Some cyclic virtually irreducible lattices

Michael Geline

Northern Illinois University, Department of Mathematical Sciences, Watson Hall, DeKalb, IL, United States

ARTICLE INFO

Article history:
Received 2 April 2010
Available online 12 February 2011
Communicated by Michel Broué

Keywords:
Virtually irreducible lattices
Brauer’s height zero conjecture
Abelian defect group

ABSTRACT

Virtually irreducible lattices which satisfy a simple rationality condition and have positive height are constructed for elementary abelian 2-groups. The non-existence of such lattices would have implied part of Brauer’s height zero conjecture. A characterization of rank 6 lattices for the elementary abelian group of size 16 is given.

© 2011 Elsevier Inc. All rights reserved.

1. Introduction

Recall that if one could prove that a virtually irreducible lattice for an abelian \( p \)-group were necessarily of \( p' \) rank, it would follow that all irreducible characters in blocks of finite groups with this particular defect group were necessarily of height zero, as conjectured by Brauer. This fact is due to Reinhard Knörr and is a main result of [5].

Unfortunately, a second result of [5] says that if \( R = \mathbb{Z}_2[2^{1/3}] \) and \( D \) is elementary abelian of order 16, then there exists a virtually irreducible \( RD \)-lattice of rank 6. Indeed Knörr displays four explicit 6-by-6 commuting involutions with entries in this \( R \), and verifies with a calculation that the resulting lattice is virtually irreducible. Nothing in [5] is said about how the example was found, the point there being that its very existence should be reason enough to pursue Brauer’s conjecture by other means.

However, it is pointed out in [3] that when invoking Knörr’s method, it suffices to restrict one’s attention to virtually irreducible \( RD \)-lattices for which the ramification of \( R/\mathbb{Z}_p \) is controlled by the exponent of \( D \). Specifically, if the exponent of \( D \) is \( p^m \), then only virtually irreducible \( KD \)-lattices in which \( R \) is the maximal unramified extension of \( \mathbb{Z}_p \) increased by the \( p^m \)th roots of unity need be considered. Thus, in the case of 2-blocks with elementary abelian defect groups, one can take \( R \) to be the maximal unramified extension of \( \mathbb{Z}_2 \). The fraction field of this ring is linearly disjoint from that of Knörr’s \( \mathbb{Z}_2[2^{1/3}] \).
The observation in [3] leads to a search for positive height virtually irreducible lattices satisfying the above rationality condition. (Recall that the height of a lattice for a \(p\)-group is simply the exponent in the power of \(p\) dividing the rank of the lattice.) While most of the work in [3] concentrates on an elementary abelian group of order 8, the main result here is that by considering larger elementary abelian 2-groups, virtually irreducible lattices of arbitrary height can be constructed. These lattices turn out to be cyclic; that is they are quotients of the regular lattice. In particular, they are defined over \(\mathbb{Z}_2\) and hence satisfy the rationality condition. The construction is given in Section 2.

The smallest such example is the same size as Knörr’s example, namely a rank 6 lattice for an elementary abelian group of order 16. A secondary result shows that, with considerable work, all virtually irreducible lattices of this size which satisfy the rationality condition are in fact cyclic. This is given in Section 3.

Quite a lot remains to be done. Of course, no substantial results about odd primes have yet been obtained although some of the techniques below and from [3] apply. Also, it is easily seen that there can be no even-rank cyclic virtually irreducible lattice for the elementary abelian group of order 8. Thus the possibility of proving Brauer’s conjecture for blocks with defect groups of size 8 directly by Knörr’s method remains open. This is quite intriguing as the conjecture was recently established in this case by Kessar, Koshitani, and Linckelmann [4] using the classification of finite simple groups.

However, the question of why the lattices constructed in Section 2 cannot arise as sources of irreducible lattices in 2-blocks with elementary abelian defect groups is, in our view, the most compelling.

2. Examples

The definition of a virtually irreducible lattice is due to Knörr [5]. Taking \(R\) to be an arbitrary discrete valuation ring of characteristic zero with residue class field of positive characteristic, an \(RG\)-lattice \(M\) is called virtually irreducible if for every endomorphism \(\alpha\) of \(M\), we have

\[
(i) \quad \nu(\text{trace}(\alpha)) \geq \nu(\text{rank}(M)), \quad \text{and} \\
(ii) \quad \text{equality holds if and only if } \alpha \text{ is invertible.}
\]

The exponential valuation of \(R\) is denoted by \(\nu\). It is clear that irreducible lattices are virtually irreducible.

**Lemma 1.** Let \(h\) be a positive integer, and let \(D\) be an elementary abelian 2-group. Let \(\chi\) be a multiplicity-free character for \(D\) such that \(\chi(g) \equiv 2^h \pmod{2^{h+1}}\) for all \(g \in D\). Then the quotient of the regular \(\mathbb{Z}_2D\)-lattice affording \(\chi\) is virtually irreducible of height \(h\).

**Proof.** Let \(M\) denote the lattice. The rank of \(M\) is congruent to \(2^h \pmod{2^{h+1}}\). The statement about the height of \(M\) follows. Also, the natural map \(\mathbb{Z}_2D \to \text{End}_{\mathbb{Z}_2D}(M)\) is surjective. If \(\alpha \in \mathbb{Z}_2D\), then regarding \(\alpha\) as an element of \(\text{End}_{\mathbb{Z}_2D}(M)\), we have

\[
\text{trace}(\alpha) \equiv 2^h \text{aug}(\alpha) \pmod{2^{h+1}}.
\]

Here, aug denotes the augmentation map from \(\mathbb{Z}_2D\) to \(\mathbb{Z}_2\). Thus \(\text{trace}(\alpha) \in 2^h\mathbb{Z}_2\), and because \(\mathbb{Z}_2D\) is a local ring, we have \(\text{trace}(\alpha) \in 2^{h+1}\mathbb{Z}_2\) if and only if \(\alpha\) is not invertible. \(\Box\)

If \(D\) has size 8 or smaller it is readily checked that no such \(\chi\) exists. However, if \(|D| = 16, h = 1, \lambda_i\) for \(1 \leq i \leq 4\) generate the group \(\text{Irr}(D)\), and \(\mu = \prod_{i=1}^{4} \lambda_i\), then we can take

\[
\chi = 1_D + \sum_{i=1}^{4} \lambda_i + \mu. \quad (1)
\]
In fact, this is the character afforded by Knörr’s example in [5]. It is somewhat cumbersome to verify directly that the character (1) indeed assumes values all congruent to 2 (mod 4). An easy way to do this is given next.

2.1. Height one

Together, Lemmas 1 and 2 reduce the question of cyclic virtually irreducible lattices of height one to combinatorics. The second condition below can be verified at a glance using a generating set for the group \( \text{Irr}(D) \). The referee has pointed out that this appears in work of Yoshida. See [6, Corollary 5.2.1].

Lemma 2. Let \( D \) be an elementary abelian 2-group, and let \( \chi \) be a character of \( D \) such that

(i) \( \chi(1) \equiv 2 \pmod{4} \), and
(ii) \( \text{Det}(\chi) = 1_D \).

Then \( \chi(g) \equiv 2 \pmod{4} \) for every \( g \in D \).

Proof. Let \( g \in D \). Then \( \chi(g) \) is the sum of \( \chi(1) \) numbers, each 1 or \(-1\). Let \( a \) be the multiplicity of 1. Let \( b \) be the multiplicity of \(-1\). Because \( \text{Det}(\chi) = 1_D \), we know that \( b \) is even. But then \( \chi(g) = a - b \equiv a + b = \chi(1) \equiv 2 \pmod{4} \) as desired. \( \square \)

As an illustration, let \( |D| = 2^{10} \). Let \( \lambda_1, \ldots, \lambda_{10} \) generate the group \( \text{Irr}(D) \). Partition the \( \lambda_i \) into three disjoint subsets such that each subset contains at least two generators. Let \( \Phi_1, \Phi_2, \) and \( \Phi_3 \) be the products of the irreducibles in each subset. Then the character

\[ \chi = 1_D + \lambda_1 + \cdots + \lambda_{10} + \Phi_1 + \Phi_2 + \Phi_3 \]

is multiplicity-free, has degree 14, and has trivial determinant. Of course there are many such ways to partition the \( \lambda_i \).

2.2. Larger height

Constructing examples of larger height seems to be more difficult, and we will not be able to construct all such. Rather, we will show that for each positive integer \( h \), there exists a cyclic virtually irreducible lattice of height \( h \) for some, possibly quite large, elementary abelian 2-group.

Regard \( h \) as fixed, and consider an elementary abelian 2-group \( D \) of size \( 2^{ml} \) where \( m \) and \( l \) are positive integers to be determined. Consider a generating set for the group \( \text{Irr}(D) \). The size of this set is \( ml \), so it can be partitioned into \( l \) disjoint subsets, each of size \( m \). Call the \( i \)th subset \( \Omega_i \), for \( 1 \leq i \leq l \).

Let \( \Phi_i \) be the sum of the nontrivial irreducible characters of \( D \) which, when expressed as words in the chosen generating set, contain only generators from \( \Omega_i \). Thus \( \Phi_i \) is a multiplicity-free character of \( D \) of degree \( 2^m - 1 \), and if \( i \neq j \), then \( \Phi_i \) and \( \Phi_j \) have no irreducible constituents in common.

Let

\[ \chi = 1_D + \Phi_1 + \cdots + \Phi_l. \]

Then \( \chi \) is multiplicity-free of degree \( 1 + l(2^m - 1) \). Assume now that \( m > h \), and \( l \equiv 2^h + 1 \pmod{2^{h+1}} \). Then we have \( \chi(1) \equiv 2^h \pmod{2^{h+1}} \) as desired. We must show that for all nonidentity \( g \in D \), we have \( \chi(g) \equiv 2^h \pmod{2^{h+1}} \) as well.

Let \( \Omega_g \) be the subset of our generating set for \( \text{Irr}(D) \) consisting of irreducible characters assuming the value \(-1\) on \( g \). Then for each \( i \), we have

\[ \Phi_i(g) = \begin{cases} 2^m - 1 & \text{if } \Omega_i \cap \Omega_g = \emptyset, \\ -1 & \text{if } \Omega_i \cap \Omega_g \neq \emptyset. \end{cases} \]
Because \( m > h \), it follows that \( \chi(g) \equiv 1 - l \equiv 2^h \pmod{2^{h+1}} \).

The simplest way to construct a lattice of height \( h \) as above is to take \( m = h + 1 \) and \( l = 2^h + 1 \). The number of generators for the group is then \( (h + 1)(2^h + 1) \). It would be interesting to know the smallest group giving rise to such a lattice.

3. Uniqueness for rank 6

From this point on we will assume that \( D \) is elementary abelian of order 16, and that \( R \) is an unramified extension of \( \mathbb{Z}_2 \). The main result here shows that, up to taking tensor products with rank 1 lattices and twisting by automorphisms of \( D \), the lattice constructed using the character (1) is the only rank 6 virtually irreducible \( RD \)-lattice.

**Theorem 3.1.** If \( M \) is a virtually irreducible \( RD \)-lattice of rank 6, then \( M \) is cyclic.

The results and techniques of [3] will be used. Among the more important facts is [3, Lemma 4], which says that if some lattice between \( 2M \) and \( M \) has an odd rank direct summand, then multiplication by 2 followed by projection gives a non-invertible endomorphism of \( M \) whose trace generates \( 2R \). If \( M \) is virtually irreducible, it follows that \( M \) has odd rank. This observation is due to Akaki Tikaradze.

A general technique of Butler [2] on (local integral) lattices for abelian \( p \)-groups will also play an important role, particularly in Lemma 5 and throughout Section 3.4. In fact, [2] is a direct generalization of earlier work of Butler [1] which was used heavily in [3]. It is what takes real advantage of our assumption that \( R/\mathbb{Z}_2 \) be unramified.

In Section 3.1 we will show that it is no loss of generality to assume \( \chi_M \), the character afforded by \( M \), is exactly the one given by (1). In Section 3.2 a subgroup \( H \) of \( D \) of size four will be found with the property that the restriction \( M_H \) will have a projective summand. The decomposition of this restriction will allow us to show that the length of the \( R \)-module \( M/M_{ss} \) is at most 6. Here \( M_{ss} \) is the maximal semisimple \( RD \)-sublattice of \( M \), as in [3]. In Section 3.3 the information gathered about \( M \) will be combined with the work of Butler [2]. This will allow us in Section 3.4 to find an element \( v \in M \) such that \( M = RDv \).

3.1. The character afforded by \( M \)

Let \( \chi_M \) denote the character afforded by \( M \). The main result of [3] implies that \( M \) is faithful. Now if \( \lambda \) is an irreducible character of \( D \) afforded by the rank 1 \( RD \)-lattice \( L_\lambda \), then \( M \otimes L_\lambda \) is also virtually irreducible. The main result of [3] implies that \( M \otimes L_\lambda \) is faithful as well.

Thus, by tensoring \( M \) with a rank 1 lattice if necessary, we can assume that \( \chi_M \) contains the trivial character \( 1_D \) as well as a generating set for the abelian group \( \text{Irr}(D) \). This determines five of the six irreducibles appearing in \( \chi_M \).

By [3, Lemma 7], we know that \( \text{Det}(M) \) is the trivial rank 1 \( RD \)-lattice. It follows that the sixth irreducible constituent of \( \chi_M \) must be the product of the four irreducibles appearing in the generating set. Thus \( \chi_M \) has the form (1).

3.2. Semisimple sublattices and Klein four subgroups

Recall that in [3] an \( RD \)-lattice is referred to as semisimple if it is generated by rank 1 sublattices. It is clear that the sum of two semisimple sublattices of a lattice is again semisimple. Thus, every finitely generated lattice has a maximal semisimple sublattice. The maximal semisimple \( RD \)-sublattice of \( M \) will be denoted \( M_{ss} \). Observe that if \( M = W \oplus V \) for sublattices \( W \) and \( V \) of \( M \), then \( M_{ss} = W_{ss} \oplus V_{ss} \).

It must be noted that we are using a definition of semisimple that is tailored to abelian groups. A definition that applies to finitely generated lattices for arbitrary finite groups (and probably even more generally) can be found in [5, Proposition 1.13].
The $RD$-module $M/M_{ss}$ is an important invariant of the lattice $M$. It is an $R$-module of finite length. We will use a special Klein four subgroup of $D$ to estimate the $R$-length of $M/M_{ss}$. If $H$ is a Klein four subgroup of $D$, we have the maximal semisimple $RH$-sublattice of the restriction $M_{H}$. This is denoted $(M_{H})_{ss}$. Because $D$ is abelian, $(M_{H})_{ss}$ is an $RD$-lattice.

The following inclusions are clear:

$M_{ss} \subseteq (M_{H})_{ss} \subseteq M$.

If the decomposition of $M_{H}$ into indecomposable $RH$-lattices is known, then the length of $M/(M_{H})_{ss}$ can be determined exactly, and the length of $(M_{H})_{ss}/M_{ss}$ can be estimated. The result is an estimate for the $R$-length of $M/M_{ss}$.

**Lemma 3.** There is a subgroup $H$ of $D$ of size four such that

$$M_{H} \cong P \oplus L_{1} \oplus L_{2}$$

where $P$ is the regular $RH$-lattice, and $L_1$ and $L_2$ are isomorphic rank 1 $RH$-lattices.

**Proof.** To begin, assume first that $H$ has been found such that $M_{H}$ has a projective summand. The rest of the decomposition of $M_{H}$ follows. If $M_{H}$ had an indecomposable summand of rank 2, then $\chi_{M}$ would vanish on some element of $H$. A similar contradiction is encountered if $M_{H}$ has two nonisomorphic nonprojective summands. It thus suffices to find a Klein four subgroup $H$ such that $M_{H}$ has a projective summand.

Let $H$ be an arbitrary Klein four subgroup of $D$. If $M_{H}$ has no projective summand, then a result of Butler (see [3, Proposition 3.1]) implies that $2M \subseteq (M_{H})_{ss}$. Then [3, Lemma 4] implies that $(M_{H})_{ss}$ has no summands of odd rank as an $RD$-lattice. It follows that $(M_{H})_{ss}$ has at most three indecomposable summands as an $RD$-lattice. Now recall that $D$ must preserve each isotypic component of the $RH$-lattice $(M_{H})_{ss}$. It follows that at most three irreducible characters of $H$ appear in $\chi_{M_{H}}$. (In fact, exactly three must appear, by an argument similar to that in Section 3.2.)

It thus suffices to find an $H$ such that all four irreducibles of $H$ appear in $\chi_{M_{H}}$. This can be done because $\chi_{M}$ is given by the formula (1). For example, $H$ can be chosen as $\text{Ker}(\lambda_{1}) \cap \text{Ker}(\lambda_{2})$. □

It follows from Lemma 3 that as $R$-modules, we have $M/(M_{H})_{ss} \cong P/P_{ss}$, where $P$ is the regular $RH$-lattice. The following isomorphism of $R$-modules is readily computed:

$$P/P_{ss} \cong R/(4) \oplus R/(2) \oplus R/(2).$$

Thus, the length of $M/(M_{H})_{ss}$ is 4. Alternatively, the length of $P/P_{ss}$ is the length of the $R$-module $R/(2^{n})$, where $2^{n}$ is the determinant of the character table of $H$.

**Lemma 4.** The $R$-module $(M_{H})_{ss}/M_{ss}$ has length at most 2.

**Proof.** As an $RH$-lattice, $(M_{H})_{ss}$ decomposes as the sum of four isotypic components. Three of these have rank 1, and the fourth has rank 3. Let $H'$ be a complement to $H$ in $D$. The action of $H'$ on $(M_{H})_{ss}$ preserves this decomposition. So as $RH'$-lattices, $(M_{H})_{ss}$ decomposes as the sum of three rank 1 lattices and a (possibly decomposable) lattice of rank 3. The result follows because $M_{ss}$ is recognized as the maximal semisimple $RH'$-sublattice of $(M_{H})_{ss}$. Only the rank 3 summand of $(M_{H})_{ss}$ can contribute to $(M_{H})_{ss}/M_{ss}$. □

Finally, we conclude that the length of the $R$-module $M/M_{ss}$ is less than or equal to 6.
3.3. Butler’s framework

We apply the construction of [2] to $M$. For this purpose, let $K$ denote the quotient field of $R$, and if $\lambda \in \text{Irr}(D)$, let $e_\lambda$ denote the corresponding idempotent of $KD$. Define

$$e_\lambda M = \sum_{\lambda \in \text{Irr}(D)} e_\lambda M.$$ 

Then $M$ is a sublattice of $e_\lambda M$ because $\sum e_\lambda = 1$.

Let $V_\lambda = e_\lambda M / e_\lambda M \cap M$. Butler’s result is that (because $M$ has no projective or rank 1 summands), $M$ can be recovered from the $RD$-module $e_\lambda M / M$, together with its submodules $V_\lambda$.

However, this is not exactly the result that we need. We must observe that

$$M_{ss} = \sum_{\lambda \in \text{Irr}(D)} e_\lambda M \cap M,$$

so that $M / M_{ss}$ can be examined as it sits inside $e_\lambda M / M_{ss}$. Then an argument of Butler’s can be adapted to our situation directly. The fact that we need follows.

Lemma 5. For each $\lambda \in \text{Irr}(D)$, we have

$$e_\lambda M \cap M = (2e_\lambda M) \cap M = (4e_\lambda M) \cap M.$$ 

In particular, if $V_\lambda \neq 0$, then $2V_\lambda \neq 0$.

Proof. First, assume that $x \in e_\lambda M \cap M$, but $x \notin (2e_\lambda M) \cap M$. Then we have a decomposition

$$e_\lambda M = Rx \oplus W$$

for some $RD$-lattice $W$. It follows that $M = Rx \oplus (W \cap M)$ is a decomposition of $RD$-lattices. This contradicts that fact that $M$ is indecomposable. (Recall that Knörr showed virtual irreducibility implies absolute indecomposability [5, Corollary 1.5].)

Now assume that $x \in (2e_\lambda M) \cap M$, but $x \notin (4e_\lambda M) \cap M$. Then we have a decomposition

$$2e_\lambda M = Rx \oplus W$$

for some $RD$-lattice $W$. Now

$$(2e_\lambda M) \cap M = Rx \oplus (W \cap M)$$

is a decomposition of $RD$-lattices. However, $2M \subseteq (2e_\lambda M) \cap M$, so this contradicts [3, Lemma 4].

Because $M$ obviously has no projective summands as an $RD$-lattice, an argument similar to [3, Proposition 3.1] (which is also due to Butler [1]) can be used to show that $8e_\lambda M \subseteq e_\lambda M \cap M$. At this point it is not clear whether equality holds. Thus, we know that if $V_\lambda \neq 0$, than as an $R$-module, $V_\lambda$ is isomorphic to either $R/4$ or $R/8$. However, in the next section we will see that our knowledge of $M/M_{ss}$ forces $V_\lambda \cong R/4$ for nonzero $V_\lambda$. 


3.4. Commutators

In this subsection we will find an element $v \in M$ such that $M = RDv$.

Observe that $M/M_{ss}$ sits naturally as a submodule of $e_s M/M_{ss} = \oplus \lambda V_\lambda$. We will examine the length of $M/M_{ss}$ by looking at elements of the form $(1 \pm g)m$ for various $m \in M/M_{ss}$.

For this purpose, it helps to make things more explicit. Thus, let us say

$$D = \langle g_1, g_2, g_3, g_4 \rangle$$

where the $g_i$ are commuting involutions. Further, we can identify the kernels of the irreducible constituents of $\chi_M$ (recall the formula (1)) as follows:

$$\text{Ker}(\lambda_i) = \langle g_j \mid j \neq i \rangle, \quad \text{for } 1 \leq i \leq 4,$$

$$\text{Ker}(\mu) = \langle g_1 g_2 g_3, g_1 g_4 \rangle.$$  

If $m \in M/M_{ss}$, then we can regard $m$ as an element of $\oplus \lambda V_\lambda$ and hence speak of projections of $m$ onto each $V_\lambda$. Let $m_\lambda$ denote such a projection. Thus

$$m = \sum_{\lambda \in \text{Irr}(D)} m_\lambda \in \bigoplus_{\lambda \in \text{Irr}(D)} V_\lambda.$$  

It is important to note that every nonzero element of $M/M_{ss}$ has at least two nonzero projections.

Now let $g \in D$. For $m \in M/M_{ss}$, observe that

$$(1 - g)m = 2 \sum_{\lambda : \lambda(g) = -1} m_\lambda$$

where only irreducible characters $\lambda$ of $D$ for which $\lambda(g) = -1$ appear in the sum. Similarly,

$$(1 + g)m = 2 \sum_{\lambda : \lambda(g) = 1} m_\lambda$$

where only irreducible characters $\lambda$ of $D$ for which $\lambda(g) = 1$ appear in the sum.

We now examine certain nonzero elements of $M/M_{ss}$. If $m \in M/M_{ss}$, then for $1 \leq i \leq 4$, the element $(1 - g_i)m$ either vanishes or has nonzero projections only in $V_{\lambda_i}$ and $V_{\mu_i}$. In fact

$$(1 - g_i)m \in (2 V_{\lambda_i} \oplus 2 V_{\mu_i}) \cap M/M_{ss}. \quad (4)$$

As $m$ varies over $M/M_{ss}$, the $R$-submodule of $M/M_{ss}$ generated by the above elements has length at least 4. Adding an element of the form

$$(1 + g_1 g_2 g_3)m \in (2 V_{1, \lambda} \oplus 2 V_{\lambda, 4}) \cap M/M_{ss} \quad (5)$$

yields an $R$-module of length at least 5.

Because this module is contained in $\oplus \lambda 2 V_\lambda = 2 e_s M/M_{ss}$, we can add an additional element, $\tilde{v} \notin 2 e_s M/M_{ss}$, to get a submodule of $M/M_{ss}$ of length at least 6. Now we already know that $M/M_{ss}$ has length at most 6, so this module must be the whole of $M/M_{ss}$, and so $M/M_{ss}$ has length exactly 6.

Choose an element $v \in M$ such that $v + M_{ss} = \bar{v} \in M/M_{ss}$. We will show that $M = RDv$. Because $M \subseteq e_s M$, we can speak of the projection $v_\lambda = e_\lambda v \in e_\lambda M$ of $v$, just as we did for elements of $M/M_{ss}$.

**Lemma 6.** For each $\lambda \in \text{Irr}(D)$, the projection $v_\lambda$ generates $e_\lambda M$. 
Proof. For each $\lambda \in \text{Irr}(D)$, there is an element of $M$ whose projection to $e_\lambda M$ generates $e_\lambda M$. If for some $\lambda$ this element cannot be chosen to be $v$, then the space $M/M \cap 2e_\lambda M$ is at least two-dimensional. However, it has been shown that $M \cap 2e_\lambda M/M_{ss}$ has a submodule of length at least 5. This makes the length of $M/M_{ss}$ too large. \qed

Lemma 6 implies that the elements $m$ appearing in (4) and (5) can be taken to be $v$. So $RDv = M/M_{ss}$.

To show that $RDv = M$ and thus conclude Theorem 3, observe that the general relations (2) and (3) can be viewed as valid for $m$ in $e_\lambda M$. Then consider the following double commutators:

$$4v_{\lambda_4} = (1 + g_1 g_2 g_3)(1 - g_4)v \in 4e_{\lambda_4}M,$$

$$4v_{\lambda_3} = (1 + g_1 g_2 g_4)(1 - g_3)v \in 4e_{\lambda_3}M,$$

$$4v_{\lambda_2} = (1 + g_1 g_3 g_4)(1 - g_2)v \in 4e_{\lambda_2}M,$$

$$4v_{\lambda_1} = (1 + g_2 g_3 g_4)(1 - g_1)v \in 4e_{\lambda_1}M.$$

Also

$$4v_{1_D} = (1 + g_1 g_2 g_3)(1 + g_2 g_3 g_4)v \in 4e_{1_D}M,$$

$$4v_{\mu} = (1 - g_1)(1 - g_2)v \in 4e_{\mu}M.$$

By Lemma 5, these elements generate $M_{ss}$. So $M_{ss} \subseteq RDv$, and $M/M_{ss} = RDv$. It follows that $M = RDv$.

References


