Perturbation Theory of Dense Point Spectra

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The property of a self-adjoint operator having pure point spectrum is stable under certain random compact perturbations. In particular, a compact perturbation of a self-adjoint operator whose spectrum has Lebesgue measure zero has pure point spectrum, almost surely.

1. INTRODUCTION

This paper is devoted to a problem of the perturbation theory of the singular spectrum: Given a self-adjoint operator H with pure point spectrum, and a compact perturbation V, when does $H + V$ also have pure point spectrum?


Worse yet, the classic paper [3] of Donoghue (cf. [18, Sect. 3] and [19, Sect. 3]) gives examples of rank one perturbations

$$H(\kappa) = H + \kappa \langle \cdot, \varphi \rangle \varphi$$

in which (a) $H$ is pure point, but $H(\kappa)$ is purely singular continuous for $\kappa \neq 0$, and (b) $H$ is purely singular continuous; but $H(\kappa)$ is pure point for $\kappa \neq 0$.

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PERTURBATION OF DENSE POINT SPECTRA

The way out is to consider, not perturbations \( V \) which are small in some abstract sense, but those which are small relative to \( H \) in the sense that they do not move the eigenvectors of \( H \) very much. Roughly, if \( e_n \) is a complete orthonormal set of eigenvectors of the pure point operator \( H \), we shall require that

\[
\sum_n |V^{1/2}e_n| < \infty.
\]

This condition is quite strong by the standards of scattering theory, reflecting the fact that absolutely continuous spectrum is much more stable than dense pure point.

We shall prove that the "generic" perturbation of this type preserves pure point spectra. The term "generic" will be given a precise meaning in terms of probability, not in terms of category, so that we shall be concerned with a class of random compact perturbations of pure point operators. We shall also consider the case in which \( V \) is fixed, and the eigenvalues of \( H \) are random.

The genesis of our methods is a recent paper of Simon and Wolff [7, 18], who, following work of Kotani [11] on random boundary conditions, achieve a significant extension of Donoghue's theory of perturbations of rank one [3]. Their chief concern is to couple this theory with \textit{a priori} estimates concerning the Anderson model and like systems to obtain new and simpler proofs of the existence of pure point spectra. The main such estimate, derived by them from estimates in the paper of Fröhlich and Spencer [5] on Anderson localization, is what has been abstracted below as \( H\)-finiteness. The present paper is an attempt to pursue the functional analytic ideas of the rank one theory to their natural generality.

Let us summarize briefly the results of the paper. Section 2 is an abstract version of the main result of [18]. There are two essential ideas: (1) the notion of \( H \)-finiteness abstracted from [5], and (2) recognition of the importance of the spectral absolute continuity of the multiplication operator

\[
\mathcal{U}u(\kappa) = (H + \kappa V)u(\kappa)
\]

on \( L_2(\mathbb{R}; \mathcal{H}) \). The explicit recognition of (2) led to the replacement of an argument of [18] by one involving the Putnam–Kato positive commutator theorem, which is more easily extended to very general situations.

Section 3 analyzes a real variable problem closely connected with \( H \)-finiteness for pure point operators \( H \), and obtains a rather complete answer. Sections 4 and 5 contain the main theorems on random compact perturbations of pure point operators.
A rather remarkable result is proved in Section 6. If $H$ is self-adjoint, and the spectrum of $H$ has Lebesgue measure zero, then the "generic" compact perturbation of $H$ is pure point.

In Section 7, we turn to the case considered in the interesting recent paper of Thomas and Wayne [19] of a fixed compact perturbation of a pure point operator $H(o)$ with random eigenvalues. A serious technical problem is overcome by a probabilistic "differentiation" method based on a theorem of Kakutani.

The last two sections discuss details. The author wishes to thank Ira Herbst for pointing out that the argument in an earlier version of Section 9 actually proved the more satisfactory result now stated there.

5. Functional Analysis

Throughout this paper $\mathcal{H}$ and $\mathcal{K}$ will denote Hilbert spaces, which will always be separable. The inner product of $x$ and $y$ is $\langle x, y \rangle$, and the norm of the vector $x$ is $|x|$. If $A$ is an operator, $\|A\|$ denotes its operator norm, $D(A)$ its domain, $R(A)$ its range, $\overline{R}(A)$ the closure of $R(A)$, and ker $A$ the kernel of $A$. If $A_n$ is a sequence of operators, $s$-lim $A_n$ denote the strong limit. Let $H = \int \lambda E(d\lambda)$ be a (possibly unbounded) self-adjoint operator on $\mathcal{H}$, and define $E[S] = \int_S E(d\lambda)$ for any Borel set $S$. By $H^p$, $H^s$, $H^sc$, and $H^ac$, we denote respectively the point, singular, singular continuous, and absolutely continuous parts of $H$, and by $E^p$, $E^s$, $E^{sc}$, and $E^{ac}$ the corresponding spectral measures. If $x \in \mathcal{H}$, $m_x$ is the measure

$$m_x(S) = \langle E(S)x, x \rangle$$

and $m^p_x$, $m^{sc}_x$, and $m^{ac}_x$ are similarly defined. The spectrum of $H$ is $\sigma(H)$, the essential spectrum $\sigma_{ess}(H)$. By the point spectrum $\sigma_p(H)$, we shall mean the set of all eigenvalues of $H$, which may not be a closed set. We say that $H$ is pure point (in an interval $J$) if and only if $\mathcal{H}$ (or $R(E[J])$) has a basis of eigenvalues of $H$.

Let $\mathcal{F}$ be a subset of $\mathcal{H}$, and $\mathcal{M}(H, \mathcal{F})$ the smallest closed reducing subspace of $\mathcal{H}$ containing $\mathcal{F}$. If $\mathcal{M}(H, \mathcal{F}) = \mathcal{H}$, we say that $\mathcal{F}$ is cyclic for $H$. This amounts to saying that the span of vectors of the form $f(H)s$, with $f$ bounded and $s \in \mathcal{F}$ is dense in $\mathcal{H}$. If $V$ is bounded and $H = H_0 + V$, and then $\mathcal{M}(H, R(V)) = \mathcal{M}(H_0, R(V))$, so that $R(V)$ is cyclic for $H$ if and only if it is cyclic for $H_0$. In any case, the orthogonal complement of $\mathcal{M}(H, R(V))$ reduces both $H$ and $H_0$, and $H = H_0$ on this space, so it is of little interest in perturbation theory.

If $\mathcal{H} = l_2$ (or more generally, if $\mathcal{H}$ is a direct sum), we shall write

$$\text{diag}\{a_n\}$$
for the operator (or operator matrix) with \( a_n \) on the diagonal. We write \( \{a_{ij}\} \) for a matrix acting on \( l_2 \).

Finally, we write \( \chi_S(x) \) for the characteristic function of the set \( S \), and \( \text{meas}(S) \) for the Lebesgue measure of \( S \). An "i.i.d." sequence of random variables is independent and identically distributed.

Let \( H \) be self-adjoint on \( \mathcal{H} \). Let \( \mathcal{H} \) be an auxiliary Hilbert space, and \( A \) a closed operator from \( \mathcal{H} \) into \( \mathcal{H} \) with dense domain \( D(A) \). For example, \( A \) can simply be a bounded operator on \( \mathcal{H} \).

Define, for \( \lambda \) real and \( \varepsilon > 0 \),

\[
F_\varepsilon(\lambda; H, A) = A[(H - \lambda)^2 + \varepsilon^2]^{-1} A^*.
\]

Let \( J \) be an open interval.

2.1. DEFINITION. The operator \( A \) is \( H \)-finite on \( J \) if and only if \( F_\varepsilon(\lambda; H, A) \) has a bounded extension to \( \mathcal{H} \), and

\[
F(\lambda; H, A) = \text{s-lim } F_\varepsilon(\lambda; H, A) \quad (2.1)
\]

exists for a.e. \( \lambda \) in \( J \).

For each fixed \( \lambda \), \( F_\varepsilon(\lambda; H, A) \) is a family of bounded nonnegative operators on \( \mathcal{H} \), which is increasing as \( \varepsilon \downarrow 0 \). Since for increasing sequences, weak convergence implies strong, (2.1) is equivalent to the requirement that

\[
\lim_{\varepsilon \downarrow 0} \langle F_\varepsilon(\lambda; H, A)x, x \rangle < \infty \quad (2.2)
\]

for every \( x \in \mathcal{H} \).

We shall write \( F_\varepsilon(\lambda) \) for \( F_\varepsilon(\lambda; H, A) \) and \( F(\lambda) \) for \( F(\lambda; H, A) \) when no confusion can arise.

Note that if we define the function

\[
\delta_\varepsilon(t) = \frac{1}{\pi} \frac{\varepsilon}{t^2 + \varepsilon^2}
\]

then

\[
A \delta_\varepsilon(H - \lambda) A^* = \frac{\varepsilon}{\pi} F_\varepsilon(\lambda; H, A). \quad (2.3)
\]

The existence of a nontrivial \( H \)-finite operator has consequences for the spectral theory of \( H \). Define the set

\[
N(H, A, J) = \{ \lambda \in J: \text{s-lim } F_\varepsilon(\lambda; H, A) \text{ does not exist} \}.
\]
2.2. **Theorem.** If $A$ is $H$-finite on $J$, and $R(A^*)$ is cyclic for $H$, then

(a) $H$ has no absolutely continuous part in $J$, and

(b) the singular part of $H$ in $J$ is supported by $N(H, A, J)$.

**Proof.** (a) The spectral density for $m^\alpha_x$ is the a.e. limit of $\langle \delta_\varepsilon(H - \lambda)x, x \rangle$. If $x = A^*y$ is in $R(A^*)$, then

$$\lim_{\varepsilon \to 0} \langle \delta_\varepsilon(H - \lambda)x, x \rangle = \lim_{\varepsilon \to 0} \langle F_\varepsilon(\lambda)y, y \rangle = 0$$

for a.e. $\lambda$ in $J$. The set of vectors $x$ for which $m^\alpha_x(J) = 0$, which is a closed reducing subspace of $H$, therefore contains the cyclic set $R(A^*)$ and so must be $\mathcal{H}$.

(b) By a theorem of de la Vallée Poussin [16] $m^\alpha_x$ is supported on the set where

$$\lim_{\varepsilon \to 0} \langle \delta_\varepsilon(H - \lambda)x, x \rangle = \infty.$$

But, as shown above, if $x = A^*y$, this limit is zero for $\lambda \in J$, $\lambda \notin N$, so the part of $m^\alpha_x$ in $J$ is a fortiori supported by $N$. Again, the set of vectors $x$ with $m^\alpha_x(J \cap N^\varepsilon) = 0$ is closed, reducing, and contains $R(A^*)$.

In order to spare the reader some ultimately superfluous technicalities, we shall assume for the remainder of this section that $A$ is bounded. For $\Im z \neq 0$, define

$$Q(z) = A(H - z)^{-1}A^*.$$

2.3. **Proposition.** Let $A$ be bounded. If $\lambda \in J$, but $\lambda \notin N(H, A, J)$, then

(a) the operator

$$Q(\lambda) = A(H - \lambda)^{-1}A^*$$

is a bounded operator on $\mathcal{H}$, and

(b) one has

$$s\lim_{\varepsilon \to 0} Q(\lambda \pm i\varepsilon) = Q(\lambda). \quad (2.4)$$

**Proof.** For simplicity, take $\lambda = 0$. By Theorem 2.2, $0 \notin \sigma_p(H)$, so $H^{-1}$ exists as a densely defined operator. If $y = A^*x$, then

$$\int \frac{1}{\lambda^2 + \varepsilon^2} \langle E(d\lambda)y, y \rangle = \langle F_\varepsilon(0)x, x \rangle \leq C|x|^2.$$
Taking \( \varepsilon \) to zero gives

\[
\int \lambda^{-2} \langle E(\lambda) y, y \rangle \leq C|x|^2 < \infty,
\]

which implies that \( y \in D(H^{-1}) \). Then \( Q(0) \) is defined on all \( \mathcal{X} \) and bounded.

If \( y \notin D(H^{-1}) \), then

\[
| (H \pm i\varepsilon)^{-1} y - H^{-1} y |^2 = \int \frac{1}{\lambda^2 + \varepsilon^2} \langle E(\lambda) y, y \rangle.
\]  \hspace{1cm} (2.5)\]

The first factor is bounded by one and tends to zero for \( \lambda \neq 0 \), while the measure \( \lambda^{-2} \langle E(\lambda)x, x \rangle \) is finite, and gives zero mass to the origin. Hence (2.5) tends to zero as \( \varepsilon \downarrow 0 \), and part (b) follows.

2.4. \textbf{Theorem.} \ Let \( A \) be bounded and \( H \)-finite on \( J \), with \( R(A^*) \) cyclic for \( H \). Let \( W \) be bounded and self-adjoint on \( \mathcal{X} \), and define

\[ H_1 = H + A^* WA \]

Assume that for \( \text{Im } z \neq 0 \), \( Q(z) \) is compact, and that \( Q(\lambda \pm i\varepsilon) \) converges to \( Q(\lambda) \) in operator norm as \( \varepsilon \downarrow 0 \) for a.e. \( \lambda \) in \( J \). Let

\[ M(H, A, J) = \{ \lambda \in J : Q(\lambda \pm i0) \text{ does not exist in norm} \}. \]

Then

(a) \( H_1 \) has no absolutely continuous part in \( J \); and

(b) the singular continuous part of \( H_1 \) in \( J \) is supported by \( N(H, A, J) \cup M(H, A, J) \).

\textbf{Proof.} \ The proof is based on the standard perturbation formulas for factorized perturbations [8]. Let \( Q_1(z) = A(H_1 - z)^{-1}A^* \) for \( \text{Im } z \neq 0 \). One then has

\[ 1 - WQ_1(z) = [1 + WQ(z)]^{-1} \]  \hspace{1cm} (2.6)\]

and

\[ A\delta_\varepsilon(H_1 - \lambda) A^* = [1 - WQ_1(\lambda + i\varepsilon)]^* A\delta_\varepsilon(H - \lambda) A^* [1 - WQ_1(\lambda + i\varepsilon)]. \]

Write \( N \) and \( M \) for \( N(H, A, J) \) and \( M(H, A, J) \), respectively. If \( \lambda \in (J \sim N) \sim M \), and \( 1 + WQ(\lambda) \) fails to be invertible, then by compactness

\[ x + WA(H - \lambda)^{-1} A^* x = 0 \]
for some $x \neq 0$. As proved above, $A^*x \in D((H - \lambda)^{-1})$, so

$$\varphi = (H - \lambda)^{-1}A^*x$$

(2.8)

is a well-defined vector of $\mathcal{H}$, and

$$x = -WA\varphi.$$  

(2.9)

By (2.8), $x \neq 0$ implies $\varphi \neq 0$, and one has

$$(H - \lambda)\varphi = A^*x = -A^*W A\varphi.$$  

or $H_1\varphi = \lambda\varphi$. Thus, $\lambda \in \sigma_p(H_1)$. Note that the multiplicity of $\lambda$ is equal to the dimension of the kernel of $1 + WQ(\lambda)$ and so is finite.

Therefore, if $\lambda \in J \sim (N \cup M \cup \sigma_p(H_1))$, which is a set of full Lebesgue measure in $J$, then the vector

$$x(\varepsilon) = [1 - WQ(\lambda + i\varepsilon)]x$$

is bounded in norm as $\varepsilon \downarrow 0$. Hence, for $y = A^*x \in R(A^*)$, the quantity

$$\langle \delta_\varepsilon(H_1 - \lambda), y \rangle = \frac{\varepsilon}{\pi} \langle F_\varepsilon(\lambda), x(x) \rangle$$

tends to zero as $\varepsilon \downarrow 0$. Part (a) follows, since $R(A^*)$ is also cyclic for $H_1$.

Let $N_1 = N(H_1, A, J)$. By (2.3) and (2.7), $\lambda \in J \sim (N \cup M \cup \sigma_p(H_1))$ implies $\lambda \notin N_1$, which is to say that

$$N_1 \subset N \cup M \cup \sigma_p(H_1).$$

Thus, $N_1$ has measure zero. By Theorem 2.2, $N_1$ supports the singular part of $H_1$. But $\sigma_p(H_1)$ is countable, so $(N \cup M) \sim \sigma_p(H_1)$ contains $N_1 \sim \sigma_p(H_1)$, which supports the singular continuous part of $H_1$.

The following result of L. de Branges is often useful for ensuring existence of the norm limit.

2.5 LEMMA (de Branges). Let $Q(z)$ be a trace class valued analytic function in the upper half-plane, with nonnegative imaginary part. Then for a.e. $\lambda$

$$\lim_{\varepsilon \downarrow 0} Q(\lambda + i\varepsilon) = Q(\lambda + i0)$$

(2.10)

exists in Hilbert–Schmidt norm.
The most convenient reference for a proof is [10, pp. 149-150].

Let \((\Omega, \mu)\) be a separable measure space, and \(H(\omega) = \int \lambda E_\omega(d\lambda)\) a measurable family of self-adjoint operators on \(\mathcal{K}\). Denote by

\[
\mathbb{H} = \int \lambda E(d\lambda)
\]

the multiplication operator

\[
(\mathbb{H}u)(\omega) = H(\omega)u(\omega)
\]

and \(L^2(\Omega, \mu; \mathcal{K})\). The following is an abstract version of an argument of Simon and Wolff [18].

2.6. Theorem. Let \(E_{-\mathbb{I}}\) be absolutely continuous, and assume that there is a fixed set \(S\) of Lebesgue measure zero which supports the singular continuous part of \(H(\omega)\) in the interval \(J\) for \(\mu\)-a.e. \(\omega\). Then \(H(\omega)\) has no singular continuous part in \(J\) for \(\mu\)-a.e. \(\omega\).

Proof: For fixed \(\omega \in \mathcal{K}\), and any measurable subset \(\Gamma\) of \(\Omega\), let \(u(\omega) = x_{\Gamma}(\omega)x\). Then

\[
\int_{\Gamma} |E_{\omega}^{sc}[J]x|^2 \mu(d\omega) \leq \int_{\Gamma} |E_{\omega}[S]x|^2 \mu(d\omega) = |E[S]u|^2 = 0.
\]

If \(x\) is restricted to a countable dense subset \(\mathcal{D}\) of \(\mathcal{K}\), this yields that for \(\mu\) a.e. \(\omega\),

\[
E_{\omega}^{sc}[J]x = 0
\]

for every \(x \in \mathcal{D}\).  

The following is the chief result of this section. When \(A = \langle \cdot, \varphi \rangle \varphi\) is of rank one, it reduces to the main theorem of Simon and Wolff [18, Theorem 2]. It will be generalized several times in subsequent sections.

2.7. Theorem. Let \(H\) and \(A\) satisfy the hypotheses of Theorem 2.4 and define for real \(\kappa\).

\[
H(\kappa) = H + \kappa A^2.
\]

Then \(H(\kappa)\) is pure point in \(J\) for \(\kappa\)-a.e. \(\kappa\).

Proof. By Proposition 2.2, with \(W = \kappa I\), \(H(\kappa)\) has no absolutely continuous part in \(J\), and its singular continuous part is supported on the fixed
set $S = N \cup M$. It therefore suffices to prove that the multiplication operator

$$\mathbb{H}u(\kappa) = H(\kappa)u(\kappa)$$

on $L^2(\mathbb{R}, \mathcal{H})$ is absolutely continuous.

Simon and Wolff prove this, in their case, by a contour integration argument. Our argument is quite different and is much more easily generalized to more complicated cases. It is based on the following theorem of Putnam and Kato [9; 15, p. 157] on positive commutators.

2.8 Proposition (Putnam and Kato). Let $H$ and $D$ be self-adjoint and $D$ bounded. If $C = i[H, D] \geq 0$, then $H$ is absolutely continuous on $R(C)$. Hence, if $R(C)$ is cyclic for $H$, then $H$ is absolutely continuous.

The idea here is simply to take $D = i(d/d\kappa)$, so that

$$i[H, D] = A^2 \geq 0.$$ 

Thus, the proof of absolute continuity of $\mathbb{H}$ depends on finding a direction in \( \omega \)-space in which $H(\omega)$ is increasing.

The problem with this proof is that boundedness of $D$ is essential in Proposition 2.8. (Compare, for example, the proof of R. Lavine's theorem on Schrödinger operators with repulsive potentials [14; 15, p. 159].) To get around this, we resort to a trick which turns out to be very useful.

2.9 Lemma. On $L^2(\mathbb{R})$, consider the operators $T = \tanh t$ and $D = \arctan(p/2)$, where $p = -i d/dt$. Then $i[T, D] = C$ is positive definite.

Proof. The operator $D$ is convolution by the Fourier transform of $\arctan(x/2)$, which is $-i\pi t^{-1}e^{-2|r|}$ [4, p. 87]. This is a singular (principal value) integral operator, because $\arctan(x/2)$ does not vanish at infinity. Thus,

$$Du(t) = -i\pi P \int \frac{e^{-2|t-y|}}{t-y} u(y) dy.$$ 

We therefore calculate that

$$Cu(t) = \pi \int e^{-2|t-y|} \left\{ \frac{\tanh t - \tanh y}{t-y} \right\} u(y) dy,$$

which is no longer singular. Now an easy calculation gives

$$Cu(t) = \int \text{sech } tp(t-y) \text{ sech } yu(y) dy.$$
where \( \varphi(t) = \pi e^{-2i\lambda t} \sinh t \), so that

\[
C = S\Phi S
\]

where \( S \) is the positive definite operator of multiplication by \( \text{sech} \ t \), and \( \Phi \) is convolution by \( \varphi \). But \( \Phi \) is also positive definite, because \( \varphi(t) \) has the positive Fourier transform:

\[
\hat{\varphi}(x) = \pi/2 \log \left[ \frac{9 + x^2}{1 + x^2} \right].
\]

(See [4, p. 163, (12)].) □

To complete the proof of Theorem 2.7, first observe that since we may scale the norm of \( A \), it suffices to prove that

\[
H(t) = H + (\tanh t) A^2
\]

is pure point for a.e. \( t \). Hence, we are reduced to proving absolute continuity of the multiplication

\[
H u(t) = H(t) u(t)
\]

on \( L^2(\mathbb{R}, \mathscr{H}) \). But now, using the bounded operator \( D = \arctan(p/2) \otimes I \), we obtain

\[
\imath [H, D] = C \otimes A^2 \geq 0.
\]

It remains to argue that \( C \otimes A^2 \) has cyclic range. Since \( C > 0 \), the range of \( C \) is dense in \( L^2(\mathbb{R}) \), so the closure of \( R(C \otimes A^2) \) contains every vector of the form \( \alpha(t) A^2 x \), where \( \alpha(t) \) is scalar-valued, and hence, every vector of the range of

\[
\mathcal{V} u(t) = (\tanh t) A^2 u(t).
\]

But \( R(\mathcal{V}) \) is cyclic for \( \mathcal{H}_0 u(t) = Hu(t) \) because \( R(A) \) is cyclic for \( H \), and hence, it is also cyclic for \( \mathcal{H} = \mathcal{H}_0 + \mathcal{V} \).

This concludes the proof of Theorem 2.7. □

3. A Real Variable Problem

Let \( a_n > 0 \) be a sequence of positive numbers, and \( \{t_n\} \) a sequence of points in the unit interval \([0, 1] \).
Define
\[ F(t) = \sum_{n=1}^{\infty} \frac{a_n^2}{(t-t_n)^2}. \]

The question we wish to consider is: When is \( F(t) \) finite for a.e. \( t \) in \( J=[0,1] \)? We are thinking primarily of a sequence \( \{t_n\} \) dense in \( [0,1] \). This question is, of course, directly related to \( H \)-finiteness for operators \( H \) with dense point spectrum.

3.1 Theorem. If \( \sum_{n=1}^{\infty} a_n < \infty \), then \( F(t) < \infty \) a.e., regardless of the sequence \( \{t_n\} \).

Proof. Let \( S_n = \{ t \in J : |t-t_n| \geq a_n \} \). Let \( \chi_n(t) \) be the characteristic function of \( S_n \), and \( \chi_n^c = 1 - \chi_n \). Write
\[ F(t) = \sum_{n=1}^{\infty} \frac{a_n^2}{(t-t_n)^2} (\chi_n(t) + \chi_n^c(t)). \]

Integrate the first term:
\[ \int_0^1 \sum_{n=1}^{\infty} \frac{a_n^2}{(t-t_n)^2} \chi_n(t) \, dt = \sum_{n=1}^{\infty} a_n^2 \int_{S_n} \frac{dt}{(t-t_n)^2} \leq \sum_{n=1}^{\infty} \frac{4}{a_n} a_n < \infty. \]

Hence, the first term converges a.e.

Write the second term as
\[ \sum_{n=1}^{N} \frac{a_n^2}{(t-t_n)^2} \chi_n^c(t) + \sum_{n=N+1}^{\infty} \frac{a_n^2}{(t-t_n)^2} \chi_n^c(t). \]

The first of these terms is finite except at the points of the sequence \( \{t_n\} \), and the last term is identically zero except on the set
\[ E_N = \bigcup_{n=N+1}^{\infty} S_n^c. \]

But \( E_N \) has measure
\[ \text{meas}(E_N) \leq \sum_{n=N+1}^{\infty} \text{meas}(S_n^c) \leq \sum_{n=N+1}^{\infty} 2a_n, \]
which is less than \( \varepsilon \) if \( N \) is large enough.
Clearly, if \( F(t) \) is finite for even one \( t_0 \), then because \( |t_0 - t_n| \leq 1 \)

\[
\sum_n a_n^2 \leq \sum_n a_n^2 (t_0 - t_n)^{-2} = F(t_0) < \infty.
\]

Thus \( \{a_n\} \) must be in \( l_2 \). It is possible that \( F(t) \) be finite a.e. for an \( l_2 \) sequence \( \{a_n\} \). For example, if \( t_n \to 0 \), \( F(t) \) is finite except at the points \( t_n \) and possibly at \( t = 0 \). However, for a "generic" sequence \( \{t_n\} \), the condition \( \sum_n a_n < \infty \) is both necessary and sufficient. To be precise, we have

3.2 Theorem. If \( \sum_{n=1}^\infty a_n = \infty \), and \( t_n(\omega) \) are independent random variables, uniformly distributed on \( [0, 1] \), then

(a) for every fixed \( t \), \( F(t, \omega) = \infty \) a.s., and

(b) for a.e. \( \omega \), \( F(t, \omega) = \infty \) for a.e. \( t \) in \( [0, 1] \).

Proof. Let \( I_k(\omega) \) be the interval \( (t_k(\omega) - a_k, t_k(\omega) + a_k) \), and \( S(\omega) = \{k: t \in I_k(\omega)\} \). If \( S(\omega) \) is infinite, then

\[
F(t, \omega) = \sum_k \frac{a_k^2}{(t - t_k(\omega))^2} \geq \sum_{k \in S(\omega)} \frac{a_k^2}{(t - t_k(\omega))^2} \geq \sum_{k \in S(\omega)} \frac{a_k^2}{a_k^2} = \sum_{k \in S(\omega)} \frac{a_k^2}{a_k^2} = 1 = \infty.
\]

But \( S(\omega) \) is finite if and only if \( t \in I_k(\omega) \) infinitely often. By independence and the second Borel–Cantelli lemma, this is true if and only if

\[
\sum_k P\{x \in I_k(\omega)\} = \infty.
\]

But if \( k \) is large enough that \( (x - a_k, x + a_k) \subset [0, 1] \), then \( P\{x \in I_k(\omega)\} = P\{\lambda_\omega(-x) \cap [x, x + a_k)\} = 2a_k \), so that (3.1) holds.

Part (b) follows by Fubini's theorem.

Remark. Theorem 3.2 also follows from Kolmogorov's Three Series Theorem, but the present argument, following Donoghue and M. Riesz [3, p. 565], is more elementary.

4. Finite Rank Perturbations

In this section, we give a generic condition on a finite rank perturbation which ensures that it preserves pure point spectrum.

4.1. Definition. Let \( H \) be a self-adjoint operator with pure point spec-
A bounded operator $A: \mathcal{H} \to \mathcal{H}$ is strongly $H$-finite if and only if
\[
\sum_{n=1}^{\infty} |Ae_n| < \infty. \tag{4.1}
\]

With no loss of generality, $H$ may be thought of as a diagonal matrix on $l_2$, and $A$ as an infinite matrix \{\(a_{ij}\}\}, in which case (4.1) simply says that
\[
\sum_{n} \left[ \sum_{i} |a_{in}|^2 \right]^{1/2} < \infty.
\]

4.2 PROPOSITION. If $H$ is pure point, and $A$ satisfies (4.1), then

(a) $A$ is trace class, and
(b) $A$ is $H$-finite.

**Proof.** For (a), simply note that the series
\[
A = AI = \sum_{n} \langle \cdot, e_n \rangle Ae_n
\]
converges in trace norm. For (b), write
\[
F(\lambda) = \sum_{n} \frac{\langle \cdot, Ae_n \rangle Ae_n}{(\lambda - \lambda_n)^2}. \tag{4.2}
\]
The trace norm of the positive operator $F(\lambda)$ is then
\[
\text{tr } F(\lambda) = \sum_{n} \frac{|Ae_n|^2}{(\lambda - \lambda_n)^2}.
\]
By (4.1) and Theorem 3.1, this is finite a.e.

Thus we obtain, in addition, that $F(\lambda)$ is trace class a.e.

4.3 THEOREM. Let $H$ be a pure point self-adjoint operator, and let $A_1, \ldots, A_N$ be strongly $H$-finite. Then for a.e. $\kappa = (\kappa_1, \ldots, \kappa_N)$ in $\mathbb{R}^N$,
\[
H(\kappa) = H + \sum_{j=1}^{N} \kappa_j A_j^* A_j
\]
is pure point.

**Proof.** Let
\[
\mathcal{H} = \bigoplus_{j=1}^{N} \mathcal{H}(A_j)
\]
with the elements of $\mathcal{X}$ represented as column vectors. Define $A: \mathcal{X} \to \mathcal{X}$ by

$$Ax = \begin{bmatrix} A_1x \\ \vdots \\ A_Nx \end{bmatrix}$$

so that $A^*: \mathcal{X} \to \mathcal{X}$ is given by

$$A^*y = A_1^*y_1 + \cdots + A_N^*y_N.$$ 

Then $F(\lambda)$ is the operator matrix on $\mathcal{X}$ given by

$$A(H-\lambda)^{-2}A^* = \{ A_j(H-\lambda)^{-2}A_j^* \}_{j=1}^{N}.$$ 

The diagonal terms are finite a.e. by Proposition 4.2, and Schwartz’s inequality therefore forces the off-diagonals to be finite as well. Hence, $A$ is $H$-finite.

Write

$$H(\kappa) = H + A^*W(\kappa)A$$

where $W(\kappa) = \text{diag}\{\kappa_j\}$. Note that $Q(z) = A(H-z)^{-1}A^*$ is of trace class, so by the Branges’ lemma $Q(\lambda \pm i0)$ exists a.e. in norm. Theorem 2.6 therefore applies if we can prove that the multiplication operator

$$\mathbb{H} u(\kappa) = H(\kappa)u(\kappa)$$

on $L^2(\mathbb{R}^N; \mathcal{X})$ is absolutely continuous. Formally, this can be done by choosing

$$D = i \sum_{j=1}^{N} \frac{\partial}{\partial \kappa_j}$$

and computing that

$$i[\mathbb{H}, D] = \sum_{j=1}^{N} A_j^*A_j \geq 0.$$ 

Instead, we proceed as in the proof of Theorem 2.7. We replace $H(\kappa)$ by

$$H(t) = H + \sum_{j=1}^{N} \tanh(t_j)A_j^*A_j,$$

define

$$D = \sum_{j=1}^{N} \arctan(p_j/2)$$
and compute that
\[ i[\mathbb{H}, D] = \sum_{j=1}^{N} C_j \otimes A_j^* A \geq 0. \]

It remains to argue that the commutator has cyclic range. We may assume initially that \( R(A^*) \) is cyclic for \( H \) and \( H(t) \), since the perturbation \( A^* W A \) vanishes on the orthocomplement of \( \mathcal{M}(H, R(A^*)) \), and \( H \) is already pure point there. This implies that \( R(1 \otimes A^*) \) is cyclic for \( \mathbb{H} \). The commutator can be written as
\[ K = (1 \otimes A^*) C (1 \otimes A) \]
where \( C = \text{diag}\{C_j \otimes I\} \) on \( \mathcal{H} \). Because \( C \) is positive definite,
\[ \ker(K) = \ker(1 \otimes A) \]
and hence, by complementation
\[ \overline{R(K)} = \overline{R(1 \otimes A^*)}. \]

It follows that \( R(K) \) is cyclic for \( \mathbb{H} \).

We now specialize to operators of finite rank.

4.4 Definition. Define \( l_1(H) \) to be the space of all \( x \in \mathcal{H} \) with norm
\[ \|x\|_1 = \sum_{n} |\langle x, e_n \rangle| < \infty. \]

If \( H \) is considered to be a diagonal matrix on \( l_2 \), then \( l_1(H) \) is just \( l_1 \).

A rank one projection \( A = \langle \cdot, \varphi \rangle \varphi \) is strongly \( H \)-finite if and only if \( \varphi \in l_1(H) \). This yields immediately:

4.5 Corollary. Let \( H \) be pure point and \( \varphi \in l_1(H) \). Then
\[ H(\kappa) = H + \kappa \langle \cdot, \varphi \rangle \varphi \]
is pure point a.e. \( \kappa \).

More generally, one has:

4.6 Corollary. Let \( H \) be pure point and \( \varphi_1, \ldots, \varphi_N \in l_1(H) \). Then for a.e.
\( \kappa = (\kappa_1, \ldots, \kappa_N) \) in \( \mathbb{R}^N \),
\[ H(\kappa) = H + \sum_{j=1}^{N} \kappa_j \langle \cdot, \varphi_j \rangle \varphi_j \]
is pure point.

Thus, the generic finite rank perturbation \( V \) with \( R(V) \subset l_1(H) \) preserves pure point spectrum.
5. Compact Perturbations

As in the preceding section, \( H \) will denote a pure point self-adjoint operator, \( \{ e_n \} \) a complete orthonormal set of eigenvectors, and \( \lambda_n \) the eigenvalue with \( He_n = \lambda_n e_n \).

We wish to consider operators of the form

\[
H(t) = H + \sum_{j=1}^{\infty} \tanh t_j A_j^* A_j
\]

for a.e. infinite vector \( t = (t_1, t_2, \ldots) \). This can mean many things, so we shall need a few preliminaries.

Let \( w(x) \) be a density function on \( \mathbb{R} \) which is positive for all \( x \). Thus

\[
\int_{-\infty}^{+\infty} w(x) \, dx = 1.
\]

The mapping

\[
W^{-1/2} u(x) = w(x)^{-1/2} u(x)
\]

is then a unitary map of \( L_2(\mathbb{R}) \) onto \( L_2(\mathbb{R}, w(x) \, dx) \). Let \( T = \tanh t \) and \( D = \arctan(p/2) \) be the previously considered operators on \( L^2(\mathbb{R}) \). Then

\[
T_w = W^{-1/2} T W^{1/2}
\]

and

\[
A_w = W^{-1/2} A W^{1/2}
\]

are bounded operators on \( L^2(\mathbb{R}, w(x) \, dx) \) whose commutator

\[
i[ A_w, T_w ] = W^{-1/2} C W^{1/2} > 0
\]

is again positive with dense range. The operator \( T_w \) is just multiplication by \( \tanh t \), while \( A_w \) is the singular integral operator

\[
A_w u(x) = \pi w^{-1/2}(x) \int \frac{e^{-2|x-y|}}{(x-y)} w^{1/2}(y) u(y) \, dy.
\]

5.1 Theorem. Let \( X_1, X_2, \ldots \) be i.i.d. absolutely continuous random variables with density \( w(x) > 0 \) for all \( x \). Let \( A_1, A_2, \ldots \) be bounded operators on \( \mathcal{H} \) with

\[
\sum_{n=1}^{\infty} \left( \sum_{j=1}^{\infty} |A_j e_n|^2 \right)^{1/2} < \infty.
\]
Then

\[ H(\omega) = H + \sum_{j=1}^{\infty} (\tanh X_j(\omega)) A_j^* A_j \]

is a.s. pure point.

Remarks. (1) In contrast to the finite case, if the density \( w(x) \) is changed, the notion of a.s. may be entirely different.

(2) Equation (5.5) implies that each \( A_j \) is strongly \( H \)-finite. By choosing \( A_j = a_j \langle \cdot, \varphi_j \rangle \varphi_j \) with \( a_j \geq 0 \), we obtain.

5.2 Corollary. Let \( \{ \varphi_j \} \) be a complete orthonormal set, and \( a_j \geq 0 \) a sequence satisfying

\[ \sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{\infty} a_j^2 |\langle \varphi_j, e_n \rangle|^2 \right\}^{1/2} < \infty. \quad (5.6) \]

Then

\[ H(\omega) = H + \sum_{j=1}^{\infty} a_j^2 (\tanh X_j) \langle \cdot, \varphi_j \rangle \varphi_j \]

is pure point a.s.

Remarks. (1) Note that (5.6) implies that every \( \varphi_j \in l_1(H) \).

(2) Let \( U = \{ \langle \varphi_j, e_k \rangle \} \) and \( A = \text{diag}\{a_j\} \). Then \( U \) is a unitary matrix on \( l_2 \), and (5.6) takes the form

\[ \sum_n \{\langle U*A^2 U \rangle_{nn}\}^{1/2} < \infty. \]

(3) Since some \( a_j \)'s may be zero, the set \( \varphi_n \) need not really be complete.

Proof of Theorem 5.1. Let

\[ \mathcal{H} = \bigoplus_{j=1}^{\infty} \tilde{R}(A_j) \]

and define \( A : \mathcal{H} \to \mathcal{H} \) by

\[ Ax = \begin{bmatrix} A_1 x \\ A_2 x \\ \vdots \end{bmatrix}. \]
Let $W = \text{diag}\{\tanh X_j(\omega)\}$ on $\mathcal{H}$. Then

$$H(\omega) = H + A^* W(\omega) A.$$  

The condition that $A$ be strongly $H$-finite is exactly (5.6). An appeal to deBranges' lemma and Theorem 2.4 yields a uniform nullset $S$ supporting the singular continuous part of $H(\omega)$ for all $\omega$.

Let $\mu$ be the finite Borel measure

$$\mu(dt) = w(t) \, dt$$

and $\Omega$ the space of sequences with infinite product measure

$$P = \mu \times \mu \times \cdots.$$  

The coordinate functions then furnish a model for the process $X_j$; that is, $X_j(\omega) = \omega_j$.

The operator

$$D_j = w(\omega_j)^{1/2} \arctan(p_j/2) w(\omega_j)^{-1/2}$$

is a bounded operator on $L^2(\Omega, P; \mathcal{H})$ acting on one factor of the tensor product. The sum,

$$D = \sum_{j=1}^{\infty} \eta_j D_j,$$

where $\eta_j > 0$ and $\sum_{j=1}^{\infty} \eta_j = 1$, converges in operator norm, and one computes

$$i[H, D] = \sum_{j=1}^{\infty} \eta_j C_j \otimes A_j^* A_j \geq 0.$$  

Here, $C_j$ acts on the $\omega_j$ variable, and $A_j^* A_j$ on the fibre space $\mathcal{H}$. Cyclicity of the range of the commutator is handled as in the proof of Theorem 4.3. 

6. SPECTRA OF MEASURE ZERO

In this section, we shall prove that a generic compact perturbation of a self-adjoint operator whose spectrum has Lebesgue measure zero is pure point. More precisely, we shall prove the following:

6.1 Theorem. Let $\varphi_n$ be a complete orthonormal set and $a_n > 0$ a positive sequence with $\lim a_n = 0$. Let $X_n(\omega)$ be i.i.d. absolutely continuous random
variables with density \( w(x) > 0 \) positive for all \( x \). Let \( \mathcal{H} \) be a self-adjoint operator.

If \( \sigma(H) \) has Lebesgue measure zero, then

\[
H(\omega) = H + \sum_n a_n (\tanh X_n(\omega)) \langle \cdot, \phi_n \rangle \phi_n
\]

is pure point a.s.

**Proof.** By invariance of the essential spectrum, the absolutely continuous part of \( H(\omega) \) vanishes, and the singular continuous part is contained in the fixed null set \( S = \sigma(H) \). We again choose \( \eta_n > 0 \) with \( \sum_n \eta_n = 1 \), define

\[
D = \sum_n \eta_n \arctan(p_n/2)
\]

and compute

\[
[i[\mathbb{H}, D] = \sum_n a_n \eta_n C_n \otimes \langle \cdot, \phi_n \rangle \phi_n > 0.
\]

The commutator is positive definite, so its range is not merely cyclic, but dense. \( \blacksquare \)

**Remarks.**

1. The same proof yields a local version of the theorem: Let \( J \) be an open interval. If \( \sigma(H) \cap J \) has Lebesgue measure zero, then \( H(\omega) \) is a.s. pure point in \( J \).

2. The hypothesis that \( w(x) \) be strictly positive for all \( x \) can also be dropped. See Section 8.

3. Donoghue [3, p. 565, Example 23] (cf. [18, Sect. 3, Example 21]) gives an example of a rank one perturbation, which turns a pure point operator into a singular continuous operator for \( \kappa \neq 0 \). Theorem 6.1 implies that the closure of the eigenvalues must have positive measure, and, in fact, they are dense in \([0, 1]\).

7. RANDOM UNPERTURBED OPERATORS

We next consider the case in which the eigenvalues of the diagonal unperturbed operator are random variables.

**7.1 Theorem.** Let \( Y_j(\omega) \) be i.i.d., absolutely continuous random variables with density \( g(x) > 0 \) for all \( x \). Let

\[
H(\omega) = \text{diag} \{ Y_j(\omega) \}
\]
and assume that
\[ \sum_{n=1}^{\infty} |Ae_n| < \infty \]
and that \( W \) is bounded. Then
\[ H(\omega) + A^*WA \]
is pure point a.s.

Note that \( H(\omega) \) is unbounded a.s. However, we also have:

7.2 Theorem. The same result holds if \( Y_j(\omega) \) are independent and uniformly distributed on \([-1, 1]\).

Before giving the proof, we obtain two corollaries. The first concerns the problem considered by Thomas and Wayne [19, Theorem 2.11, who assume (more or less) that \( \alpha > 4 \).

7.3 Corollary. Let \( Y_j(\omega) \) be i.i.d. and uniformly distributed on \([-1, 1]\). Let \( H(\omega) = \text{diag}\{Y_j(\omega)\} \), and let \( B_n \) be self-adjoint with range contained in \( \mathcal{M}_n = \text{sp}\{e_1, \ldots, e_n\} \). If
\[ \|B_n\| = O(n^{-\alpha}) \] (7.2)
for some \( \alpha > 3 \), then
\[ H(\omega) + \sum_{n=1}^{\infty} B_n \]
is pure point a.s.

Proof. Factor \( B_n = A_n W_n A_n \), where \( A_n = |B_n|^{1/2} \) and \( W_n \) is unitary on \( \mathcal{M}_n \). Let
\[ \mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{M}_n. \]
and define \( A: \mathcal{H} \to \mathcal{H} \) by
\[ Ax = \begin{bmatrix} A_1x \\ A_2x \\ \vdots \end{bmatrix} \]
and \( W: \mathcal{H} \to \mathcal{H} \) by
\[ W = \text{diag}\{W_n\}. \]
To verify condition (7.1), we simply note that
\[
\sum_{n=1}^{\infty} |A e_n| = \sum_{n=1}^{\infty} \left[ \sum_{j=1}^{\infty} |A_j e_n|^2 \right]^{1/2} = \sum_{n=1}^{\infty} \left[ \sum_{j=n}^{\infty} |A_j e_n|^2 \right]^{1/2}
\leq \sum_{n=1}^{\infty} \left[ \sum_{j=n}^{\infty} \|A_j\|^2 \right]^{1/2} = \sum_{n=1}^{\infty} \left[ \sum_{j=n}^{\infty} \|B_j\|^2 \right]^{1/2}
\leq C \sum_{n=1}^{\infty} \left[ \sum_{j=n}^{\infty} j^{-s} \right]^{1/2} < \infty.
\]

Our next result concerns Jacobi matrices. Let \( \mathcal{H} = l_2(Z) \) be the space of bilateral square-summable sequences, and \( e_n \) the usual coordinate basis vectors. The Jacobi matrix
\[
J(y, b) = \begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & y_{-1} & b_0 & 0 & \\
\cdots & \cdots & b_0 & y_0 & b_1 \\
\cdots & \cdots & \cdots & b_1 & y_1 \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\tag{7.3}
\]
can be written as follows. Let \( a_j = |b_j|^{1/2} \),
\[
W_j = \text{sgn}(b_j)\{ \langle \cdot, e_j \rangle e_{j+1} + \langle \cdot, e_{j+1} \rangle e_j \},
\]
and let \( P_j \) be the projection onto \( \text{sp}\{e_j, e_{j+1}\} \). Let \( \mathcal{H} = \bigoplus_{j=\infty}^{\infty} R(P_j) \) and define \( A: \mathcal{H} \to \mathcal{H} \) by
\[
Ax = \begin{bmatrix}
\cdots \\
a_0 P_0 x \\
a_1 P_1 x \\
\cdots
\end{bmatrix}
\]
and \( W: \mathcal{H} \to \mathcal{H} \) by \( W = \text{diag}\{W_j\} \). Then
\[
J(y, b) = \text{diag}\{y_j\} + A^* WA. \tag{7.4}
\]
Condition (7.1) then amounts to
\[
\sum_n |A e_n| = \sum_n \sum_j a_j |P_j e_n| = \sum_n \sum_j a_j (\delta_{n,j} + \delta_{n-1,j}) = 2 \sum_n a_n = \sum_n |b_n|^{1/2}.
\]
We therefore obtain
7.4. COROLLARY. Let $Y_j(\omega)$ be i.i.d., and uniformly distributed on $[-1, 1]$, and
\[ \sum_{n = -\infty}^{n} |b_n|^{1/2} < \infty. \] (7.5)

Let $y(\omega) = \{Y_j(\omega)\}$ and $b = \{b_n\}$. Then $J(y(\omega), b)$ is pure point a.s.

Let us note a few facts in preparation for the proofs. Let $Y$ and $V$ be independent, absolutely continuous random variables, with densities $g(x)$ and $v(x)$. Assume that $v(x)$ is supported in $[-1, 1]$. The random variable $Z_\varepsilon = Y + \varepsilon V$

has the density
\[ g_\varepsilon(t) = \int_{-1}^{1} g(t - \varepsilon x) v(x) \, dx \] (7.6)

which converges to $g(x)$ in $L_1(\mathbb{R})$ as $\varepsilon \downarrow 0$. One has
\[ \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} \sqrt{g(x)} g_\varepsilon(x) \, dx = 1. \] (7.7)

To prove this, let $\varepsilon_n \downarrow 0$, and $g_n(x) = g_{\varepsilon_n}(x)$. By passing to a subsequence, assume that $g_n \rightarrow g$ a.e. Since $2(\varepsilon_n g_n)^{1/2} \leq g + g_n$, the sequence $h_n = (\varepsilon_n g_n)^{1/2}$ is uniformly integrable on $[-N, N]$, and so [1, pp. 295–297],
\[ \lim_{n \rightarrow \infty} \int_{-N}^{N} h_n(x) \, dx = \int_{-N}^{N} g(x) \, dx \]

for any $N$. Therefore,
\[ 0 \leq 1 - \int_{-\infty}^{+\infty} h_n = \int_{-\infty}^{+\infty} (g - h_n) = \int_{|x| \leq N} (g - h_n) + \int_{|x| \geq N} (g - h_n) \]
so that, because $2h_n \leq g + g_n$,
\[ 0 \leq \lim \sup \left( 1 - \int_{-\infty}^{+\infty} h_n \right) \leq \lim \sup \int_{|x| \geq N} (g - h_n) \]
\[ \leq \lim \sup \int_{|x| \geq N} (3/2) g + (1/2) g_n = 2 \int_{|x| \geq N} g. \]

This can be made arbitrarily small by taking $N$ large.
Next assume that $g(x)$ is also supported in $[-1, 1]$. The scaled random variable $\alpha Z_\varepsilon$ has density

$$g_\varepsilon(x, \alpha) = \alpha^{-1} g(x/\alpha).$$

(7.8)

If we choose $\alpha = (1 + \varepsilon)^{-1}$, then $g_\varepsilon(x, \alpha)$ will also be supported on $[-1, 1]$. For any Borel set $S$,

$$\int_S g_\varepsilon(t, \alpha) \, dt = \alpha^{-1} \int_{-1}^{+1} g \left[ \frac{t - \varepsilon x}{\alpha} \right] v(x) \, dx \, dt$$

$$= \int_{-1}^{+1} \int_{S(t, \varepsilon x)} g(s) \, ds \, v(x) \, dx$$

where $S(t, \varepsilon x) = \{ s: s = \alpha^{-1}(t - \varepsilon x), t \in S \} = \alpha^{-1}(S - \varepsilon x)$. Now, since $\text{meas}(S(t, \varepsilon x)) = \alpha^{-1} \text{meas}(S)$, the family $g_\varepsilon(x, \alpha)$ has uniformly absolutely continuous integrals for, say, $0 < \varepsilon < 1$, and $1/2 < \alpha < 2$. The same holds for $(g_\varepsilon(x, \alpha) g(x))^{1/2}$, as above. Since $[-1, 1]$ is finite, Vitali's convergence theorem implies that

$$\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{+\infty} \left[ g(x) g_\varepsilon(x, (1 + \varepsilon)^{-1}) \right]^{1/2} \, dx = 1. \quad (7.9)$$

The chief difficulty in the proof of Theorem 7.1 is that, although $A$ is $H(\omega)$-finite for every $\omega$, the exceptional set $N(H(\omega), A)$ depends on $\omega$. We get around this by a sort of differentiation procedure based on the following theorem of Kakutani [6, 12, p. 1161, which we state in a special case.

7.5 Theorem (Kakutani). Let $v_j(dx) = f_j(x) \, dx$ and $\mu_j(dx) = g_j(x) \, dx$ be absolutely continuous probability measures on $\mathbb{R}$ for $j = 1, 2, \ldots$. Assume that $\mu_j$ and $v_j$ are mutually absolutely continuous. Then the product measures $\mu = \mu_1 \times \mu_2 \times \cdots$ and $v = v_1 \times v_2 \times \cdots$ are equivalent if and only if

$$\prod_{j=1}^{\infty} \int_{-\infty}^{+\infty} \sqrt{f_j(x)} \, g_j(x) \, dx > 0. \quad (7.10)$$

Remark. If this product, in which all the factors do not exceed 1, diverges to zero, then $\mu$ and $v$ are not only not equivalent but are mutually singular. That is why different choices of density functions in the theorems of the preceding sections lead to entirely different notions of "generic."

We begin the proof by proving:

7.6 Proposition. Let $X_j(\omega')$ be i.i.d. absolutely continuous random
variables with density \( w(x) \) strictly positive for all \( x \), and let \( \epsilon_j > 0 \) be a positive sequence with

\[
\sum_{j=1}^{\infty} \epsilon_j^{1/2} < \infty. \tag{7.11}
\]

Then for any any diagonal operator \( H \), the operator

\[
H(\omega') = H + \text{diag}\{\epsilon_j \tanh X_j(\omega')\} + A^*WA \tag{7.12}
\]

is pure point a.s.

**Proof.** The proof is the same as before. The commutator here is

\[
\text{diag}\{\epsilon_j \otimes C_j\}
\]

which has dense range. \( \square \)

**Proof of Theorem 7.1.** Let \( X_j(\omega') \) be a process like that of Proposition 7.6, and independent of the process \( Y_j(\omega) \). For every fixed \( \omega \), the operator

\[
H(\omega, \omega') = \text{diag}\{Y_j(\omega)\} + \text{diag}\{\epsilon_j \tanh X_j(\omega')\} + A^*WA \tag{7.13}
\]

is pure point for a.e. \( \omega' \), which says that the operator

\[
\text{diag}\{Z_j\} + A^*WA \tag{7.14}
\]

is a.s. pure point, where \( Z_j = Y_j + \epsilon_j \tanh X_j \). We will have proved the theorem if the processes \( Z_j \) and \( Y_j \) define equivalent measures on sequence space. Kakutani's theorem tells us this is true if \( \epsilon_j \) tends to zero sufficiently fast.

In fact, taking \( Y = Y_j \) and \( V = \tanh X_j \) in our previous discussion, \( Z_j \) has the density \( g_{\epsilon_j}(x) \). If, as we assume, \( g(x) > 0 \), then \( g_{\epsilon_j}(x) > 0 \), so that the distributions of \( Y_j \) and \( Z_j \) are equivalent (mutually absolutely continuous). Choose, by (7.7), the sequence \( \epsilon_j \) so that \( \sum_j \epsilon_j < \infty \) and

\[
\int_{-\infty}^{+\infty} \sqrt{g_{\epsilon_j}(x)g(x)} \, dx = 1 - \eta_j
\]

with \( \sum_j \eta_j < \infty \). Then (7.10) holds.

This completes the proof. \( \square \)

**Proof of Theorem 7.2.** Using (7.9), choose \( \epsilon_j \) so that \( \sum_j \epsilon_j < \infty \) and

\[
\int_{-\infty}^{+\infty} \left[ g_{\epsilon_j}(x, (1 + \epsilon_j)^{-1}) g(x) \right]^{1/2} \, dx = 1 - \eta_j
\]
with $\sum_j \eta_j < \infty$. Let $\alpha_j = (1 + \varepsilon_j)^{-1}$. By Proposition 7.6, for every $\omega$ the operator

$$\text{diag}\{\alpha_j Z_j\} + A^* W A$$

$$= \text{diag}\{\alpha_j y_j(\omega)\} + \text{diag}\{\alpha_j \varepsilon_j \tanh X_j(\omega')\} + A^* W A$$

is pure point for a.e. $\omega'$. But, since we have rescaled, $Y_j$ and $\alpha_j Z_j$ have equivalent densities. The proof concludes as before. 

8. Generalizations

We want to mention very briefly some generalizations of the preceding results in two directions. We first consider the question of local spectral results, and conclude with remarks on the condition of strictly positive densities.

All the results of Section 2 were phrased locally in terms of a spectral interval $J$. The main theorems in question are therefore Theorems 4.3, 5.1, and 6.1. We have already noted the local version of 6.1. We shall ignore the results of Section 7. It is clearly possible to formulate something rather artificial, but this seems pointless.

Thus, we are looking for local versions of Theorems 4.3 and 5.1. All that is really required is a definition.

8.1 Definition. Let $J$ be an open interval, and $H$ a self-adjoint operator which has pure point spectrum in $J$. Let $e_n$ be an orthonormal basis of $R(E[\cdot J])$ consisting of eigenvectors of $H$. We say that a bounded operator $A$ is strongly $H$-finite in $J$ if and only if

$$\sum_{n=1}^{\infty} |Ae_n| < \infty. \quad (8.1)$$

8.2 Proposition. Let $H$ be pure point in $J$ and $A$ strongly $H$-finite in $J$. Then

(a) $A$ is $H$-finite in $J$,

(b) $AE[J]$ is trace class, and

(c) $Q(\lambda \pm i0)$ exists in operator norm for a.e. $\lambda$ in $J$.

Proof. (a) Write

$$F(\lambda) = \sum_n (\lambda - \lambda_n)^{-2} \langle \cdot, Ae_n \rangle Ae_n + AE[J^*](H - \lambda)^{-2} A^*. \quad (8.2)$$
The first term is finite and in trace class for a.e. \( \lambda \) by (8.1), while the second is norm continuous in \( J \).

(b) One has \( AE[J] = \sum_n \langle \cdot, e_n \rangle A e_n \).

(c) Write

\[
Q(z) = AE[J](H-z)^{-1}A^* + AE[J'](H-z)^{-1}A^*. \quad (8.3)
\]

The first term is trace class with positive imaginary part, and therefore has Hilbert–Schmidt boundary values a.e. by DeBranges' Lemma. The second term is norm analytic on \( J \).

Our previous methods then lead, \textit{mutatis mutandis}, to the following results.

8.3 \textbf{Theorem.} In Theorem 4.3, assume only that \( H \) is pure point in an open interval \( J \), and that \( A_1, \ldots, A_N \) are strongly \( H \)-finite in \( J \). Then \( H(\kappa) \) is pure point in \( J \) for a.e. \( \kappa \).

8.4 \textbf{Theorem.} In Theorem 5.1, assume only that \( H \) is pure point in an open interval \( J \) and that (5.5) holds where \( e_n \) are eigenvectors of \( H \) forming an orthonormal basis of \( R(E[J]) \). Then \( H(\omega) \) is a.s. pure point in \( J \).

In a number of places, we have assumed that the density functions of certain random variables is strictly positive. In Section 4, this does not matter, since only a finite number of random variables are involved, but in Sections 5 and 6, it leads to an entirely different version of “almost surely.” In Section 7, there is a serious problem in dropping strict positivity because convolution may change the support of the measure, and Kakutani's theorem requires that the factors be mutually absolutely continuous. In Theorem 7.2, where \( g(x) \) was supported on \( [-1, 1] \), we were able to rescale and avoid this problem, but the general case appears difficult.

In Sections 5 and 6, we have better luck.

8.5 \textbf{Theorem.} The hypothesis that \( w(x) \) is strictly positive may be dropped in Theorems 5.1, 6.1, and 8.4.

\textbf{Proof.} The only use of strict positivity is in the use of the pair of operators. \( T = \tanh x \) and \( D = W^{-1/2} \arctan(p/2) W^{1/2} \) on \( L^2(\mathbb{R}, w(x) \, dx) \). The point is that the commutator

\[
C = i[ T, D ] > 0
\]

is positive definite. We simply need to construct such a pair on \( L^2(\mathbb{R}, v(x) \, dx) \) for an arbitrary \( v(x) \).
This is easily done by compression. Let \( E = \{ x : v(x) > 0 \} \), and let \( w(x) \) be a strictly positive density such that
\[
v(x) = kw(x) \chi_E(x)
\]
for some constant \( k > 1 \). Let \( P \) be the projection
\[
Pu(x) = \chi_E(x) u(x)
\]
on \( L^2(\mathbb{R}, w(x) \, dx) \). Then \( P \) commutes with \( T \), so
\[
i[T, PDP] = PCP
\]
where \( PCP \) is positive definite on \( R(P) \). But \( R(P) \) is isometrically isomorphic to \( L^2(\mathbb{R}, v(x) \, dx) \), after renorming to get rid of the trivial factor \( k \).

The proofs now proceed using \( PDP \) instead of \( D \).  

One possible generalization which we have not considered, but which might be of some interest, would be to replace independence by some version of approximate asymptotic independence.

9. REMARKS ON H-FINITENESS

In order to prove finiteness a.e. of
\[
F(\lambda) = \sum_n (\lambda - \lambda_n)^{-2} \langle \cdot, \psi_n \rangle \psi_n
\]
with \( \psi_n = A e_n \), we have resorted to taking the trace of \( F(\lambda) \), and using the scalar Theorem 3.1 to obtain the result. This led to the condition
\[
\sum_n |Ae_n| < \infty
\]
which we have termed strong H-finiteness.

At first glance, taking the trace seems rather heavy-handed, and one would expect a much weaker condition to suffice. However, the following shows that, in a sense, (9.1) is optimal for generic sequences.

9.1 THEOREM. Let \( \lambda_n(\omega) \) be i.i.d. uniform r.v. on \([0, 1]\) and
\[
H(\omega) = \sum_n \lambda_n(\omega) \langle \cdot, e_n \rangle e_n
\]
where \( e_n \) is a complete orthonormal set. Then \( A \) is \( H(\omega) \)-finite for a.e. \( \omega \) if and only if (9.1) holds.
Proof. We shall prove that if (9.1) fails, then $A$ is not $H(\omega)$-finite for a.e. $\omega$. To this end, let $\psi_n = Ae_n$, $w_n = |\psi_n|$ and assume that

$$\sum_n w_n = \infty.$$ 

Choose (by the closed graph theorem) a sequence $a_n > 0$ with

$$\sum_n a_n = \infty$$  \hspace{1cm} (9.2)

but such that $a_n = o(w_n)$.

If

$$F(\lambda) = \sum_n (\lambda - \lambda_n(\omega))^{-2} \langle \cdot, \psi_n \rangle \psi_n$$

is bounded, then because each term is a positive operator,

$$(\lambda - \lambda_n(\omega))^{-2} \langle \cdot, \psi_n \rangle \psi_n \leq F(\lambda)$$

and hence the sequence of norms

$$c_n(\lambda) = (\lambda - \lambda_n(\omega))^{-2} w_n^2$$

is bounded. Let

$$S_n(\omega) = \{ \lambda : |\lambda - \lambda_n(\omega)| \leq a_n \}.$$ 

If $\lambda \in S_n(\omega)$ infinitely often, then

$$c_n(\lambda) \geq \frac{w_n^2}{a_n^2}$$

infinitely often, so that $c_n(\lambda)$ is unbounded. However, the second Borel–Cantelli lemma and (9.2) imply that for every fixed $\lambda$,

$$P\{ \lambda \in S_n \text{ infinitely often} \} = 1.$$ 

A Fubini argument then gives that

$$\text{meas}\{ \lambda : \lambda \in S_n(\omega) \text{ infinitely often} \} = 1 \quad \text{a.s.} \]$$

References

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