Discrete Mathematics 58 (1986) 93–95
North-Holland

NOTE

A DECOMPOSITION OF COMPLETE BIPARTITE GRAPHS INTO EDGE-DISJOINT SUBGRAPHS WITH STAR COMPONENTS

Yoshimi EGAWA and Masatsugu URABE
Department of Applied Mathematics, Faculty of Science, Science University of Tokyo, Tokyo 160, Japan

Toshihito FUKUDA and Seiichiro NAGOYA
Department of Information Mathematics, University of Electro-Communications, Tokyo 182, Japan

Received 20 December 1983
Revised 9 August 1984

A subgraph F of a graph G is called a star-subgraph if each component of F is a star. In this note we show that the minimum number of star-subgraphs that partition the edges of $K_{2n-1,2n-1}$ or $K_{2n,2n}$, respectively, is $n+2$.

The notions of star-subgraph and star-index are introduced by Akiyama and Kano in [1]. A subgraph F of a graph G is called a *star-subgraph* if each component of F is a star. Here by a star, we mean a complete bipartite graph of the form $K_{1,m}$ with $m \geq 1$. The *star-index* of G, denoted by $*(G)$, is the minimum number of star-subgraphs that partition the edges of G. Kano posed the following question verbally:

"What is the star-index of a complete bipartite graph?"

In this paper, we show:

Theorem. $*(K_{2n-1,2n-1}) = *(K_{2n,2n}) = n+2, \quad n \geq 4.$

In order to prove this result, it clearly suffices to prove the following two lemmas.

Lemma 1. $*(K_{2n-1,2n-1}) \geq n+2, \quad n \geq 4.$

Lemma 2. $*(K_{2n+2,2n}) \leq n+2, \quad n \geq 2.$

Proof of Lemma 1. We proceed by induction. By way of contradiction, suppose that there exists a decomposition

$$E(K_{2n-1,2n-1}) = \bigcup_{i=1}^{n+1} F_i$$

of the edge set $E(K_{2n-1,2n-1})$ such that each $\langle F_i \rangle$ is a star-subgraph. For each i, let $S_{i,1}, S_{i,2}, \ldots, S_{i,k_i}$ be the connected components of $\langle F_i \rangle$, and, for each $1 \leq h \leq k_i$, let $v_{i,h}$ be a vertex of $S_{i,h}$ for which the degree in $S_{i,h}$ is maximum, and set $W_i = \{v_{i,h} \mid 1 \leq h \leq k_i\}$. (Unless $S_{i,h} \cong K_{1,1}$, $v_{i,h}$ is what is called the center of $S_{i,h}$.) Let X denote the set of all vertices x such that x belongs to only one of the W_i. We first remark that if $e \in F_i$, then exactly one of the endpoints of e belongs to W_i.

We next argue that $|X \cap W_i| \leq 1$, $1 \leq i \leq n+1$. By way of contradiction, suppose that $x, y \in W_i$, $x \neq y$, and let S and T be the connected components of $\langle F_i \rangle$ containing x and y, respectively. Let $V(K_{2n-1,2n-1}) = A \cup B$ be the bipartition of $K_{2n-1,2n-1}$. We may assume $x, y \in A$, for if x and y were adjacent, then the edge xy could not belong to any of the F_j by the above remark, which would be absurd. Since the numbers of the edges of F_j incident to x and y, respectively, are both at most 1 for all $j \neq i$, $|V(S) \cap B|$ and $|V(T) \cap B|$ are at least $(2n-1) - n = n-1$. Consequently, if we set

$$U = ((V(S) \cap B) \cup (V(T) \cap B)) \cup (A - \{x, y\}),$$

$\langle U \rangle$ contains $K_{2n-3,2n-3}$. Since the F_j, $j \neq i$, must cover the edges of $\langle U \rangle$, this contradicts the inductive hypothesis. Thus $|X \cap W_i| \leq 1$ for all i, and so $|X| \leq n+1$. Since each vertex belongs to at least one of the W_i,

$$\sum_{i=1}^{n+1} k_i \geq 2\,|V(K_{2n-1,2n-1})| - |X| \geq 7n - 5.$$

Hence

$$\begin{aligned}\sum_{i=1}^{n+1} |F_i| &\leq \sum_{i=1}^{n+1} (|V(K_{2n-1,2n-1})| - k_i) \\ &\leq (n+1)(4n-2) - (7n-5) \\ &< 4n^2 - 4n + 1 = |E(K_{2n-1,2n-1})|.\end{aligned}$$

This contradiction proves the Lemma. □

Proof of Lemma 2. Let

$$V(K_{2n+2,2n}) = \{u_i, v_i \mid 0 \leq i \leq n\} \cup \{x_i, y_i \mid 1 \leq i \leq n\}$$

be the bipartition of $K_{2n+2,2n}$. Set

$$F_k = \{u_k y_i,\ v_k x_i \mid 1 \leqslant i \leqslant n,\ i \neq k\} \cup \{u_j x_k,\ v_j y_k \mid 0 \leqslant j \leqslant n,\ j \neq k\},\quad k = 1, \ldots, n$$

$$F_{n+1} = \{u_0 y_i,\ u_i x_i,\ v_i x_i \mid 1 \leqslant i \leqslant n\},$$

$$F_{n+2} = \{v_0 x_i,\ u_i y_i,\ v_i y_i \mid 1 \leqslant i \leqslant n\}.$$

Then this decomposition yields the desired upper bound. □

Acknowledgment

The authors wish to thank Professor Jin Akiyama, Professor Mikio Kano and Professor Hikoe Enomoto for their valuable suggestions.

Reference

[1] J. Akiyama and M. Kano, Path factors of a graph, in: Graphs and Applications (Wiley, New York, 1984).