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Marginal and Weakly Nonlinear Stability in Spatially Developing Flows

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Abstract—This work is devoted to revealing the essence of near-critical phenomena in nonlinear problems with nonparallel effects. As a generalization of the well-known concept of linear stability in Fourier space for a parallel basic state, we introduce a new concept valid for nonparallel flows as well. The new picture allows one to demonstrate the possible singular limit to the parallel case. Also, on its basis we derive a weakly nonlinear model valid near criticality. The damped Kuramoto-Sivashinsky equation with variable coefficients is used to illustrate the application of the theory.

Keywords—Linear stability, Nonparallel flows, Spatially developing flows, Critical phenomena, Global modes.

Linear stability of systems with nonparallel, or spatially developing, basic states has been the subject of many studies. Most works are based on regular perturbation procedures: this constitutes the essence of a *local* approach; refer to [1] for a *temporal* formulation, and to [2] for a *spatial* one. Because of insufficient agreement between these theories and experiments, a *global* approach [3] has been pursued. At present, a mathematically rigorous connection between these two approaches has not been established. Weakly nonlinear studies range from near critical model problems, lacking rigorous mathematical relation to physically significant formulations [4], to those far from critical regime [5].

Let us consider a general system of evolution type, defined on $(\mathbf{x}, \mathbf{y}) \in \bar{\Omega} = \bar{\Omega}_1 \times \bar{\Omega}_2 \subseteq \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ and decomposed into a stationary spatially inhomogeneous basic state $U(\mathbf{x}, \mathbf{y}; \epsilon^*)$, whose stability we investigate, and disturbance $\mathbf{u}(t, \mathbf{x}, \mathbf{y}; \sigma)$. The parameter $\epsilon^* \in \mathbb{R}^+$ measures the degree of nonparallelism of the basic state, while $\sigma \in \Sigma \subseteq \mathbb{R}^n$ is a multidimensional real bifurcation parameter. The form of the system in wavenumber space, after Fourier transformation in the

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extended dimensions $\mathbf{x} \in \mathbb{R}^{m_1}$ is

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{M}\hat{\mathbf{u}} &= \mathcal{L}\hat{\mathbf{u}} + \mathcal{N}(\hat{\mathbf{u}}), && \text{in } \Omega \times (0,+\infty), \\ \mathcal{B}\hat{\mathbf{u}} &= 0, && \text{on } \partial\Omega_2 \times \bar{\Omega}_1 \times [0,+\infty), \\ \hat{\mathbf{u}} &= \hat{\mathbf{u}}_0(\mathbf{y};\mathbf{k}), && \text{on } \Omega \times \{t=0\}, \end{aligned} \tag{1}$$

where $\mathbf{k} \in \mathbb{R}^{m_1}$, and the linear operator is given by $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \otimes$ with the following definition for convolution (suppressing other arguments) resulting from the spatial inhomogeneity:

$$\mathcal{L}_2 \otimes \hat{\mathbf{u}} = \frac{1}{(2\pi\epsilon^*)^{m_1}} \int_{-\infty}^{+\infty} \mathcal{L}_2\left(\frac{\mathbf{k}-\mathbf{k}'}{\epsilon^*}, \mathbf{k}'\right) \hat{\mathbf{u}}(\mathbf{k}')\, d\mathbf{k}'. \tag{2}$$

Operators $\mathcal{M}$, $\mathcal{L}$, $\mathcal{B}$, and $\mathcal{N}$ are assumed not to contain time derivatives or time explicitly. The nonlinear terms have the form

$$\mathcal{N}(\hat{\mathbf{u}}) = \sum_i (\mathsf{p}_i + \check{\mathsf{p}}_i \otimes)\, \mathcal{P}_i^{(1)}(\hat{\mathbf{u}}) \otimes \mathcal{P}_i^{(2)}(\hat{\mathbf{u}}) + \cdots,$$

where $\mathcal{P}_i^{(k)}(\mathbf{u})$, etc., are linear operators.

The *global modes* approach applied to analyze the linear stability of (1) entails nothing more than the implementation of the usual spectral analysis of the associated eigenvalue problem in which all spatial directions are considered as eigendirections. As a result, the critical value of the bifurcation parameter σ_c is determined when the most unstable eigenvalue crosses the imaginary axis from left to right. However, global analysis does not reveal the intrinsic wavenumber structure of the solution in the unstable region as a local approach would. This information is essential in the rigorous investigation of the weakly nonlinear regime.

In the case $\epsilon^* = 0$, the *local* approach results in the classical picture, where for $\sigma > \sigma_c$ the solution grows for ranges of wavenumbers forming the *instability regions* in the space $(\sigma, \mathbf{k})$ with envelope(s), referred to as *marginal stability curve(s)* that correspond to the set of wavenumbers that neither grow nor decay. Thus, we have the following.

DEFINITION 1. *The marginal stability curve $\tilde{\sigma}(\mathbf{k})$ consists of the set of points in the space $(\sigma, \mathbf{k})$, such that the norm of the solution $\|\hat{\mathbf{u}}\|(t;\tilde{\sigma})$ for the linear part of (1) neither grows nor decays with time (i.e., it is either constant or oscillatory), and is defined from the linear part of (1) with $\hat{\mathbf{u}}_0 = \delta(\mathbf{k} - \tilde{\sigma}^{-1}(\mathbf{k}))$ and $\mathcal{L}_2|_{\epsilon^*=0}$.*

The solution, in view of the fact that the system is local in $\mathbf{k}$, has a point support in wavenumber space (Dirac delta-function) which corresponds to a harmonic function in physical space, thus, justifying *normal mode analysis*.

As soon as $\epsilon^* \neq 0$, Definition 1 no longer applies since one cannot expect, in general, a solution with point support in wavenumber space, or, more transparently, a solution of harmonic form in physical space, in view of the lack of translational invariance in $\mathbf{x}$. On the contrary, the assumption of existence of a solution with point support for the linearization of (1) immediately leads to a contradiction since the convolution integral (2) results in a continuous function of $\mathbf{k}$, while the other terms give rise to delta-functions. Subsequently, this necessitates a new definition of marginal stability which includes the case $\epsilon^* = 0$ as a limit.

Before elaborating on this, let us consider a simple example demonstrating all necessary important features:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + \sigma e^{2ax} u + N(u), \tag{3}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}^+$, $a \in \mathbb{C}$, $|u| < \infty$. For convenience let us assume that if a is real then $\operatorname{sign} a = \operatorname{sign} x$. The linear stationary neutral solution is written as

$$u_0 = C_1 e^{-ax} e^{i(\sigma^{1/2}/a)[e^{ax}-1]} + C_2 e^{-ax} e^{-i(\sigma^{1/2}/a)[e^{ax}-1]}. \tag{4}$$

(i) If $a \equiv 0$, then the neutral solution in physical space has the well-known form

$$u_0(x) = C_1 e^{i\sigma^{1/2}x} + C_2 e^{-i\sigma^{1/2}x}, \tag{5}$$

and, correspondingly, in Fourier space it has a singular character (Dirac delta-functions). Subsequently, the neutral curve is of the shape $\sigma = k^2$, where k is wavenumber.

(ii) If $a \equiv i\epsilon^*$, then it is easy to observe that the solution (4) tends to (5) nonuniformly in x as $\epsilon^* \to +0$. The last fact may have an effect in wavenumber space, since the Fourier transformation is nonlocal. If $\sigma^{1/2} \gg \epsilon^*$, expression (4), in view of its periodicity with period $T = 2\pi/\epsilon^*$ and slow divergence of nonparallel effects, has the asymptotic form for $\epsilon^* \ll 1$

$$u_0(k;\sigma) \sim C_1\sqrt{\frac{\pi}{\sigma^{1/2}\epsilon^*}}\, e^{-(\sqrt{\sigma}-k^2)/2\sigma^{1/2}\epsilon^*} + C_2\sqrt{\frac{\pi}{\sigma^{1/2}\epsilon^*}}\, e^{-(\sqrt{\sigma}+k)^2/2\sigma^{1/2}\epsilon^*}.$$

which corresponds to a *delta-sequence* according to Sobolev's *sequential approach* [6]. The result is shown in Figure 1.

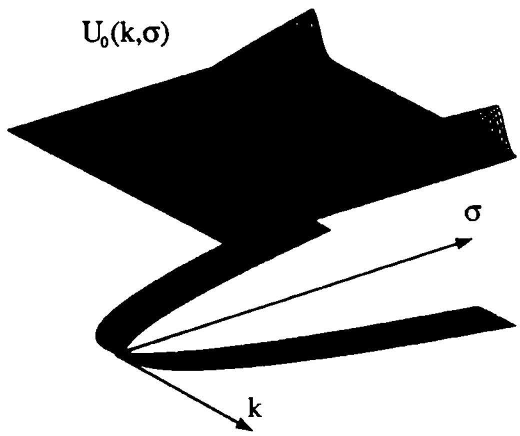

Figure 1. Neutral solution (4) for $a \equiv i\epsilon^*$, $\epsilon^* \ll 1$.

(iii) If $\operatorname{Im} a = \epsilon^* \equiv 0$ in (3), solution (4) is absolutely integrable for $a \neq 0$, but due to the exponentially divergent properties of nonparallel effects, the limit $\epsilon^* \to 0$ does not produce a Dirac delta-function sequence

$$u_0(k;\sigma) \sim C\,\frac{\sigma^{1/2}}{k}\sqrt{\frac{\pi}{\sigma^{1/2}\epsilon^*}}\, e^{i\{(\pi/4)+(k-\sigma^{1/2}-k\ln(k/\sigma^{1/2}))/\epsilon^*\}} + O(\epsilon^*).$$

The fast divergence properties are reflected in the presence of the $O(\epsilon^{*-1})$ region in which the difference in phase with the harmonic function $e^{i\sqrt{\sigma}x}$ is significant and leads to the above asymptotic behavior.

From the example, we conclude that it is possible to have singular behaviors: one solution corresponds to that of the originally parallel solution obtained by setting $\epsilon^* = 0$, and the other is obtained by taking the limit $\epsilon^* \to +0$.

Historically, it is accepted that in the case $\epsilon^* \neq 0$ there exists a marginal stability curve $\tilde{\sigma}(\mathbf{k})$ in the space $(\mathbf{k}, \sigma)$, on which disturbances neither grow nor decay. In the parallel case ($\epsilon^* \equiv 0$) the situation is very simple since disturbances of different wavenumbers are decoupled, thus, corresponding to the *local* case. In our case, the disturbance of a particular wavenumber is coupled *nonlocally* to all wavenumbers as indicated by (2). As one may conclude from the above example, the case $\epsilon^* \neq 0$ does not conform to the standard concept of a marginal stability curve: the support of the neutral solution $\mathbf{u}_0(\mathbf{k}, \sigma)$ (more precisely, the support contributing to the norm) is a region, as shown in Figure 1, which we call the *marginal stability region*.

DEFINITION 2. *Assume that there exists an initial condition* $\hat{\mathrm{u}}_0(\mathrm{y};\mathrm{k})$ *such that the linearized problem (1) has a nontrivial solution* $\hat{\mathrm{u}}(t,\mathrm{y};\mathrm{k},\sigma)$, *the norm of which* $\|\hat{\mathrm{u}}\|(t;\sigma)$ *neither grows nor decays with time. Then the marginal stability region* $\mathcal{R}_\epsilon$ *defined with accuracy* ϵ *consists of a set of points in the space* (σ,k) *formed by the sequence of filter functions* Φ_ϵ *with finite support* $\mathcal{R}_\epsilon = \mathrm{Supp}\,\Phi_\epsilon \subseteq (\sigma,\mathrm{k})$, *such that* $\lim_{\epsilon\to 0}\|\hat{\mathrm{u}} - \Phi_\epsilon\hat{\mathrm{u}}\| = 0$.

Obviously, this definition, as illustrated in Figure 2, includes as a subset Definition 1 of the marginal stability curve.

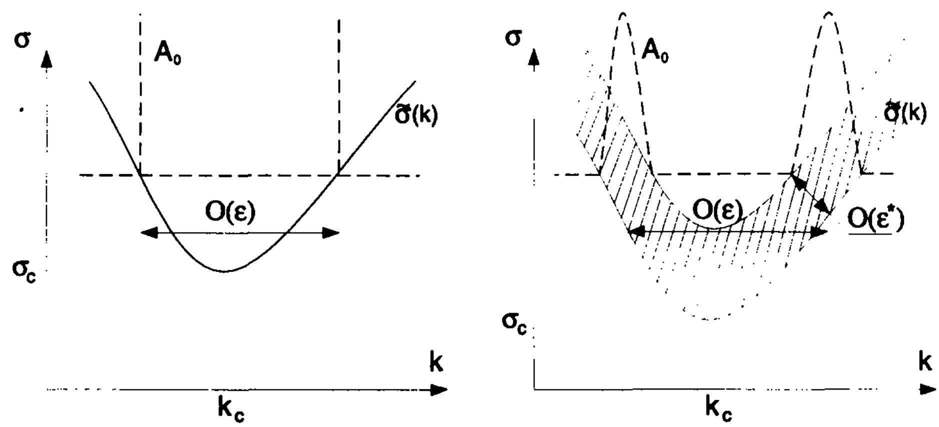

Figure 2. Topology of near-critical behavior.

The weakly nonlinear behavior of the solution is governed by the order parameter equation

$$\frac{dA}{dt} = \mathcal{L}A + \mathcal{N}(A), \tag{6}$$

where $\mathcal{L} = \omega + \alpha\otimes$, $\alpha = \alpha(k,(k-k')/\epsilon^*,k',\sigma)$, and the nonlinear term $\mathcal{N}(A)$ is nonlocal. This equation corresponds to a truncation of a general pseudodifferential equation which is rigorously derived in [7] through reduction of the phase-space domain of (1) using attracting manifold theory.

The solution of the linear part of (6) in general either decays, grows, or remains neutral. In this work, we investigate the structure of the solution only for the systems (6) having critical phenomena, and furthermore, we restrict ourselves to weakly nonparallel flows with $\epsilon^* \ll 1$. Obviously, all real flows which are considered for simplicity as parallel are in reality almost parallel. This means that a proper mathematical formulation should provide a smooth transition of the neutral solution to the parallel case under the limit $\epsilon^* \to 0$ if the stability analysis in the parallel approximation agrees well with a practically-parallel flow experiment. As a result, one can expect for $\epsilon^* \ll 1$ the analogous mode-clustered form of the solution as in the parallel case [8]. The existence of a critical point entails the following consequences for weakly nonlinear analysis:

- the support of $A(t;k,\sigma)$ is narrow, of the order of ϵ, around k_c (wavepacket),
- the amplitude $A(t;k,\sigma)$ is small in physical space,
- the evolution of $A(t;k,\sigma)$ results from a balance of both linear and nonlinear effects.

In the parallel case, for a nonzero critical wavenumber there exists a mode-clustered form of the solution whose leading amplitudes enable rigorous coarsening, i.e., construction of a simplified model with the same dynamics as one of the original evolution system. In the nonparallel case, in view of the nonlocal linear part in Fourier space, the exchange of information among different wavenumbers occurs not only due to nonlinearity, but also owing to the convolving linear part. One may conclude from Figure 2 that the most interesting situation corresponds to $\epsilon^*/\epsilon \sim 1$, since $\epsilon^*/\epsilon \ll 1$ is akin to the usual parallel case. If $A(t;k,\sigma) = O(1)$, then it follows that $\mathcal{L}(k,\sigma)A(t;k,\sigma) = O(\epsilon^2)$. Here we consider only the case of a stationary neutral solution which is defined in Fourier space by $\mathcal{L}A_0(t;k,\sigma) = 0$. Clearly, assuming existence of a critical point (k_c, σ_c) with distinct k_c: $A_0(t;k,\sigma_c) \sim \delta(k - k_c)$, we obtain the following relations:

$$\omega(k_c,\sigma_c) + \frac{1}{2\pi\epsilon^*}\int_{-\infty}^{+\infty} \alpha\left(\kappa + k_c, \frac{\kappa}{\epsilon^*}, k_c, \sigma_c\right) e^{i\kappa x}\, d\kappa = 0,$$

$$\frac{\partial\omega}{\partial k}(k_c,\sigma_c) + \frac{1}{2\pi\epsilon^*}\int_{-\infty}^{+\infty} \frac{\partial\alpha}{\partial k'}\left(\kappa + k_c, \frac{\kappa}{\epsilon^*}, k_c, \sigma_c\right) e^{i\kappa x}\, d\kappa = \epsilon i\mathcal{F},$$

where $\mathcal{F} = -\delta\sigma\mathcal{L}_\sigma\,(\mathcal{A}_0/\mathcal{A}_{0\xi}) + (1/2)\,\mathcal{L}_{kk}\,(\mathcal{A}_{0\xi\xi}/\mathcal{A}_{0\xi})$, with definitions

$$\mathcal{L}_\sigma = \frac{\partial\omega}{\partial\sigma}(k_c,\sigma_c) + \frac{1}{2\pi\epsilon^*}\int_{-\infty}^{+\infty} \frac{\partial\alpha}{\partial\sigma}\left(\kappa + k_c, \frac{\kappa}{\epsilon^*}, k_c, \sigma_c\right) e^{i\kappa x}\, d\kappa,$$

$$\mathcal{L}_{kk} = \frac{\partial^2\omega}{\partial k^2}(k_c,\sigma_c) + \frac{1}{2\pi\epsilon^*}\int_{-\infty}^{+\infty} \frac{\partial^2\alpha}{\partial k'^2}\left(\kappa + k_c, \frac{\kappa}{\epsilon^*}, k_c, \sigma_c\right) e^{i\kappa x}\, d\kappa.$$

$\mathcal{A}_0$ denotes the inverse Fourier transform of A_0, which along with its derivatives, is evaluated at $\sigma = \sigma_c + \epsilon^2\,\delta\sigma$. Note that the important difference between parallel and nonparallel conditions on neutrality consists in the independence of wavenumber k and bifurcation parameter σ in the latter case versus the functional dependence $\tilde{\sigma}(k)$, which defines the marginal stability curve, in the first instance. Therefore, the usual relations at the critical point in the parallel case may be recovered from the above two relations at the critical point only by simultaneously taking the limits $\epsilon^* \to 0$ and $\sigma \to \tilde{\sigma}(k)$.

Further analysis is in the spirit of [9], where the fact that the solution near criticality is of mode-clustered form has been used. In the case of quadratic nonlinearity, $\mathcal{N} \equiv \beta A \otimes A$, from (6) one obtains a system for the mode clusters analogous to that given in [9], which after applying the inverse Fourier transform, takes the form of an *envelope* equation for amplitude $\mathcal{A}(t,x,\xi,\sigma)$:

$$\frac{\partial\mathcal{A}}{\partial\tau} + a_0(x,\xi)\mathcal{A} + ia_1(x,\xi,\mathcal{A}_0)\,\frac{\partial\mathcal{A}}{\partial\xi} + a_2(x,\xi)\,\frac{\partial^2\mathcal{A}}{\partial\xi^2} = \gamma(x,\xi)\mathcal{A}|\mathcal{A}|^2, \tag{7}$$

where $\xi = \epsilon x$, $\tau = \epsilon^2 t$, $\delta\sigma = (\sigma - \sigma_c)/\epsilon^2$ and the coefficients are defined by $a_0 = -\delta\sigma\mathcal{L}_\sigma$, $a_1 = \mathcal{F}$, $a_2 = (1/2)\,\mathcal{L}_{kk}$,

$$\gamma = -\frac{\bar{\beta}(k_c,0,k_c)\,\bar{\beta}(0,-k_c,k_c)}{\omega(0,\sigma_c) + (1/2\pi\epsilon^*)\int_{-\infty}^{+\infty}\alpha\,(\kappa,\kappa/\epsilon^*,0,\sigma_c)\,e^{i\kappa x}\,d\kappa} - \frac{\bar{\beta}(k_c,2k_c,-k_c)\beta(2k_c,k_c,k_c)}{\omega(2k_c,\sigma_c) + (1/2\pi\epsilon^*)\int_{-\infty}^{+\infty}\alpha(\kappa+2k_c,\kappa/\epsilon^*,2k_c,\sigma_c)e^{i\kappa x}\,d\kappa},$$

with $\bar{\beta}(k,k-k',k') = \beta(k,k-k',k') + \beta(k,k',k-k')$. The Landau constant γ is written in an *adiabatic* (or quasisteady) approximation analogous to Haken's definition [10]. The notable features of the above coarse model, which differentiates it from the conventional complex Ginzburg-Landau equation (GLE), consist in the spatial inhomogeneity with *fast* and *slow* scales and coupling to the neutral solution for the same value of the bifurcation parameter. Hereafter, we will call (7) simply as the *variable-coefficient complex Ginzburg-Landau equation* (VCGLE). The advantage of the VCGLE is its universality regardless of the complexity of the original evolution system. The situation resembles the case of the constant coefficient GLE, but the dynamics of the VCGLE is much richer. In general, $A(t;k,\sigma)$ may not have narrow support and/or $\alpha(\kappa + k, \kappa/\epsilon^*, k, \sigma)$ may be nonexpandable in the vicinity of the critical wavenumber k_c, thus, leading to a *nonlocal* "low-dimensional" model even if one starts from the commonly accepted *principle of locality* in the original variables.

To illustrate the basic idea of the derivation of the amplitude equation in the case where effects of nonparallelism and nonlinearity are of equal order, i.e., $\epsilon^* = \epsilon$, we use the damped Kuramoto-Sivashinsky equation with variable coefficients in analogy with the derivation of the Ginzburg-Landau equation in the constant coefficient case [11]. The problem can be cast in the form

$$\begin{gathered} \partial_t u = \left[\sigma - \nu(\epsilon x)\left(\partial_x^2 + 1\right)^2\right] u - u\partial_x u, \\ u, \partial_x u \longrightarrow 0, \qquad \text{as } |x| \longrightarrow \infty, \\ u(0,x) = u_0(x), \end{gathered} \tag{8}$$

where σ is the bifurcation parameter related to a driving force. In the parallel limit $\epsilon \to 0$, bifurcation occurs at $\sigma_c = 0$, and the most unstable wavenumbers are $k_c = \pm 1$.

Application of our theory starts with the reformulation of (8) in Fourier space (the case $\nu(\epsilon x) = e^{-\epsilon^2 x^2}$ is considered) to obtain (6) with

$$\omega(k,\sigma) = \sigma, \qquad \alpha\left(\frac{k-k'}{\epsilon}, k'\right) = -\sqrt{\pi}\left(k'^2 - 1\right)^2 e^{-(k-k')^2/4\epsilon^2}.$$

Subsequently, one immediately obtains the VCGLE (7) with the following coefficients:

$$a_0 = -\delta\sigma, \qquad a_1 = \delta\sigma\,\frac{\mathcal{A}_0}{\mathcal{A}_{0\xi}} + 4e^{-\xi^2}\,\frac{\mathcal{A}_{0\xi\xi}}{\mathcal{A}_{0\xi}}, \qquad a_2 = 4e^{-\xi^2}, \qquad \gamma = -\frac{1}{9}\,e^{\xi^2}.$$

For intermediate times, $0 \ll t \ll \epsilon^{-2}$, (7) simplifies to

$$\begin{gathered} \frac{\partial \mathcal{A}}{\partial \tau} - \delta\sigma\mathcal{A} + \frac{2i}{\xi}\left[\delta\sigma + 2\left(\frac{\xi^2}{2} - 1\right)e^{-\xi^2}\right]\frac{\partial \mathcal{A}}{\partial \xi} - 4e^{-\xi^2}\frac{\partial^2 \mathcal{A}}{\partial \xi^2} = -\frac{1}{9}e^{\xi^2}\mathcal{A}|\mathcal{A}|^2, \\ \mathcal{A}(0,\xi) = \mathcal{A}_0(\xi) \quad \text{and} \quad \mathcal{A} \longrightarrow 0, \quad \text{as } |\xi| \longrightarrow \infty. \end{gathered} \tag{9}$$

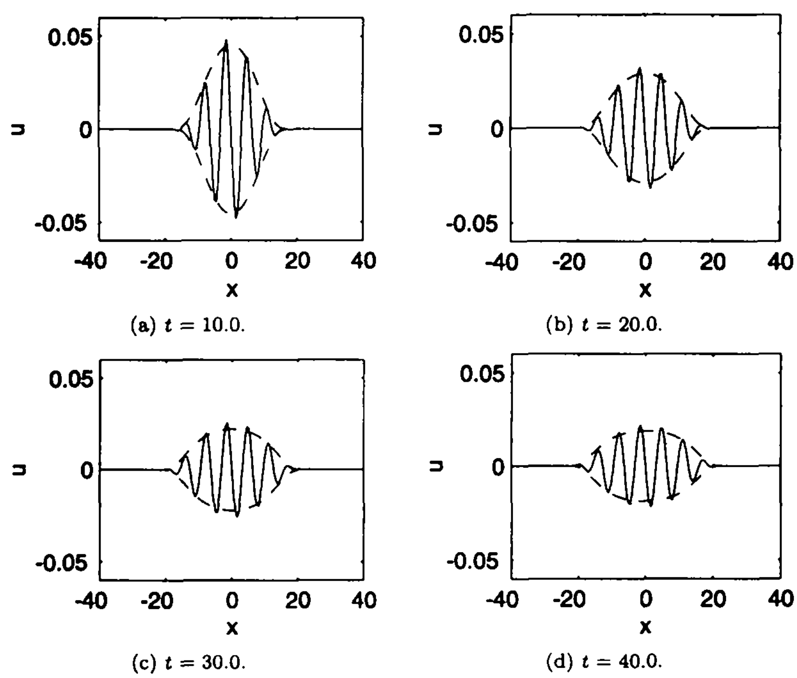

Figure 3. Comparison of the solution of (8) with the wavepacket envelope solution of (9).

The relation between solutions of (8) and (7) is established by $u(t,x) = 2\epsilon[\mathcal{A}^R(\tau,\xi)\cos x - \mathcal{A}^I(\tau,\xi)\sin x]$. To demonstrate the accuracy of the derived model, (8) and (9) have been solved numerically with identical initial conditions and compared at different times in Figure 3. Numerical simulations demonstrate that the relative error between them decreases monotonically after $t \simeq 10$, confirming that (7) corresponds to the global attractor that captures the long-time behavior of (8).

Several comments need to be made on the stability of nonparallel flows. As follows from linear analysis, if initial conditions lie on the marginal solution $\hat{\mathrm{u}}_0(\mathbf{y};\mathbf{k},\sigma)$, then the resulting solution again neither grows nor decays with time globally. As opposed to the parallel case, here the amplitude distribution of initial conditions plays a crucial role. Furthermore, an arbitrary small change in the form of the initial conditions, even having support within the marginal stability region, will lead to linear (in)stability—this property can be interpreted as the lack of attractiveness in the linear formulation. But, for the case $\epsilon^* \ll 1$ considered here, one should note the unimportance (in an asymptotic sense) of the neutral wavepacket form and the fact that the complete nonlinear problem possesses the property of attractiveness, in particular allowing for the existence of modulation equations near criticality and usually resulting in loss of memory of the details of initial conditions [12]. Thus, one may conclude that initial conditions of arbitrary form with support in the marginal stability region lead to neutral solutions separating stable and unstable regions in wavenumber space. As a result, the stability of weakly nonparallel flows can be understood only in conjunction with nonlinear effects, which attract all solutions with initial conditions having support in the marginal stability region.

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