpha \otimes e_{i\alpha} := \sigma_i.$$  

Moreover, let $u: P_0 \to \Lambda$ be the multiplication map.

**Theorem [S].** We have an exact sequence of $L-L$ bimodules

$$0 \to \Lambda_1 \to P_2 \xrightarrow{R} P_1 \xrightarrow{\delta} P_0 \xrightarrow{u} \Lambda \to 0,$$

where $\Lambda_1$ is the bimodule structure on $\Lambda$ where the action on the right is twisted by an automorphism $\nu$ of $\Lambda$ of order 2.

Now one observes that $\Lambda_1 \otimes (\Lambda_1) \cong \Lambda$ and hence if one tensors the above exact sequence with $\Lambda_1$, one obtains the other half of a projective resolution of $\Lambda$ and it has length 6. The automorphism $\nu$ is closely related to $\pi$; for type $A_n$ we will define it in 3.3.

1.3. **Notation.** For any finite-dimensional algebra $A$, we let $\text{Hom}(-,-)$ or just $(-,-)$ denote $A-A$ bimodule homomorphisms. We denote the Hochschild cohomology groups by $HH^i(A)$. We write $\Omega M$ for the kernel of a projective cover of $M$, and $\Omega^{-1}M$ is the cokernel of an injective hull, as an $A-A$ bimodule. We denote by $J$ the radical of $A$.

2. **GENERAL PROPERTIES**

2.1. For the first part of this section $A$ is a finite-dimensional self-injective basic algebra. Suppose $N$ is the $A-A$ bimodule $N = \Lambda_{\rho}$ where $\rho$ is an automorphism of the algebra $A$.

**Lemma.** Assume $N \otimes_A N = A$; then for any $A-A$ bimodule $X$

$$\text{Hom}(X, N) \cong \text{Hom}(X \otimes_A N, A).$$
Proof. We have $- \otimes N$ is the same as twisting on the right by $\rho$, and $\Hom(X, N) \cong \Hom(X' \rho, N' \rho)$ which gives the statement.

We will use later that $\phi \in (X, N)$ corresponds to $f \circ (\phi \otimes 1)$ if $f : N \otimes N \to A$ is an isomorphism.

2.2. We will now study the map $R$; this can be done more generally. Let $A = KQ/I$ where $Q$ is a quiver and $I$ is an admissible ideal of $KQ$. Then we can always define an $A - A$ bimodule homomorphism $R : \bigoplus_i A(e_i \otimes e_i) A \to \bigoplus_{i,j} A(e_i \otimes e_j) A$ by setting

$$R(e_i \otimes e_i) := \sum_{ia=i} a \otimes e_i + \sum_{ib=i} e_i \otimes b$$

which generalizes the map defined in 1.2. (Here $\otimes$ means $\otimes_K$; this should not cause confusion.) Since $A$ is self-injective, there is a nonsingular bilinear form $(-, -) : A \times A \to K$ such that $(xy, z) = (x, yz)$ for $x, y, z \in A$ (see for example [CR; Section 9]). Given any $K$-basis $B$ of $A$, there is a corresponding dual basis $B^*$ of $A$ such that $(v_i, v_j^*) = \delta_{ij}$. Let $\nu$ be the Nakayama automorphism associated to this nonsingular bilinear form.

Thus $\nu(e_i) = \pi(e_i)$ for all vertices $i$.

Now we assume that in addition $A$ is graded by the powers of the radical, and equivalently, that the ideal $I$ is homogeneous. Then we fix a basis $B$ consisting of homogeneous elements such that each $v \in B$ belongs to $e_i A e_j$ for some $i, j$, and moreover we assume that the basis contains the $e_i$. Note that $v \in e_i A e_j$ if and only if $v^* \in e_j B^* e_{\nu(v)}$. We define elements $\zeta_i \in \bigoplus \Lambda(e_i \otimes e_i) \Lambda$ by the formula

$$\zeta_i := \sum_{x \in e_i B} (-1)^{\deg(x)} x \otimes x^*.$$

2.3. Proposition. We have that $R(\zeta_i) = 0$. Moreover for each $i$,

$$\zeta_i A \cong e_{\nu(i)} A \quad \text{and} \quad A \zeta_i \cong A e_i$$

as right (respectively left) $A$-modules.

Proof. We fix some $i$, and we will show that the terms of $\sum (-1)^{\deg(x)} R(x \otimes x^*)$ cancel out for which the first component has a fixed degree $d$ and which corresponds to a fixed arrow $a : j \to k$. These are of the form

$$(-1)^d x a \otimes x^*, \quad x \in e_j A e_j, \deg(x) = d - 1$$

$$(-1)^{d+1} y \otimes ay^*, \quad y \in e_i A e_k, \deg(y) = d.$$
Let \((x_1, \ldots, x_s) = (e_iBc_j)_{d-1}\) and \((y_1, \ldots, y_u) = (e_iBc_k)_{d}\), then we are done if we show that

\[
(\ast) \sum_{l=1}^{s} x_l a \otimes x_l^* = \sum_{t=1}^{u} y_t \otimes a(y_t)^*.
\]

Since \(x_l a \in e_iAe_k\) and has degree \(d\), there are \(b_{il} \in K\) such that

\[
x_l a = \sum_{t=1}^{u} b_{il} y_t, \quad 1 \leq l \leq s
\]

(that is, the coefficients of \(x_l a\) are given by the \(l\)th columns of the matrix \([b_{ij}]\)). Next we determine \(a(y_t)^*\) for \(1 \leq t \leq u\), by using \((-,-)\). Let \(x' \in B\) and suppose that \((x', a(y_t)^*) \neq 0\); recall that \((x', a(y_t)^*) = (x'a, y_t)^*\); therefore we must have that \(x' \in e_iBc_j\), and since \(x'\) is homogeneous, it must have degree \(d-1\). Hence \(x' \in \{x_1, \ldots, x_s\}\). On the other hand, we have

\[
(x_l a, y_t) = \left( \sum_{m=1}^{u} b_{ml} y_m, y_t^* \right) = b_{il}.
\]

Therefore \(a(y_t^*) = b_{12}(x_2^*) + b_{22}(x_3^*) + \cdots + b_{s2}(x_s^*)\), hence is given by the \(r\)th row of the matrix \([b_{ij}]\). Substituting this into \((\ast)\) we obtain

\[
\left( \sum_{t=1}^{u} y_t \otimes a(y_t)^* \right) = \sum_{t=1}^{u} \left[ y_t \otimes \left( \sum_{l=1}^{s} b_{il} (x_l^*) \right) \right] = \sum_{l=1}^{s} \left[ \sum_{t=1}^{u} b_{il} y_t \otimes (x_l^*) \right]
\]

which is equal to \(\sum_{l=1}^{s} (x_l a) \otimes (x_l^*)\), as required.

For the last part, observe that \(\zeta_i = \zeta_i e_{v(i)}\). Hence we define a surjective (right) \(A\)-homomorphism \(\pi\): \(e_{v(i)}A \to \zeta_iA\) by \(\pi(x) = \zeta_i x\). The socle of \(e_{v(i)}A\) is simple, and it is spanned by \((e_{v(i)})^*\). Moreover, let \(\omega_i \in e_iB\) be the element which spans the socle of \(e_iA\), then since \((e_{v(i)})^*\) annihilates the radical, we have

\[
\pi(e_{v(i)}^*) = \zeta_i (e_{v(i)}^*) = \pm (\omega_i \otimes (\omega_i)^*)(e_{v(i)}^*).
\]

This is nonzero since \(\omega_i^*\) involves a nonzero scalar multiple of \(e_{v(i)}\). Hence \(\pi\) is \(1-1\) and is therefore an isomorphism. This proves the first part; the other is similar.

2.3.1. Note that if \(m\) is a monomial in the algebra and \(a\) is an arrow with \(ma \neq 0\) then from the above proof we have

\[
a(ma)^* = m^*.
\]

2.4 Now let \(P := \bigoplus A(e_i \otimes e_i)A\) and let \(N := \langle \zeta_i \rangle\), the sub-bimodule generated by the \(\zeta_i\). Let also \(W = \rho A\) where \(\rho\) is some automorphism of
$A$ (which could be the identity); we denote the automorphism induced on the quiver by $\rho$ as well. Let $F := \{ i \in I : \rho(i) = i \}$, the fixed points of $\rho$.

We denote by $d(i, j)$ the distance of the vertices $i, j$ in the quiver. If the underlying graph $\Delta$ is a tree then for any monomial $\nu \in e_i Be_j$, $\deg(\nu) = d(i, j) \pmod{2}$.

Let $C = [c_{ij}]$ be the Cartan matrix of $A$. We define the signed truncated Cartan matrix to be

$$C^s := \left[ (-1)^{d(i, j)} c_{ij} \right]_{j \in F}.$$  

We will view this as a linear map from the space $\bigoplus_{j \in k} e_j K$ into $\text{soc} A$, with respect to the bases $\{e_j : j \in F\}$ and $\{\omega_i\}$ where $\omega_i = e_i^k$. If $\rho = id$ we write $C^s$ instead of $C^s_i$.

2.4.1. Lemma. Assume that $\Delta$ is a tree. Then as vector spaces

$$\text{Hom}(P/N, W) \cong \bigoplus_i e_i J e_{\rho(i)} \oplus \text{Ker} C^s,$$

where $C^s$ is the signed truncated Cartan matrix of $A$.

Proof. We identify $\text{Hom}(P/N, W)$ with $\{(\theta: P \to W : \theta(\xi_i) = 0 \text{ for all } i)\}$. Let $\theta$ belong to this space; then we note that $\theta(e_i \otimes e_i)$ lies in $e_i W e_i$ which is as a vector space isomorphic to $e_i A e_{\rho(i)}$. Define

$$T_0 := \{\theta : P \to W : \theta(e_i \otimes e_i) \in e_i J e_{\rho(i)} , \quad 0 \leq i \leq n - 1\}.$$

Then $T_0 \cong \bigoplus_i e_i J e_{\rho(i)}$ as a vector space since $P$ is projective.

1. We claim that $T_0 \subseteq \text{Hom}(P/N, W)$: Fix $j$.

Let $m \geq 1$ such that $e_j J^m \neq 0$ but $e_j J^{m+1} = 0$. Then for each $\nu \in e_j B$ if $v$ has degree $d$ we have $\nu \in J^d$ and $v^* \in J^{m-d}$. Consequently

$$\theta(\xi_j) = \sum_{i=0}^{n-1} \sum_{v \in e_j B e_i} (-1)^{\deg(v)} v \theta(e_i \otimes e_i) v^* \in \sum_{d} e_j J^d J^{m-d} \subseteq e_j J^{m+1} = 0.$$

Let $T_1 := \{\theta \in \text{Hom}(P/N, W) : \theta(e_i \otimes e_i) = \lambda_i e_i, \lambda_i \in K\}$.

2. We claim that $\text{Hom}(P/N, W) = T_0 \oplus T_1$ as vector spaces: Clearly $T_0 \oplus T_1 \subseteq \text{Hom}(P/N, W)$. Conversely if $\phi \in \text{Hom}(P/N, W)$ let $\phi(e_i \otimes e_i) = \gamma_j + \lambda_i e_i$ where $\lambda_i \in K$ and $\gamma_j \in e_j J e_{\rho(i)}$. Then $\phi = \phi_0 + \phi_1$ where $\phi_0 : P \to W$ is the $A^s$-homomorphism taking $e_i \otimes e_i$ to $\gamma_j$ and $\phi_1$ is the $A^s$-homomorphism taking $e_i \otimes e_i$ to $\lambda_i e_i$. By (1), $\phi_0$ belongs to $\text{Hom}(P/N, W)$. Since $\phi_0 \in \text{Hom}(P/N, W)$, it follows that $\phi_1$ is an element of $\text{Hom}(P/N, W)$, that is, it lies in $T_1$. 

We claim that $T_1 \cong \text{Ker } C^e_\rho$. Let $\theta : P \rightarrow W$ such that $\theta(e_i \otimes e_i) = \lambda_i e_i$ for $i \in F$. Note that $\lambda_i \neq 0$ implies that $i$ is a fixed point under $\rho$. Then $\theta \in T_1$ if and only if for all $j$

$$0 = \theta(\zeta_j) = \sum_{v \in e_j B, w \in F} (-1)^{\deg(v)} \lambda_v u_{v,*} = \left( \sum_{v \in e_j B, w \in F} (-1)^{\deg(v)} \lambda_v \right) \omega_j,$$

that is, $(\sum_{v \in e_j B, w \in F} (-1)^{\deg(v)} \lambda_v) = 0$. Hence $\theta \in T_1$ if and only if $((\lambda_j)_{j \in F})$ is a solution of

$$\sum_{j \in F} (-1)^{d(i,j)} c_{ij} x_j = 0, \quad (0 \leq i \leq n - 1).$$

3. SOME GENERAL FACTS ON PREPROJECTIVE ALGEBRA $\Lambda$, IN PARTICULAR FOR TYPE $A_n$

Let $\Lambda$ be a finite-dimensional preprojective algebra and $N = \Lambda \Delta$, as in 1.2. The automorphism $\tilde{\nu}$ will be defined explicitly for $\Lambda$ of type $A_n$ in 3.3. At this stage all we require is that $\tilde{\nu}(e_i) = \nu(e_i)$ for all vertices $i$. In general $\text{Hom}(P_0, M) \cong \bigoplus_{i=1}^{n-1} e_i Me_i$ and $\text{Hom}(P_1, M) \cong \bigoplus_a e_ia Me_i$. As vector spaces we may identify $e_i Ne_i$ with $e_ia \Lambda e_{\nu(i)}$ and $e_i Ne_ia$ with $e_i \Lambda e_{\nu(i)}$.

3.1. Proposition. For a finite-dimensional preprojective algebra $\Lambda$, $(P_0, \Lambda) \cong (P_0, N)$ and $(P_1, \Lambda) \cong (P_1, N)$.

Proof. Let $(-, -)$ be the bilinear form as in 2.2. Then for each $i, j$ the restriction of $(-, -)$ to the space $e_i \Lambda e_j \times e_j \Lambda e_{\nu(i)}$ is nonsingular and hence

$$\dim e_i \Lambda e_j = \dim(e_j \Lambda e_{\nu(i)}) = \dim e_j Ne_i.$$

Take $i = j$ and sum over $i$ to give the first part. Next, take $i = ia$ and $j = ta$ for an arrow $a$; since $ia = ia$ and $ia = ta$ we have $e_i Ne_i = e_i Ne_i$. Sum over all arrows; since $a \rightarrow \bar{a}$ is bijective, one gets the second statement. (If $B, B^*$ are bases as in 2.2 then $\nu \rightarrow \nu^*$ induces $K$-linear isomorphisms in 3.1.)

3.2. We describe the strategy which we will here follow for $\Lambda$ of type $A_n$. We use the first part of the projective resolution of $\Lambda$ as given in 1.2.

To find $(\text{Im } \phi, \Lambda)$ and $(\text{Im } \phi \otimes N, \Lambda)$ where $\phi = u$ or $\bar{\delta}$ or $R$, it is convenient to use $\text{Im } \phi = P/\text{Ker } \phi$ where $\phi$ starts at $P$. When $\phi = \bar{\delta}$, this gives the identification of $(\text{Im } \bar{\delta}, \Lambda) = (\text{Ker } u, \Lambda)$ with the set of
homomorphisms $P_1 \to \Lambda$ which map the generators $\sigma_i$ of $\ker \delta$ to zero. Then $(\ker u, \Lambda)$ is considered both as an image of $(P_0, \Lambda)$ and as a submodule of $(P_1, \Lambda)$ as appropriate to the particular context. Similar identifications are made when $\phi = u$ or $R$.

To find $HH^2(\Lambda)$ we use the short exact sequence $0 \to \ker u \to P_0 \to \Lambda \to 0$ and apply $\hom_{\Lambda^e}(\_, \Lambda)$ to give

$$0 \to (\Lambda, \Lambda) \to (P_0, \Lambda) \xrightarrow{i^*} (\ker u, \Lambda) \to HH^2(\Lambda) \to 0.$$  

Then $HH^2(\Lambda)$ is isomorphic to a complement of $\im i^*$ in $(\ker u, \Lambda)$. Now consider $(\ker u, \Lambda)$ as a submodule of $(P_1, \Lambda)$. The map $\delta: P_1 \to P_0$ induces a map $\delta^*: (P_0, \Lambda) \to (P_1, \Lambda)$. This map $\delta^*$ is the composition of $i^*$ with the embedding $0 \to (\ker u, \Lambda) \to (P_1, \Lambda)$ induced by $P_1 \to \im \delta \to 0$. Thus $\im \delta^* = \im i^*$, and so $HH^2(\Lambda)$ is isomorphic to a complement of $\im \delta^*$ in $(\ker u, \Lambda)$. We use this identification to give a basis for $HH^2(\Lambda)$. Similar identifications are made to find the other $HH^i(\Lambda)$'s.

The spaces $HH^i(\Lambda)$ for $i \geq 4$ are determined by applying the functor $\hom(\_, \Lambda)$ to the first part of the projective resolution, using techniques as before (see 2.1).

The preprojective algebra $\Lambda$ is added with respect to the powers of the radical. This grading is preserved by the maps $u$, $\delta$, and $R$. Thus, for each $HH^i(\Lambda)$, every term in the exact sequence ending at $HH^i(\Lambda)$ is graded, and may be expressed as a direct sum of homogeneous components.

3.3. From now, we assume that $\Lambda$ is preprojective of type $A_n$. We will fix the labeling of the vertices and arrows of $\mathcal{E}$ as follows.

$$0 \xleftarrow{\sigma_0} \sigma_1 \xleftarrow{\sigma_1} \cdots \xleftarrow{n-2} \sigma_{n-1} \xrightarrow{\sigma_{n-1}} \sigma_0.$$  

We set

$$m := \begin{cases} (n-1)/2 & n \text{ odd} \\ (n-2)/2 & n \text{ even.} \end{cases}$$

3.3.1. For $\Lambda$ of type $A_n$, $\sigma_j(a_j a_{j+1} \cdots a_{j+s}) = (-1)^{s+1}(a_{j+1} \cdots a_{j+s+1}) \sigma_j a_{j+s+1}.$

The proof, by induction on $s$, is straightforward. We shall use this tacitly.

Let $0 \neq y$ be a monomial of fixed length starting at $i$. By 3.3.1, there are the same number of "bar" letters in all expressions for $y$. Thus $y$ has at most $i$ bar letters and at most $n-i-1$ "nonbar" letters. Consequently, if $y$ has exactly $i$ bar letters then $a_{i-1}y = 0$ (for $a_{i-1}y$ starts at $i-1$ and has $i$ bar letters). This also shows that $J^n = 0$. 

\[\]
Suppose \( y \) is a monomial. We define \( y \) to be right-normalized if it is of the form \( y = a_{i_1} \cdots a_{i_s} \bar{a}_{i_{s+1}} \cdots \bar{a}_{i_{t}} \), that is, if all the bar letters are to the right. Similarly \( y \) is said to be left-normalized if it is of the form \( \bar{a}_{i_{s-1}} a_{i_{s}} \cdots a_{i_1} \).

3.3.2. We describe now the automorphism \( \tilde{\nu} \) for type \( A_n \); in particular this also confirms that \( N \) is a twisted bimodule. On idempotents we have \( \tilde{\nu}(e_i) = \nu(e_i) = \pi(e_i) = e_{n-i-1} \), and then it suffices to give the image of the arrows. Define

\[
\tilde{\nu}(a_i) = \begin{cases} 
-\bar{a}_{n-i-2} & i \leq m-1 \\
(1)^{n-1}a_{n-i-2} & i \geq m
\end{cases}
\]
\[
\tilde{\nu}(\bar{a}_i) = \begin{cases} 
-\bar{a}_{n-i-2} & i \leq m-1 \\
-a_{n-i-2} & i \geq m
\end{cases}
\]

Then \( \tilde{\nu} \) has order 2.

**Proof.** It follows from 2.3 and 1.2 that \( N = \langle \zeta_i \rangle \) where \( \zeta_i \) are as defined in Section 2. By 2.3 we must show that \( a_i\zeta_{i+1} = \zeta_i \tilde{\nu}(a_i) \) and \( \bar{a}_i \zeta_i = \zeta_{i+1} \tilde{\nu}(\bar{a}_i) \) where \( \tilde{\nu}(a_i), \tilde{\nu}(\bar{a}_i) \) are as stated. Fix a nonzero element \( \omega_i \) of the socle of \( e_i \Lambda \) as follows. If \( n \) is even, take \( \omega_i \) to be a right-normalized monomial. If \( n \) is odd, take \( \omega_i \) right-normalized for \( i \leq m \) and if \( i \geq m \) take \( \omega_i \) to be \((1)^{m-i}\omega_i^*\) where \( \omega_i^* \) is right-normalized. Then one finds that for \( n \) even

\[
a_i \zeta_{i+1} = -\zeta_i \bar{a}_{n-i-2}, \quad \bar{a}_i \zeta_i = -\zeta_{i+1} a_{n-i-2}.
\]

Moreover if \( n \) is odd we have

\[
a_i \zeta_{i+1} = \begin{cases} 
-\zeta_i \bar{a}_{n-i-2} & i \leq m-1 \\
\zeta_i a_{n-i-2} & i \geq m
\end{cases}
\]
\[
\bar{a}_i \zeta_i = \begin{cases} 
\zeta_{i+1} a_{n-i-2} & i \leq m-1 \\
-\zeta_{i+1} a_{n-i-2} & i \geq m
\end{cases}
\]

4. THE BASIS OF THE PREPROJECTIVE ALGEBRA \( \Lambda \) OF TYPE \( A_n \)

In this section we give a basis for each \( e_i \Lambda e_i \), where \( \Lambda \) is of type \( A_n \), the union of which gives a basis \( B \) of \( \Lambda \). This information is used to give the Cartan matrix of \( \Lambda \) and \( \text{dim}(P_i, \Lambda) \) for \( i = 0, 1 \).
From 3.3, each $e_i \Lambda e_j$ has a basis $e_i B e_j$ of right-normalized monomials. Moreover for each $i, j$, there is at most one right-normalized monomial of a given length in $e_i \Lambda e_j$. Let $B = \bigcup_{i, j} e_i B e_j$.

4.1. Proposition. The set $B$ is a basis for the preprojective algebra $\Lambda$ of type $A_n$ which consists of right-normalized monomials.

Let $0 \leq i \leq m$. Then the basis $e_i B e_j$ of $e_i \Lambda e_j$ is given by

$$
e_i B e_0 = \{ \bar{a}_{i-1} \ldots \bar{a}_0 \}
$$

$$
e_i B e_1 = \{ \bar{a}_{i-1} \ldots \bar{a}_1, (a_i \bar{a}_i) \bar{a}_{i-1} \ldots \bar{a}_1 \}
$$

$$
\vdots
$$

$$
e_i B e_{i-1} = \{ \bar{a}_{i-1}, (a_i \bar{a}_i) \bar{a}_{i-1}, \ldots, (a_i \ldots a_{2i-2} \bar{a}_{2i-2} \ldots \bar{a}_i) \bar{a}_{i-1} \}
$$

$$
e_i B e_i = \{ e_i, a_i \bar{a}_i, \ldots, a_i \ldots a_{2i-1} \bar{a}_{2i-1} \ldots \bar{a}_i \}
$$

$$
e_i B e_{i+1} = \{ a_i, a_i (a_{i+1} \bar{a}_{i+1}), \ldots, a_i (a_{i+1} \ldots a_q \bar{a}_q \ldots \bar{a}_{i+1}) \}
$$

where $q = \min(2i, n - 2)$

$$
\vdots
$$

$$
e_i B e_{n-1} = \{ a_i \ldots a_{n-1} \}. 
$$

Thus, for $0 \leq i \leq m$,

$$
\dim e_i \Lambda e_j = \begin{cases} j + 1 & \text{for } j = 0, \ldots, i \\ i + 1 & \text{for } j = i + 1, \ldots, n - i - 1 \\ n - j & \text{for } j = n - i, \ldots, n - 1. \end{cases}
$$

Observe that $\dim e_i \Lambda e_j = \dim e_i \Lambda e_{e(j)}$ for $0 \leq i \leq m$ and $0 \leq j \leq n - 1$. As in the proof of 3.1, $\dim e_i \Lambda e_{e(j)} = \dim e_{e(j)} \Lambda e_{e(j)} = \dim e_{e(j)} \Lambda e_j$ and so $\dim e_{e(j)} \Lambda e_j = \dim e_i \Lambda e_j$ for $0 \leq i \leq m$. Now that $\dim e_i \Lambda e_j$ is known for $m + 1 \leq i \leq n - 1$, specific basis elements could be given for the remaining $e_i \Lambda e_j$.

4.2. The Cartan matrix $C$ of $\Lambda$ is the matrix $[c_{ij}]$ with $c_{ij} = \dim e_i \Lambda e_j$ for $0 \leq i, j \leq n - 1$. For $0 \leq i \leq m$ the $i$th row of $C$ is

$$
(1 \ 2 \ 3 \ldots i + 1 \ i + 1 \ldots i + 1 \ldots 2 \ 1),
$$

where the entry $i + 1$ occurs $n - 2i$ times. We also have that $c_{ij} = c_{e(j)}$ and so, for $m + 1 \leq i \leq n - 1$ the $i$th row of $C$ is the same as the $(n - i - 1)$th row. Finally we describe a typical element of $(P_0, \Lambda)$ and of $(P_1, \Lambda)$ and give the dimensions of these spaces.
4.3. Let \( b_{i,0} = e_i \) and \( b_{i,j} = a_i a_{i+1} \ldots a_{i+j-1} \bar{a}_{i+j-1} \ldots \bar{a}_{i+1} \bar{a}_i \), the unique right-normalized monomial in \( e_i \Lambda e_i \) with \( j \) bar letters. Observe that \( b_{i,j} \) has length \( 2j \) and has \( j \) nonbar letters.

A typical element \( z \) of \( \Phi^{n-1}_{i=0} e_i \Lambda e_i \) may be expressed in the form

\[
z = \sum_{r=0}^{m} \sum_{j=0}^{r} c_{r,j} b_{r,j} + \sum_{r=m+1}^{n-1} \sum_{j=0}^{r-1} c_{r,j} b_{r,j},
\]

where the coefficients \( c_{r,j} \) are in \( K \). Changing the order of summation gives

\[
z = \sum_{j=0}^{m} \sum_{r=j}^{n-1} c_{r,j} b_{r,j},
\]

Now \( (P_0, \Lambda) \equiv \Phi^{n-1}_{i=0} e_i \Lambda e_i \), and so the component of \( (P_0, \Lambda) \) of degree \( 2j \) has dimension \( n - 2j \). Hence \( \dim(P_0, \Lambda) = \sum_{j=0}^{m}(n - 2j) = (m + 1) \cdot (n - m) \).

4.4. Let \( \theta \) be any element of \( (P_1, \Lambda) \). We define

\[
w := \sum_{i=0}^{n-2} e_i \otimes e_{i+1}, \quad \bar{w} := \sum_{i=0}^{n-2} e_{i+1} \otimes e_i,
\]

Then \( \theta \) is determined by its action on \( w \) and \( \bar{w} \). We note that \( e_i \Lambda e_{i+1} = a_i e_{i+1} \Lambda e_{i+1} \) and \( e_{i+1} \Lambda e_i = e_{i+1} \Lambda e_{i+1} \bar{a}_i \), so that \( a_i b_{r+1,j}, b_{r+1,j} \bar{a}_i \) are typical basis elements of \( e_i \Lambda e_{i+1}, e_{i+1} \Lambda e_i \), respectively. Thus we may write

\[
\theta(w) = \sum_{r=0}^{n-2} \theta(w_{a_r}) = \sum_{r=0}^{m-1} \sum_{j=0}^{r} c_{r,j} a_r b_{r+1,j} + \sum_{r=m}^{n-2} \sum_{j=0}^{r-2} c_{r,j} a_r b_{r+1,j},
\]

where the coefficients \( c_{r,j} \) are in \( K \). Changing the order of summation gives

\[
\theta(w) = \sum_{j=0}^{m} \sum_{r=j}^{n-2} c_{r,j} a_r b_{r+1,j}.
\]

Similarly there are coefficients \( d_{r,j} \) in \( K \) such that

\[
\theta(\bar{w}) = \sum_{j=0}^{m} \sum_{r=j}^{n-2} d_{r,j} b_{r+1,j} \bar{a}_r.
\]

It follows that the dimension of the component of \( (P_1, \Lambda) \) of degree \( 2j + 1 \) is \( 2(n - 2j - 1) \) and thus \( \dim(P_1, \Lambda) = \sum_{j=0}^{m} 2(n - 2j - 1) = 2(m + 1) \cdot (n - m - 1) \).
4.4.1. Let $\sigma_i$ be the generators of $\ker \delta$ as in 1.2. These generators may also be given in terms of the elements $w_i, \bar{w}_i$ of $P_1$ by

$$\sigma_i = w_i a_i + a_i \bar{w}_i + \bar{w}_i a_{i-1} + a_{i-1} w_i$$

with the convention that $a_i = 0 = \bar{a}_i$ if $r < 0$ or $r > n - 2$.

We shall also use the notation $\eta := \sum_i a_i$ and so $\bar{\eta} = \sum_i \bar{a}_i$.

5. $HH^0(\Lambda)$ AND $HH^6(\Lambda)$ FOR $\Lambda$ OF TYPE $A_n$

First we determine $HH^0(\Lambda)$ by using the fact that $HH^0(\Lambda) \equiv Z(\Lambda)$, the center of $\Lambda$. Recall from 4.3 that $h_{r,j}$ is the unique right-normalized basis element in $e_r \Lambda e_r$ of degree $2j$. The following identity is easily verified.

5.1. For $\Lambda$ of type $A_n$, $0 \leq j \leq r \leq n - 2$ and $r + j \leq n - 2$, we have $a_i h_{r+1,j} = (-1)^j h_{r,j} a_i$.

5.2. Proposition. For $\Lambda$ of type $A_n$, define $z_j := \sum_{r=0}^{n-1-j} (-1)^j h_{r,j}$, where $j = 0, \ldots, m$. Then the set $\{z_j; j = 0, \ldots, m\}$ is a basis for $Z(\Lambda)$.

Proof. It is clear that this set consists of linearly independent elements and so it is enough to prove that this set is a spanning set for $Z(\Lambda)$. We start by showing that each element of $Z(\Lambda)$ is a linear combination of elements in this set.

Let $z$ be an element of the center of $\Lambda$. Then $z \in \mathfrak{g}_i \Lambda \mathfrak{g}_i$, so we may use the grading of $(P_0, \Lambda)$ to denote the component of the element $z$ of degree $2j$ by $\xi_j$. From 4.3 there are coefficients $c_{r,j}$ in $K$ such that $\xi_j = \sum_{r=0}^{n-1-j} (-1)^j c_{r,j} h_{r,j}$.

For $j \leq n - j - 2$, we have $a_i c_{r+1,j} h_{r+1,j} = a_i \xi_j = a_i \xi_j a_i = c_{r,j} h_{r,j} a_i$, and so, from 5.1, $c_{r+1,j} = (-1)^j c_{r,j}$. Then, by induction, $c_{r,j} = (-1)^{r+j} c_{0,j}$ for $r = 1, \ldots, n - j - 1$. Thus $\xi_j = (-1)^{r+j} c_{0,j} \sum_{r=0}^{n-1-j} (-1)^j h_{r,j}$. Hence $Z(\Lambda) \subseteq \text{sp}(z_j; j = 0, \ldots, m)$.

For the reverse inclusion, note that $h_{r,j} = h_{r,j}$, so $h_{r,j} = z_j$ and it is enough to show that $a_i z_j = z_j a_i$ for all $r$, since this implies $z_i a_r = a_r z_i = z_i a_r$. If $j = 0$ then $z_0 = \sum_{r=0}^{n-2} e_r = 1$ and so $z_0 \in Z(\Lambda)$. So suppose $j \geq 1$.

If $0 \leq j \leq r - 2$ or $n - j \leq r \leq n - 2$ then $a_i z_j = 0 = z_j a_i$. If $r = j - 1$ then $a_i z_j = a_i z_{j-1} (-1)^j h_{j-1} = 0$ since this monomial starts at $j - 1$ and has $j$ bar letters. So $a_i z_j = (1) = z_j a_i$. Similarly, if $r = n - j - 1$ then $a_i z_j = 0 = z_j a_i$. In the remaining case $j \leq r \leq n - j - 2$, 5.1 shows that $z_j$ commutes with $a_i$. Hence $z_j \in Z(\Lambda)$ for all $j (j = 0, \ldots, m)$.

5.3. Corollary. For $\Lambda$ of type $A_n$, $HH^0(\Lambda) \equiv Z(\Lambda)$ and $\dim HH^0(\Lambda) = m + 1$. 
From 2.1 we have \((N, N) \cong (\Lambda, \Lambda)\) and so we may determine \(HH^6(\Lambda)\) using the exact sequence

\[
0 \to (\text{Im } R, N) \to (P_0, N) \to (\Lambda, \Lambda) \to HH^6(\Lambda) \to 0.
\]

5.4. Lemma. For \(\Lambda\) of type \(A_n\), \((\text{Im } R, N) = \bigoplus_i e_i J_{e_i(i)}\), hence

\[
\dim(\text{Im } R, N) = \begin{cases} 
\dim(P_0, N) & \text{if } n \text{ is even} \\
\dim(P_0, N) - 1 & \text{if } n \text{ is odd}.
\end{cases}
\]

Proof. Using 2.4.1 with \(\rho = \tilde{\nu}, W = N\), and \(P = P_2\), we must show that \(\text{Ker } C_\rho^e = 0\) in this case. From 2.4, if \(n\) is even then \(F = B\) and so \(\text{Ker } C_\rho^e = 0\). On the other hand if \(n\) is odd then \(F = \{m\}\) and \(\text{Ker } C_\rho^e\) consists of the \((m + 1)\)th column of \(C^e\); so \(\text{Ker } C_\rho^e = 0\).

5.5. Proposition. Assume \(\Lambda\) is of type \(A_n\). If \(n\) is even then \(HH^6(\Lambda) \cong Z(\Lambda)\) and if \(n\) is odd then \(HH^6(\Lambda) \cong Z(\Lambda)/(\omega_m)\) (factored out by the socle of \(e_m\)).

Proof. If \(n\) is even, then \((\text{Im } R, N) = (P_0, N)\) and so \(HH^6(\Lambda) \cong Z(\Lambda)\). If \(n\) is odd then \((\text{Im } R, N)\) has codimension 1 in \((P_0, N)\) and the result follows.

6. \(HH^3(\Lambda)\) for \(\Lambda\) of type \(A_n\)

We use the exact sequence

\[
0 \to (\Lambda, \Lambda) \to (P_0, \Lambda) \to (\text{Ker } u, \Lambda) \to HH^3(\Lambda) \to 0
\]

to find \(HH^3(\Lambda)\). The map \(\delta: P_1 \to P_0\) induces a map \(\delta^*: (P_0, \Lambda) \to (P_1, \Lambda)\) given by \(\phi \mapsto \phi \delta\). Then \(\text{Im } \delta^* \subseteq (\text{Ker } u, \Lambda)\) and \(\dim(\text{Im } \delta^*) = \dim(P_0, \Lambda) - \dim(\Lambda, \Lambda) = (m + 1)(n - m - 1)\). The following characterization of \(\text{Im } \delta^*\) enables us to give a decomposition of \((\text{Ker } u, \Lambda)\). We use the identification of \((\text{Ker } u, \Lambda)\) as a submodule of \((P_1, \Lambda)\) as in 3.2 and recall that the generators of \(\text{Ker } \delta\) are given in 4.4.1. Then \(HH^3(\Lambda)\) is isomorphic to a complement of \(\text{Im } \delta^*\) in \((\text{Ker } u, \Lambda)\).

6.1. Lemma. Let \(\gamma \in (P_1, \Lambda)\). Then \(\gamma \in \text{Im } \delta^*\) if and only if \(\gamma(\overline{\bar{e}}) + \gamma(\overline{\bar{w}}) = 0\).

Proof. First we show that \(\phi \delta(\overline{\bar{e}}) + \overline{\phi \delta(\overline{\bar{w}})} = 0\) whenever \(\phi \in (P_0, \Lambda)\). We have \(\phi \delta(\overline{\bar{w}}) = \sum_{i=1}^{n-1} (\pi, \phi(e_i \otimes e_{i+1}) - \phi(e_i \otimes e_{i+1})\overline{\bar{a}})\) and \(\phi \delta(\overline{\bar{e}}) = \sum_{i=1}^{n-1} (\pi, \phi(e_i \otimes e_{i+1}) - \phi(e_i \otimes e_{i+1})\overline{\bar{a}})\). The anti-automorphism induced by \(\overline{\bar{\cdot}}\) acts as the identity on the basis we have given for \(e_i \Lambda e_i\) and so
\[ \bar{\phi}(\overline{w}) = \sum_{i=0}^{n-2} (\phi(e_{i+1} \otimes e_{i+1}) \overline{a}_i - \overline{a}_i \phi(e_i \otimes e_i)). \]

Hence \( \bar{\phi}(\overline{w}) + \overline{\phi(w)} = 0. \)

Suppose \( \gamma \in \text{Im} \, \delta^* \). Then there is some \( \phi \in (P_0, \Lambda) \) such that \( \phi \delta = \gamma \). Thus \( \gamma(\overline{w}) = \gamma(w) = 0. \)

Conversely, suppose that \( \gamma \in (P_1, \Lambda) \) with \( \gamma(\overline{w}) + \overline{\gamma(w)} = 0. \) Since \((P_1, \Lambda)\) is graded, write \( \gamma = \sum_{j=0}^m \gamma_j \) with \( \gamma_j \) of degree \( 2j + 1 \). Then there are coefficients \( c_{r,j} \in K \) such that \( \gamma_j(w) = \sum_{r=-2}^{r=-2} c_{r,j} a_r b_{r,j} \) \((4.4)\). Define the map \( \phi: P_0 \to \Lambda \) by \( \phi = \sum_{j=0}^m \phi_j \) where \( \phi_j: P_0 \to \Lambda \) is given by
\[
\phi_j: e_i \otimes e_i \mapsto \begin{cases} s_{i-1,j} b_{i,j} & \text{if } i = j + 1, \ldots, n - j - 1 \\ 0 & \text{otherwise}, \end{cases}
\]

where \( s_{i,j} = \sum_{r=0}^{r=0} (-1)^{(r-1)} c_{r,j} \). It can be shown by direct calculation that \( \phi_j \delta(w) = \gamma_j(w) \) for \( j = 0, \ldots, m \) and hence \( \phi \delta(w) = \gamma(w) \).

Since \( \phi \in (P_0, \Lambda) \), the first part of this proof gives \( \phi \delta(\overline{w}) = - \overline{\phi(w)} \) and thus, using the hypothesis, \( \phi \delta(\overline{w}) = - \overline{\gamma(w)} = \gamma(\overline{w}) \). Hence \( \phi \delta = \gamma \) and so \( \gamma \in \text{Im} \, \delta^* \) as required.

6.2. Proposition. \( (\text{Ker } u, \Lambda) = \text{Im} \, \delta^* \otimes \{ \theta \in (P_1, \Lambda): \theta(w) = 0, \theta(\overline{w}) \in Z(\Lambda, \overline{\eta}) \} \).

Proof. Let \( V = \{ \theta \in (P_1, \Lambda): \theta(w) = 0, \theta(\overline{w}) \in Z(\Lambda, \overline{\eta}) \} \).

It is clear that \( \text{Im} \, \delta^* \subseteq (\text{Ker } u, \Lambda) \). To show that \( V \subseteq (\text{Ker } u, \Lambda) \) let \( \theta \in V \) and write \( \theta(\overline{w}) = z \overline{\eta} \) for some \( z \in Z(\Lambda, \Lambda) \). For \( r = 1, \ldots, n - 2, \)
\[ \theta(\sigma_r) = a_r \theta(\overline{w}) + \theta(\overline{w}) a_r = a_r z \overline{\eta} + z \overline{\eta} a_r = z(a_r \overline{\eta} + \overline{\eta} a_r) = 0. \]
Similarly, we also have \( \theta(\sigma_0) = \theta(\sigma_{n-1}) = 0 \). Hence \( V \subseteq (\text{Ker } u, \Lambda) \).

Hence \( \theta = 0 \) and \( \text{Im} \, \delta^* \cap V = \{0\} \). Thus \( \text{Im} \, \delta^* \otimes V \subseteq (\text{Ker } u, \Lambda) \).

For the reverse inclusion, let \( \theta \in (\text{Ker } u, \Lambda) \). Define \( \psi: P_1 \to \Lambda \) by
\[ \psi(w) = 0, \psi(\overline{w}) = \theta(\overline{w}) + \overline{\theta(w)}. \]

Let \( \gamma = \theta - \psi \) so \( \gamma \in \text{Hom}(P_1, \Lambda) \). Then \( \theta = \gamma + \psi \). We claim that \( \gamma \in \text{Im} \, \delta^* \) and \( \psi \in V \).

First, we consider \( \gamma \). We have \( \gamma(w) = \theta(w) \) and \( \gamma(\overline{w}) = \overline{\theta(w)} \) so \( \gamma(\overline{w}) + \overline{\gamma(w)} = 0 \). Thus, by \( 6.1 \), we know \( \gamma \in \text{Im} \, \delta^* \). Now both \( \theta \) and \( \gamma \) are in \( (\text{Ker } u, \Lambda) \) and so \( \psi \in (\text{Ker } u, \Lambda) \). Use the grading of \( (\text{Ker } u, \Lambda) \) to write \( \psi = \sum \psi_j \) with each \( \psi_j \in (\text{Ker } u, \Lambda), \psi_j \) of degree \( 2j + 1 \) and \( \psi_j(w) = 0 \).

Now there are coefficients \( d_{r,j} \in K \) such that \( \psi_j(\overline{w}) = \sum_{r=0}^{r=0} d_{r,j} a_r \) \((4.4)\). For \( r = j + 1, \ldots, n - j - 2 \), we have \( 0 = \psi_j(\sigma_r) = a_r d_{r,j} b_{r,j+1} + d_{r-1,j} b_r a_r = (d_{r,j} - (1)^{r-1} d_{r-1,j}) b_{r,j+1} \) (by applying \( r \) to \( 5.1 \)). Thus \( d_{r,j} = (-1)^{r-1} d_{r-1,j} \) for \( r = j, \ldots, n - j - 2 \). Then
\[
\begin{align*}
\psi_j(\overline{w}) &= (-1)^{j+1} d_{j,j} \sum_{r=j+1}^{r=0} (-1)^{r+j} b_{r,j+1} \overline{a}_r = \sum_{r=j+1}^{r=0} (-1)^{r+j} d_{r,j+1} \overline{a}_r \\
&= (-1)^{j+1} d_{j+1,j+1} \overline{a}_{j+1} = 0.
\end{align*}
\]
This gives \( \psi_j(\overline{w}) = 0 \) where \( z_j \) is the element of degree \( 2j \) in the basis for
Z(Λ) given in 5.2. Thus \( \psi(\overline{w}) \in Z(\Lambda)\overline{\eta} \) and so \( \psi \in V \). Hence \( \psi \in V \) and we have \( \theta \in \text{Im} \ \delta^* \oplus V \). This gives the decomposition \( (\text{Ker} \ u, \Lambda) = \text{Im} \ \delta^* \oplus V \).

It is now immediate from 6.12 that \( HH^1(\Lambda) \cong V \).

6.3. **Corollary.** \( HH^1(\Lambda) \cong \{ \theta \in (P_1, \Lambda): \theta(w) = 0, \theta(\overline{w}) \in Z(\Lambda)\overline{\eta} \} \) and

\[
\dim HH^1(\Lambda) = n + m - 1 = \begin{cases} m & \text{if } n \text{ is odd} \\ m + 1 & \text{if } n \text{ is even.} \end{cases}
\]

**Proof.** It is enough to show that \( \dim V = n - m - 1 \). Define \( \chi_j: P_1 \to \Lambda \) by \( \chi_j(w) = 0, \chi_j(\overline{w}) = z_j\overline{\eta} \) for \( j = 0, \ldots, m \) where \( z_j \) is the basis element of \( Z(\Lambda) \) of degree \( 2j \) (from 5.2). Then the nonzero elements in \( \{ \chi_j \} \) form a basis for \( V \). By direct calculation we see that \( \chi_j = 0 \) if and only if \( j = m \) where \( n \) is odd. Thus \( \chi_j \neq 0 \) whenever \( j = 0, \ldots, m - 1 \) for \( n \) odd, and 

\[
j = 0, \ldots, m \text{ for } n \text{ even. Hence } \dim V = n - m - 1.
\]

6.4. **Remark.** Using 6.2 and 6.3, \( \dim(\text{Ker} \ u, \Lambda) = \dim(\text{Im} \ \delta^*) + \dim V = (m + 2)(n - m - 1) \). A basis of \( HH^1(\Lambda) \) is given by \( g_0, \ldots, g_{n - m - 2} \) where \( g_i \) is defined by

\[
g_i(\delta(w)) = 0, \quad g_i(\delta(\overline{w})) = z_i\overline{\eta}
\]

with \( z_i \) as in 5.2.

7. **ON HH^2 AND HH^3 FOR TYPE A_n**

To determine \( HH^2(\Lambda) \), we use the exact sequence

\[
0 \to (\text{Ker} \ u, \Lambda) \to (P_1, \Lambda) \to (\text{Im} \ R, \Lambda) \to HH^2(\Lambda) \to 0.
\]

It follows from 6.4, 4.3, and 4.4 that both \( \text{Im}(j^*) \) and \( \bigoplus_i e_i J e_i \) have dimension equal to \( m(n - m - 1) \).

7.1. **Lemma.** As a vector space, \( HH^2(\Lambda) \) is isomorphic to the kernel of the signed Cartan matrix \( C^* \) of \( \Lambda \).

**Proof.** We have \( (\text{Im} \ R, \Lambda) = (P/\Lambda) \) where \( P \) is as in Section 2. By 2.4.1 applied with \( W = \Lambda \) and \( \rho = \text{id} \), we get

\[
(\text{Im} \ R, \Lambda) \cong \bigoplus_i e_i J e_i \oplus \text{Ker} \ C^*.
\]

Next, observe that if \( \phi \in (\text{Im} R, \Lambda) \) and \( \phi \) factors through \( P_1 \) then \( \text{Im} \phi \subseteq J \). Namely, let \( \phi = \psi j \), then \( \text{Im} \phi \subseteq \text{Im} \psi \) and this is a homomor-
phic image of $P$, which is clearly contained in $J$. Hence with the identification we have $\text{Im}(j^*) \subseteq \bigoplus e_i Ne_i$, and we must have equality since both spaces have the same dimension.

7.2. Lemma. The general solution of $C^*x = 0$ is

(i) For $n$ even, $x = (\lambda_0, \lambda_1, \ldots, \lambda_m, \lambda_{m-1}, \ldots, \lambda_0)^t$ of dimension $m + 1$.

(ii) For $n$ odd, $x = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1}, 0, -\lambda_m, -\lambda_{m-2}, \ldots, -\lambda_0)^t$ of dimension $m$.

7.2.1. The space $HH^2(\Lambda)$ has a basis $\{f_i: 0 \leq i \leq n - m - 2\}$ where

$$f_i(\sigma_j) = \begin{cases} e_i & j = i \\ (-1)^n e_{v(i)} & j = v(i) \\ 0 & \text{otherwise}. \end{cases}$$

Proof of 7.2. We use 4.2. Obviously rows $m + 1, \ldots, n - 1$ of $C^*$ are multiples of the first $m + 1$ rows; so let $C_1$ be the matrix consisting of the first $m + 1$ rows of $C^*$. Write $R_k$ for the $k$th row of $C_1$. Replacing $R_k$ by $R_k + R_{k-1}$ for $k = m, m - 1, \ldots, 1$ brings $C_1$ to the form

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & \cdots & 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 \\ \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 & \cdots & \cdots & -1 & 1 \\ 0 & 1 & \cdots & \cdots & 1 & 0 \\ \cdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \end{pmatrix},$$

$n$ even, $n$ odd,

and the Lemma follows.

Now we shall determine $HH^3(\Lambda)$, using the exact sequence

$$0 \rightarrow (\text{Im} R, \Lambda) \rightarrow (P_0, \Lambda) \rightarrow (N, \Lambda) \rightarrow HH^3(\Lambda) \rightarrow 0.$$

7.3. Lemma. As vector spaces, $(\Lambda, N) \cong \text{soc} \Lambda$.

Proof. We identify $(\Lambda, N)$ with $(\phi \in (P_0, N): \phi(x_a) = 0$ for all arrows $a)$; recall that $(P_0, N) = \otimes e_i Ne_i$, as vector spaces. Clearly, if $\phi \in (P_0, N)$ and $\text{Im} \phi \subseteq \text{soc} \Lambda$ then $\phi(x_a) = 0$ for $a$ an arrow. Conversely, let $\phi(e_i \otimes e_i) = z_i \in e_i Ne_i$, and suppose $\phi(x_a) = 0$ for all arrows $a$; that is
Let $a_{i-1}z_i = z_{i-1}a_{i-1}$ and $\bar{a}_iz_i = z_{i+1}\bar{a}_i$ for all $i$. From 4.1 it is clear that $z_0$ and $z_{n-1}$ must lie in the socle. This gives $a_0z_1 = 0$ and also $a_1\bar{a}_1z_1 = -a_0\bar{a}_0z_1 = 0$ which implies that $\bar{a}_1z_1 = 0$ (using 4.1); and $z_1$ lies in the socle; and the same argument applies to $z_{n-2}$. The Lemma follows by induction.

7.4. Lemma. As vector spaces we have that $HH^3(\Lambda)$ is isomorphic to $\text{soc}\Lambda/\text{Im} C^*$ where $C^*$ is viewed as a linear map $\langle e_i \rangle \to \text{soc} \Lambda$, as described in 2.4.

Proof. We know that $(N, \Lambda) \cong (\Lambda, N) \cong \text{soc} \Lambda$ and $(\text{Im} R, \Lambda)$ was given in the proof of 7.1. We factor out $\oplus e_i R$ from the first two terms of the above sequence and get

$$0 \to \text{Ker} C^* \to \text{sp}\{e_i\} \to (N, \Lambda) \to HH^3(\Lambda) \to 0.$$ 

Denote by $\hat{\zeta}_i$ the image of the map induced by $e_i$ in $(N, \Lambda)$. This takes $\zeta_j$ to

$$\sum_{v \in e_i \text{Re}_i} (-1)^{i+j} u^*_v = (-1)^{i+j} c_{ij} \omega_j.$$ 

So the map $\hat{\zeta}_i$ is described by the $i$th column of $C^*$ and hence the image of $j^*$ is the column space of $C^*$.

7.5. Corollary. The space $(N, \Lambda)$ has basis $h_i$ for $0 \leq i \leq n - 1$ where $h_i(\zeta_j) = \delta_{ij} \omega_j$. The classes of $h_i$, $0 \leq i \leq n - m - 2$, form a basis for $HH^3(\Lambda)$. Moreover $[h_i] = (-1)^i[h_{i+1}]$ in $HH^3(\Lambda)$ and if $n$ is odd then $[h_m] = 0$.

Suppose $(-, -)$ is the nonsingular bilinear form on $(\Lambda, N)$ such that the $\omega_i$ form an orthonormal basis; then we have the following.

Lemma. Assume $\text{char} K \neq 2$, then

$$(N, \Lambda) = \text{Im} C^* \oplus (\text{Im} C^*)^\perp.$$ 

In particular in this case $HH^3(\Lambda) \cong (\text{Im} C^*)^\perp$. Moreover $(\text{Im} C^*)^\perp$ has a basis given by

$$\{\omega_i + (-1)^n \omega_{i+1}: 0 \leq i \leq n - m - 2\}.$$ 

Proof. The column space of $C^*$ is the same as the row space, so we see from the proof of 7.2 that the listed elements form a basis of $(\text{Im} C^*)^\perp$. If $\text{char} K \neq 2$ then the intersection with $\text{Im} C^*$ is zero.
8. $\text{HH}^3(\Lambda)$ for $\Lambda$ of type $A_n$

For finding $\text{HH}^3(\Lambda)$ we will need appropriate elements of $N$; here we take $N$ explicitly in the form $N = \langle \zeta \rangle$. We need the following fact.

8.1. Lemma. Fix $i$ with $0 \leq i \leq n - 2$, and let $l$ be such that $e_iN_i^{(l)} \neq 0$. Then there are nonzero elements $\kappa = \kappa_{i,j} \in e_iN_i^{(l)}$ and $\mu = \mu_{i,j} \in e_{i+1}N_{i+1}^{(l)}$ such that

$$\kappa a_i = a_i \mu \quad \text{and} \quad a_i \kappa = \mu a_i.$$ 

Moreover, these are nonzero and $(\kappa, \mu)$ is unique up to scalar multiples.

Proof. The uniqueness follows since $\dim e_iN_i^{(l)} = 1$. We start with a reduction.

1. Suppose $i$ is fixed; then it suffices to prove this for the shortest possible length. Namely, suppose we have found $\kappa_0, \mu_0$ of shortest length, $l_0$ say, satisfying the stated relations. If $l$ is arbitrary then $l = l_0 + 2k$ for $k \geq 0$ and we take

$$\kappa = \kappa_0(a_i a_i)^k, \quad \mu = \mu_0(a_i a_i)^k$$

which clearly satisfy the relations. By using 2.3 one sees that $\kappa a_i$ and $\mu a_i$ are nonzero.

2. Suppose $n$ is odd. Assume first that $i \leq m - 1$. Let $L$ be the shortest monomial $e_{i+1}L \zeta_m L$ and take $\kappa = a_i L \zeta_m L$ and $\mu = L \zeta_m L a_i$. Then clearly $\kappa a_i = a_i \mu$. Moreover, if $L$ has length $u \geq 0$ then we have using 3.3 that

$$\kappa a_i = (-1)^{u+1}(L a_m a_m \zeta_m L) = (-1)^u(L \zeta_m a_{m-1} a_{m-1} L) = \mu a_i.$$ 

The case when $i \geq m$ is similar.

3. Now assume that $n$ is even. If $i = m$ take $\kappa = \zeta_m$ and $\mu = -\zeta_{m+1}$. Now suppose $i < m$. Let $L$ be the shortest path from $i + 1$ to $m$; then take $\kappa = a_i L \zeta_m L \zeta_m L$ and $\mu = L \zeta_m L a_i$. It is straightforward to check that this works. The case $i \geq m + 1$ is similar.

8.2. We have the exact sequence

$$0 \to (\text{Ker} u, N) \to (P_1, N) \to (\text{Im} R, N) \to \text{HH}^3(\Lambda) \to 0.$$ 

We know from 5.4 that $(\text{Im} R, N)$ is $(P_0, N)$ if $n$ is even; and if $n$ is odd it has codimension 1 in $(P_0, N)$ and consists of all maps $\theta$ such that $\zeta_m$ is not involved in $\theta(e_m \otimes e_m)$. 
Consider now $\text{Im}(i^*)$. We identify this space with $(\phi \circ R; \phi \in (P_1, N))$. Suppose $a$ is an arrow, let $\phi_a, \ell \in (P_1, N)$ be the element which takes $e_a \otimes e_{\ell a}$ to $\kappa_n, \ell \in e_n, Ne_{\ell a}$ as defined in 8.1 and which takes all other generating idempotents to zero. The nonzero maps of this form are a $K$-basis for $(P_1, N)$.

8.3. LEMMA. The set $(\phi_a, \ell \circ R; \phi_a, \ell \neq 0)$ is a $K$-basis for $\text{Im}(i^*)$.

Proof. Let $\phi = \phi_a, \ell$ be nonzero.

(1) Assume first $a = a_j$. Let $\kappa = \phi(e_i \otimes e_{i+1})$; suppose this is a monomial of length $l$. We claim that

$$\phi \circ R(e_k \otimes e_{\ell k}) = \begin{cases} \kappa a_i & k = i \\
\bar{a}_{k, \ell} & k = i + 1 \\
0 & \text{otherwise.} \end{cases}$$

Since $\phi(0) = 0$ in this case we have $\phi \circ R(e_k \otimes e_{\ell k}) = \bar{a}_{k-1} \phi(w) + \phi(w) \bar{a}_{k}$ and the claim follows since $\phi(w) \in e_i Ne_{i+1}$ and $\bar{a}_{k} = e_{k+1} \bar{a}_{k}$ and $\bar{a}_{k-1} = \bar{a}_{k-1} e_{k+1}$.

(2) Now assume $a = a_i$. Let $\mu = \phi(e_{i+1} \otimes e_{i})$. Then we have similarly that

$$\phi \circ R(e_k \otimes e_{\ell k}) = \begin{cases} \mu a_i & k = i + 1 \\
\bar{a}_{k, \ell} & k = i \\
0 & \text{otherwise.} \end{cases}$$

It follows now from 8.1 that $\phi_a, \ell \circ R = \phi_a, \ell \circ R$, and we deduce that the given set spans $\text{Im}(i^*)$. By (1) and 8.1 it is also linearly independent.

8.4. COROLLARY. We have

$$\dim(\text{Im}(i^*)) = \dim(Ker u, N) = (1/2) \dim(P_1, N) = (m + 1)(n - m - 1).$$

Proof. In 8.3, we could have also taken the $\phi_{a, \ell}$ for constructing the basis, hence $\text{Im}(i^*)$ has dimension $(1/2) \dim(P_1, N)$; and by the exact sequence this is also equal to the dimension of $(\text{Ker} u, N)$. The last statement follows from 3.1 and 4.4.

8.5. We give a complement of $\text{Im}(i^*)$ in $\text{Im}(R, N)$ (see 5.4 for $\text{Im}(R, N)$). Let $y = \sum_{i \leq m} e_i \otimes e_i$; this is an element in $P_{\ell}$. Define

$$Y := \{ \theta \in (\text{Im} R, N); \theta \circ R(y) = 0 \}.$$ 

Then $\dim(Y) = \dim(e_m Je_{\ell(m)})$ which is $m + 1$ if $n$ is even and $m$ otherwise. By considering the above basis for $\text{Im}(i^*)$ it is clear that $\text{Im}(i^*) \cap Y$
Proposition. We have $HH^3(\Lambda)$ is isomorphic to

$$\{ \theta \in (\text{Im } R, N) : \theta \circ R(y) = 0 \},$$

where $y = \sum_{i=0}^{m} e_i \otimes e_i \in P_0$.

8.5.1. Recall $e_m N e_m = e_m \Lambda \zeta_{(m)}$ (2.3). A basis of $e_m J e_{n(m)}$ is given by $a_m z_i, 0 \leq i \leq m$ if $n$ is even, and by $a_m \overline{a}_m z_i, 0 \leq i \leq m - 1$ in case $n$ odd. Here $z_i \in Z(\Lambda)$ are as in 5.2. It follows that $HH^3(\Lambda)$ has a basis $\theta_i$, $0 \leq i \leq n - m - 2$, where for $n$ even, $\theta_i(\sigma_j) = \delta_{jm} a_m \overline{a}_m z_i \zeta_{m+1}$. If $n$ is odd then define $\theta_i(\sigma_j) = \delta_{jm} a_m \overline{a}_m z_i \zeta_m$.

9. $HH^4(\Lambda)$ FOR $\Lambda$ OF TYPE $A_n$

The exact sequence

$$0 \rightarrow (\Lambda, N) \rightarrow (P_0, N) \rightarrow (\text{Ker } u, N) \rightarrow HH^4(\Lambda) \rightarrow 0$$

is used to find $HH^4(\Lambda)$. We determine $\dim(\text{Ker } u, N)$ and hence obtain a decomposition of $(\text{Ker } u, N)$ from which $HH^4(\Lambda)$ is found. We take $N$ explicitly as $\langle \zeta \rangle$.

9.1. We know already from 8.4 that $\dim(\text{Ker } u, N) = (m + 1)(n - m - 1)$. We shall now give an explicit basis; we use the notation of 8.1. Fix $l$ for each $l$ such that $e_l N e_{l+1} \neq 0$ define $h_l^{(l)}$. $P_1 \rightarrow N$ by $h_l^{(l)}(\mu_l) = \kappa_l$ and $h_l^{(l)}(w_l) = -\mu_l$. Then one gets by 8.1 that $h_l^{(l)}(\theta) = 0$ for all $j$, so the map belongs to $(\text{Ker } u, N)$. These are clearly linearly independent since the elements $\kappa_l$ run through a basis, and the number of maps is the same as the dimension.

$HH^4(\Lambda)$ is isomorphic to a complement of $\text{Im } \delta^*$ in $(\text{Ker } u, N)$ where the map $\delta^*: (P_0, N) \rightarrow (P_1, N)$ is given by $\phi \mapsto \phi \delta$, and $\dim(\text{Im } \delta^*) = \dim(P_0, N) - \dim(\Lambda, N) = m(n - m - 1)$. The next result gives a necessary condition for an element of $(P_1, N)$ to be in $\text{Im } \delta^*$.

9.2. Proposition. Let $\theta \in \text{Im } \delta^*$ with $\theta(w) \neq 0$. Then there are distinct primitive idempotents $e, e'$ such that $e \theta(w) \neq 0$ and $e' \theta(w) \neq 0$.

Proof. There is some $\phi \in (P_0, N)$ with $\theta = \phi \delta$. Then $\phi(e_i \otimes e_i) \in e_l N e_{l+1}$ and $\theta(w) = \sum_{i=0}^{m-1} a_i \phi(e_{i+1} \otimes e_{i+1}) = \phi(e_i \otimes e_i) a_i$. Since $\theta(w) \neq 0$, there is some $i$ with $\phi(e_i \otimes e_i) \in \text{soc } \Lambda$, and $i \neq 0, n - 1$ (since $e_i \Lambda e_i \subseteq \text{soc } \Lambda$ for $i = 0, n - 1$). Choose $r$ minimal and $s$ maximal such that $1 \leq r, s \leq n - 2$ and $\phi(e_r \otimes e_s) \in \text{soc } \Lambda$ and $\phi(e_s \otimes e_r) \in \text{soc } \Lambda$. Then $e_{r-1} \theta(w) = a_{r-1} \phi(e_r \otimes e_r) \neq 0$ and $e_s \theta(w) = -\phi(e_s \otimes e_r) a_r \neq 0$. Since $r = 1 < r' \leq s$ the primitive idempotents $e_{r-1}$ and $e_r$ are distinct.
9.3. Let \( X = (\psi \in (P_\lambda, N); \psi(w) = z\zeta_m \eta, \psi(\bar{w}) = z\bar{\zeta}_m \eta^* \) for some \( z \in Z(\Lambda) \) if \( n \) is odd, and let \( X = (\psi \in (P_\lambda, N); \psi(w) = z\zeta_m, \psi(\bar{w}) = z\bar{\zeta}_m \) for some \( z \in Z(\Lambda) \) if \( n \) is even. We show that \( X \) forms a complement of \( \text{Im} \delta^* \in \text{Ker} u, N \) and is thus isomorphic to \( HH^4(\Lambda) \).

**Proposition.** With the above notation, \( \text{Ker} u, N = \text{Im} \delta^* \oplus X \) and \( \dim X = n - m - 1 \).

**Proof.** Take \( \psi \in X \). Then it is clear that \( \psi(\sigma) = 0 \) for \( r = 0, \ldots, m - 1 \) and for \( r = m + 2, \ldots, n - 1 \).

We consider separately the cases \( n \) odd and \( n \) even. First, suppose that \( n \) is odd. Then \( \psi(\sigma_m) = z\zeta_m \eta \bar{a}_m + a_m \zeta_m \eta^* = z\zeta_m a_m \bar{a}_m + z\bar{\zeta}_m \eta \).

Second, suppose that \( n \) is even. Then \( \psi(\sigma_m) = z\zeta_m \bar{a}_m + a_m \zeta_m \bar{a}_m = z\zeta_m a_m \bar{a}_m + z\bar{\zeta}_m \eta \).

Hence \( \psi(\sigma) = 0 \) for \( r = 0, \ldots, n - 1 \) and so \( \psi \in (\text{Ker} u, N) \).

From 9.2, this contradicts \( \psi \in \text{Im} \delta^* \). Thus \( \text{Im} \delta^* \cap X = \{0\} \) and so \( \text{Im} \delta^* \oplus Y \subseteq (\text{Ker} u, N) \).

The nonzero maps \( \psi_j : P_\lambda \to N \) given by \( \psi_j \in X \) and \( z = z_j \), the basis element in \( Z(\Lambda) \) of degree \( 2j \), form a basis for \( X \) (see 5.2). Now \( \psi_j(w) \neq 0 \) precisely for \( j = 0, \ldots, m - 1 \) if \( n \) is odd and for \( j = 0, \ldots, m \) if \( n \) is even. Thus \( \dim X = m \) if \( n \) is odd and \( m + 1 \) if \( n \) is even, that is, \( \dim X = n - m - 1 \). Hence \( \dim \text{Im} \delta^* \cap X = m(n - m - 1) + (n - m - 1) = \dim \text{Ker} u, N \) (9.1) and so \( (\text{Ker} u, N) = \text{Im} \delta^* \oplus X \).

9.4. **Corollary.** \( HH^4(\Lambda) \) is isomorphic to \( \{\psi \in (P_\lambda, N); \psi(w) = z\zeta_m \eta, \psi(\bar{w}) = z\bar{\zeta}_m \eta^* \) for some \( z \in Z(\Lambda) \) if \( n \) is odd, and \( \{\psi \in (P_\lambda, N); \psi(w) = z\zeta_m, \psi(\bar{w}) = z\bar{\zeta}_m, \psi \} \) for some \( z \in Z(\Lambda) \) if \( n \) is even. Moreover \( \dim HH^4(\Lambda) = n - m - 1 \).

9.4.1. The space \( HH^4(\Lambda) \) has a basis \( \psi_i \) where \( \psi_i \) is given by the element \( z_i \in Z(\Lambda) \) as in 5.2, \( 0 \leq i \leq n - m - 2 \).

We have now proved the following.

**Theorem.** Let \( \Lambda \) be preprojective of type \( A_n \). Then

1. \( HH^4(\Lambda) \equiv Z(\Lambda) \).

2. \( HH^4(\Lambda) \equiv \{0 \in (P_\lambda, \Lambda); \theta(w) = 0, \theta(\bar{w}) = \bar{Z(\Lambda)\eta}) \}

3. \( HH^4(\Lambda) \) is isomorphic to the kernel of the signed Cartan matrix \( C^e \) of \( \Lambda \).

4. \( HH^4(\Lambda) \) is isomorphic to \( (N, \text{soc} \Lambda)/\text{Im} C^e \) (7.4).
(4) \( HH^4(\Lambda) \) is isomorphic to \( \{ \psi \in (P_1, N): \psi(w) = z \zeta_m \eta, \psi(\overline{w}) = z \overline{\zeta}_m \eta \text{ for some } z \in Z(\Lambda) \} \) if \( n \) is odd, and \( \{ \psi \in (P_1, N): \psi(w) = z \zeta_m, \psi(\overline{w}) = z \overline{\zeta}_m \text{ for some } z \in Z(\Lambda) \} \) if \( n \) is even (9.4).

(5) \( HH^2(\Lambda) \cong \{ \phi \in (\text{Im } R, N): \phi \circ R(\sum_i e_i \otimes e_j) = 0 \} \) (8.5).

(6) \( HH^6(\Lambda) \) is isomorphic to \( Z(\Lambda)/(z_m) \) if \( n \) is odd, and to \( Z(\Lambda) \) if \( n \) is even (5.5).

Moreover, \( HH^{6+k}(\Lambda) \cong HH^k(\Lambda) \) for \( k \geq 1 \).

A summary of explicit bases of the Hochschild cohomology groups may be found at the beginning of part II.

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