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# Zeros of linear combinations of Laguerre polynomials from different sequences

Kathy Driver[a,*], Kerstin Jordaan[b]

[a] Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag X3, Rondebosch 7701, Cape Town, South Africa

[b] Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria, 0002, South Africa

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ABSTRACT

We study interlacing properties of the zeros of two types of linear combinations of Laguerre polynomials with different parameters, namely $R_n = L_n^{\alpha} + aL_n^{\alpha'}$ and $S_n = L_n^{\alpha} + bL_{n-1}^{\alpha'}$. Proofs and numerical counterexamples are given in situations where the zeros of $R_n$, and $S_n$, respectively, interlace (or do not in general) with the zeros of $L_k^{\alpha}, L_k^{\alpha'}$, $k = n$ or $n-1$. The results we prove hold for continuous, as well as integral, shifts of the parameter $\alpha$.



## 1. Introduction

Let $\mu$ be a positive Borel measure supported on a finite or infinite interval $[a, b]$ and let $\{p_n\}_{n=0}^{\infty}$ be the sequence of polynomials, uniquely determined up to normalization, orthogonal with respect to $\mu$. Then $\int_a^b x^k p_n(x) \mathrm{d}\mu(x) = 0$ for $k = 0, 1, \ldots, n-1$ and it is well known that the zeros of $p_n$ are real and simple and lie in $(a, b)$. Moreover, if $a < x_1 < x_2 < \cdots < x_n < b$ and $a < y_1 < y_2 < \cdots < y_{n-1} < b$ are the zeros of $p_n$ and $p_{n-1}$ respectively, then

$$a < x_1 < y_1 < x_2 < y_2 < \cdots < x_{n-1} < y_{n-1} < x_n < b,$$

a property usually called the interlacing of the zeros of $p_n$ and $p_{n-1}$.

The interlacing of zeros of polynomials is particularly important in numerical quadrature (cf. [1]) and also arises, inter alia, in the context of extremal Zolotarev–Markov problems (cf. [2]), the Korous–Peebles problem (cf. [3]) and the Gelfond interpolation problem (cf. [4]).

In [5], Alfaro, Marcellán, Peña and Rezola derived necessary and sufficient conditions for the orthogonality of $\{Q_n\}_{n=0}^{\infty}$, where $Q_n(x) = p_n(x) + a_1 p_{n-1}(x) + \cdots + a_k p_{n-k}(x)$, $a_k \neq 0$, $n \geq k$ and $\{p_n\}_{n=0}^{\infty}$ is a sequence of monic orthogonal polynomials. Their work extends the results of Peherstorfer (cf. [6]) who established sufficient conditions, when $\mathrm{supp}(\mu) = (-1, 1)$, on the real numbers $\{a_j\}_{j=1}^{k}$ such that $p_n + a_1 p_{n-1} + \cdots + a_k p_{n-k}$ has $n$ simple zeros in $(-1, 1)$. Marcellán raised a more general question at OPSFA 2007: Given two different orthogonal sequences $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$, under what conditions does a linear combination $r_n = p_n + aq_n$, $a \neq 0$, form an orthogonal sequence $\{r_n\}_{n=0}^{\infty}$? A related question, relevant for applications, is whether and when the zeros of $r_n$ interlace with the zeros of $p_n$, $p_{n-1}$, $q_n$ or $q_{n-1}$. One starting point for answering these general questions is to consider linear combinations of classical orthogonal polynomials from the same family but from different sequences.

* Corresponding author.
*E-mail addresses:* Kathy.Driver@uct.ac.za (K. Driver), kjordaan@up.ac.za (K. Jordaan).

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In this paper, we consider linear combinations of Laguerre polynomials $L_n^\alpha$ of the form $R_n^{\alpha,t} = L_n^\alpha + a\,L_n^{\alpha+t}$ and $S_n^{\alpha,t} = L_n^\alpha + b\,L_{n-1}^{\alpha+t}$ where $\alpha > -1$, $t > 0$ and $a,\ b \neq 0$. We recall that the Laguerre polynomials (cf. [7]) are orthogonal with respect to the weight function $e^{-x}x^\alpha,\ \alpha > -1$, on the interval $(0, \infty)$.

For $0 < t \leq 2$, we give proofs (or counterexamples) for the interlacing of the zeros of $R_n^{\alpha,t}$ and $S_n^{\alpha,t}$ with the zeros of $L_n^\alpha,\ L_n^{\alpha+t},\ L_{n-1}^\alpha$ and $L_{n-1}^{\alpha+t}$.

We will make use of two well known identities (cf. [8], 22.7.30 and 22.7.29)

$$L_n^\alpha = L_n^{\alpha+1} - L_{n-1}^{\alpha+1} \tag{1}$$

$$\text{and}\quad xL_n^{\alpha+1}(x) = (x-n)L_n^\alpha(x) + (\alpha+n)L_{n-1}^\alpha(x). \tag{2}$$

## 2. Linear combinations of Laguerre polynomials of the same degree

Let

$$R_n^{\alpha,t} = L_n^\alpha + aL_n^{\alpha+t}, \quad a \neq 0,\ \alpha > -1. \tag{3}$$

**Theorem 2.1.** *For $0 < t \leq 2$, the zeros of $R_n^{\alpha,t}$ interlace with the zeros of* (i) $L_n^\alpha$, (ii) $L_n^{\alpha+t}$.

**Proof.** We know from [9, Theorem 2.3] that the zeros of $L_n^\alpha$ interlace with the zeros of $L_n^{\alpha+t}$ for $0 < t \leq 2$ which implies that $L_n^\alpha$ has a different sign at successive zeros of $L_n^{\alpha+t}$ and vice versa. Evaluating (3) at succesive zeros $x_i$ and $x_{i+1}$ of $L_n^\alpha$ we obtain

$$\begin{aligned} R_n^{\alpha,t}(x_i)R_n^{\alpha,t}(x_{i+1}) &= a^2L_n^{\alpha+t}(x_i)L_n^{\alpha+t}(x_{i+1}), \quad 1 = 1, 2, \ldots .n-1 \\ &< 0 \quad \text{for all } a \neq 0. \end{aligned}$$

Therefore $R_n^{\alpha,t}$ has a different sign at successive zeros of $L_n^\alpha$ and so the zeros interlace. The same argument shows that the zeros of $R_n^{\alpha,t}$ interlace with those of $L_n^{\alpha+t}$ by evaluating (3) at successive zeros of $L_n^{\alpha+t}$. □

**Remark:** For the integer values $t = 1$ and $t = 2$, $R_n^{\alpha,1}$ and $R_n^{\alpha,2}$ are in fact each a linear combination of orthogonal polynomials from the same sequence. Indeed, using the identity (1), we see that

$$R_n^{\alpha,1} = (a+1)L_n^{\alpha+1} - L_{n-1}^{\alpha+1}, \tag{4}$$

and the restrictions on $a$ to ensure that $\{R_n^{\alpha,1}\}_{n=0}^\infty$ has all its zeros in $(0, \infty)$ can be deduced from ([10], Theorem 3(v)). Similarly, applying (1) iteratively, we obtain

$$R_n^{\alpha,2} = (a+1)L_n^{\alpha+2} - 2L_{n-1}^{\alpha+2} + L_{n-2}^{\alpha+2} \tag{5}$$

and the zeros of this type of linear combination are discussed in ([10], Theorem 5, and [1]).

Evaluating (4) at successive zeros of $L_{n-1}^{\alpha+1}$, one can also prove that the zeros of $R_n^{\alpha,1}$ interlace with the zeros of $L_{n-1}^{\alpha+1}$. However, the zeros of $R_n^{\alpha,t}$ do not interlace with the zeros of $L_{n-1}^\alpha$ even in the simple special case when $t = 1$, as illustrated by the following example: For $n = 5$, $a = 2.33$, $\alpha = 1.45$ and $t = 1$, the zeros of $L_4^{1.45}$ are

$$\{0.954365,\ 2.94834,\ 6.26071,\ 11.6366\}$$

while those of $R_5^{1.45,1}$ are

$$\{1.17057,\ 3.01797,\ 5.80288,\ 9.83574,\ 15.9213\}.$$

## 3. Linear combinations of Laguerre polynomials of different degree

Next we consider linear combinations of the type

$$S_n^{\alpha,t} = L_n^\alpha + b\,L_{n-1}^{\alpha+t}, \quad b \neq 0,\ \alpha > -1. \tag{6}$$

We will need information on the interlacing properties of the two polynomials $L_n^\alpha$ and $L_{n-1}^{\alpha+t}$ in the linear combination.

**Theorem 3.1.** *Let $\alpha > -1$ and let*

$0 < x_1 < x_2 < \cdots < x_n$ *be the zeros of* $L_n^\alpha$,

$0 < y_1 < y_2 < \cdots < y_{n-1}$ *be the zeros of* $L_{n-1}^\alpha$,

$0 < t_1 < t_2 < \cdots < t_{n-1}$ *be the zeros of* $L_{n-1}^{\alpha+t}$ *and*

$0 < X_1 < X_2 < \cdots < X_{n-1}$ *be the zeros of* $L_{n-1}^{\alpha+2}$

*where $0 < t < 2$. Then*

$$0 < x_1 < y_1 < t_1 < X_1 < x_2 < \cdots < x_{n-1} < y_{n-1} < t_{n-1} < X_{n-1} < x_n.$$

**Proof.** A simple computation using (1) and (2) leads to

$$(\alpha+1)L_n^{\alpha+1}(x)=(\alpha+n+1)L_n^{\alpha}(x)+xL_{n-1}^{\alpha+2}(x). \tag{7}$$

Evaluating (7) at successive zeros $x_k$ and $x_{k+1}$ of $L_n^{\alpha}(x)$, we obtain

$$x_kx_{k+1}L_{n-1}^{\alpha+2}(x_k)L_{n-1}^{\alpha+2}(x_{k+1})=(\alpha+1)^2L_n^{\alpha+1}(x_k)L_n^{\alpha+1}(x_{k+1}).$$

The expression on the right is negative since the zeros of $L_n^{\alpha}$ and $L_n^{\alpha+1}$ interlace (cf. [9, Theorem 2.3]) and therefore

$$x_k<X_k<x_{k+1}\quad \text{for each fixed } k,\ k=1,\dots,n-1.$$

The zeros of $L_{n-1}^{\alpha}$ increase as $\alpha$ increases (cf. [7], p. 122), hence

$$y_k<t_k<X_k\quad \text{for each fixed } k,\ k=1,\dots,n.$$

Finally, since the zeros of $L_n^{\alpha}$ and $L_{n-1}^{\alpha}$ separate each other, we know that

$$x_k<y_k<x_{k+1}\quad \text{for each fixed } k,\ k=1,\dots,n-1$$

and this completes the proof. □

Note that this result extends Theorem 2.4 in [9] to the case of polynomials of different degree with continuously varying parameters.

**Theorem 3.2.** *For $0<t\le 2$, the zeros of $S_n^{\alpha,t}$ interlace with the zeros of* (i) $L_n^{\alpha}$, (ii) $L_{n-1}^{\alpha+t}$.

**Proof.** We know from Theorem 3.1 that the zeros of $L_n^{\alpha}$ interlace with the zeros of $L_{n-1}^{\alpha+t}$ for $0<t\le 2$ which implies that $L_{n-1}^{\alpha+t}$ has a different sign at successive zeros of $L_n^{\alpha}$ and vice versa. Evaluating (6) at successive zeros $x_i$ and $x_{i+1}$ of $L_n^{\alpha}$ we obtain

$$\begin{aligned} S_n^{\alpha,t}(x_i)S_n^{\alpha,t}(x_{i+1}) &= b^2L_{n-1}^{\alpha+t}(x_i)L_{n-1}^{\alpha+t}(x_{i+1}),\quad i=1,2,\dots .n-1\\ &<0\quad \text{for all } b\neq 0.\end{aligned}$$

Therefore $S_n^{\alpha,t}$ has a different sign at successive zeros of $L_n^{\alpha}$ and so the zeros interlace. The same argument shows that the zeros of $S_n^{\alpha,t}$ interlace with those of $L_{n-1}^{\alpha+t}$ by evaluating (6) at successive zeros of $L_{n-1}^{\alpha+t}$. □

It is interesting to note that in the case of linear combinations of Laguerre polynomials of different degree, the zeros of $S_n^{\alpha,t}$ do not interlace with the zeros of $L_{n-1}^{\alpha}$. Indeed, even in the simplest case when $t=1$ and $n=5$, $b=2.33$, $\alpha=1.45$ in (6), the zeros of $S_5^{1.45,1}$ are

$$\{1.34638,\ 3.48132,\ 6.74108,\ 11.6384,\ 20.6928\}$$

while those of $L_4^{1.45}$ are

$$\{0.954365,\ 2.94834,\ 6.26071,\ 11.6366\},$$

and interlacing does not occur. The zeros of $S_n^{\alpha,t}$ and $L_n^{\alpha+t}$ are interlacing when $t=1$ since $S_n^{\alpha,1}=L_n^{\alpha+1}+(b-1)L_{n-1}^{\alpha+1}$. However, when $t=2$, the zeros of $S_5^{1.45,2}$ are

$$\{1.94417,\ 4.47751,\ 8.08954,\ 12.6085,\ 16.7802\}$$

while those of $L_5^{1.45+2}$ are

$$\{1.70945,\ 3.92167,\ 7.07942,\ 11.5061,\ 18.0334\},$$

and interlacing fails in this case.

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