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Symmetric units satisfying a group identity ☆

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Abstract

Let K be an infinite field of characteristic $p \neq 2$, G a locally finite group and KG its group algebra. Let $\varphi : KG \to KG$ denote the K-linear extension of an involution φ defined on G. In this paper we prove, under some assumptions, that if the set of φ-symmetric units of KG satisfies a group identity then KG satisfies a polynomial identity. Moreover, in case the prime radical of KG is nilpotent we characterize the groups for which the φ-symmetric units satisfy a group identity.

Keywords: Group algebra; Unit group; Group identities; Symmetric units

1. Introduction

A subset H of the unit group $\mathcal{U}(R)$ of a ring R with unity 1 is said to satisfy a group identity (we will say that H satisfies a GI for short) if there exists a non-trivial word $w(x_1, \ldots, x_n)$ in the free group generated by $x_1, \ldots, x_n$ such that $w(u_1, \ldots, u_n) = 1$ for all $u_1, \ldots, u_n \in H$.

Hartley (see [17, Problem 52]) conjectured that if the whole unit group $\mathcal{U}(KG)$ of the group algebra KG of a periodic group G over a field K satisfies a group identity, then KG satisfies a polynomial identity. An affirmative answer was proved in [6,8,12,13]. Recall that an algebra A over a field K satisfies a polynomial identity (we say that A is PI for short) if there exists a

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non-zero polynomial $f(x_1, x_2, \dots, x_n)$ in the free associative K-algebra on $x_1, x_2, \dots, x_n$ such that $f(a_1, a_2, \dots, a_n) = 0$ for all $a_1, a_2, \dots, a_n \in A$.

The answer to Hartley's Conjecture shows that properties on the unit group give properties on the whole group algebra. Even special subsets of the unit group can determine such information. Indeed, in [9] was proved that for a group algebra of a torsion group over an infinite field K with $\mathrm{char}(K) \neq 2$, a group identity for the symmetric units under the classical involution $*$ also gives a polynomial identity on the whole group algebra.

Now let φ be an arbitrary involution on G and let K be a field. Then the K-linear extension of φ is an involution on KG. Denote by $(KG)_\varphi^+$ the symmetric elements of KG, i.e. elements x in KG for which $x = \varphi(x)$ and $\mathcal{U}_\varphi^+(KG)$ stands for the symmetric units. In this paper we prove that if for a locally finite group G over an infinite field K with $\mathrm{char}(K) \neq 2$ the symmetric units $\mathcal{U}_\varphi^+(KG)$ for an arbitrary involution φ satisfy a GI, it again follows, under some assumptions, that the group algebra KG is PI. Moreover, in case the prime radical of KG is nilpotent we characterize the groups for which the symmetric units satisfy a group identity.

2. Semisimple algebras

In the investigation of group identities central idempotents are crucial. In [9, Theorem 2] the following useful fact is proved.

Theorem 2.1. *Let R be a semiprime ring and φ an involution on R. If $\mathcal{U}_\varphi^+(R)$ satisfies a GI, then every symmetric idempotent of R is central.*

Let K be a field with $\mathrm{char}(K) = p \geqslant 0$. Let G be a group such that KG is semiprime and $\mathcal{U}_\varphi^+(KG)$ satisfies a GI. In case the group is endowed with the classical involution one immediately gets that a cyclic subgroup generated by a torsion element whose order is not divisible by p is normal in G. This instantly determines the structure of G. In case φ is an arbitrary involution, this property is lost. So we need other tools to derive information on G.

To deal with group algebras of finite groups we first characterize when the symmetric units of a semisimple algebra A over an infinite field of characteristic not 2 satisfy a GI. As in most group ring problems on units, division algebras in the Wedderburn decomposition of A form an obstacle. To overcome this, we can exclude them or put suitable conditions on K such that, for example, there exist free subgroups generated by symmetric units in the division algebra. Recall that involutions on an algebra that fix the center elementwise are said to be of first kind. The other ones are of second kind.

We will denote by $[x_1, x_2] = x_1x_2 - x_2x_1$ the additive commutator of $x_1, x_2 \in R$ and by $(x_1, x_2) = x_1x_2x_1^{-1}x_2^{-1}$ the (multiplicative) commutator of $x_1, x_2 \in \mathcal{U}(R)$. Recursively, one defines $(x_1, x_2, \dots, x_n) = ((x_1, x_2, \dots, x_{n-1}), x_n)$ where $x_1, \dots, x_n \in \mathcal{U}(R)$.

Theorem 2.2. *Let A be a semisimple K-algebra where K is an infinite field with* $\mathrm{char}(K) \neq 2$ *and φ be an involution on A. Suppose one of the following conditions holds*:

(1) *K is uncountable,*
(2) *A has no simple components that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}_\varphi^+(A)$ satisfies a GI if and only if A_φ^+ is central in A.

Proof. Suppose that $\mathcal{U}_\varphi^+(A)$ satisfies a GI, then by Theorem 2.1 symmetric idempotents of A are central. Let e_i be a primitive central idempotent of A.

If $\varphi(e_i) = e_i$, then φ induces an involution on $Ae_i = M_{n_i}(D_i)$ where D_i is a division algebra. There are two types of involutions on a matrix ring over a division algebra [5, Lemma 3.1]. In case φ induces an involution of transpose type, $E_{j,j}$ is a symmetric idempotent, thus central, for all $1 \leqslant j \leqslant n_i$. Hence $n_i = 1$ and $Ae_i \cong D_i$, a division algebra. In case φ induces an involution of symplectic type, then n_i is even and D_i is a field K_i. In this case $E_{1,1} + E_{\frac{n_i}{2}+1,\frac{n_i}{2}+1}$ is a symmetric idempotent, thus central, for all $1 \leqslant j \leqslant n_i$. Hence $n_i = 2$ and $Ae_i \cong M_2(K_i)$.

If $\varphi(e_i) \neq e_i$, then Ae_i and $A\varphi(e_i)$ are anti-isomorphic. Since $\varphi(e_i)$ is also a primitive central idempotent, we get that $Ae_i \oplus A\varphi(e_i) \cong M_{n_i}(D_i) \oplus M_{n_i}(D_i)$ with exchange involution given by $\varphi((a,b)) = (\varphi(b), \varphi(a))$. Hence $(Ae_i \oplus A\varphi(e_i))_\varphi^+ \cong \{(a, \varphi(a)) \mid a \in M_{n_i}(D_i)\}$. Since $\mathcal{U}_\varphi^+(A)$ satisfies a GI, also $GL_{n_i}(D_i) = \mathcal{U}(M_{n_i}(D_i))$ satisfies a GI. Since K is an infinite field, we have by [4, Corollary 1.4(2)] that $GL_{n_i}(D_i)$ has to be commutative, thus $n_i = 1$ and D_i is a field K_i.

Hence we have proved that $A \cong D_1 \oplus \cdots \oplus D_k \oplus M_2(K_1) \oplus \cdots \oplus M_2(K_l)$, a direct sum of division algebras (some of which are fields) and 2×2-matrices over a field, where φ induces the symplectic involution in the latter case and hence all symmetric elements of $\bigoplus_{j=1}^{l} M_2(K_j)$ are central.

Assume that condition (1) holds. Let D_i be a non-commutative simple component, then $(D_i \setminus \{0\})_\varphi^+$ satisfies a GI. Suppose first that φ induces an involution of the first kind. Since K is uncountable with $\mathrm{char}(K) \neq 2$ [5, Corollary 5.5] gives symmetric units in D_i that generate a free group (which contradicts GI) unless D_i is a quaternion algebra with anti-symmetric basis $\{1, i, j, k\}$. This means that $\varphi(i) = -i, \varphi(j) = -j, \varphi(k) = -k$ and hence φ induces the quaternion conjugation on D_i. Therefore the symmetric elements of D_i are in the center. Finally, suppose φ induces an involution of the second kind. Since K is uncountable with $\mathrm{char}(K) \neq 2$ [5, Theorem 6.1] also gives symmetric units in D_i that generate a free group (which contradicts GI) unless all symmetric elements of D_i are central or D_i is a quaternion algebra with anti-symmetric basis. Since the involution is of the second kind, the latter case cannot occur as [5, Lemma 2.6] again gives symmetric units that generate a free group.

In case (2) holds, suppose that D_i is a quaternion algebra. Then [5, Lemma 2.6] gives us symmetric units that generate a free group unless D_i is a quaternion algebra with anti-symmetric basis and φ induces an involution of the first kind. As shown above, φ induces the quaternion conjugation on D_i. Hence we can conclude that A_φ^+ is central in A.

The other implication is trivial. $\square$

Remark. By [7, Theorem 1], under the above assumptions, it follows that $\mathcal{U}_\varphi^+(A)$ satisfies a GI is equivalent to A_φ^+ is Lie nilpotent.

We state the following well-known fact which we will often make use of.

Lemma 2.3. *Let A be a K-algebra and let φ be a K-linear involution on A. Suppose that I is a nil ideal of A which is invariant under φ. Denote by $\bar{a}$ the image of a in A/I. Let $\bar{\varphi}$ be the involution on A/I defined by $\bar{\varphi}(\bar{a}) = \overline{\varphi(a)}$ with $a \in A$. Then every $\bar{\varphi}$-symmetric unit of A/I can be lifted to a φ-symmetric unit of A.*

Let R be a ring, then denote by $\mathcal{J}(R)$ the Jacobson radical of R. For a finite-dimensional algebra over a non-absolute field with $\mathrm{char}(K) \neq 2$ we can, under certain assumptions, obtain a relation between group identities and the existence of free groups in the symmetric units. Recall that a field K is non-absolute if it is not an algebraic extension of a finite field.

Proposition 2.4. *Let K be a non-absolute field with* $\mathrm{char}(K) \neq 2$. *Let A be a finite-dimensional K-algebra with involution φ. Suppose one of the following conditions holds*:

(1) *K is uncountable,*
(2) *$A/\mathcal{J}(A)$ has no simple components that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}_\varphi^+(A)$ does not contain a free group of rank 2 if and only if $\mathcal{U}_\varphi^+(A)$ satisfies the group identity $(x_1, x_2, \ldots, x_m) = 1$, for some positive integer m.

Proof. Suppose that $\mathcal{U}_\varphi^+(A)$ does not contain a free group of rank 2 and let $J = \mathcal{J}(A)$ be the Jacobson radical of A. Then $\varphi(J) = J$ and since A is finite dimensional we have that $J^m = 0$ for some positive integer m. Denote by $\bar{A} = A/J$. Now $\mathcal{U}_{\bar{\varphi}}^+(\bar{A})$ is a quotient of $\mathcal{U}_\varphi^+(A)$, hence it cannot contain free groups of rank 2. Since K is non-absolute, by [5, Proposition 3.3] we get a free group in $\mathcal{U}(M_{n_i}(D_i))_{\bar{\varphi}}^+$ unless $n_i = 2$ and $\bar{\varphi}$ is the symplectic involution. Hence we can follow the lines of the proof of Theorem 2.2 and we again get that $\bar{A} \cong D_1 \oplus \cdots \oplus D_k \oplus M_2(K_1) \oplus \cdots \oplus M_2(K_l)$, where $\bar{\varphi}$ is the symplectic involution on $M_2(K_i)$ for $1 \leqslant i \leqslant l$. Therefore $\bar{A}_{\bar{\varphi}}^+$ is in the center of $\bar{A}$ and $\mathcal{U}_{\bar{\varphi}}^+(\bar{A})$ satisfies the group identity $(x_1, x_2) = 1$ and hence $(x_1, x_2, \ldots, x_m) = 1$ is an identity for $\mathcal{U}_\varphi^+(A)$.

The converse is obvious. □

3. Group algebras

Let K be a field of characteristic $p \geqslant 0$ and G be a group. Let $\varphi : KG \to KG$ denote the K-linear extension of an involution φ defined on G. In [11] a characterization of the groups is given of which the symmetric elements $(KG)_\varphi^+$ of KG commute, that is, form a ring. It turns out that the non-commutative groups satisfying this property when $p \neq 2$ are precisely the groups with a unique non-trivial commutator and that satisfy the lack of commutativity property (LC for short). The latter means that for any pair of elements $g, h \in G$, it is the case that $gh = hg$ if and only if $g \in \mathcal{Z}(G)$ or $h \in \mathcal{Z}(G)$ or $gh \in \mathcal{Z}(G)$. By [10, Proposition III.3.6] such groups are precisely those non-commutative groups with $G/\mathcal{Z}(G) \cong C_2 \times C_2$, where C_2 denotes the cyclic group of order 2. Groups with the LC property and unique non-trivial commutator will be called LCUC.

Recall that a ring R (with identity) is regular if for each $x \in R$ there is a $y \in R$ such that $xyx = x$. Let K be a field of characteristic p and G a group. By [3] we have that the group algebra KG is regular if and only if G is locally finite and the order of each finite subgroup of G is a unit in K. Note that in this case KG is semiprime and the set of p-elements P is trivial (in case $\mathrm{char}(K) = 0$, we agree that $P = 1$).

We now classify the groups with a regular group algebra over an infinite field K with $\mathrm{char}(K) \neq 2$ for which the symmetric units satisfy a GI.

Theorem 3.1. *Let K be an infinite field with* $\mathrm{char}(K) = p \neq 2$ *and let G be a non-abelian group such that KG is regular. Let φ be an involution on G. Suppose one of the following conditions holds*:

(1) *K is uncountable,*
(2) *all finite non-abelian subgroups of G which are invariant under φ have no simple components in their group algebra over K that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}_\varphi^+(KG)$ satisfies a GI if and only if G has the lack of commutativity property and a unique non-trivial commutator s and the involution $\varphi : G \to G$ is given by

$$\varphi(g) = \begin{cases} g & \text{if } g \in \mathcal{Z}(G), \\ sg & \text{if } g \notin \mathcal{Z}(G). \end{cases}$$

Moreover, in this case $(KG)_\varphi^+$ is a ring contained in $\mathcal{Z}(KG)$.

Proof. Suppose that $\mathcal{U}_\varphi^+(KG)$ satisfies a GI. Let $g, h \in G$ be such that $gh \neq hg$. Write $G = \bigcup_i H_i$, where H_i runs through all finite (non-abelian) subgroups of G which are invariant under φ and that contain g and h. Since KG is regular, we have that for all i, KH_i is semisimple and $\mathcal{U}_\varphi^+(KH_i)$ satisfies a GI. Because of Theorem 2.2 we have that $(KH_i)_\varphi^+$ is central in KH_i. By [11, Theorem 2.4] the latter is equivalent with H_i having the LC property, a unique non-trivial commutator $s = ghg^{-1}h^{-1}$ and the involution $\varphi : H_i \to H_i$ is as given in the statement.

It follows that G has a unique non-trivial commutator, namely s. Indeed, assume the contrary, then there exist $x, y \in G$ with $xyx^{-1}y^{-1} \neq s$. Now x and y are contained in some H_i, for example,

$$H_i = \langle x, y, g, h, \varphi(x), \varphi(y), \varphi(g), \varphi(h) \rangle.$$

Therefore $xyx^{-1}y^{-1} = ghg^{-1}h^{-1} = s$, a contradiction.

As for all i we have that $\mathcal{Z}(H_i) = (H_i)_\varphi^+$, we have that $G_\varphi^+ = \bigcup_i (H_i)_\varphi^+ = \bigcup_i \mathcal{Z}(H_i)$. If $x \in \mathcal{Z}(G)$, then it is clear that $x \in G_\varphi^+$. When $x \in G_\varphi^+$, x is contained in the center of some H_i and hence x commutes with g. As g is an arbitrary non-central element, we have that $x \in \mathcal{Z}(G)$. Therefore we have that G is LCUC with involution as stated and $(KG)_\varphi^+$ is a ring contained in $\mathcal{Z}(KG)$. Moreover, $s = g\varphi(g)^{-1}$ for all $g \in G \setminus \mathcal{Z}(G)$.

The converse is obvious, because $(KG)_\varphi^+$ is a ring contained in $\mathcal{Z}(KG)$. □

Remark. For example, condition (2) is fulfilled when $\mathrm{char}(K) = 0$ and G is a locally finite group such that all finite non-abelian subgroups H which are invariant under φ are such that the order of every root of unity in each $K(\chi_i) = K\{\chi_i(h) \mid h \in H\}$, where χ_i is an irreducible character of H, is at most 2. Indeed, then every simple component that is a non-commutative division algebra is a quaternion algebra [1,2].

Another case in which condition (2) holds is when $K = \mathbb{Q}$ and the subgroups H mentioned above do not have non-abelian homomorphic images that are fixed-point free, because then by [16] it follows that the rational group algebra of such finite groups has no non-commutative division algebras as simple components. Recall that a group H is said to be fixed-point free if it has a complex irreducible representation ρ such that for every non-identity element h of H, $\rho(h)$ has all eigenvalues different from one.

To deal with groups for which the group algebra is not regular we need some extra machinery. The following lemma shows that there are a lot of symmetric elements contained in the Jacobson radical. This will lead to information on the p-elements.

Lemma 3.2. *Let A be a finite-dimensional K-algebra with K an infinite field with* $\mathrm{char}(K) \neq 2$. *Let φ be a K-linear involution on A and suppose that $\mathcal{U}_{\varphi}^{+}(A)$ satisfies a GI. Then if $x \in A$ is such that $x^n \in \mathcal{J}(A)$, then $x\varphi(x), x + \varphi(x) \in \mathcal{J}(A)$.*

Proof. Let $x \in A$ be such that $x^n \in \mathcal{J}(A)$, then $\bar{x}^n = 0$ in $\bar{A} = A/\mathcal{J}(A)$. By Lemma 2.3, $\mathcal{U}_{\bar{\varphi}}^{+}(\bar{A})$ satisfies a GI and thus by Theorem 2.2 $\bar{A}$ is isomorphic to a direct sum of division algebras and 2×2-matrices over a field. Hence we get that $\bar{x}^2 = 0$. By [9, Lemma 2] we have that there exists a positive integer m such that $(\bar{x}\bar{\varphi}(\bar{x}))^m = 0$. Since $\bar{A}_{\bar{\varphi}}^{+}$ is central in the semisimple algebra $\bar{A}$, we have that $\bar{x}\bar{\varphi}(\bar{x}) = 0$ and similarly we get that $\bar{\varphi}(\bar{x})\bar{x} = 0$. Then $\bar{x} + \bar{\varphi}(\bar{x})$ is symmetric with square zero and thus also $\bar{x} + \bar{\varphi}(\bar{x}) = 0$, which finishes the proof of the lemma. □

Lemma 3.3. *Let K be an infinite field with* $\mathrm{char}(K) = p \neq 2$ *and let G be a locally finite group. Let φ be an involution on G. If $\mathcal{U}_{\varphi}^{+}(KG)$ satisfies a GI, then P is a normal subgroup of G.*

Proof. Since G is locally finite, in order to show that P is a subgroup, we may assume that G is finite. If $p = 0$, then $P = 1$. Suppose that $p > 2$, then because of [9, Lemma 5], we only need to prove that for all $g \in P$, $(g-1)^2 \in \mathcal{J}(KG)$.

Take $g \in P$, then $g - 1$ is nilpotent. Hence there exists a positive integer n such that $(g-1)^n = 0$. Then by Lemma 3.2

$$(g-1) + \bigl(\varphi(g) - 1\bigr) = g + \varphi(g) - 2 \in \mathcal{J}(KG). \tag{1}$$

Hence multiplying by g we have that

$$g^2 + g\varphi(g) - 2g \in \mathcal{J}(KG). \tag{2}$$

On the other hand, again by Lemma 3.2,

$$(g-1)\bigl(\varphi(g) - 1\bigr) = g\varphi(g) - g - \varphi(g) + 1 \in \mathcal{J}(KG). \tag{3}$$

Then by subtracting (2) from (3) and adding (1) we get that $(g-1)^2 \in \mathcal{J}(KG)$. Hence P is a normal subgroup in G. □

When we work under the conditions of Lemma 3.3, we from now on implicitly use that P is a (normal) subgroup of G. If N is a normal subgroup of G, we denote by $\Delta(G, N)$ the kernel of the map $KG \to K(G/N)$ defined by

$$\sum_{g \in G} \lambda_g g \to \sum_{g \in G} \lambda_g gN$$

and $\Delta(G, G) = \Delta(G)$ is the augmentation ideal.

Theorem 3.4. *Let K be an infinite field with* $\mathrm{char}(K) = p \neq 2$. *Let G be a locally finite group and φ an involution on G. Suppose that $\mathcal{U}_\varphi^+(KG)$ satisfies a GI and that one of the following conditions holds*:

(1) *K is uncountable,*
(2) *all finite non-abelian subgroups of G/P which are invariant under φ have no simple components in their group algebra over K that are non-commutative division algebras other than quaternion algebras,*

then we have that

(1) *G/P is abelian, or*
(2) *G/P has the LC property, a unique non-trivial commutator $\bar{s}$ and the involution $\overline{\varphi} : G/P \to G/P$ induced by φ is given by*

$$\overline{\varphi}(\bar{g}) = \overline{\varphi(g)} = \begin{cases} \bar{g} & \text{if } \bar{g} \in \mathcal{Z}(G/P), \\ \overline{sg} & \text{if } \bar{g} \notin \mathcal{Z}(G/P). \end{cases}$$

Moreover, KG is PI.

Proof. By Lemma 3.3, we have that P is a (normal) subgroup of G.

Now $K(G/P) \cong KG/\Delta(G, P)$ and $\Delta(G, P)$ is nil. Hence we have by Lemma 2.3 that $\mathcal{U}_\varphi^+(K(G/P))$ satisfies a GI. As $K(G/P)$ is regular, it follows by Theorem 3.1 that G/P is abelian or G/P has the LC property and a unique non-trivial commutator $\bar{s}$ and the involution $\overline{\varphi} : G/P \to G/P$ is as in the statement.

Hence $K(G/P)$ is PI. Since $\Delta(G, P)$ is a nil subring of KG invariant under φ, by [9, Remark 2] we have that $\Delta(G, P)$ is PI. As being PI is closed under ideal extensions we have that KG is PI. $\square$

Finally we can characterize under certain assumptions the locally finite groups for which the symmetric units of the group algebra satisfy a GI. For the converse of Theorem 3.4 we need the following lemma.

Lemma 3.5. *Let K be a field with* $\mathrm{char}(K) = p \neq 2$. *Let G be a locally finite group and φ an involution on G. If P is a subgroup of bounded exponent and G/P is abelian or LCUC with involution as in Theorem* 3.1, *then $\mathcal{U}_\varphi^+(KG)$ satisfies a GI.*

Proof. Assume that P is a subgroup of bounded exponent and that G/P and φ are as in the statement. Then by [11, Theorem 2.4], $\mathcal{U}_\varphi^+(K(G/P))$ is abelian. Hence $(\mathcal{U}_\varphi^+(KG), \mathcal{U}_\varphi^+(KG)) \subset 1 + \Delta(G, P)$. Now $\Delta(G, P)$ is nil of bounded exponent and thus $(\mathcal{U}_\varphi^+(KG), \mathcal{U}_\varphi^+(KG))^{p^n} = 1$ for some $n \geqslant 0$. Hence $\mathcal{U}_\varphi^+(KG)$ satisfies a GI. $\square$

Remark. Note that, under the assumptions of Lemma 3.5, in case KG is PI and G/P is abelian, we get that $G' \subset P$ is of bounded exponent. Hence by [15] even $\mathcal{U}(KG)$ satisfies a GI. Moreover, if G/P is LCUC one easily deduces that also in this case G' is of bounded exponent, but not necessarily a p-group.

Let $C_G(g)$ be the centralizer of g in G. Denote by $\phi(G) = \{g \in G \mid [G : C_G(g)] < \infty\}$ the finite conjugacy subgroup of G and $\phi_p(G) = \langle P \cap \phi(G)\rangle$.

Theorem 3.6. *Let K be an infinite field with* $\mathrm{char}(K) = p \neq 2$. *Let G be a locally finite group and φ an involution on G. Suppose that the prime radical of KG is nilpotent and that one of the following conditions holds*:

(1) *K is uncountable,*
(2) *all finite non-abelian subgroups of G/P which are invariant under φ have no simple components in their group algebra over K that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}_\varphi^+(KG)$ satisfies a GI if and only if P is a finite normal subgroup and G/P is abelian or LCUC with involution as in Theorem 3.1.

Proof. Suppose $\mathcal{U}_\varphi^+(KG)$ satisfies a GI, then by Theorem 3.4 we have that P is a normal subgroup, G/P is abelian or LCUC with involution as in Theorem 3.1 and KG is PI. Thus by [14, Theorem 5.2.14] $\phi(G)$ is a normal subgroup of finite index in G (and $\phi(G)'$ is finite). Since the prime radical is nilpotent [14, Theorem 8.1.12] gives us that $\phi_p(G) = P \cap \phi(G)$ is a finite normal p-subgroup. As $P\phi(G)/\phi(G) \cong P/\phi_p(G)$ is finite, we get that P is finite. The converse follows from Lemma 3.5. □

We can now easily characterize the locally finite groups with semiprime group algebra for which the symmetric units of the group algebra satisfy a GI. Since the only groups for which the classical involution equals the involution from Theorem 3.1 are Hamiltonian 2-groups, we get the same result as stated in [9, Theorem 5].

Corollary 3.7. *Let K be an infinite field with* $\mathrm{char}(K) = p \neq 2$ *and let G be a locally finite group such that KG is semiprime. Let φ be an involution on G. Suppose one of the following conditions holds*:

(1) *K is uncountable,*
(2) *all finite non-abelian subgroups of G/P which are invariant under φ have no simple components in their group algebra over K that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}_\varphi^+(KG)$ satisfies a GI if and only if G is abelian or LCUC with involution as in Theorem 3.1.

Proof. Suppose $\mathcal{U}_\varphi^+(KG)$ satisfies a GI, then KG is semiprime PI. Hence $\phi_p(G) = 1$ and following the lines of the previous proof we get that $P = 1$ and thus KG is regular. The result now follows from Theorem 3.1. □

In case of the classical involution we get in general the following result.

Theorem 3.8. *Let K be an infinite field with* $\mathrm{char}(K) \neq 2$ *and let G be a locally finite group. Let $*$ denote the classical involution on G. Suppose one of the following conditions holds*:

(1) *K is uncountable,*
(2) *all finite non-abelian subgroups of G/P have no simple components in their group algebra over K that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}_*^+(KG)$ satisfies a GI if and only if P is a normal subgroup of bounded exponent and G/P is abelian or an Hamiltonian 2-group.

Proof. Suppose $\mathcal{U}_*^+(KG)$ satisfies a GI, then by Theorem 3.4 we have that P is a normal subgroup and G/P is abelian or an Hamiltonian 2-group. From the proof of [9, Theorem 7] it follows that P is of bounded exponent. The argument uses Lemmas 6 and 9 from [9] which strongly rely on the classical involution. The converse follows from Lemma 3.5. $\square$

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