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Variations on a theme of Steinberg

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Abstract

The Steinberg tensor product theorem is a fundamental tool for studying irreducible representations of simple algebraic groups over fields of positive characteristic. This paper is concerned with extending the result, replacing the target group $SL(V)$ by an arbitrary simple algebraic group.

1. Introduction

Let K be an algebraically closed field of characteristic $p > 0$, and let X be a simple, simply connected algebraic group over K. The Steinberg tensor product theorem [14] is fundamental to the analysis of irreducible rational representations of X. In this paper we establish similar results for morphisms from X into simple algebraic groups of arbitrary type.

Steinberg's theorem shows that if $\phi : X \to SL(V)$ is an irreducible rational representation, then we can write $V = V_1^{(q_1)} \otimes \cdots \otimes V_k^{(q_k)}$, where the V_i are restricted KX-modules and the q_i are distinct powers of p. The result can be reformulated in terms of a factorization of ϕ:

$$X \to X \times \cdots \times X \to GL(V),$$

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where the first map is a twisted diagonal map $x \to (x^{(q_1)}, \dots, x^{(q_k)})$, where $x^{(q_i)}$ denotes the image of x under a standard Frobenius q_i-map, and the second map restricts to a completely reducible representation on each simple factor, with restricted composition factors. Under the assumption $q_1 < \cdots < q_k$, one has a uniqueness result as well. With the above formulation the result extends to completely reducible representations.

Our goal is to generalize this result, replacing the target group $SL(V)$ by an arbitrary simple algebraic group G, assuming p is a good prime for G. The extension to classical groups is relatively minor. On the other hand, obtaining such a result for exceptional groups is much deeper and the results rest on the analysis of subgroups of exceptional groups along with results from [13].

The formulation requires two ingredients: a generalization of the usual notion of complete reducibility and a suitable analog for the notion of a restricted representation. We shall develop intrinsic versions of these concepts.

Throughout the paper, G denotes a connected simple algebraic group over an algebraically closed field K of characteristic p which is assumed to be a good prime for G. (Recall that this means $p > 2$ for groups of type B_n $(n \geqslant 2)$, C_n $(n \geqslant 2)$, D_n $(n \geqslant 4)$ and $p > 3$ for exceptional groups, except E_8, where $p > 5$.)

The following notion was introduced by Serre.

Definition. A subgroup $D < G$ is called *G-completely reducible* (G-cr for short), if whenever D is contained in a parabolic subgroup P of G, it is contained in a Levi subgroup of P.

For $G = SL(V)$ this notion agrees with the usual notion of complete reducibility. In fact, if G is any of the classical groups then the notions coincide, although for symplectic and orthogonal groups this requires our assumption that p is a good prime for G.

Complete reducibility of representations and the notion of G-cr subgroups have been the focus of several recent articles. The following result provides conditions which guarantee that certain subgroups satisfy the G-cr condition. In particular, the result shows that this is quite often the case when G is an exceptional group.

G-cr Theorem (McNinch [11], Liebeck–Seitz [7]). *Let X be a connected simple subgroup of G. Then X is G-cr if either of the following hold*:

(i) *G is classical with natural module V, and $p \geqslant \dim V/\operatorname{rank}(X)$.*
(ii) *G is of exceptional type and $p > 7$.*

In particular, if $p \geqslant h(G)$, the Coxeter number of G, then all closed, connected simple subgroups of G are G-cr.

We remark that [7] establishes results stronger than what is asserted in (ii) above. The characteristic requirements depend on the pair (G, X); for example $p > 7$ is needed only when $G = E_7, E_8$ with X of rank 1 or 2.

We next aim at a suitable notion of a restricted morphism. A few preliminary remarks are required. If X is a simple, simply connected algebraic group and $\phi : X \to G$ is a morphism, then ϕ lifts to a morphism $\hat{\phi}: X \to \widehat{G}$, where $\widehat{G}$ is the simply connected cover of G.

Next, we extend the usual notion of irreducible restricted representation by defining a (not necessarily irreducible) representation $X \to GL(V)$ to be *restricted* if all composition factors are restricted.

If G is of classical type, by the *natural* $\widehat{G}$-module we mean the usual classical module (of high weight λ_1). We allow more than one natural module in a few cases. For $G = A_n$, we also allow the dual of the usual module and for $G = D_4$ we define as natural each of the three 8-dimensional modules of high weights $\lambda_1, \lambda_3, \lambda_4$. Also, B_2 has two natural modules, of dimensions 4 and 5, because of the isomorphism $B_2 \cong C_2$; likewise $A_3 \cong D_3$ has two natural modules of dimensions 4, 6.

Definition. Let X be simple and simply connected. A morphism $\phi: X \to G$ is *restricted* if either of the following holds:

(i) $X = SL_2$, and composing ϕ with the adjoint representation of G, all weights of a maximal torus of X are at most $2p - 2$.

(ii) $X \neq SL_2$ and $X \xrightarrow{\hat{\phi}} \widehat{G} \to GL(V)$ is a restricted representation, where V is a natural $\widehat{G}$-module if G is of classical type and $V = L(\widehat{G})$ if G is of exceptional type.

Condition (i) says that $\phi(X)$ is a *good* A_1 in the sense of [13]. For classical groups these are just A_1's which have restricted action on the natural $\widehat{G}$-module. The definition in (ii) does not depend on the natural module chosen in those cases where there is more than one natural module (see Lemma 5.1).

The next result provides a more uniform criterion for a restricted morphism.

Restricted Morphism Theorem. *Let X be simple and simply connected, and let $\phi: X \to G$ be a morphism such that the image $\phi(X)$ is G-cr. Then ϕ is restricted if and only if $C_G(\phi(X))^0 = C_G(d\phi(L(X)))^0$.*

A connected simple subgroup of G is called *restricted* if it is the image of a restricted morphism. (So with this definition, the good A_1's of [13] are also called restricted A_1's of G.) We extend this to semisimple groups X and morphisms $\phi: X \to G$, by saying that ϕ is *restricted* if its restriction to each simple factor is restricted.

We now state our generalization of the Steinberg tensor product theorem. In the following we fix X with an $\mathbb{F}_p$-structure and corresponding Frobenius p-power maps. The morphism $x \to x^{(q)}$ refers to the Frobenius q-power map.

Theorem 1. *Let G be a simple algebraic group over K in good characteristic p. Assume X is a simply connected, simple algebraic group over K and $\phi: X \to G$ is a nontrivial morphism with image group G-cr. Then there is a unique integer k, unique powers q_i of p with $q_1 < \cdots < q_k$, and unique morphisms ψ and μ, such that ϕ factors $X \xrightarrow{\psi} X \times \cdots \times X \xrightarrow{\mu} G$, where $\psi(x) = (x^{(q_1)}, \ldots, x^{(q_k)})$ and μ is restricted with finite kernel.*

Theorem 1 can be formulated in terms of subgroups of G, where there are significant applications, especially for exceptional groups.

Corollary 1. *If X is a connected simple G-cr subgroup of G, then there is a uniquely determined commuting product $E_1 \cdots E_k$ with $X \leqslant E_1 \cdots E_k \leqslant G$, such that each E_i is a simple restricted subgroup of the same type as X, and each of the projections $X \to E_i/Z(E_i)$ is nontrivial and involves a different field twist.*

It will be shown in Lemma 7.2 and Proposition 9.2 that the commuting product $E_1 \cdots E_k$ given by Corollary 1 and each of its simple factors are G-cr. We also remark that there is a unique i such that $L(X) = L(E_i)$. The other projections involve nontrivial and distinct field twists. These projections may also involve twists by graph automorphisms and in the case of B_2, G_2, F_4 with $p = 2, 3, 2$, respectively, exceptional isogenies may also be present.

Steinberg's theorem also applies to finite groups of Lie type, $Y(q)$, where q is a power of p. Take $Y(q)$ of universal type so that $Y(q) = Y_\sigma$ for a simply connected, simple algebraic group Y, with σ a Frobenius morphism. Here the Steinberg theorem shows that any irreducible representation $Y(q) \to SL(V)$, for V finite-dimensional over the algebraic closure of $\mathbb{F}_q$, extends to an irreducible representation of Y.

Our next result extends this to arbitrary simple algebraic groups. However, to obtain a result covering exceptional groups, we require an assumption on the underlying finite field $\mathbb{F}_q$ defining the finite group.

Consider a homomorphism $\phi : Y(q) \to G$, where G is a simple exceptional group in (good) characteristic p. In [9, Theorem 1] it is shown that for q sufficiently large, there is a connected subgroup of G, containing $\phi(Y(q))$, which stabilizes all $\phi(Y(q))$ invariant subspaces of $L(G)$. Usually $q > 9$ is sufficient, but a larger bound is required for the case where $Y(q)$ is a rank 1 group. This field restriction is required for our next theorem.

In order to formulate a uniqueness result, we need the following terminology. If $Y > Y(q)$ are as above, a morphism $\psi : Y \to G$ is said to be *q-restricted* if $\psi(Y)$ is G-cr and in the factorization given by Theorem 1, each of the field twists q_i is less than q.

In the special cases $Y(q) = {}^2B_2(q), {}^2G_2(q), {}^2F_4(q)$, with $p = 2, 3, 2$, respectively, we must modify the above definition slightly. We are assuming that p is good, so these cases only occur when G is classical. If V is the natural module for G, we say that ψ is (q, s)-*restricted* if ψ is q-restricted and the high weights of all composition factors of Y on V have support on the short fundamental roots.

Theorem 2. *With notation as above, let $\phi : Y(q) \to G$ be a homomorphism with image group G-cr. If G is of exceptional type, suppose also that q satisfies the lower bounds in the hypothesis of* [9, Theorem 1]. *Then ϕ factors uniquely as $Y(q) \hookrightarrow Y \xrightarrow{\psi} G$, where the first map is inclusion, and ψ is a q-restricted morphism ((q, s)-restricted if $Y(q) = {}^2B_2(q), {}^2G_2(q), {}^2F_4(q)$), with image group G-cr.*

Theorem 2 can also be formulated in terms of subgroups of G along the lines of Corollary 1. We define a connected, simple subgroup of G to be q-restricted (respectively (q, s)-restricted), if it is the image of a q-restricted (respectively (q, s)-restricted) morphism.

Corollary 2. *Let $Y(q)$ be a G-cr subgroup of G. If G is of exceptional type, suppose also that q satisfies the lower bounds in the hypothesis of* [9, Theorem 1]. *Then there is a unique connected, simple subgroup Y of G such that Y contains $Y(q)$, Y is of the same type as $Y(q)$, and Y is q-restricted ((q,s)-restricted if $Y(q) = {}^2B_2(q), {}^2G_2(q), {}^2F_4(q)$).*

When studying a subgroup $X < G$, it is important to have information on the action of X on certain modules for G, in particular, the adjoint module and, for G of classical type, the natural module. For G classical and X a G-cr subgroup, this is relatively easy, since one can obtain the precise action of X on the classical module from knowledge of high weights of composition factors. A result for G-cr subgroups of exceptional groups, giving the precise action on the adjoint module is highly desirable, but has until now proved elusive. Results exist (e.g., [5,7]) which determine the composition factors of X on $L(G)$, but not the precise action. The difficulty is that even though the subgroup X is G-cr, complicated indecomposable modules may occur within $L(G) \downarrow X$. In the following we establish results that resolve this problem.

We fix notation as follows to be used in Theorems 3 and 4 below. As before, X will denote a connected simple G-cr subgroup of G, a simple algebraic group in good characteristic. Let $E_1, \ldots, E_k$ and $1 = q_1 < \cdots < q_k$ be the corresponding subgroups and prime powers given by Corollary 1.

Theorem 3 is a tensor product theorem in the case where $X = A_1$ in its representation on the adjoint module, $L(G)$. Here tilting modules are the basic objects.

Recall that a tilting module is one which has filtrations both by Weyl modules and also by dual Weyl modules. For each non-negative integer c, there is a unique indecomposable tilting module $T(c)$ for A_1 of highest weight c, and every tilting module is a direct sum of these. Some basic information on tilting A_1-modules can be found in [13, Section 2].

The results in [13] highlight the importance of tilting modules for restricted (i.e., good) A_1's in G. It is shown in [13, Theorem 1.1(iii)] that with one exception $L(G) \downarrow A_1$ is a tilting module for such an A_1. The exception occurs only for G of type A_n with $p \mid n+1$ and even here we get a tilting module if we replace G by GL_{n+1}.

Theorem 3. *Let G be a simple algebraic group in good characteristic p, except for the case where G is of type A_n with $p \mid n+1$, in which case assume that $G = GL_{n+1}$. Let $X = A_1$ be a connected simple, G-cr subgroup of G. Then $L(G) \downarrow X$ is a direct sum of modules of the form $T(c_1)^{(q_1)} \otimes \cdots \otimes T(c_k)^{(q_k)}$, where for $1 \leqslant i \leqslant k$, $T(c_i)$ is a tilting module for E_i of high weight $c_i \leqslant 2p-2$.*

The tilting decomposition of Theorem 3 does not extend to groups of rank greater than 1, as can be easily seen by looking at classical groups. However, for exceptional groups it is still possible to obtain a tensor product theorem with information on tensor factors. The result is as follows.

Theorem 4. *Let G be a simple exceptional group in good characteristic p and let X be a connected simple G-cr subgroup of rank at least 2. Then $L(G) \downarrow X$ is a direct sum of modules of the form $V_1^{(q_1)} \otimes \cdots \otimes V_k^{(q_k)}$, where each V_i is a restricted module for E_i. Moreover, one of the following holds*:

(i) *each V_i is a Weyl module, a dual Weyl module, or a tilting module*;
(ii) $p = 7, X = G_2$ *and either* X *is maximal in an* F_4 *subgroup of* G, *or* $X < F_4G_2 < E_8 = G$ *with* X *projecting to a maximal subgroup of the* F_4 *factor.*

We remark that (ii) is a real exception. Indeed, if $p = 7$ and $G_2 < F_4$ is maximal, then $L(F_4) \downarrow G_2$ is a direct sum of two irreducibles $V_{G_2}(01) \oplus V_{G_2}(11)$, while the Weyl module $W_{G_2}(11)$ is reducible with irreducible maximal submodule of high weight 20 (see [12]).

Corollary 3. *Assume* G *is an exceptional group and* $p > 7$. *If* X *is a connected simple subgroup of* G *of rank at least* 2, *then* $L(G) \downarrow X$ *is completely reducible with each irreducible summand a twisted tensor product of* (*irreducible*) *Weyl modules.*

Corollary 3 combines with Theorem 1 to yield a tensor product theorem with respect to the adjoint representation of G. This tensor product theorem contains much more information than what is provided by the Steinberg tensor product theorem for the representation $X \to G \to GL(V)$, with $V = L(G)$. Indeed, the latter shows that the image of X is contained in a certain product of subgroups of $GL(V)$. Theorem 1 implies that these subgroups are actually contained in the image of G.

Corollary 1 reduces the problem of determining connected simple G-cr subgroups of G to the problem of determining commuting products of restricted subgroups. In the last section of the paper we establish results which should be useful in determining all such commuting products (see, for example, Corollary 9.5).

The paper is organized as follows. In Section 2 we discuss material on subgroups of algebraic groups which will be required for work on exceptional groups. Theorem 1 is proved in Sections 3 and 4, the former for the uniqueness assertion and the latter establishing existence of the factorization. The Restricted Morphism Theorem is deduced in Section 5, and Theorems 2–4 are proved in Sections 6–8, respectively. The paper concludes with a section containing applications of the results of this paper to the analysis of subgroups of exceptional algebraic groups.

Notation. We shall use the following notation for representations: if X is a reductive algebraic group and λ a dominant weight, then $V_X(\lambda)$, $W_X(\lambda)$, $T_X(\lambda)$ denote the corresponding irreducible module, Weyl module, or indecomposable tilting module of high weight λ, respectively. If $\lambda_1, \dots, \lambda_k$ are dominant weights, then $\lambda_1/\lambda_2/\cdots/\lambda_k$ will denote a module having the same composition factors as $W_X(\lambda_1) \oplus \cdots \oplus W_X(\lambda_k)$. Finally, $\lambda_1|\lambda_2|\cdots|\lambda_k$ denotes a module having composition factors $V_X(\lambda_1), \dots, V_X(\lambda_k)$.

2. G-cr and restricted subgroups of exceptional groups

When G is of exceptional type, the results of this paper ultimately rely on a major analysis of the subgroup structure of exceptional algebraic groups. Indeed the results of [7] are key to finding the commuting product required for Theorem 1. In this section we derive results from this analysis which will be required later. The main result of the section is

Proposition 2.3, which is not only used in the proof of Theorem 1, but is also fundamental to the proof of the Restricted Morphism Theorem.

The maximal connected reductive subgroups of exceptional algebraic groups were determined in [12], under certain mild assumptions on the characteristic p of the underlying field. These assumptions are slightly stronger than the assumption that p is a good prime. Then in [7] the authors analyzed arbitrary reductive subgroups under roughly the same characteristic restrictions. More recently, in [10] the authors have extended the results of [12], removing all characteristic restrictions. Parts of this work together with the results and arguments of [7] will be needed in what follows.

The following theorem is the final result on maximal subgroups. It is considerably stronger than what we need here, as we are assuming p is a good prime for G.

Theorem 2.1 [10,12]. *Let G be an exceptional algebraic group in arbitrary characteristic $p > 0$, and let M be a maximal connected subgroup of G. Then either M is parabolic, reductive of maximal rank, or G, M are as in Table* 1. *Maximal subgroups of each type indicated in the table exist, subject to the indicated restrictions on p, and are unique up to* $\mathrm{Aut}(G)$*-conjugacy.*

Remarks. 1. For $G = E_7, E_8$, Table 1 has repetitions for groups of type A_1. This is done to indicate distinct conjugacy classes of subgroups of this type.

2. We shall be using Theorem 2.1 only in the case where p is a good prime for G; in this case Theorem 2.1 is already proved in [12], except when $X = A_1,\ p \leqslant 7$, or when $(X, G, p) = (A_2, E_7, 5)$. For these cases it is proved in [10] that only $X = A_2$ occurs as a maximal subgroup.

With a description of the maximal subgroups in hand, the next step is to try to understand the embedding of semisimple subgroups in the maximal subgroups. Under the hypothesis that the subgroup in question is G-cr, this ultimately comes down to embeddings in certain reductive subgroups. For this we need the notion of *essential embedding*.

Let Y be a semisimple algebraic group, and let X be a semisimple subgroup of Y. For a subgroup A of Y write $\overline{A} = AZ(Y)/Z(Y)$, and for a simple factor S of Y, let $\pi_S : \overline{X} \to \overline{S}$ be the projection map. The connected preimage of $\pi_S(\overline{X})$ in S is called the *projection* of X in S. We say that X is *essentially embedded* in Y if, for each exceptional simple factor Y_0 of Y, the projection of X in Y_0 is either Y_0 or maximal connected but not of maximal rank in Y_0, and for each classical factor Y_1 of Y, the projection of X in Y_1 is either irreducible on the natural Y_1-module, or $Y_1 = D_n$ and the natural module splits under X into a sum of two non-isomorphic irreducible summands of odd dimension.

Table 1

G	M
G_2	A_1 $(p \geqslant 7)$
F_4	A_1 $(p \geqslant 13)$, G_2 $(p = 7)$, A_1G_2 $(p \geqslant 3)$
E_6	A_2 $(p \geqslant 5)$, G_2 $(p \neq 7)$, F_4, C_4 $(p \geqslant 3)$, A_2G_2
E_7	A_1 $(p \geqslant 17)$, A_1 $(p \geqslant 19)$, A_2 $(p \geqslant 5)$, A_1A_1 $(p \geqslant 5)$, A_1G_2 $(p \geqslant 3)$, A_1F_4, G_2C_3
E_8	A_1 $(p \geqslant 23)$, A_1 $(p \geqslant 29)$, A_1 $(p \geqslant 31)$, B_2 $(p \geqslant 5)$, A_1A_2 $(p \geqslant 5)$, G_2F_4

Recall also from [7] that a *subsystem* subgroup of G is a connected semisimple subgroup which is normalized by a maximal torus of G.

Proposition 2.2. *Let G be an exceptional algebraic group over K in good characteristic p, and let X be a connected semisimple subgroup of G. Assume that X is G-cr. Choose a subsystem subgroup Y of G, minimal subject to containing X (possibly $Y = G$). Then one of the following holds*:

(i) *X is essentially embedded in Y*;
(ii) *X has a factor G_2, $p = 7$, $Y = E_6$ or E_8, and $X < F_4 < E_6$ or $X < G_2F_4 < E_8$, respectively, with X projecting to a maximal subgroup G_2 of the F_4 factor*;
(iii) *X has a factor A_1, and there is a subgroup $Y_0 = F_4, E_6, E_7$ or E_8 of G, a maximal connected subgroup Z of Y_0 not containing a maximal torus, and a semisimple subgroup Y_1 of $C_G(Y_0)$, such that either X is essentially embedded in ZY_1, or $X = Y_0Y_1$.*

Proof. This follows from the proofs of [7, Theorems 5, 7, pp. 53–55], where the result is proved under the assumption that $p > N(X,G)$, where $N(X,G)$ is as defined on [7, p. 2] (this excludes a few good characteristics in some cases). The only points to note are that the use of [7, Theorem 1] is replaced by our hypothesis that X is G-cr; use of [12] is replaced by use of Theorem 2.1; and extra subgroups $X < G_2F_4 < E_8$ $(p = 7)$ show up under (ii), which do not appear in [7, Theorem 5], because of the stronger characteristic assumption there. □

Remark. In Proposition 2.2(iii), the possibilities for Z are given by Theorem 2.1, and the possibilities for $C_G(Y_0)$ are as follows:

Y_0	$C_G(Y_0)$ $(G = E_8, E_7, E_6, F_4)$
F_4	$G_2, A_1, 1, 1$ (respectively)
E_6	$A_2, T_1, 1, –$
E_7	$A_1, 1, –, –$
E_8	$1, –, –, –$

Let E be a simple algebraic group. We introduce the following notation to deal with cases where $L(E)$ has nontrivial ideals. Let $L(E)^+$ denote the subalgebra of $L(E)$ generated by all nilpotent elements. We note that $L(E) = L(E)^+$ if E is simply connected, and, of course, this also holds if $L(E)$ is simple. With the exception of some orthogonal groups in characteristic 2, $L(E)^+$ has codimension at most 1 in $L(E)$.

The next proposition is the main result of the section.

Proposition 2.3. *Let G be an exceptional algebraic group over K in good characteristic p, and let E be a connected simple subgroup of G.*

(i) *If E is a restricted A_1, then E is G-cr.*

(ii) *If* $\mathrm{rank}(E) \geqslant 2$, *then* E *is* G*-cr, except possibly when* $E = G_2$, $p = 7$ *and* $G = E_7$ *or* E_8.

(iii) *Suppose that* E *is restricted, and also that either* E *is* G*-cr or* $C_G(E)$ *contains a connected simple subgroup of the same type as* E. *Then* $C_G(E)^0$ *is reductive,* $C_G(E)^0 = C_G(L(E)^+)^0$, *and* $C_{L(G)}(E) = C_{L(G)}(L(E)^+)$.

Proof. (i) This follows from [13, Theorem 1.1(iv)].

(ii) Assume $\mathrm{rank}(E) > 1$. Theorem 1 of [7] shows that E is G-cr provided the prime p satisfies $p > N(E, G)$, where $N(E, G)$ is defined in the table in [7, p. 2]. The only cases where this inequality is stronger than p being a good prime are $(E, G, p) = (A_2, E_7, 5)$, $(G_2, E_7, 5)$, $(G_2, E_7, 7)$, and $(G_2, E_8, 7)$. The last two possibilities appear in the conclusion of (ii), so we must show that in the first two cases E is G-cr.

For this we follow the proof of [7, Theorem 1]. Let $P = QL$ be a parabolic subgroup of G, minimal subject to containing E, with unipotent radical Q and Levi subgroup L. Using Theorem 2.1 and arguing as in [7, 3.2], we see that either L' is a commuting product of classical groups, or $L' = E_6$ and E projects to a maximal subgroup of L' or is diagonal in a subsystem of type $A_2A_2A_2$. Now we see as in the proof of [7, 3.3, 3.4] that the possible high weights for E acting on composition factors of Q are as listed on p. 36 of [7]. In our cases, $p = 5$, and the rest of the proof of [7, 3.4] gives the conclusion.

(iii) Here we are assuming that E is a restricted subgroup. If $E = A_1$ then the hypothesis implies that E is a good A_1 in G. The first equality follows from [13, Theorem 1.2]. For the second equality, first use [13, Theorem 1.1] to see that $L(G) \downarrow E$ is a tilting module and then apply [13, Lemma 2.3(d)] to get the equality on fixed points.

Suppose now that $\mathrm{rank}(E) \geqslant 2$. Assume first that E is G-cr. Letting Y be a minimal subsystem subgroup of G containing X, the embedding of X in Y is given by (i) or (ii) of Proposition 2.2.

In case of Proposition 2.2(ii) we have $p = 7$ and either $E = G_2 < F_4 < E_6 \leqslant G$ or $E = G_2 < G_2F_4 < E_8 = G$. In either case $L(E) = L(E)^+$. In the first case, we have, using [12],

$$L(F_4) \downarrow E = V_E(01) \oplus V_E(11), \qquad V_{F_4}(\lambda_4) \downarrow E = V_E(20).$$

Moreover, $L(G) \downarrow F_4$ is the sum of an adjoint module, a fixed space of dimension $\dim C_G(F_4)$, and a number of copies of $V_{F_4}(\lambda_4)$. It follows that $C_G(E)^0 = C_G(F_4)^0 = 1$, A_1, G_2 for $G = E_6, E_7, E_8$ (see [12]). Further, since E is restricted, only trivial composition factors of $L(G) \downarrow E$ can be centralized by $L(E)^+$, and so it follows that $C_G(L(E)^+)^0 = C_G(E)^0$ and $C_{L(G)}(E) = C_{L(G)}(L(E)^+)$, as required.

In the second case above, $E = G_2 < G_2F_4 < E_8$, we have

$$L(E_8) \downarrow E = V_E(01)^2 \oplus V_E(11) \oplus \big(V_E(10) \otimes V_E(20)\big).$$

To understand the last summand we first consider $V_E(10) \otimes T_E(20)$, where the second factor is the indecomposable tilting module of high weight 20, which has shape $00/20/00$. The tensor product of tilting modules is again a tilting module and using this we find that $V_E(10) \otimes V_E(20) = V_E(30) \oplus V_E(01) \oplus T_E(11)$, where $T_E(11)$ has socle length 3 with layers $20, 11 \oplus 00, 20$. It follows that $C_G(E)^0 = C_G(L(E))^0 = 1$ and $C_{L(G)}(E) = C_{L(G)}(L(E)) = 0$.

Next consider the situation of Proposition 2.2(i). Here E is essentially embedded in the subsystem subgroup Y. The possibilities for Y, E, and $L(G) \downarrow E$ are worked out explicitly in [7, pp. 56–68 and Tables 8.1–8.4], under the assumption that $p > N(E,G)$. In this situation we have

$$C_G(E) \leqslant C_G\big(L(E)\big) \leqslant C_G\big(L(E)^+\big),$$
$$C_{L(G)}(E) \leqslant C_{L(G)}\big(L(E)\big) \leqslant C_{L(G)}\big(L(E)^+\big),$$
$$\dim C_G\big(L(E)^+\big) \leqslant \dim C_{L(G)}\big(L(E)^+\big), \quad \text{and}$$
$$\dim C_G(E) = \dim L\big(C_G(E)\big) \leqslant \dim C_{L(G)}(E) \leqslant \dim C_{L(G)}\big(L(E)^+\big).$$

Hence to prove that $C_G(E)^0 = C_G(L(E)^+)^0$ and $C_{L(G)}(E) = C_{L(G)}(L(E)^+)$, it suffices to show that $\dim C_G(E) = \dim C_{L(G)}(L(E)^+)$.

As noted above, only trivial composition factors of $L(G) \downarrow E$ can be centralized by $L(E)^+$.

Assume $p > N(E,G)$. As observed in [7, p. 90], Tables 8.1–8.4 of [7] show that in all but three cases, the number of trivial composition factors in $L(G) \downarrow E$ is equal to $\dim C_G(E)$, hence $\dim C_G(E)) = \dim C_{L(G)}(L(E)^+)$; in the exceptional cases $E = A_4$ $(p = 5)$ or A_6 $(p = 7)$, and the same conclusion holds, by an argument in [7, p. 90]. Finally, $C_G(E)^0$ is reductive by [7, Theorem 2].

Now assume $p \leqslant N(E,G)$. As p is good, this means that $(E,G,p) = (A_2, E_7, 5)$, $(G_2, E_7, 5 \text{ or } 7)$ or $(G_2, E_8, 7)$. In each case $L(E)$ is simple, and, in particular, $L(E) = L(E)^+$. The possibilities for Y, E and $L(G) \downarrow E$ can be worked out exactly as in [7] (p. 62 for G_2, pp. 64–67 for A_2), and are just as in Tables 8.1–8.4 of [7]. In particular the maximal A_2 in E_7 satisfies $L(E_7) \downarrow A_2 = L(A_2) \oplus V_{A_2}(44)$, so there are no fixed points. In all but one case we find that the number of trivial composition factors in $L(G) \downarrow E$ is equal to $\dim C_G(E)$, and $C_G(E)^0$ is reductive, giving the conclusion as above. The exceptional case occurs when $E = G_2$, $Y = A_6$, and $p = 7$; here

$$L(E_7) \downarrow E = 01/10^5/20^3/00^3, \qquad L(E_8) \downarrow E = 01^5/10^{13}/20^3/00^6,$$

where (as in [7]) the notation ab^n indicates the presence of the composition factors of n copies of the Weyl module $W_E(ab)$. Now $W_E(20)$ has a trivial one-dimensional submodule when $p = 7$; this means that the number of trivial composition factors in $L(G) \downarrow E$ is 6 or 9, for $G = E_7$ or E_8, respectively. The restrictions $L(G) \downarrow E$ can be calculated precisely by first restricting to $A_6T_1 = GL_7$, where we see that the action is a direct sum of modules of the form V, $\bigwedge^2 V$, $\bigwedge^3 V$, duals of these modules, trivial modules, and $V \otimes V^*$, where V denotes a usual 7-dimensional module. It follows that $L(G) \downarrow E$ is a tilting module.

In particular, for each occurrence of the composition factor 20, there is a direct summand which is an indecomposable tilting module of shape $00/20/00$. Hence the dimension of the fixed point space of E (or $L(E)$) on $L(G)$ is 3 or 6, according as $G = E_7$ or E_8. If $G = E_7$ then $C_G(E) = A_1$, as shown in [12, pp. 34–35]. And if $G = E_8$ then $E < A_6 < E_7$, so that

$C_G(E) \geqslant C_{E_7}(A_1)C_G(E_7) = A_1A_1$, and by consideration of dimension $C_G(E)^0 = A_1A_1$. This gives the assertion here.

We have now proved part (iii) of the proposition under the assumption that E is G-cr. It remains to prove it under the assumption that $\mathrm{rank}(E) \geqslant 2$, E is restricted, not G-cr, and $C_G(E)$ contains a connected simple subgroup of the same type as E.

By part (ii), the assumption that E is not G-cr forces $E = G_2$, $p = 7$, and $G = E_7$ or E_8. Moreover, the proof of [7, Theorem 1] shows that E must lie in a parabolic subgroup $P = QL$ of G, such that the unipotent radical Q, when restricted to E, has a composition factor $V_E(\lambda)$ such that the Weyl module $W_E(\lambda)$ has a trivial composition factor. Choose P minimal for this. From [7, p. 36], we see that the only possibilities are $L = A_6$ or E_6, with $\lambda = 20$. As in [7] we calculate the composition factors of $L(G) \downarrow E$ in these cases; it turns out that the number of trivial composition factors is less than $\dim E$, except when $L = E_6$ and $G = E_8$, in which case this number is precisely $14 = \dim E$. Hence by our hypothesis, this case must occur, and we must have $C_G(E)^0 \cong E = G_2$ and $\dim C_{L(G)}(L(E)) = C_{L(G)}(L(E)^+) = 14 = \dim C_G(E)$ also. (Such a configuration exists as $E_8 \geqslant F_4G_2 \geqslant G_2G_2$.) This completes the proof. □

3. Theorem 1: uniqueness

In this section we prove the uniqueness part of Theorem 1. Suppose then that G is a simple algebraic group in characteristic p, a good prime, and that X is a simple, simply connected group and $\phi : X \to G$ is a morphism whose image is G-cr. Let $k, q_1, \ldots, q_k, \psi$ and μ be as in Theorem 1. Now let $k', q'_1, \ldots, q'_{k'}, \psi'$, and μ' correspond to another factorization of ϕ.

If $\mathrm{d}\phi = 0$, then ϕ can be factored through a Frobenius morphism of X which induces the p-power map on a maximal torus (see [7, Lemma 1.2]). Repeating this we see that there is a unique power q of p such that $\phi = \mu \circ F$, where F is a Frobenius morphism inducing the q-power map on a maximal torus and $\mathrm{d}\mu \neq 0$.

The assumption $\mathrm{d}\phi = 0$ implies both $q_1 > 1$ and $q'_1 > 1$. Moreover, the uniqueness of q forces $q = q_1 = q'_1$. We can then factor off a q-power map and assume $q_1 = q'_1 = 1$.

For $1 \leqslant i \leqslant k$, let μ_i be the restriction of μ to the ith simple factor of $X \times \cdots \times X$ (k factors). Thus $\phi(x) = \prod_1^k \mu_i(x^{(q_i)})$ for $x \in X$. Similarly, $\phi(x) = \prod_1^{k'} \mu'_j(x^{(q'_j)})$.

We aim to show that $k = k'$, $q_i = q'_i$, and $\mu_i = \mu'_i$ for all i. For convenience we may assume $k \geqslant k'$ and proceed by induction on k. The base case $k = k' = 1$ is trivial. Assume $k \geqslant 2$. Write $E_i = \mu_i(X)$ and $F_j = \mu'_j(X)$; these are connected, simple, restricted subgroups of G. We have $\phi(X) \leqslant E_1 \cdots E_k$ with a q_i-field twist in the projection to $E_i/Z(E_i)$, and likewise $\phi(X) \leqslant F_1 \cdots F_{k'}$ with a q'_j-twist in the jth projection. Since $q_1 = q'_1 = 1$ and recalling the notation given just before Proposition 2.3, we have

$$L\big(\phi(X)\big)^+ = L(E_1)^+ = L(F_1)^+.$$

The following is a key result for the uniqueness proof.

Lemma 3.1. (i) $C_G(E_1)^0 = C_G(L(E_1)^+)^0$.
(ii) $C_G(E_1)^0$ *is reductive.*

Proof. Assume first that G is of exceptional type. Since $k \geqslant 2$, the hypothesis of Proposition 2.3(iii) is satisfied by E_1, so both (i) and (ii) follow from that result.

Suppose now that G is of classical type. We first claim that for purposes of proving (i) we may work with the actual classical group (i.e. with $G = SL, Sp$, or SO). To see this let $\widehat{G}$ be the simply connected cover of G, $\pi : \widehat{G} \to G$ the natural surjection, and $\widehat{E_1}$ the connected preimage of E_1 in $\widehat{G}$. Then $Z = \ker(\pi)$ is finite and $S = \ker(\mathrm{d}\pi)$ is of dimension at most one and consists of semisimple elements. Indeed, since p is good $S = 0$ unless $\widehat{G} = SL_n$ and $p \mid n$.

Set $\widehat{C} = C_{\widehat{G}}(\widehat{E}_1)^0$ and $C = C_G(E_1)^0$. Similarly, set $\widehat{D} = C_{\widehat{G}}(L(\widehat{E}_1))^0$ and $D = C_G(L(E_1))^0$. To prove the claim it will suffice to show that $C = D$ if and only if $\widehat{C} = \widehat{D}$.

Now $\widehat{E}_1$ and E_1 are generated by unipotent elements while $L(\widehat{E}_1)^+$ and $L(E_1)^+$ are generated by nilpotent elements. Therefore $\pi : \widehat{E}_1 \to E_1$ and $\mathrm{d}\pi : L(\widehat{E}_1)^+ \to L(E_1)^+$ are surjective. For $u \in \widehat{G}$ a unipotent element and $n \in L(\widehat{G})$ a nilpotent element it follows from the Jordan decomposition that $C_{\widehat{G}}(uZ) = C_{\widehat{G}}(u)$ and $C_{\widehat{G}}(n + S) = C_{\widehat{G}}(n)$.

It follows from the previous paragraph that $\pi^{-1}(C) = \widehat{C} \cdot Z$ and $\pi^{-1}(D) = \widehat{D} \cdot Z$. We get the claim by taking connected components.

Thus to prove (i), we may work with any image of $\widehat{G}$ and we choose the actual classical group. Indeed it will suffice to establish the result for $G = SL(V)$. By hypothesis X is completely reducible in its action on V. Let Y be the direct factor mapping under the morphism μ of the theorem to E_1. Then Y acts homogeneously on each irreducible summand of $V \downarrow X$. Hence $V \downarrow Y$ is completely reducible with all irreducibles restricted. It follows that Y and $L(Y)$ leave invariant precisely the same subspaces of V. Also, $\mu(Y) = E_1$ and since $L(Y) = L(Y)^+$ we have $\mathrm{d}\mu(L(Y)) = \mathrm{d}\mu(L(Y)^+) \leqslant L(E_1)^+$.

Now consider centralizers. Clearly $C_G(E_1) \leqslant C_G(L(E_1)^+)$, so we must establish the reverse containment. We first observe that E_1 and $L(E_1)^+$ leave invariant the same subspaces of V. Surely any subspace invariant under E_1 is invariant under $L(E_1)$ and hence $L(E_1)^+$. Conversely, suppose $L(E_1)^+$ leaves W invariant. By the above $\mathrm{d}\mu(L(Y))$ also leaves W invariant and we have seen that Y and $L(Y)$ leave invariant the same subspaces. Hence W is Y-invariant, and hence E_1-invariant, as $\mu(Y) = E_1$.

Decompose V into homogeneous components with respect to $L(E_1)^+$. Each is invariant under the action of E_1 as well as $C_G(L(E_1)^+)$, so we may assume that V is homogeneous under the action of $L(E_1)^+$. Now [8, Lemma 2.3] shows that there is a decomposition $V = V_1 \otimes V_2$ such that $C_{GL(V)}(L(E_1)^+) = 1 \otimes GL(V_2)$ and $C_{GL(V)}(C_{GL(V)}(L(E_1))^+) = GL(V_1) \otimes 1$. Hence $E_1 \leqslant N_{GL(V)}(GL(V_1) \otimes GL(V_2))^0 = GL(V_1) \otimes GL(V_2)$. Now $L(E_1)^+ \leqslant L(GL(V_1))$ and E_1 is restricted, so this forces $E_1 \leqslant GL(V_1)$. But then E_1 centralizes the second factor, establishing (i).

It follows from the above that $C_{GL(V)}(E_1)$ is a product of smaller GL's. This implies (ii) for $G = SL(V)$. If G is a symplectic or orthogonal group we must take fixed points of this centralizer with respect to an involution. As $p > 2$ here (p is good), this centralizer is reductive, giving (ii). □

We are now in position to complete the uniqueness argument. Set $D = C_G(E_1)^0$, so that by Lemma 3.1(i) we have $D = C_G(L(E_1)^+)^0 = C_G(L(F_1)^+)^0$. Applying Lemma 3.1 again, this time to the second factorization, $\phi(X) \leqslant F_1 \cdots F_{k'}$ yields $C_G(F_1)^0 = C_G(L(F_1)^+)^0 = D$. Now $E_2 \cdots E_k$ and $F_2 \cdots F_{k'}$ are contained in D, so that $E_1 \cdots E_k =$

$\phi(X)(E_2\cdots E_k)$ and $F_1\cdots F_{k'} = \phi(X)(F_2\cdots F_{k'})$ are contained in $\phi(X)D = E_1\circ D = F_1\circ D$. It follows that $E_1 = F_1$.

Now for $x\in X$ we have $\prod \mu_i(x^{(q_i)}) = \phi(x) = \prod \mu'_j(x^{(q'_j)})$, and hence

$$\bigl(\mu'_1\bigl(x^{(q'_1)}\bigr)\bigr)^{-1}\mu_1\bigl(x^{(q_1)}\bigr) = z(x)\in E_1\cap D.$$

Since $E_1\cap D\leqslant Z(E_1)$, the map $x\to z(x)$ is a group homomorphism $X\to Z(E_1)$. However, $X = X'$ so this map must be trivial; in other words, $z(x) = 1$ for all $x\in X$, whence $\mu_1 = \mu'_1$.

We now have $\prod_{i>1}\mu_i(x^{(q_i)}) = \prod_{j>1}\mu'_j(x^{(q'_j)})$. View this as an equality between two factorizations of another morphism from X to G, where the intermediate direct product has one less factor in each case. The inductive hypothesis now yields the result.

4. Theorem 1: existence

Let G be a simple algebraic group over an algebraically closed field K of good characteristic p.

To establish the existence part of Theorem 1, we may replace X by its image in G, so we take $X\leqslant G$, a connected simple subgroup which is G-cr. We need to prove the existence of a commuting product $E_1\cdots E_r$ of restricted subgroups of the same type as X, such that $X\leqslant E_1\cdots E_r$ and the projections $X\to E_i/Z(E_i)$ are nontrivial and involve distinct field twists.

The case where G is of classical type is fairly easy due to Steinberg's theorem. This is settled in the following lemma.

Lemma 4.1. *Theorem* 1 *holds if G is a classical group.*

Proof. We may assume $X\leqslant G\leqslant SL(V)$. If G is a symplectic or orthogonal group, then we are assuming $p\neq 2$, so that $G = SL(V)_\tau$ for a suitable involutory automorphism τ of G. Moreover, X is completely reducible in its action on V.

First assume $G = SL(V)$. Here the Steinberg tensor product theorem provides the required (twisted diagonal) embedding $X < E_1\cdots E_r$, corresponding to field twists $1 = q_1 < \cdots < q_r$.

Now suppose $G = SL(V)_\tau$. From the uniqueness result we see that τ normalizes each E_i while centralizing the projection of X. However, for each i, E_i and X are of the same type, so it follows that τ centralizes E_i and the commuting product is contained in G. □

From now on we assume that G is an exceptional group. Here the most complicated case is that in which $X = A_1$ (i.e. $X = SL_2$ or PSL_2), and we settle this case in the following subsection. The higher rank cases will be settled in Section 4.2.

4.1. The case $X = A_1$

Assume $X = SL_2$ or PSL_2. We must find suitable restricted groups E_i. These restricted A_1's are good A_1's of G, in the sense of [13]. Theorem 1.2 of [13] provides a strong connection between good A_1's and unipotent classes. We will use this result to show that restricted A_1's of certain subgroups of G are also restricted for G. Combining this with Proposition 2.2 we are in position to carry out an inductive proof of Theorem 1.

We begin with a general result on reductive subgroups of G of maximal rank (i.e. containing a maximal torus).

Proposition 4.2. *Let G be a simple algebraic group in characteristic p, a good prime for G, and let M be a proper connected reductive subgroup of G of maximal rank. Then $Z(M) \neq 1$ and $M = C_G(Z(M))^0$.*

Proof. As p is good, an inspection of subsystem groups (using the Borel–de Siebenthal algorithm) shows that $Z(M) \neq 1$. Let $D = C_G(Z(M))^0$, so $M \leqslant D$ and $Z(M) \leqslant Z(D)$. Choose a maximal torus T of M containing $Z(M)$. Then $Z(D) \leqslant C_G(M) \leqslant C_G(T) = T \leqslant M$, and hence $Z(D) = Z(M) = Z$, say. If $M < D$ then $M/Z < D/Z$. But M/Z is a maximal rank subgroup of D/Z, so must have a nontrivial center, whereas $Z(M/Z) = 1$, a contradiction. Therefore $M = D = C_G(Z(M))^0$. $\square$

Recall that if X is an A_1 subgroup of a connected reductive group M, we will say X is *restricted* in M provided all weights of X on $L(M)$ are at most $2p-2$. If $X \leqslant M \leqslant G$ and if X is G-restricted, then clearly X is also M-restricted. The following result is a remarkable converse for certain particularly nice subgroups M of G.

Proposition 4.3. (i) *Let M be a connected reductive subgroup of G of maximal rank. Then restricted A_1's in M are also restricted in G.*

(ii) *Let τ be a semisimple automorphism of G. Then restricted A_1's in $C_G(\tau)$ are also restricted in G.*

Proof. (i) Suppose X is a restricted A_1 in M. Let u be a non-identity unipotent element of X. Theorem 1.2 of [13] implies that $C_G(u) = QC_G(A)$, where Q is normal and unipotent and A is a restricted A_1 in G containing u. As $u \in M$ we have $Z = Z(M) \leqslant C_G(u)$.

We claim that there exists $x \in Q$ such that $Z \leqslant C_G(A)^x$. Certainly Z^0 lies in a maximal torus of $QC_G(A)$, hence $Z^0 \leqslant C_G(A)^y$ for some $y \in Q$. Write $C = C_G(u)$, so $C_C(Z^0) = Q_0R_0$ where $Q_0 = C_Q(Z^0)$ and $R_0 = C_{C_G(A^y)}(Z^0)$. Now $Z = Z^0 \times Z_1$ with Z_1 a finite abelian p'-group. Then $Z_1 \leqslant R_0^z$ for some $z \in Q_0$, and hence $Z \leqslant C_G(A)^{yz}$, proving the claim.

Replacing A by A^x (which still contains u), we have $C_G(u) = QC_G(A)$, $u \in A$, and $Z \leqslant C_G(A)$. Then $u \in A \leqslant C_G(Z)^0$, and so by the previous proposition, $u \in M$. By [13], u lies in a unique $C_M(u)$-class of restricted A_1's in M, and hence X is $C_M(u)$-conjugate to A. In particular, X is restricted in G, proving (i).

(ii) Let X be a restricted A_1 in $C_G(\tau)$ and $u \in X$ a non-identity unipotent element. Let A be a restricted A_1 of G containing u. Then A^τ is another such, and so by [13, 1.1] there exists $x \in Q = R_u(C_G(u))$ with $A^{\tau x} = A$. Now, τ normalizes $C_G(u)$ so it follows that $\tau x \in Q\tau$ and so the semisimple part of τx is conjugate to τ by an element of Q. As τx normalizes A, so does its semisimple part. Hence, we may assume τ normalizes A, while centralizing u. But then τ induces a unipotent automorphism of A, whereas τ is semisimple. It follows that τ centralizes A and so X and A are good A_1's of $C_G(\tau)$ containing u. From the conjugacy result in [13, 1.1], we conclude that X is restricted in G. □

Notice that parts of the above result can be combined. For example, if $G = E_8$ and D is a group of type F_4 or C_4 contained in a subsystem subgroup E_6 of G, then it follows that restricted A_1's in D are also restricted in G.

We proceed with the existence part of Theorem 1 by induction. So we assume that the result holds for A_1 subgroups of simple algebraic groups of dimension smaller than that of G.

Lemma 4.4. *Theorem* 1 *holds if X is contained in a proper connected reductive subgroup of maximal rank in G, or in a proper parabolic subgroup of G, or in $C_G(\tau)$ for τ a nontrivial semisimple automorphism of G.*

Proof. Suppose X is contained in one of these types of subgroups. As X is G-cr, we then have $X \leqslant M < G$, with M connected reductive of maximal rank or $M = C_G(\tau)$. By induction the theorem holds for the projection of X in each simple factor of M. So for each simple factor there is a commuting product of A_1's which are restricted for that factor, such that the projection of X is a diagonal subgroup of this product, with distinct field twists.

Fix a particular field twist and consider the corresponding A_1's associated to this twist in various simple factors of M. It is obvious from a consideration of weights that a diagonal A_1 (no twists) in the product of these A_1's is restricted for M, and so Proposition 4.3 shows it is restricted for G as well. Finally, X is diagonal in a product of these A_1's, with distinct field twists, giving the conclusion. □

Recall the assumption that G is of exceptional type. Since p is a good prime for G, it is not 2 or 3 and also is not 5 when $G = E_8$.

If $G = G_2$ then using Lemma 4.4 we may assume that X is maximal in G, and hence by [12], we have $p \geqslant 7$ and $L(G) \downarrow X$ has highest weight 10. Consequently X is good in G, giving the existence conclusion of Theorem 1. Thus we assume from now on that $G \neq G_2$.

At this point we combine Proposition 2.2 with Lemma 4.4 to obtain precise information about the possible embeddings of X in G.

Lemma 4.5. *Theorem* 1 *holds unless one of the following occurs*:

(i) *there is a subgroup $Y_0 = F_4$ of G, a maximal connected subgroup Z of Y_0 not containing a maximal torus, and a semisimple subgroup Y_1 of $C_G(Y_0)$, such that X is essentially embedded in ZY_1*;

(ii) *there is a maximal connected subgroup Z of G not containing a maximal torus, such that X is essentially embedded in Z.*

The possibilities for Z in (i) *and* (ii) *are as listed in Table* 2, *and the possibilities for $C_G(Y_0)$ in* (i) *are given in the remark following Proposition* 2.2.

Table 2

Case in Lemma 4.5	Possibilities for Z
(i)	A_1, G_2 ($p=7$), A_1G_2
(ii), $G=E_6$	A_2, G_2 ($p\neq 7$), A_2G_2
(ii), $G=E_7$	A_2, A_1A_1, A_1G_2, A_1F_4, G_2C_3
(ii), $G=E_8$	B_2, A_1A_2, G_2F_4

Proof. This follows from Proposition 2.2 and Lemma 4.4, noting that in Table 2 we have omitted the cases $Z = F_4, C_4$ when $G = E_6$, since these are involution centralizers, and we have also omitted the maximal A_1's in E_7, and E_8, since these are restricted in G (see [13]). □

Lemma 4.6. *Theorem* 1 *holds in the case of Lemma* 4.5(ii)*.*

Proof. Assume that Lemma 4.5(ii) holds, so that X is essentially embedded in a maximal connected subgroup Z of G as in Table 2. Moreover, Z is a product of at most two simple factors, and with one possible exception, the essentiality implies that the projection of X in each factor is either equal to, or maximal in the factor; the exception is for the factor C_3 (of G_2C_3 in E_7), when the projection of X could be an irreducible but non-maximal A_1 in C_3 (lying in a subgroup A_1A_1 of C_3 acting on the natural module as $1\otimes 2$).

We have either $X \leqslant A_1^k$, where $k \leqslant 2$ is the number of simple factors of Z, or $X \leqslant A_1^3$ with $Z = G_2C_3$. There are possibly field twists in some projections. Let X_1 denote a diagonal A_1 in this product without any field twists.

The composition factors of $L(G) \downarrow Z$ are given in [7, Section 2]. We summarize the information in Table 3. In the third column, we give the highest weight of X_1 on $L(G)$. If this highest weight is at most $2p-2$ then X_1 is restricted in G, from which it follows that the conclusion of Theorem 1 holds; the remaining cases are listed in the last column of the table. Note that the conditions on p given in the first column follow either from the

Table 3

$Z<G$	$(L(G)/L(Z))\downarrow Z$	Highest weight of X_1 on $L(G)$	Open cases
$B_2<E_8$ ($p\geqslant 5$)	$06/32$	18	$p=7$
$A_1A_2<E_8$ ($p\geqslant 5$)	$6\otimes 11/4\otimes 30/$ $4\otimes 03/2\otimes 22$	10	
$G_2F_4<E_8$ ($p\geqslant 13$)	$10\otimes 0001$	22	
$A_2<E_7$ ($p\geqslant 5$)	44	16	$p=5,7$
$A_1A_1<E_7$ ($p\geqslant 5$)	$2\otimes 8/4\otimes 6/6\otimes 4/$ $2\otimes 4/4\otimes 2$	10	$p=5$
$A_1G_2<E_7$ ($p\geqslant 7$)	$4\otimes 10/2\otimes 20$	14	$p=7$
$A_1F_4<E_7$ ($p\geqslant 13$)	$2\otimes 0001$	18	
$G_2C_3<E_7$ ($p\geqslant 7$)	$10\otimes 010$	14	$p=7$
$A_2<E_6$ ($p\geqslant 5$)	$41/14$	10	$p=5$
$G_2<E_6$ ($p\geqslant 11$)	11	16	
$A_2G_2<E_6$ ($p\geqslant 7$)	$11\otimes 10$	10	

existence of maximal A_1's in the simple factors of Z, or simply from the fact that p is good.

First assume $G = E_8$. The only open case is $Z \cong B_2$ with $p = 7$. Here X is a maximal A_1 of B_2 and it follows from [12, p. 193] that the labeled diagram of a maximal torus of X is 00020020. This yields all weights on $L(G)$, and we calculate that the composition factors of X on $L(G)$ are as follows:

$$L(G) \downarrow X = 18^2 \big| 16 \big| 14^3 \big| 12^6 \big| 10^4 \big| 8^5 \big| 6^5 \big| 4^4 \big| 2^6 \big| 0^3.$$

It is proved in [10] that a subgroup $X \cong A_1$ with these composition factors on $L(G)$ is G-conjugate to an A_1 which lies in a maximal rank subgroup A_8 of G, acting indecomposably on the usual 9-dimensional module with composition factors $4|1 \otimes 1^{(7)}$. But then X is contained in a proper parabolic subgroup of A_8 and hence one of G. So the result follows from Lemma 4.4. (Actually this A_1 fails to be G-cr.)

Assume next that $G = E_7$, and consider first the case where $Z = A_2$ with $p = 5$ or 7. For $p = 7$, restricting $V_{A_2}(44)$ to X, we find that

$$L(E_7) \downarrow X = 16 \big| 14 \big| 12^3 \big| \ldots \big| 0^3.$$

By [1], of the composition factors appearing, only $12 = 5 \otimes 1^{(7)}$ extends the trivial module, and $\mathrm{Ext}_X(12, 0)$ has dimension 1. Since $L(E_7)$ is self-dual, it follows that X fixes a nonzero vector $v \in L(E_7)$. By [12, 1.3], the stabilizer of v in E_7 lies in a proper subgroup of E_7 which is either parabolic or reductive of maximal rank. In either case the result follows from Lemma 4.4. When $p = 5$, a similar argument applies: here we find

$$L(E_7) \downarrow X = 16 \big| 14 \big| 12^3 \big| 10^2 \big| 8^5 \big| \ldots \big| 0^4,$$

and the only composition factor present which extends the trivial module is $8 = 3 \otimes 1^{(5)}$. From the extension theory of SL_2 we can write $L(E_7) \downarrow X = A \oplus B$, where A contains all the composition factors of high weight $\sum c_i p^i$ for which $c_0 = 0$ or $p - 2$. Here A has composition factors $10^2|8^5|0^4$. It then follows from the proof of [10, 3.6(i)] that X fixes a nonzero vector in A. The conclusion follows as before.

The remaining cases for $G = E_7$ each have Z the product of two simple factors. From the information in the table it is clear that Theorem 1 holds except for the case where X is diagonal in Z with no field twist in either factor. Consequently we now assume that $X = X_1$.

First consider $Z = A_1A_1$ with $p = 5$. Let T be a maximal torus of X_1. From $L(E_7) \downarrow Z$ we calculate that the non-negative weights of T on $L(E_7)$ are $10^3, 8^6, 6^5, 4^4, 2^{11}, 0^3$. We check also that these weights agree with those of a one-dimensional torus lying in a maximal rank subgroup A_2A_5 of E_7, projecting to a torus of a regular A_1 in each factor. Therefore $T < A_2A_5$. Now let V_{56} be the 56-dimensional irreducible E_7-module $V(\lambda_7)$. By [7, 2.3] we have

$$V_{56} \downarrow A_2A_5 = \lambda_1 \otimes \lambda_1/\lambda_2 \otimes \lambda_5/0 \otimes \lambda_3.$$

Hence the non-negative weights of T on V_{56} are $9, 7^3, 5^6, 3^9, 1^9$, and so the composition factors of $L(E_7) \downarrow X$ are $9|7^2|5^3|3^6|1^2$. Of these composition factors, only $7 = 2 \otimes 1^{(5)}$ extends 1. Since $L(E_7)$ is self-dual, we conclude that $L(E_7) \downarrow X$ has a submodule $W \cong 1$ (of dimension 2). The variety of all 2-spaces in V_{56} has dimension 108, and hence $N_{E_7}(W)$ is a closed subgroup of E_7 containing X_1 and of dimension at least $\dim E_7 - 108 = 25$. Let M be a maximal connected subgroup of E_7 containing $N_{E_7}(W)^0$. If M is parabolic or reductive of maximal rank, we are done by Lemma 4.4. Otherwise, by [12], $M = A_1F_4$ or G_2C_3. Neither of these fixes a 2-space in V_{56} (see [7, 2.5]), so $N_{E_7}(W)^0$ is proper in M.

If X is contained in a proper parabolic of M then it is also contained in one for G and Lemma 4.4 yields the result. If X is contained in a subgroup of M of maximal rank, then $X < C_M(s) < M$ for some semisimple elements of M. But then $C_G(s)$ has maximal rank in G and contains X, and again Lemma 4.4 gives the result. Now the dimension restriction and [12] imply that the only remaining possibility is that $M = A_1F_4$ and $X < F_4$. But this is clearly impossible, since X has no fixed points on $L(E_7)$, whereas $C_M(F_4) = A_1$.

Similar considerations apply to the cases $Z = A_1G_2$ or G_2C_3 with $p = 7$. By [7, 2.5],

$$V_{56} \downarrow A_1G_2 = 1 \otimes 01/3 \otimes 10, \qquad V_{56} \downarrow G_2C_3 = 10 \otimes 100/00 \otimes 001.$$

Hence, if T denotes a maximal torus of X, we calculate that the non-negative weights of T on V_{56} are $11, 9^3, 7^4, 5^5, 3^7, 1^8$ in both cases. It follows that the composition factors of X are

$$L(E_7) \downarrow X = 11|9^2|7|5^2|3^4|1^2.$$

By [1], only $11 = 4 \otimes 1^{(7)}$ extends the module 1, and hence X fixes a 2-space W in V_{56}. Now we complete the argument as above.

Finally, let $G = E_6$ with $Z = A_2$ and $p = 5$. We consider the 27-dimensional E_6-module $V_{27} = V_{E_6}(\lambda_1)$. Let T be a maximal torus in X. By [12, p. 65], $T < A_1A_5 < E_6$, and by [7, 2.3], $V_{27} \downarrow A_1A_5 = 1 \otimes \lambda_5/0 \otimes \lambda_4$. Hence we calculate the T-weights on V_{27}, from which it follows that $V_{27} \downarrow X = 8|6|4^2|2|0^2$. Only the composition factor $8 = 3 \otimes 1^{(5)}$ extends the trivial module, so we deduce that X fixes a 1-space $\langle v \rangle$ of V_{27}. So $X < M = N_G(\langle v \rangle)$, which has dimension at least $\dim E_6 - 26 = 52$. By Lemma 4.4 we may assume X lies in no parabolic or maximal rank subgroup of E_6, so we must have $M = F_4$ by [12]. Now $V_{27} \downarrow F_4 = V_{26} \oplus 0$, where V_{26} is the irreducible F_4-module $V_{F_4}(\lambda_4)$. As $V_{26} \downarrow X = 8|6|4^2|2|0$, X must also fix a 1-space $\langle w \rangle$ of V_{26}. It now follows using [12] that X lies in a parabolic or maximal rank subgroup of F_4, and again Lemma 4.4 yields the result. $\square$

Lemma 4.7. *The conclusion of Theorem* 1 *holds in the case of Lemma* 4.5(i).

Proof. Here $X \leqslant F_4C$, where $C = C_G(F_4) = G_2, A_1, 1$ or 1, according as $G = E_8, E_7, E_6$ or F_4, respectively. Moreover, by [7, 2.4], $(L(G)/L(F_4C)) \downarrow F_4C = 0001 \otimes 10, 0001 \otimes 2$ or 0001, according as $G = E_8, E_7$ or E_6. Write V_{26} for the 26-dimensional F_4-module $V_{F_4}(0001)$.

Denote by X_1 the projection of X in F_4, and by X_2 an A_1 lying in F_4C which projects to a maximal A_1 in each factor with no twists involved in any projection.

We record the possibilities for Z, $L(F_4)\downarrow Z$, and $V_{26}\downarrow Z$, given by [12, p. 193] and [7, 2.5]:

Z	$(L(F_4)/L(Z))\downarrow Z$	$V_{26}\downarrow Z$	Highest weight of X_2 on $L(F_4), V_{26}$
A_1	$22/14/10$	$16/8/0$	$22, 16$
G_2 $(p=7)$	11	20	$16, 12$
A_1G_2 $(p\geqslant 7)$	$4\otimes 10$	$2\otimes 10/4\otimes 00$	$10, 8$

It follows from this that the conclusion holds, unless either $Z=G_2$, $p=7$ or $Z=A_1G_2$, $p=7$, $G=E_8$ and X projects to a maximal A_1 in $C=G_2$.

Suppose $Z=G_2$. By [12, p. 193], the labeling of a maximal torus T of X_1 in F_4 is 2022. Now consider an A_1 lying in a maximal rank subgroup A_1C_3 of F_4 via the embedding $1^{(7)}, 5$ (i.e., the projection to the factor C_3 is the irreducible representation of high weight 5, and the projection to the factor A_1 is a twist of the representation 1). We calculate the weights of a maximal torus T_1 of this A_1 using the restriction $L(F_4)\downarrow A_1C_3 = L(A_1C_3)/1\otimes 001$, and conclude from these weights that the labeled diagram of T_1 is also 2022. Hence by [7, Theorem 6], X_1 is F_4-conjugate to this A_1 in A_1C_3. It follows that X centralizes an involution in F_4 and hence an involution in G, so the result follows from Lemma 4.4.

A similar argument settles the case $Z=A_1G_2$, $p=7$. This time we calculate the weights of T on $L(F_4)$, and find that they agree with the weights of a maximal torus of an A_1 lying in a maximal rank subgroup A_1C_3, embedded via the untwisted representations 1, 5. Hence, again by [7, Theorem 6], we conclude that $X_1 < A_1C_3$ and hence X centralizes an involution and again Lemma 4.4 yields the assertion. □

This completes the existence proof of Theorem 1 for $X=A_1$.

4.2. *The case where* rank$(X)\geqslant 2$

We continue with the proof of Theorem 1, where it remains to treat the case of a simple group X with rank$(X)\geqslant 2$. The information provided in [7] make this a much easier task than for groups of type A_1. Indeed, except for a couple of situations in small characteristic, the possibilities for X are described explicitly in [7].

Recall that G is an exceptional group and we are trying to prove the existence of a commuting product $E_1\cdots E_r$ of restricted subgroups E_i of the same type as X, such that $X\leqslant E_1\cdots E_r$ and the projections $X\to E_i/Z(E_i)$ are nontrivial and involve distinct field twists.

The embedding of X in G is given by Proposition 2.2, (i) and (ii). First consider the case of Proposition 2.2(ii): here $p=7$, $X=G_2$, and either $X<F_4<E_6\leqslant G$, or $X<G_2F_4<E_8=G$, with X projecting to a maximal subgroup of the F_4 factor. Let λ_1,λ_6 denote the fundamental dominant weights of E_6 corresponding to the restricted 27-dimensional modules. From [12] we have

$$L(E_6)\downarrow G_2 = 01/11/20, \qquad V_{E_6}(\lambda_1)\downarrow G_2 = 20/00, \quad \text{and}$$
$$L(E_8)\downarrow G_2G_2 = L(G_2G_2)/00 \otimes 11/10 \otimes 20,$$

where in the last case the G_2G_2 lies in G_2F_4, the second factor G_2 being maximal in F_4. We note that $L(E_8)\downarrow E_6 = L(E_6) \oplus V_{E_6}(0)^8 \oplus V_{E_6}(\lambda_1)^3 \oplus V_{E_6}(\lambda_6)^3$. It follows that in the case where $X < E_6$, X is restricted; and in the case $X < G_2F_4$, if neither projection involves a twist then X is restricted, and otherwise X lies in the product of two restricted G_2's with distinct twists in the projections. Hence the result holds in the case of Proposition 2.2(ii).

Now consider the case of Proposition 2.2(i). Here there is a subsystem subgroup Y of G such that X is essentially embedded in Y. When $p > N(X, G)$ (as defined in [7, p. 2]), the possibilities for Y and $L(G)\downarrow X$ are given in [7, Tables 8.1–8.4]. And in the extra cases where p is good but $p \leqslant N(X, G)$—namely, the cases $(X, G, p) = (A_2, E_7, 5)$, $(G_2, E_7, 5)$, $(G_2, E_8, 5 \text{ or } 7)$—the possibilities for Y and $L(G)\downarrow Y$ can be calculated exactly as in [7, pp. 62, 64] (using Theorem 2.1 for the case where X is maximal in G). The outcome is that the possibilities in these cases are exactly as in [7, Tables 8.1 and 8.2].

We first settle the case where the subsystem subgroup Y has a simple factor Y_1 of exceptional type. By Theorem 2.1 there are very few possibilities; they are as follows:

$$(Y_1; X) = (E_8; B_2),\ (E_7; A_2),\ (E_6; A_2, G_2, F_4 \text{ or } C_4),\ (F_4; G_2) \quad (p = 7).$$

First suppose $X < Y_1$. Then X is a maximal subgroup of Y_1 and it is clear from Theorem 2.1 [7, 2.4] (together with the remark after Theorem 2.1) that $L(G)\downarrow X$ has all composition factors restricted. Hence X is a restricted subgroup of G and there is nothing to prove. Now suppose $X \nleqslant Y_1$. Then Y has at least two simple factors, and as $\mathrm{rank}(X) \geqslant 2$, the only remaining possibility is that $Y = E_6A_2$, $G = E_8$, and $X = A_2$. Here $X < A_2A_2$, where the first A_2 is a maximal subgroup of E_6 and the other is a subsystem group. If the embedding does not involve a field twist in either factor, then we see from the A_2E_6 row of [7, p. 100] that all composition factors of X on $L(G)$ are restricted. If a field twist is present, then we need only show that each of the A_2 factors is restricted and this information is also immediate from [7, Table 8.1].

From now on assume that $Y = Y_1 \cdots Y_k$ with each Y_i a simple group of classical type. Let X_i be the projection of X in Y_i. Recall that X is essentially embedded in Y and hence for each i, either X_i is irreducible on the natural module for Y_i or else $Y_i = D_k$ for some k and the natural orthogonal module restricts to X_i as the direct sum of two irreducible nondegenerate subspaces.

We now inspect Tables 8.1–8.4 of [7], which give the possibilities for the composition factors of $L(G)\downarrow X$. If none of these composition factors involves a q-field twist then we see from the tables that they are all restricted, so X is a restricted subgroup and there is nothing to prove.

So suppose there is a composition factor present which involves a q-twist. This can happen for a number of reasons.

First, there could be a projection $X \to Y_i$ which corresponds to an irreducible twisted tensor product representation for X on the natural Y_i-module. Since X has rank at least 2, this can only happen when $X = A_2 < A_2A_2 < A_8 = Y$ or $X = C_2 < C_2C_2 < D_8 = Y$, with

$G = E_8$ in both cases. In either case, we see from the tables that the two A_2 or C_2 factors are both restricted, and the result follows.

Second, there could be a projection $X \to Y_i$ which corresponds to a reducible representation of X on the natural Y_i-module, with different twists for each summand. This occurs only if Y_i is of type D_n; for example, $X = B_2 \to D_5 = Y_i$ via the embedding $10 \oplus 10^{(q)}$, or $X = G_2 \to D_7 = Y_i$ via $10 \oplus 10^{(q)}$. In all such cases, inspection of the tables shows that we can choose a suitable product of restricted copies of X in Y_i and the other factors of Y to give the conclusion.

Finally, there could simply be distinct twists for the projections $X \to Y_i$; such a situation is indicated by the notation $Y_1 Y_2^q \ldots$ in the tables. Let $Z_1, Z_2, \ldots$ be products of the Y_i's corresponding to the same twist. Once again, inspection of the tables shows that we can find restricted copies of X in each Z_i so that X is contained in the product of these, with different twists in each projection. This completes the proof.

The proof of Theorem 1 is now complete.

5. Proof of the Restricted Morphism Theorem

In this section we prove the Restricted Morphism theorem, using Theorem 1. Let X be a simple simply connected group and let $\phi : X \to G$ be a morphism with image group G-cr, where G is a simple algebraic group in good characteristic p. We begin with two lemmas.

The first lemma shows that in part (ii) of the definition of a restricted morphism (see Section 1), in the cases where G is classical and has more than one natural module it does not matter which natural module is chosen.

Lemma 5.1. *Let X be simple and simply connected of rank at least 2, and let $G = A_n$, B_2, A_3 or D_4 (with p a good prime for G). If $\phi : X \to G$ is a representation which is restricted on some natural module for G, then ϕ is restricted on all natural modules for G.*

Proof. The result is trivial if X and G are of the same type, so assume this is not the case. If $G = A_n$, the result is immediate using duals. For $G = B_2$ there are no possibilities with X proper. For A_3 the 6-dimensional module is the wedge square of the 4-dimensional natural module. The only possibilities with X proper are $X = A_2$ or B_2, and considering possible actions on the 4-dimensional module immediately yields the assertion.

Now let $G = D_4$. We may as well take ϕ to be restricted on the natural 8-dimensional module $V = V_G(\lambda_1)$. The possibilities for X and the high weights of the composition factors of $V \downarrow \phi(X)$ are as follows:

$$
\begin{aligned}
X &= A_2, & V \downarrow \phi(X) &= 11 \text{ or } 10/01/00^2,\\
X &= A_3, & V \downarrow \phi(X) &= 100/001 \text{ or } 010/00^2,\\
X &= B_2, & V \downarrow \phi(X) &= 01^2 \text{ or } 10/00^3,\\
X &= B_3, & V \downarrow \phi(X) &= 100/000 \text{ or } 001.\\
X &= G_2, & V \downarrow \phi(X) &= 10/00.
\end{aligned}
$$

In the irreducible A_2 case, the image centralizes a triality morphism of G which permutes the 3 modules in question. Excluding this case, we see that in each case $\bigwedge^2 V$ is also restricted for X. But this wedge is the same for any of the 3 modules, so they must also be restricted. $\square$

The second lemma shows that centralizer condition in the Restricted Morphism Theorem is independent of the isogeny type of G. The proof is very similar to an argument in the proof of Lemma 3.1, but we give details for completeness. Let $\widehat{G}$ be the simply connected group of the same type as G, and let $\pi : \widehat{G} \to G$ be the canonical surjection. As X is simply connected we can find $\hat{\phi} : X \to \widehat{G}$ such that $\phi = \pi \circ \hat{\phi}$.

Lemma 5.2. *With notation as above,* $C_G(\phi(X))^0 = C_G(\mathrm{d}\phi(L(X)))^0$ *if and only if* $C_{\widehat{G}}(\hat{\phi}(X))^0 = C_{\widehat{G}}(\mathrm{d}\hat{\phi}(L(X)))^0$.

Proof. Let $C = C_G(\phi(X))^0$ and $\widehat{C} = C_{\widehat{G}}(\hat{\phi}(X))^0$. Similarly we set $D = C_G(\mathrm{d}\phi(L(X)))^0$ and $\widehat{D} = C_{\widehat{G}}(\mathrm{d}\hat{\phi}(L(X)))^0$.

Now X is generated by unipotent elements and, as X is simply connected, $L(X)$ is generated by nilpotent elements. Similarly for the images of X under ϕ and $\hat{\phi}$ and for the images of $L(X)$ under $\mathrm{d}\phi$ and $\mathrm{d}\hat{\phi}$.

For $u \in \widehat{G}$ a unipotent element and $n \in L(\widehat{G})$ a nilpotent element it follows from the Jordan decomposition that $C_{\widehat{G}}(uZ) = C_{\widehat{G}}(u)$ and $C_{\widehat{G}}(n + S) = C_{\widehat{G}}(n)$, where $Z = \ker(\pi)$ and $S = \ker(\mathrm{d}\pi)$. It follows that $\pi^{-1}(C) = \widehat{C} \cdot Z$ and $\pi^{-1}(D) = \widehat{D} \cdot Z$, so the result follows by taking connected components. $\square$

We can now prove the Restricted Morphism Theorem. Let $\phi : X \to G$ be as above, with $\phi(X)$ a G-cr subgroup of G.

Suppose first that $C_G(\phi(X))^0 = C_G(\mathrm{d}\phi(L(X)))^0$. By Theorem 1, ϕ factors as

$$X \xrightarrow{\psi} X \times \cdots \times X \xrightarrow{\mu} G,$$

where $\psi(x) = (x^{(q_1)}, \ldots, x^{(q_k)})$, $q_1 < \cdots < q_k$, and μ is restricted with finite kernel. Let $E_1 \cdots E_k$ be the image of μ. If $q_1 > 1$ then $\mathrm{d}\phi(L(X)) = 0$, contradicting the supposition that $C_G(\phi(X))^0 = C_G(\mathrm{d}\phi(L(X)))^0$. Hence $q_1 = 1$. If $k > 1$ then $\mathrm{d}\phi(L(X)) \leqslant L(E_1)$, so $\mathrm{d}\phi(L(X))$ is centralized by E_i for $i > 1$. However, $\phi(X)$ does not centralize any E_i. Hence $k = 1$, and so $\phi = \mu$ is restricted, as required.

Conversely, suppose that $\phi : X \to G$ is restricted. We need to show that $C_G(\phi(X))^0 = C_G(\mathrm{d}\phi(L(X)))^0$. Set $E = \phi(X)$, a restricted subgroup of G.

First assume G is of exceptional type. Then as p is good for G the only proper ideals of $L(X)$ consist of semisimple elements (this could fail if X had type B_n, C_n, F_4, G_2 with $p = 2, 2, 2, 3$, respectively). Hence $\mathrm{d}\phi(L(X)) = L(E)^+$ and we must show that $C_G(E)^0 = C_G(L(E)^+)^0$. But this is immediate from Proposition 2.3(iii).

Now assume G is of classical type. By Lemma 5.2 we may assume that $G = SL(V)$, $Sp(V)$ or $SO(V)$, a classical group with natural module V. It will suffice to establish the result for $G = SL(V)$. The fact that ϕ is restricted simply means that $\phi(X)$ has restricted composition factors on V. Since $\phi(X)$ is G-cr and p is good, V is completely reducible and restricted for X. It follows that X and $L(X)$ have precisely the same irreducible subspaces on V under the representations ϕ and $\mathrm{d}\phi$, respectively. Now [8, 2.3] shows that $\phi(X)$ and $\mathrm{d}\phi(L(X))$ have the same centralizer in $GL(V)$.

This completes the proof of the Restricted Morphism Theorem.

6. Proof of Theorem 2

Assume the hypotheses of Theorem 2 where we aim for a tensor product theorem covering finite groups, $Y(q)$, of Lie type. The main difficulty is for exceptional groups G, where the argument is based on results in [9] showing that for q suitably large there is a connected subgroup $\widetilde{Y}$ of G such that $\widetilde{Y}$ and $Y(q)$ stabilize precisely the same subspaces of $L(G)$.

Throughout this section assume that G is a simple algebraic group in good characteristic and that $Y(q)$ is a finite group of Lie type over $\mathbb{F}_q$, with $Y(q) = Y_\sigma$, where Y is a simple, simply connected algebraic group and σ is a Frobenius morphism. Also we fix $\phi : Y(q) \to G$ a nontrivial homomorphism with image group G-cr.

We first establish the result for classical groups where it follows readily from the Steinberg tensor product theorem. Suppose that $G = SL(V), Sp(V)$ or $SO(V)$ is classical, with natural module V. The G-cr subgroup $\phi(Y(q))$ acts completely reducibly on V. Steinberg's theorem implies that each irreducible summand of $V \downarrow Y(q)$ extends to an irreducible q-restricted representation $Y \to SL(V)$ ((q,s)-restricted if $Y(q) = {}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$). This establishes the existence of the required factorization $Y(q) \hookrightarrow Y \xrightarrow{\psi} G$ of ϕ, in the case where $G = SL(V)$. Also, $\psi(Y)$ is completely reducible on V and stabilizes precisely the same subspaces as $\phi(Y(q))$. It follows that the images of $Y(q)$ and Y have the same centralizer in $SL(V)$.

If $\mu : Y \to SL(V)$ is another such q-restricted morphism ((q,s)-restricted if $Y(q) = {}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$) factorizing ϕ, then ψ and μ are representations of Y with the same restriction to $Y(q)$ and so it follows that there exists $g \in SL(V)$ such that for $y \in Y$, we have $\mu(y) = \psi(y)^g$. Then g centralizes the image of $Y(q)$ and hence centralizes $\psi(Y)$, as well. Therefore, $\psi = \mu$ and uniqueness is established for $G = SL(V)$.

If G is symplectic or orthogonal, then $p \neq 2$ and $G = SL(V)_\tau$ for an appropriate involutory automorphism τ of $SL(V)$. With ψ as above, the morphism $\tau \circ \psi$ is another q-restricted representation such that ψ and $\tau \circ \psi$ agree on $Y(q)$. It follows from the above that these two morphisms are equal. Then $\psi(Y) \leqslant G$ giving existence. Uniqueness is a consequence of unicity for $G = SL(V)$.

Now suppose G is exceptional. The cases $Y(q) = {}^2B_2(q), {}^2G_2(q), {}^2F_4(q)$ do not occur here as p is good. Define $Y_0 = \phi(Y(q))$. By [9, Corollary 5], there is a proper connected subgroup $\widetilde{Y}$ of G containing Y_0 and fixing the same subspaces of $L(G)$ as Y_0. Choose $\widetilde{Y}$ minimal subject to these conditions. The proof of [9, 9.4] shows that $\widetilde{Y}$ is reductive, and now the proof of [9, 9.5] and the ensuing argument shows that $\widetilde{Y}$ is simple and of the same type as Y.

We claim $\widetilde{Y}$ is G-cr. Suppose $\widetilde{Y} < P = QR$, a parabolic with unipotent radical Q and Levi subgroup R. As Y_0 is G-cr, we may assume that $Y_0 < R$. Then Y_0 fixes $L(R)$, hence so does $\widetilde{Y}$. However, $N_P(L(R))^0 = R$, as shown in the proof of [9, 9.4], so this means that $\widetilde{Y} \leqslant R$, showing that $\widetilde{Y}$ is G-cr.

From Corollary 1, we have $\widetilde{Y} \leqslant E_1 \cdots E_k$, a commuting product of connected simple restricted subgroups E_i of the same type, with distinct q_i-field twists in the projections. Consequently, we can find a morphism $\mu : Y \to G$ with image $\widetilde{Y}$ and which factors as in Theorem 1 with p-powers, $q_1, \ldots, q_k$. Adjusting μ by a morphism of $\widetilde{Y}$ we can assume that $\mu \downarrow Y(q) = \phi$.

At this point μ restricts to $Y(q)$ as ϕ, but it is possible that μ is not q-restricted. For each i, let r_i denote the reduction of q_i modulo q. Using the factorization of μ we can obtain a morphism $\psi : Y \to E_1 \cdots E_k$, where the field twists are $r_1, \ldots, r_k$ and the restriction to $Y(q)$ is still ϕ.

Suppose $r_i = r_j$ for $i \neq j$. Then Y_0 fixes the Lie algebra of a diagonal subgroup of $E_i E_j$ which is not fixed by $\widetilde{Y}$, a contradiction. Hence the r_i are distinct.

Next we show that $\overline{Y} = \psi(Y)$ is G-cr. Suppose $\overline{Y} < P = QR$, a parabolic with unipotent radical Q and Levi subgroup R. As Y_0 is G-cr we can take $Y_0 < R$. Now Y_0 fixes $L(Q)$ and $L(R)$, hence so does $\widetilde{Y}$. Therefore $\widetilde{Y} \leqslant N_G(L(Q)) = P$, and hence $\widetilde{Y} \leqslant N_P(L(R))^0 = R$. Let $Z = Z(R)$. Then Z centralizes $\widetilde{Y}$. By Lemma 9.3(ii) below, $C_G(\widetilde{Y}) = C_G(E_1 \cdots E_r)$, and hence $E_1 \cdots E_r \leqslant C_G(Z) = R$. As $E_1 \ldots E_r$ contains $\overline{Y}$, it follows that $\overline{Y} \leqslant R$. Consequently $\overline{Y}$ is G-cr.

We have now established that ψ satisfies the conclusion of Theorem 2.

It remains to prove the uniqueness of ψ. Suppose $\psi' : Y \to G$ is another such morphism. Then ψ' determines a commuting product $F_1 \cdots F_l$ of restricted simple subgroups F_i with distinct s_i-twists in the projections of Y_0, where $s_i < q$. Also, Y_0 fixes each $L(F_i)$, hence so does $\widetilde{Y}$.

Observe next that the hypothesis of Proposition 2.3(iii) holds for each F_i: this is clear if $l > 1$, and is also true if $l = 1$, since then $F_1 = \psi'(Y)$ is G-cr. Then by Proposition 2.3(iii), we have $C_G(L(F_i))^0 = C_G(F_i)^0$, and hence $N_G(L(F_i))^0 = F_i C_G(F_i)^0$. It follows that $\widetilde{Y}$ normalizes $F_1 \cdots F_l$, hence lies in $F_1 \cdots F_l D$, where $D = C_G(F_1 \cdots F_l)^0$. Since $Y_0 < F_1 \cdots F_l$, the projection from $\widetilde{Y}$ to D has kernel containing Y_0, and hence also $\widetilde{Y} \leqslant F_1 \cdots F_l$.

The projections of $\widetilde{Y}$ to the simple factors F_i involve distinct field twists, as this is already the case for Y_0. It now follows from the uniqueness assertion in Theorem 1, that $k = l$ and $E_1 \cdots E_k = F_1 \cdots F_l$, and reordering we may assume $E_i = F_i$ for each i.

The maps ψ, ψ' factor in accordance with Theorem 1. We then have an equality $\prod \psi_i(x^{(r_i)}) = \prod \psi_i'(x^{(s_i)})$ for all $x \in Y(q)$. As in the uniqueness argument of Section 3 this implies

$$\psi_i\big(x^{(r_i)}\big) = \psi_i'\big(x^{(s_i)}\big) \tag{$*$}$$

for each i and all $x \in Y(q)$. Fix i. There is an automorphism α of Y such that $\psi_i = \psi_i' \circ \alpha$. Taking $r_i \leqslant s_i$ and writing $t_i = s_i/r_i$, we then have $\psi_i'(\alpha(x)) = \psi_i(x) = \psi_i'(x^{(t_i)})$ for all $x \in Y(q)$, and hence $\alpha(x) = x^{(t_i)}$ for all $x \in Y(q)$. It follows that $\alpha(y) = y^{(t_i q^r)}$ for some $r \geqslant 0$ and all $y \in Y$. However, we know that $\psi_i = \psi_i' \circ \alpha$ and ψ_i, ψ_i' are restricted morphisms. Hence it must be the case that $r = 0$ and $t_i = 1$. In other words, $\psi_i = \psi_i'$. This establishes the uniqueness of ψ.

7. Tilting decompositions

In this section we establish Theorem 3. Let G be as in the hypothesis of Theorem 3, and let X be a connected, simple subgroup of G of type A_1 which is G-cr. Our goal is to show that $L(G) \downarrow X$ is a direct sum of modules, each of which is a twisted tensor product of tilting modules for X where the tensor factors have (untwisted) high weights at most $2p-2$.

From Theorem 1 we have $X \leqslant R_1 \cdots R_k$, with each R_i a restricted A_1 (i.e. a good A_1), and X is embedded with distinct field twists in each factor. Consequently, it will suffice to show that $L(G) \downarrow (R_1 \cdots R_k)$ is a direct sum with each summand being a tensor product of indecomposable tilting modules for the factors R_i with appropriate high weights.

We know from [13, Theorem 1(iii)] that $L(G) \downarrow R_i$ is a tilting module for each i. However, unlike the situation for completely reducible modules, this does not in general imply a tilting decomposition for $R_1 \cdots R_k$. For classical groups it is easy to get the result, but for exceptional groups we will have to work harder.

Note that by the above, we can assume that $k \geqslant 2$. The first lemma relates Weyl modules and tilting modules for $R_1 \cdots R_k$ to those of the individual R_i. If λ is a dominant weight for a semisimple group E, let $W_E(\lambda), T_E(\lambda)$ denote the corresponding Weyl module and indecomposable tilting module.

Lemma 7.1. *Let $\lambda = \lambda_1 + \cdots + \lambda_k$ be a dominant weight of $R_1 \cdots R_k$, where λ_i is a dominant weight for R_i. Then*

(i) $W_{R_1 \cdots R_k}(\lambda) = W_{R_1}(\lambda_1) \otimes \cdots \otimes W_{R_k}(\lambda_k)$.
(ii) $T_{R_1 \cdots R_k}(\lambda) = T_{R_1}(\lambda_1) \otimes \cdots \otimes T_{R_k}(\lambda_k)$.

Proof. (i) Let $V = W_{R_1}(\lambda_1) \otimes \cdots \otimes W_{R_k}(\lambda_k)$. Then V has the same character as $W_{R_1 \cdots R_k}(\lambda)$. Fix i and consider $V \downarrow R_i$. This restriction is the direct sum of copies of $W_{R_i}(\lambda_i)$ and hence all semisimple quotients are homogeneous of type $V_{R_i}(\lambda_i)$. Now letting i vary we see that any simple quotient of V has high weight λ. As λ has multiplicity 1 we conclude that V is indecomposable. The universal property of Weyl modules [3, p. 209] implies that V is the image of $W_{R_1 \cdots R_k}(\lambda)$, and these modules have the same dimension. Part (i) follows.

(ii) As each $T_{R_i}(\lambda_i)$ has a filtration by Weyl modules, (i) implies that the same holds for $S = T_{R_1}(\lambda_1) \otimes \cdots \otimes T_{R_k}(\lambda_k)$. Similarly, we see that S has a filtration by dual Weyl modules. It follows that $T_{R_1}(\lambda_1) \otimes \cdots \otimes T_{R_k}(\lambda_k)$ is a tilting module with high weight λ. Consequently we can write $T_{R_1}(\lambda_1) \otimes \cdots \otimes T_{R_k}(\lambda_k) = T_{R_1 \cdots R_k}(\lambda) \oplus T_{R_1 \cdots R_k}(\delta) \oplus T_{R_1 \cdots R_k}(\mu) \oplus \cdots$, where $\lambda > \delta \geqslant \mu \ldots$.

Suppose $T_{R_1 \cdots R_k}(\delta) \neq 0$. Inductively, (ii) holds for δ so that $T_{R_1 \cdots R_k}(\delta) = T_{R_1}(\delta_1) \otimes \cdots \otimes T_{R_k}(\delta_k)$. Fix i. Then $S \downarrow R_i$ is the direct sum of copies of $T_{R_i}(\lambda_i)$ and hence is a tilting module. Direct summands of tilting modules are again tilting modules, so that $T_{R_1 \cdots R_k}(\delta) \downarrow R_i$ is a tilting module and is thus the direct sum of copies of $T_{R_i}(\lambda_i)$. But from (ii) for δ we obtain $\delta_i = \lambda_i$. Letting i vary this gives $\delta = \lambda$, a contradiction. The result follows. □

The next lemma is presented in a more general form than is required for this section.

Lemma 7.2. *Let X be a connected simple subgroup of G which is G-cr and $R_1 \cdots R_k$ the commuting product given by Corollary* 1. *Then $R_1 \cdots R_k$ is G-cr.*

Proof. Suppose that $R_1 \cdots R_k < P$, with P a parabolic subgroup of G. Then $X < P$. As X is G-cr there is a Levi subgroup L of P containing X. Let Z be the connected center of L, a nontrivial torus.

The uniqueness assertion in Theorem 1 (or Corollary 1) implies that Z normalizes $R_1 \cdots R_k$ and connectedness of Z implies that $Z < N_G(R_i)$ for each i. As $Z \leqslant C_G(X)$ and X projects onto each R_i, we conclude that $R_1 \cdots R_k \leqslant C_G(Z) = L$, proving the lemma. □

Lemma 7.3. *Theorem* 3 *holds if G is a classical group.*

Proof. Suppose G is classical, with natural module V. Lemma 7.2 and our assumption that p is a good prime imply that $V \downarrow (R_1 \cdots R_k)$ is completely reducible, with each composition factor a tensor product of restricted modules for the various factors R_i. Thus $V \downarrow (R_1 \cdots R_k)$ is a tilting module. Now tensor products of tilting modules and direct summands of tilting modules are again tilting modules. Since $L(G)$ is a direct summand of $V \otimes V^*$, we have the result. □

For the remainder of the proof of Theorem 3 assume G is of exceptional type. As p is a good prime for G this implies $p > 3$.

Lemma 7.4. *Theorem* 3 *holds if $L(G) \downarrow R_1 \cdots R_k = \bigoplus_j V_j$, where for each j, at most one R_i fails to be completely reducible on V_j. In particular, the result holds if $L(G) \downarrow R_1 \cdots R_k$ is completely reducible.*

Proof. Assume $L(G) \downarrow R_1 \cdots R_k$ is completely reducible. Since we know that each R_i is a good A_1, this implies that each V_j is restricted and then the result is immediate. So now assume that for some fixed j one R_i, say R_k, is not completely reducible on V_j.

Consider the action of $R_1 \cdots R_k$ on V_j. Each of $R_1, \ldots, R_{k-1}$ is completely reducible on V_j. It follows (see [8, 2.3] and argue by induction) that $A = R_1 \cdots R_{k-1}$ acts completely reducibly on V_j, and by restricting to a homogeneous component we may assume that V_j is homogeneous in this action. Let $C = C_{GL(V_j)}(A)$. Another application of [8, 2.3] shows that we can write $V_j = Y \otimes W$ for some spaces Y, W, so that A induces a subgroup of $GL(Y) \otimes 1$ and $C = 1 \otimes GL(W)$; in particular, $V_j \downarrow C$ is homogeneous of type W. On the other hand, $R_k \leqslant C$ and $V_j \downarrow R_k$ is known to be a tilting module. As direct summands of tilting modules are tilting, $W \downarrow R_k$ is tilting, hence is a direct sum of indecomposable tilting modules. Moreover, A is completely reducible on Y, with each irreducible restricted and hence tilting. It follows that $V_j \downarrow R_1 \cdots R_k$ is a direct sum of submodules, each of which is a tensor product of restricted irreducibles for $R_1, \ldots, R_{k-1}$, and an indecomposable tilting module for R_k of high weight at most $2p - 2$. The result follows. □

In the ensuing argument we shall make use of Proposition 2.2, which shows that either $R_1 \cdots R_k$ is essentially embedded in a subsystem subgroup of G, or the situation of Proposition 2.2(iii) holds. With this in mind, we first establish the following.

Lemma 7.5. *Let Y be a semisimple subsystem subgroup of G.*

(i) *If Y has no factor A_{p-1} then $L(G) \downarrow Y$ is completely reducible.*
(ii) *If Y has a factor $S = A_{p-1}$, then $L(G) \downarrow Y = A \oplus B$, with B completely reducible. In addition, S is the only factor of Y acting nontrivially on A and $S = SL_p$ acts on A as on gl_p.*

Proof. (i) Write $Y = Y_1 \cdots Y_r$, a commuting product of simple subsystem groups Y_i. It is well known and easy to prove (for example, use [8, 2.3] and induction) that $L(G) \downarrow Y$ is completely reducible if and only if $L(G) \downarrow Y_i$ is completely reducible for each i. So we may assume that Y is simple. Now the high weights, λ, of composition factors for Y on $L(G)$ are given by [7, Tables 8.1–8.5]: we list below the possible nonzero high weights other than that of the adjoint module of Y:

(a) $Y = A_n$: $\lambda = \lambda_j$ or λ_{n-j} $(j = 1, 2, 3, 4)$, $2\lambda_1, 2\lambda_n, 3\lambda_1$.
(Note: $2\lambda_1, 2\lambda_n$ occur only for $G = F_4$ with $n \leqslant 2$, and $3\lambda_1$ only for $G = G_2$ with $n = 1$.)
(b) $Y = D_n$: $\lambda = \lambda_1, \lambda_{n-1}, \lambda_n$.
(c) $Y = E_6$ (respectively E_7): λ_1 or λ_6 (respectively λ_7).
(d) $Y = B_n, C_n$ $(G = F_4, n \leqslant 4, n \leqslant 3$, respectively): λ_1, λ_n.

For each of these high weights we claim that the corresponding Weyl module $W_Y(\lambda)$ is irreducible. This follows from [7, 1.11] except when $(Y, \lambda) = (A_n, \lambda_4)$ or (C_3, λ_3); in the first case $W_Y(\lambda_4)$ is the fourth wedge of the natural A_n-module, which is irreducible, and in the second the claim follows from [2]. Moreover, it is well known—see, for example, [9, 1.10]—that the adjoint module $L(Y)$ is irreducible except for the special cases of the lemma, where $(Y, p) = (A_4, 5)$ or $(A_6, 7)$. This establishes (i).

Now assume Y has a factor $S = A_{p-1}$. If $G = E_8$, only the case $p = 7$ occurs since we are assuming p to be a good prime. A consideration of subsystems implies that $Y \leqslant S \cdot T_1 \cdot R$, where R is semisimple. There is a subsystem group of type D_p containing $S \cdot T_1$ as a Levi factor. Then $L(S \cdot T_1) \cong gl_p$ as an S-module. This yields the space A, which is nondegenerate. Taking perpendicular spaces we proceed as above to get (ii). □

Lemma 7.6. *Theorem* 3 *holds if $R_1 \cdots R_k$ is essentially embedded in a subsystem subgroup Y of G such that each simple factor of Y is of classical type.*

Proof. We first argue that it suffices to consider the case where Y is simple. The previous lemma shows that either Y is completely reducible on $L(G)$ or this is true with the exception of just one summand where a single A_{p-1} factor acts nontrivially. In reducing to the case Y simple we consider one summand at a time. So we may ignore the exceptional cases for now. Consider an irreducible summand, which is the tensor product of irreducible representations for the various simple factors of Y. This yields a corresponding tensor product for the action of $R_1 \cdots R_k$. The tensor product of tilting modules is again a tilting module, so we may replace $R_1 \cdots R_k$ by its projection in a simple factor of Y. In this way, we reduce to the case Y simple.

Consider first the case where $Y = A_n$. Here the embedding of $R_1 \cdots R_k$ in Y corresponds to an irreducible representation. Moreover, each R_i is a good A_1 of G and hence of Y. Hence, the natural module, say V, for Y (or an appropriate cover) affords an irreducible restricted module for the corresponding cover of $R_1 \cdots R_k$. Thus V affords a tilting module for $R_1 \cdots R_k$. Lemma 7.5 shows that Y is completely reducible on $L(G)$, except for the cases $Y = A_4, A_6$, with $p = 5, 7$, respectively. In the exceptional cases the action is completely reducible except for a summand of type $gl_p \cong V \otimes V^*$. As tensor products of tilting modules are again tilting, this case causes no difficulty. The other direct summands of $L(G) \downarrow Y$ have high weights of irreducibles listed under case (a) in the proof of Lemma 7.5. As p is a good prime, each of these summands is a direct summand of an appropriate tensor power of the natural module. The family of tilting modules is closed under tensor products and direct summands, so the assertion follows in this case.

Next assume $Y = D_n$. Here $R_1 \cdots R_k \leqslant Y$ essential means that under the action of $R_1 \cdots R_k$, the natural orthogonal D_n-module is either irreducible or decomposes as an orthogonal sum of two irreducibles of odd degree. Since each irreducible summand of D_n is completely reducible under the action of $B_k \times B_{n-k-1}$ another reduction allows us to assume that $R_1 \cdots R_k < Y_0 = B_r$ or D_r, where $r \leqslant 7$ or 8, respectively, and the embedding corresponds to an irreducible restricted representation. From the information in (b) of the proof of Lemma 7.5 we see that $L(G) \downarrow Y_0$ is a direct sum of an adjoint module, natural modules, and spin modules. The only issue is the action of $R_1 \cdots R_k$ on the corresponding spin modules.

Recall the assumption that $k \geqslant 2$. The possibilities for the embedding $R_1 \cdots R_k < Y_0$ and the corresponding composition factors of the spin modules for Y_0 restricted to $R_1 \cdots R_k$ can be read off from the table of [7, p. 29]. If each composition factor for each R_i is restricted, then $R_1 \cdots R_k$ acts completely reducibly on the spin module and there is nothing to prove. In the remaining cases we have $k = 2$. We list the cases, indicating the possible pairs $(i \otimes j, Y_0)$, where $i \otimes j$ is the irreducible tensor product representation of $R_1 R_2$ on the natural Y_0-module:

$$(5 \otimes 1, D_6), \quad (4 \otimes 2, B_7), \quad (7 \otimes 1, D_8), \quad (3 \otimes 3, D_8).$$

In all but the last case it follows from [7, p. 29] and our assumption that p is a good prime for G, that R_2 is completely reducible on the spin modules. Since we also know that R_1 has a tilting decomposition on $L(G)$ and hence on the spin modules, consideration of homogeneous summands for R_2 gives the conclusion.

In the last case we have $R_1 R_2 < C_2 C_2 < D_8$. If W_1, W_2 denote the two restricted spin modules for D_8, then from [7, p. 30] we have

$$W_1 \downarrow C_2 C_2 = 10 \otimes 11/11 \otimes 10, \qquad W_2 \downarrow C_2 C_2 = 20 \otimes 01/01 \otimes 20/02 \otimes 00/00 \otimes 02.$$

Since $p \geqslant 7$ here (as p is good), it follows that $W_i \downarrow C_2 C_2$ is completely reducible for $i = 1, 2$.

Fix i and $R_i < C_2$. We will consider restrictions of the above representations to R_i. First note that the modules 10 and 01 are both irreducible restricted representations for R_i, hence irreducible tilting modules. Hence $01 \otimes 01, 10 \otimes 10$ and $10 \otimes 01$ are all tilting modules upon restriction to R_i. These tensor products decompose for C_2 as $02 \oplus 20 \oplus 00, 20 \oplus 01 \oplus 00, 11 \oplus 10$, respectively. Hence R_i acts on each W_j as a sum of indecomposable tilting modules and the result follows.

Finally, consider the cases where $Y = B_n$, $C_n < F_4$. Here $R_1 \cdots R_k$ is an irreducible subgroup of Y, so $k = 2$ and we indicate the possibilities for $(i \otimes j, Y)$, where $i \otimes j$ is the representation of $R_1 R_2$ on the natural Y-module:

$$(2 \otimes 2, B_4), \quad (1 \otimes 2, C_3), \quad (1 \otimes 1, C_2).$$

Now $p \geqslant 5$ and we claim in each case that $L(G) \downarrow R_i$ is restricted. In the first case this is shown in [7, 2.13]. In the other cases it follows from fact (d) given in the proof of Lemma 7.5 that the composition factors of C_n to consider are those of high weights λ_1, λ_n. These occur within the appropriate wedge of the natural module, so the claim is immediate. The conclusion now follows from Lemma 7.4. □

Lemma 7.7. *Theorem* 3 *holds if $R_1 \cdots R_k$ is contained in no subsystem subgroup having each factor of classical type.*

Proof. Under the hypothesis, Proposition 2.2 shows that there is a subgroup Y_0 of exceptional type F_4, E_6, E_7 or E_8 in G, a maximal connected subgroup Z of Y_0 not containing a maximal torus, and a semisimple subgroup Y_1 of $C_G(Y_0)$ such that $R_1 \cdots R_k$ is essentially embedded in ZY_1.

If Y_1 is not simple, then in view of the possibilities for $C_G(Y_0)$ (see the remark after Proposition 2.2), we have $Y_0 C_G(Y_0) = F_4 G_2 < E_8 = G$ and $Y_1 = A_1 A_1$. But then $R_1 \cdots R_k$ centralizes an involution and we can replace $Y_0 C_G(Y_0)$ by the centralizer $A_1 E_7$ of this involution. Hence we may assume that Y_1 is simple. As a consequence we have that the projection of $R_1 \cdots R_k$ to Y_1 is either trivial or a single A_1.

The group $Y_0 C_G(Y_0)$ acts completely reducibly on $L(G)$ with composition factors given by [7, 2.1, 2.4]. Using this we see that the projection of $R_1 \cdots R_k$ to Y_1 acts completely reducibly on $L(G)$ with each composition factor restricted. Since tensor products of tilting modules are tilting, it suffices to work with the projection to Z. That is we assume that $R \leqslant Z$, essentially embedded.

Taking into account the fact that $k \geqslant 2$, by Theorem 2.1 we have the following configurations to consider:

$$\begin{aligned}
&Y_0 = F_4, \quad Z = A_1 G_2,\\
&Y_0 = E_6, \quad Z = A_2 G_2, C_4,\\
&Y_0 = E_7, \quad Z = A_1 A_1, A_1 G_2, A_1 F_4, G_2 C_3,\\
&Y_0 = E_8, \quad Z = A_1 A_2, G_2 F_4.
\end{aligned}$$

With the exception of the cases $Z = C_4$, $G_2 C_3$, which will be settled later in the proof, the essentiality of $R_1 \cdots R_k$ in Z implies that $k = 2$, with one R_i in each simple factor of Z.

So write $Z = Z_1Z_2$ with $R_i \leqslant Z_i$, where Z_1 is the first factor in the list above. In view of Lemma 7.4 we are done if we can show that either R_1 or R_2 has all composition factors on $L(G)$ being restricted.

Consider the cases where $(Z, Y_0) = (A_1G_2, F_4)$, (A_2G_2, E_6), (A_1G_2, E_7), (A_1F_4, E_7), or (A_1A_2, E_8). For the E_8 case we have $p \geqslant 7$ as p is good; this also holds in the other cases, because maximal A_1's in G_2, F_4 require $p \geqslant 7, 13$, respectively. Using [7, 2.4 and 2.5] we check that R_1 has all composition factors on $L(G)$ being restricted, so we have the result by Lemma 7.4.

A similar argument holds for the case where $Z = G_2F_4 < E_8$. Here, $R_1 < G_2$ is irreducible and restricted on the usual 7-dimensional G_2-module, and the existence of a maximal A_1 in F_4 implies that $p \geqslant 13$. Then [7, 2.4] implies that all composition factors of R_1 on $L(G)$ are restricted, giving the result by Lemma 7.4.

Next suppose $Z = C_4 < E_6$. Here, $k = 3$ and the natural C_4-module V_8 restricts to $R_1R_2R_3$ as $1 \otimes 1 \otimes 1$. By [7, 2.4], the possible composition factors of C_4 on $L(G)$ have high weights 2000, 0100, 0001. It follows that each R_i has only restricted composition factors on $L(G)$ and again the result follows from Lemma 7.4.

Now assume $Z = G_2C_3 < E_7$. We may suppose R_1 projects nontrivially to G_2 as a maximal A_1. This forces $p \geqslant 7$. If the projection of $R_1 \cdots R_k$ to C_3 is an irreducible A_1, then $k = 2$, R_1 has trivial projection to C_3 and we are immediately done by Lemma 7.4. So assume the projection of R to C_3 corresponds to an irreducible subgroup of type A_1A_1 acting as $1 \otimes 2$ on the 6-dimensional symplectic module. Also, $k = 2$ or 3.

The composition factors of Z on $L(G)$ are $L(Z)$, $10 \otimes 010$, $10 \otimes 100$, $00 \otimes 001$, where the latter two occur only if $G = E_8$. This action is completely reducible so we can work with the individual summands. Now $R_2, (R_3) < C_3$ and from the tensor embedding on the natural module we easily see that all composition factors of R_2 (and R_3 if it occurs) on $L(G)$ are restricted (as $p \geqslant 7$). So once again Lemma 7.4 settles the issue.

The remaining case is $Z = R_1R_2 = A_1A_1 < E_7$. Here, by [7, 2.4],

$$L(E_7) \downarrow R_1R_2 = 2 \otimes 0/0 \otimes 2/2 \otimes 8/4 \otimes 6/6 \otimes 4/2 \otimes 4/4 \otimes 2.$$

If $G = E_8$ the restriction of $L(G)$ to E_7 involves $L(E_7)$ plus two copies of $V_{56} = V(\lambda_7)$. By [7, 2.5], we have $V_{56} \downarrow R_1R_2 = 6 \otimes 3/4 \otimes 1/2 \otimes 5$. If $p \geqslant 7$, then R_1 has all factors restricted, so the result follows from Lemma 7.4. The only difficulty occurs for $G = E_7$ with $p = 5$. Here we must be a little more careful.

Notice that each of R_1 and R_2 have composition factors of high weight 4. These extend no other composition factors. Consequently, we may write $L(G) \downarrow R_1R_2 = V_1 \oplus V_2 \oplus V_3$, where for $i = 1, 2$, $V_i \downarrow R_i = 4^k$. We have $V_3 \downarrow R_1R_2 = 2 \otimes 0/0 \otimes 2/2 \otimes 8$. On each factor either R_1 or R_2 is restricted, while the other restricts to a tilting module. Once again the result follows from Lemma 7.4. □

At this point we have completed the proof of Theorem 3.

8. Theorem 4

In this section we prove Theorem 4 and Corollary 3. Assume then that G is of exceptional type and $X < G$ is a connected simple G-cr subgroup of rank at least 2. Let $E_1, \ldots, E_r$ be the subgroups given in Corollary 1.

By Proposition 2.2, either X is essentially embedded in a subsystem subgroup Y of G, or $X = G_2$, $p = 7$ and conclusion (ii) of Theorem 4 holds. In the latter case the restriction $L(G) \downarrow X$ can be worked out using the following restrictions:

$$L(F_4) \downarrow G_2 = L(G_2) \oplus V_{G_2}(11), \qquad L(E_8) \downarrow G_2F_4 = L(G_2) \oplus L(F_4) \oplus (10 \otimes 0001)$$

(see [12, p. 193]), from which we see that Theorem 4 holds in this case.

Assume now Theorem 4(ii) does not hold, so that X is essentially embedded in a subsystem subgroup Y of G. As observed in the proof of Proposition 2.3, when $p > N(X,G)$ (as defined in [7, p. 2]), the possibilities for Y, X, and the composition factors of $L(G) \downarrow X$ are worked out explicitly in [7, Tables 8.1–8.4]; and when $p \leqslant N(X,G)$, we have $(X,G,p) = (A_2, E_7, 5)$, $(G_2, E_7, 5 \text{ or } 7)$, or $(G_2, E_8, 7)$, and the possibilities for Y, X and $L(G) \downarrow X$ can be worked out as in [10], and are just as in Tables 8.1–8.4 again. These tables give the composition factors of $L(G) \downarrow X$, and indicate those cases where one of the corresponding Weyl modules is reducible. Moreover, the proof of Theorem 1 shows that the product $E_1 \cdots E_r$ lies in Y and can be read off from the tables.

If all the relevant Weyl modules are irreducible, then $L(G) \downarrow E_i$ is completely reducible for each i, this shows that $L(G) \downarrow E_1 \cdots E_r$ is completely reducible and that each irreducible summand is a tensor product of (irreducible) Weyl modules for the factors. Thus Theorem 4 holds. Moreover, we see from [7, Tables 8.1–8.4] that when $p > 7$ all the relevant Weyl modules are irreducible, so this establishes Corollary 3.

It remains to consider those cases where one of the Weyl modules corresponding to a composition factor of $L(G) \downarrow X$ is reducible. From the tables in [7], these cases are in Table 4.

Table 4

X	Y	p	Reducible Weyl module in $L(G) \downarrow X$
A_6	A_6	7	$W(\lambda_1 + \lambda_6) = \lambda_1 + \lambda_6 \vert 0$
A_4	A_4	5	$W(\lambda_1 + \lambda_4) = \lambda_1 + \lambda_4 \vert 0$
B_3	A_6	7	$W(200) = 200 \vert 000$
B_3	A_7	7	$W(101) = 101 \vert 001$
C_3	D_7	7	$W(110) = 110 \vert 100$
G_2	A_6	7	$W(20) = 20 \vert 00$
G_2	D_7	7	$W(11) = 11 \vert 20$
B_2	D_7	7	$W(22) = 22 \vert 02$, $W(13) = 13 \vert 03$
B_2	D_5	5	$W(11) = 11 \vert 01$
A_2	A_5	5	$W(22) = 22 \vert 11$
A_2	A_2A_5	5	$W(22) = 22 \vert 11$, $W(31) = 31 \vert 20$
A_2	E_6	5	$W(22) = 22 \vert 11$
A_2	E_7	7	$W(44) = 44 \vert 11$

In all cases except $(X,Y)=(A_2,A_2A_5)$, the fact that X is essentially embedded in Y and there is a composition factor in $L(G)\downarrow X$ as indicated in the last column, implies that $r=1$ and hence that X is a restricted subgroup of G. Consequently, it will suffice in these cases to show that $L(G)\downarrow X$ is a direct sum of Weyl modules, dual Weyl modules, and tilting modules. In the exceptional case with $Y=A_2A_5$, either $r=1$ and X is a restricted subgroup, or $r=2$ and there is a field twist in one of the projections from X to the factors of Y.

Consider the first case $X=A_6<G$ with $p=7$. Here $G=E_7$ or E_8. Let V_7 be the usual 7-dimensional module for X. It follows from [7] that $L(G)\downarrow A_6=R\oplus S$, where S is a sum of irreducible wedge modules $\bigwedge^i(V_7)=V(\lambda_i)=W(\lambda_i)$ and their duals, and R has a single adjoint composition factor and some trivial composition factors. Now X is contained in a subgroup $GL_7\cong A_6T_1<E_7$. Indeed, there is a Levi subgroup $E=A_6T_1$ which induces GL_7 on a 7-dimensional submodule of $L(E_7)$. We have $L(E)\cong V_7\otimes V_7^*$, which is a tilting module for X. Also, $R\downarrow A_6$ is the direct some of $L(E)$ and some trivial modules, so this yields the result. The second case, $X=A_4$, $p=5$, is similar.

Now consider the third case, $X=B_3<A_6<G$ with $p=7$. As above, $L(G)\downarrow A_6=R\oplus S$. Each of the wedge modules in S is a direct summand of a tensor power of V, hence is tilting for X. And taking $E=GL_7$ as above, $L(E)\cong V\otimes V^*$ is also a tilting module for X, and the conclusion follows. The sixth case $X=G_2<B_3<A_6<G$ is entirely similar.

Next consider the cases where $(X,Y,p)=(B_3,A_7,7)$ or $(A_2,A_5,5)$. Here the embedding $X<Y$ is given by the irreducible $V_X(001)$ or $V_X(20)$, respectively, both of which are irreducible Weyl modules. From [7] we see that $L(G)\downarrow Y$ is a direct sum of $L(Y)$ with wedge modules $\bigwedge^i V$, $\bigwedge^i V^*$ and trivials, where V is the usual module for Y. Moreover, $L(Y)$ is a direct summand of $V\otimes V^*$, while $\bigwedge^i V$ is a summand of the ith tensor power of V. It follows that $L(G)\downarrow X$ is a direct sum of tilting modules, as required.

The case where $(X,Y,p)=(A_2,A_2A_5,5)$ is similar: here $G=E_7$ and $L(G)\downarrow A_2A_5=L(A_2)\oplus L(A_5)\oplus(\lambda_1\otimes\lambda_2)\oplus(\lambda_2\otimes\lambda_4)$. If $r=2$ and there is a field twist in one of the projections from X to the factors of Y, then the conclusion follows from the $Y=A_5$ case above. And if $r=1$, we see as above that each summand is tilting for X.

We next treat together the cases $X=C_3,G_2$ or B_2 with $p=7$ and $Y=D_7$. Here $X<D_7<E_8=G$ with the embedding in D_7 given by the 14-dimensional X-modules $V_X(\lambda)$ with $\lambda=\lambda_2,\lambda_2$ or $2\lambda_2$, respectively. For each of these, the Weyl module $W_X(\lambda)$ is irreducible.

It follows from [7] that $L(E_8)\downarrow D_7=\lambda_2/\lambda_1^2/\lambda_6/\lambda_7/0$. This is a direct sum, so it suffices to consider the various summands. Let V denote the natural module for D_7, an irreducible tilting module for X. Hence $V\otimes V$ and its direct summand $L(D_7)$ are also tilting for X. (We note that in B_2 case this restriction is $T_{B_2}(22)=02|22|02$.) So it suffices to consider the action of X on the two spin modules.

Let $A<X$ be a regular A_1 in X. One then checks that $V\downarrow A=T(8)$ or $T(10)$, the latter only when $X=G_2$. It follows that if $1\neq u\in A$ is unipotent, then u acts on V as the sum of two Jordan blocks of size 7. Hence u has type A_6 in the notation of the classification of unipotent classes in G (see [4]). Then [4] implies that $L(G)\downarrow u=(J_7)^{35}+(J_1)^3$, where J_r denotes a Jordan block of size r. In particular there is no Jordan block of length 6.

It is shown in [7, 2.12] that each of the spin modules restricts to X with composition factors the same as those of the Weyl module $W_{C_3}(110)$, $W_{G_2}(11)$ or $W_{B_2}(13)$. We have $W_{C_3}(110) = 110|100$, $W_{G_2}(11) = 11|20$, and $W_{B_2}(13) = 13|03$. In each case a dimension argument using the action of u implies that the spin module must be indecomposable for X, hence must be isomorphic to one of these Weyl modules, and the conclusion follows.

Next consider $X = B_2$ with $p = 5$. Here $X < D_5$ with embedding given by the 10-dimensional adjoint module $V_X(02)$. Now $G = E_6$ or E_7 (as 5 is not a good prime for E_8). As above, let $1 \neq u \in A < X$, where A is of type A_1 embedded in X via an irreducible restricted representation. As $V_X(02)$ is a direct summand of $V_X(01) \otimes V_X(01)$, it is tilting, so it follows that $V_X(02) \downarrow A = T_X(6)$. Consequently, u acts as J_5^2 and is hence a unipotent element of type A_4 in G.

Now $L(G) \downarrow D_5$ is a direct sum of $L(D_5)$, trivial modules, natural modules (only in E_7), and spin modules, so we work with each of these. Observe that $L(D_5)$ is a direct summand of the tensor square of the natural module, so its restriction to B_2 is a direct summand of $02 \otimes 02$, a tilting module. So we need only consider the spin modules. Now [7, 2.12] shows that restrictions to B_2 of the spin modules have composition factors $11|01$. By [4] unipotent elements of type A_4 have Jordan form on $L(G)$ of type $J_5^a + J_1^b$. On the other hand, the action on 01 is J_4. Hence the spin modules must be indecomposable upon restriction to B_2, as required.

Next consider the case where $(X, Y, p) = (A_2, E_7, 7)$. Here X is a maximal subgroup of E_7. We first consider the action of E_7 on $V = V_{E_7}(\lambda_7)$, an irreducible 56-dimensional module. It follows from [7] that $V \downarrow X = 60 + 06$, the sixfold symmetric power of the natural module plus its dual.

Let A be a regular A_1 subgroup of X and u a nontrivial unipotent element of A. As $W_X(60)$ is irreducible, it is a direct summand of the sixfold tensor power of the natural module 10, and similarly for the dual. Restricting to A, we see that $W_X(60) \downarrow A$ is a tilting module for A, and a consideration of weights shows this to be $T(12) + T(8)$. It follows that $V \downarrow u = J_7^8$. Consequently, it follows from [4] that u is of type A_6. This implies that $L(E_7) \downarrow u = J_7^{19}$.

The composition factors of $L(E_7) \downarrow X$ are $44|11^2$. Since $L(E_7)$ is self-dual, the only possibilities are $L(E_7) \downarrow X = T_X(44)$ or $V_X(44) \oplus V_X(11)^2$. But the latter case is impossible, as this would contradict the action of u. Therefore, $L(E_7) \downarrow X = T(44)$ and $L(E_8) \downarrow X = T(44) \oplus 60^2 \oplus 06^2 \oplus 00^3$.

It remains to handle the case $(X, Y, p) = (A_2, E_6, 5)$. Here X is maximal in Y, and $L(E_6) \downarrow X = 11 \oplus 41 \oplus 14$, a sum of irreducible Weyl modules. Hence we can assume that $G = E_7$ (not E_8, as p is a good prime). We have $L(E_7) \downarrow E_6 = L(E_6) \oplus L(T_1) \oplus V_{27} \oplus V_{27}^*$, where V_{27} is the 27-dimensional module $V_{E_6}(\lambda_1)$.

Let A be regular A_1 in X. As above, $V_X(40) \downarrow A$ is tilting, hence so is the restriction to A of the tensor product $V_X(40) \otimes V_X(01)$. A calculation with weights shows that $V_X(40) \otimes V_X(01) = 41|30$. As these composition factors do not extend each other, this is a direct sum.

We conclude that the direct summand $V_X(41)$ is tilting on restriction to A, and further calculation with weights implies that $V_X(41) \downarrow A = T(10) \oplus T(6) \oplus T(4)$. Hence if $1 \neq u \in A$ is a unipotent element, it acts on $V_X(41)$ as J_5^7. Therefore u acts on $L(E_6)$

as $J_5^{15} + J_3$; it follows by [4] that u lies in the class $A_4 + A_1$. Consequently, by [4] again, we have $L(E_7) \downarrow u = J_5^{25} + J_3 + J_2^2 + J_1$.

Finally, from [7] we have $V_{27} \downarrow X = 22|11$. The action of u shows that this must be indecomposable. Therefore $V_{27} \downarrow X = W_X(22)$, and the conclusion follows.

This completes the proof of Theorem 4.

9. Additional results

Theorem 1 and its corollary are of considerable importance for the analysis of subgroups of exceptional algebraic groups. In this section we establish additional results on subgroups.

We first extend Corollary 1 so as to cover semisimple groups. Then, returning to the case where X is simple, we show that the restricted subgroups E_i given by Corollary 1 are themselves G-cr and we determine $C_G(X)$ as the intersection of the groups $C_G(E_i)$. Finally we describe a procedure for constructing all commuting products $E_1 \cdots E_k$ as given in Corollary 1.

Let $X = X_1 \cdots X_r$ be a commuting product of connected simple G-cr subgroups of G. Corollary 1 shows that for each i there is a uniquely determined family $E_{i,1}, \ldots, E_{i,n_i}$ of commuting restricted subgroups of G such that X_i is contained in $E_{i,1} \cdots E_{i,n_i}$ with distinct field twists in each projection.

Proposition 9.1. *If each X_i is a G-cr subgroup of G, then the corresponding restricted subgroups $E_{i,k}$ and $E_{j,l}$ commute for $i \neq j$. Hence X is contained in the commuting product $(E_{1,1} \cdots E_{1,n_1}) \cdots (E_{r,1} \cdots E_{r,n_r})$.*

Proof. Fix $i \neq j$ and let $\widetilde{X}_i, \widetilde{X}_j$ be the corresponding covering groups. The groups $E_{i,s}, E_{j,t}$ arise from Theorem 1. Let $\phi_i : \widetilde{X}_i \to G$ have image X_i and factor as in Theorem 1 with certain field morphisms and a uniquely determined restricted morphism μ_i.

Let $x_j \in X_j$. Then composing μ_i with conjugation by x_j yields another such morphism and corresponding factorization of ϕ_i. Uniqueness implies that these morphisms agree and hence x_j centralizes $E_{i,s}$ for all $1 \leqslant s \leqslant n_i$.

Now start with $\phi_j : \widetilde{X}_j \to G$ with image X_j and factor this using a unique restricted morphism μ_j. Conjugating by elements of $E_{i,1} \cdots E_{i,n_i}$ and using uniqueness from Theorem 1, we have the result. □

For the next two results fix X a simple G-cr subgroup of G and let $X \leqslant E_1 \cdots E_k$ be as in Corollary 1. So each E_i is a restricted subgroup of G. The next result shows that these restricted subgroups are also G-cr.

Proposition 9.2. *With notation as above, E_i is G-cr for $i = 1, \ldots, k$.*

Proof. If $X = A_1$, then each E_i is a good A_1 of G, so by [13, 1.1(iv)] each E_i is G-cr. So now assume X has rank at least 2. Let $\widehat{X}$ be the simply connected cover of X and

$\phi : \widehat{X} \to X$ be the natural surjection. Factor $\phi = \mu \circ \psi$ (viewed as a morphism from $\widehat{X}$ to G) as in Theorem 1.

First suppose G is of classical type. The issue of being G-cr is independent of the isogeny type of G, so we may take $G = SL(V)$, $Sp(V)$, or $SO(V)$. As p is a good prime for G, the issue is whether or not the E_i act completely reducibly on V. Now μ is uniquely determined. So if τ is an automorphism of G centralizing X, then $\tau \circ \mu = \mu$, hence τ centralizes $E_1 \cdots E_k$.

Write $V = V_1 \perp \cdots \perp V_s$, where each summand is X-invariant. Moreover, we can make the choice such that for $G = SL(V)$ each V_i is irreducible for X and for $G = Sp(V)$ or $SO(V)$ each summand is either irreducible of the sum of two dual irreducible singular spaces. It is now clear that we can choose suitable semisimple automorphisms, τ_j, of G such that the intersection of the centralizers of the τ_j must stabilize each V_i and both summands of V_i in case V_i is the sum of two X-invariant singular spaces. Hence $E_1 \cdots E_k$ is completely reducible and thus so are each of the summands.

Now assume G is an exceptional group. Then Proposition 2.3(ii) gives the result except when $X = G_2$ and $p = 7$. In this case the argument of Section 4.2 (which is based on Proposition 2.2) shows that $k \leqslant 2$ and describes the containment $X \leqslant E_1 \cdots E_k$. If $k = 1$, the assertion is immediate since then $X = E_1$ which is assumed to be G-cr. Suppose $k = 2$. Then either $E_1E_2 = G_2G_2 < B_3B_3 < D_7$ or $E_1E_2 = G_2G_2 < G_2F_4 < G = E_8$. We must show that in either case both G_2 factors are G-cr.

If E is a G_2 subgroup with E contained in a D_4 subsystem subgroup of G, then the high weights of composition factors of E on $L(G)$ are 00, 10, 01. None of these extend the trivial module, so the arguments of [7] show that E is G-cr. This settles the issue except for $E = E_2$ in the second case which we now consider.

Using [12, p. 193] we have $L(G) \downarrow G_2F_4 = L(G_2) \oplus L(F_4) \oplus (10 \otimes 0001)$ and $L(F_4) \downarrow E_2 = L(E_2) \oplus 11$. Also, using the labeled diagram in this reference we have $0001 \downarrow E_2 = 20$. So $L(G) \downarrow E_2 = 20^7 \oplus 11 \oplus 10 \oplus 00^{14}$, which is completely reducible. We cannot immediately conclude that E_2 is G-cr because E_2-composition factors of high weight 20 do extend the trivial module. Note however, that the decomposition does imply that $C_G(E_2)^0 = E_1$.

Suppose that $E_2 < P$, a parabolic subgroup of G. Comparing composition factors of P on $L(G)$ with those of E_2 it is clear that the Levi factor of P must contain an E_6 factor. In fact, with suitable choice of root system $P = P_7$ or $P_{7,8}$. Hence $P \leqslant P_7 = N_G(U_\alpha U_\beta)$ where α is the high root and $\beta = \alpha - \alpha_8$. Let $L = E_6A_1T_1$ be the Levi factor of P_7 and $W = U_\alpha U_\beta$. Then W is centralized by $R_u(P)$ and by the E_6 component of L and W affords an irreducible module for the A_1T_1 part of L. Hence $E_2 < C_G(W)$ and so $W < E_1$. We now argue from [6, 2.2(i)] that all elements of W are long root elements of $E_1 = G_2$ and so $N_{E_1}(W)$ is a maximal parabolic subgroup of E_1. In particular, there is a one-dimensional torus Z in $C_G(E_2)$ inducing scalars on $U_\alpha U_\beta$. Then Z is a torus in P_7 centralizing the projection of E_2 and inducing scalars on W. It follows that Z is P-conjugate to the central torus of L and hence $E_2 < C_G(Z) = E_6A_1Z$, from which we conclude $E_2 < E_6$, so that E_2 is G-cr. $\square$

We next state a useful result on centralizers which follows easily from what has already been established.

Proposition 9.3. *Let $X \leqslant E_1 \cdots E_k \leqslant G$ be as in Corollary 1. Then*

(i) $C_G(E_i)$ *is reductive for* $i = 1, \ldots, k$.
(ii) $C_G(X) = \bigcap_i C_G(E_i)$.

Proof. (i) The previous proposition shows that each E_i is G-cr. If G is of exceptional type than Proposition 2.3(iii) yields the result. For G of classical type this is proved at the end of the proof of Lemma 3.1.

For (ii) first note that $\bigcap_i C_G(E_i) \leqslant C_G(X)$. For the other containment, let $g \in C_G(X)$ and let inn_g denote the corresponding inner automorphism of G. Let $\widehat{X}$ be the simply connected cover of X and $\phi : \widehat{X} \to X$ the natural surjection. Factor $\phi = \mu \circ \psi$ as in Theorem 1, so that $\mu(X \times \cdots \times X) = E_1 \cdots E_k$. Now consider the map $\mu' \circ \psi$, where $\mu' = \mathrm{inn}_g \circ \mu$. As g centralizes X this is another factorization of ϕ, so the uniqueness assertion of Theorem 1 implies that $\mu = \mu'$. But this implies that g centralizes each E_i, as required. □

We next establish results for G of exceptional type which can be used to determine commuting products of restricted simple subgroups.

Assume then that G is a simple algebraic group of exceptional type over an algebraically closed field of good characteristic p. The simple restricted subgroups of G are reasonably well understood. The restricted A_1's are determined in [13] and closely linked to unipotent elements of prime order; the higher rank subgroups are determined explicitly in [10].

If X is a connected, restricted, simple subgroup of G, then by definition X is also a restricted subgroup of any connected group containing it. The following remarkable result shows that the converse often holds, and is a key result for determining commuting products. Recall the definition of $N(X, G)$ taken from [7, p. 2].

Proposition 9.4. *Let S be any closed subgroup of the exceptional group G such that $C_G(S)$ is reductive. If R is a connected simple restricted subgroup of $C_G(S)$ and $p > N(R, G)$, then R is also restricted in G.*

Proof. By assumption $D = C_G(S)^0$ is reductive. Let R be a simple restricted subgroup of D. Suppose R fails to be G-restricted. As $p > N(R, G)$, [7, Theorem 1] implies R is G-cr. Consequently we may apply Theorem 1 of this paper to R, obtaining a containment $R \leqslant R_1 \cdots R_k$, where each R_i is restricted in G and the embedding is diagonal with distinct field twists in each projection. The result is trivial if $k = 1$, so assume that $k \geqslant 2$.

Reorder if necessary, so that $L(R) = L(R_1)$. Of course, $S \leqslant C_G(L(R))$. Using Proposition 2.3 we then have $S \leqslant C_G(L(R))^0 = C_G(L(R_1))^0 = C_G(R_1)^0$. Therefore, $R_1 \leqslant C_G(S)^0$. So then R, R_1 are both restricted subgroups of D having the same Lie algebra.

We claim that R, R_1 are D-cr. If D has an exceptional simple factor D_i, then $N(R, G) \geqslant N(R, D_i)$ and so the projection to this simple factor is D_i-cr by [7, Theorem 1]. For classical factors the same follows from [7, Theorem 3.8] (as p is a good prime for G).

At this point Lemma 3.1 shows that $C_D(R)^0 = C_D(L(R))^0 = C_D(L(R_1))^0 = C_D(R_1)^0$. Call this group E. Then $R \circ E = N_D(L(R))^0 = N_D(L(R_1))^0 = R_1 \circ E$. It follows that $R = R_1$, so that R is restricted in G. $\square$

Corollary 9.5. *Let A be a restricted, connected, simple subgroup of G and assume $p > N(A, G)$. If B is a simple restricted subgroup of $C_G(A)$ of the same type as A, then B is G-restricted.*

Corollary 9.5 provides an algorithm for determining commuting products of restricted subgroups of given type. The procedure is to choose one such subgroup and find its centralizer. Choose a restricted subgroup of the required type in the (reductive) centralizer, and repeat the process. It is hoped that the conjugacy classes of such commuting products will be calculated in future work.

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