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Projective Differential Geometrical Structure of the Painlevé Equations

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The necessary and sufficient conditions that an equation of the form $y''=f(x, y, y')$ can be reduced to one of the Painlevé equations under a general point transformation are obtained. A procedure to check these conditions is found. The theory of invariants plays a leading role in this investigation. The reduction of all six Painlevé equations to the form $y''=f(x, y)$ is obtained. The structure of equivalence classes is investigated for all the Painlevé equations. Following Cartan the space of the normal projective connection which is uniquely associated with any class of equivalent equations is considered. The specific structure of the spaces under investigation allows us to immerse them into $\mathbb{R}\mathbf{P}^3$. Each immersion generates a triple of two-dimensional manifolds in $\mathbb{R}\mathbf{P}^3$. The surfaces corresponding to all the Painlevé equations are presented.

1. INTRODUCTION

It is well known that P. Painlevé [P1] looked for nonlinear second order ordinary differential equations which were free from movable critical points. He investigated the equations of the type

$$y''=F(x, y, y'), \tag{1}$$

where F was rational in y' and in y, and analytic in x. There do exist six canonical equations nonequivalent to the known special function equations for which the absence of movable critical points is characteristic. Only the

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first three types were discovered by P. Painlevé. The last three ones were subsequently added by B. Gambier [Gm]. Now, all these six equations are known as the Painlevé equations:

$$y'' = 6y^2 + x, \tag{2}$$

$$y'' = 2y^3 + xy + \alpha, \tag{3}$$

$$y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{(\alpha y^2 + \beta)}{x} + \gamma y^3 + \frac{\delta}{y}, \tag{4}$$

$$y'' = \frac{y'^2}{2y} + \frac{3y^3}{2} + 4xy^2 + 2(x^2 - \alpha)\, y + \frac{\beta}{y}, \tag{5}$$

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right) y'^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1}, \tag{6}$$

$$\begin{aligned} y'' = {} & \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) y'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) y' \\ & + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha - \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \left(\frac{1}{2} - \delta\right)\frac{x(x-1)}{(y-x)^2}\right). \end{aligned} \tag{7}$$

At present, the theory of Painlevé equations is a rapidly developing area of mathematics due to the close relations to the integrable nonlinear systems [B, ARS, IN, LW].

Our investigation is devoted to those properties of the Painlevé equations that are invariant under general point transformation

$$x = u(t, z), \qquad y = v(t, z), \tag{8}$$

where v and u are arbitrary smooth functions and t is the new independent variable. Let us first introduce some notions and definitions. A function $\tau_n(x, y)$ depending only on the data of Eq. (1) is called a semi invariant of a weight n if after the transformation (8)

$$\tilde{\tau}_n(t, z) = \left(\frac{\partial(v, u)}{\partial(t, z)}\right)^n \tau_n(u(t, z), v(t, z)), \tag{9}$$

where $\partial(v, u)/\partial(t, z)$ is the Jacobi determinant of the transformation. Such a function is called an absolute invariant if either $n = 0$ or if it vanishes identically.

The simplest form of an ordinary differential equation of the second order (1) conserved under general point transformations (8) is

$$y'' + a_1(x, y)\, y'^3 + 3a_2(x, y)\, y'^2 + 3a_3(x, y)\, y' + a_4(x, y) = 0, \tag{10}$$

due to the fact that $\partial^4 F(x, y, y')/\partial y'^4 = 0$ is an absolute invariant. The Painlevé equations are just of this type. We shall look for differential invariants for the equation of the type (10) satisfying the condition (9), which are combinations of the coefficients of Eq. (10) and their partial derivatives only.

The first investigation of invariants of Eq. (10) was done by R. Liouville [L1]. The construction of such quantities is not unique but there exist a basis of differential invariants which characterize the equation completely. We will take the first set of differential invariants obtained by Liouville in his earlier work because the invariants described in later works are either the same or they can easily be represented by the previous ones.

Liouville found some series of absolute and semi invariants and discovered a procedure for the construction of other invariants of higher weights if the initial semi invariant for this series does not vanish. Two quantities L_1, L_2, (14), (15) which will be defined next played a major role in Liouville's investigations. Let us introduce

$$\Pi^0_{11} = 2(a_3^2 - a_2 a_4) + a_{3x} - a_{4y}, \tag{11}$$

$$\Pi^0_{22} = 2(a_2^2 - a_1 a_3) + a_{1x} - a_{2y}, \tag{12}$$

$$\Pi^0_{12} = \Pi^0_{21} = a_2 a_3 - a_1 a_4 + a_{2x} - a_{3y}, \tag{13}$$

where by the subscripts x, y we denote the corresponding partial derivatives. Then L_1, L_2 are defined by

$$L_1 = -\frac{\partial \Pi^0_{11}}{\partial y} + \frac{\partial \Pi^0_{12}}{\partial x} - a_2 \Pi^0_{11} - a_4 \Pi^0_{22} + 2a_3 \Pi^0_{12}, \tag{14}$$

$$L_2 = -\frac{\partial \Pi^0_{12}}{\partial y} + \frac{\partial \Pi^0_{22}}{\partial x} - a_1 \Pi^0_{11} - a_3 \Pi^0_{22} + 2a_2 \Pi^0_{12}. \tag{15}$$

The case when both L_1 and L_2 are equal to zero is not interesting for us because in that case Eq. (10) is equivalent to $y'' = 0$ [L1] and we shall exclude it. The value v_5, the most important semi invariant discovered by R. Liouville, is

$$\begin{aligned} v_5 = {} & L_2(L_1 L_{2x} - L_2 L_{1x}) + L_1(L_2 L_{1y} - L_1 L_{2y}) \\ & - a_1 L_1^3 + 3a_2 L_2^1 L_2 - 3a_3 L_1 L_2^2 + a_4 L_2^3. \end{aligned} \tag{16}$$

If we have

$$v_5 = 0 \tag{17}$$

then v_5 is an absolute invariant. It can not be used for the construction of any other invariants higher weights because it is equal to zero. R. Liouville discovered an another initial semi-invariant of weight 1 for which (17) holds, it is

$$w_1 = \frac{1}{L_1^4}[-L_1^3(\Pi_{12}^0 L_1 - \Pi_{11}^0 L_2) - R_1(L_1^2)_x - L_1^2 R_{1x} + L_1 R_1(a_3 L_1 - a_4 L_2)], \tag{18}$$

where

$$R_1 = L_1 L_{2x} - L_2 L_{1x} + a_2 L_1^2 - 2a_3 L_1 L_2 + a_4 L_2^2$$

and $L_1 \neq 0$. For the case $L_2 \neq 0$ we have a similar formula. The first non-vanishing semi-invariant in case $w_1 = 0$ found by R. Liouville has the weight 2 and is expressed as

$$i_2 = \frac{3R_1}{L_1} + \frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial y}. \tag{19}$$

It can be used for the construction of the last series of invariants investigated by R. Liouville. During the investigation of some dynamical systems L. A. Bordag and V. S. Dryuma [BD] calculated all three types of Liouville invariants v_5, w_1, i_2 for the equations PI–PVI and found that $v_5 = 0$ and $w_1 = 0$ for all the Painlevé equations and additionally $i_2 = 0$ for the first Painlevé equation. R. Liouville had some problems with the equations for which the invariants v_5, w_1, i_2 vanish. He stated that all such equations are equivalent to $y'' = 0$ under the general point transformation (8). This incorrectness was noticed later by P. Painlevé and was the subject of a controversial discussion between P. Painlevé [P2] and R. Liouville [L2]. P. Painlevé had heavy doubts about the method of R. Liouville and as a result this voluminous and mostly correct work was neither cited nor used (to our knowledge). A further consideration of the same problem from the point of view of groups of infinitesimal transformations was introduced by S. Lie [Li1, Li2] and completed by A. Tresse. In his first work [T1] A. Tresse looked for complete series of invariants for the equations of type (10). Later [T2] he investigated the general second order differential Eq. (1) with an arbitrary smooth function $F(x, y, y')$. A. Tresse got an award for this work because he found the complete set of invariants for the Eq. (1) under general point transformations (8). The work of A. Tresse was almost forgotten perhaps in cause of his inconvenient notations and we did not find an application of his work. Note that the results of A. Tresse are inapplicable to the case $v_5 = w_1 = 0$.

At the beginning of our century, G. Thomsen [Th] has found the series of invariants for the Eq. (10) under the point transformation (8) using differential geometry methods. His invariants were quite the same as were found by R. Liouville. He also excluded the case $v_5 = w_1 = 0$ from the consideration.

The underlying geometrical theory of Eqs. (10) was developed by E. Cartan [C1, C2]. He introduced the concept of the space of the projective connection (SPC). Equation (10) can be considered as the equation on the geodesics in this space. E. Cartan found some special classes of SPC, so called spaces of the normal projective connection (SNPC) that are in one-to-one correspondence with the equivalence classes of (10) under the general point transformations (8).

The normal projective connection corresponding to (10) can be assigned by the matrix of 1-forms ω:

$$
\begin{aligned}
&\omega_0^0 = 0, && \omega_0^1 = \mathrm{d}x, && \omega_0^2 = \mathrm{d}y,\\
&\omega_1^0 = \Pi_{11}^0\,\mathrm{d}x + \Pi_{12}^0\,\mathrm{d}y, && \omega_1^1 = -a_3\,\mathrm{d}x - a_2\,\mathrm{d}y, && \omega_1^2 = a_4\,\mathrm{d}x + a_3\,\mathrm{d}y,\\
&\omega_2^0 = \Pi_{21}^0\,\mathrm{d}x + \Pi_{22}^0\,\mathrm{d}y, && \omega_2^1 = -a_2\,\mathrm{d}x - a_1\,\mathrm{d}y, && \omega_2^2 = -\omega_1^1
\end{aligned}
\tag{20}
$$

where $\Pi_{11}^0, \Pi_{22}^0, \Pi_{12}^0, \Pi_{21}^0$ are defined by (11)–(13).

In this context Cartan introduced the tensor $R_{\beta ij}^{\alpha}$ of the projective torsion and curvature $R_{\beta ij}^{\alpha}$, $\alpha, \beta = 0, 1, 2$, $i, j = 1, 2$

$$
\begin{aligned}
&R_{0lk}^i \omega_0^l \wedge \omega_0^k = \omega_0^h \wedge \omega_h^i - d\omega_0^i\\
&R_{jlk}^i \omega_0^l \wedge \omega_0^k = \omega_j^0 \wedge \omega_0^i + \omega_j^h \wedge \omega_h^i - \delta_j^i \omega_0^h \wedge \omega_h^0 - d\omega_j^i\\
&R_{jlk}^0 \omega_0^l \wedge \omega_0^k = \omega_j^h \wedge \omega_h^0 - d\omega_j^0,
\end{aligned}
\tag{21}
$$

where summation over the repeated index h is assumed, $h = 1, 2$, and δ_j^i is the Kronecker symbol.

For SNPC the tensor R has only two non-vanishing components which coincide with Liouville's L_1, L_2

$$
R_{112}^0 = -L_1, \qquad R_{212}^0 = -L_2.
$$

E. Cartan defined and investigated the holonomy group of SNPC corresponding to (10). He proved that there is only one special case when the holonomy group is non-trivial. It has a fixed point on $\mathbb{R}\mathbf{P}^2$ and it can be shown that in this case the equation is equivalent to $y'' = f(x, y)$ for some $f(x, y)$. In other cases the holonomy group is either the projective group $PGL(2, \mathbb{R})$ or it is trivial (for equations equivalent to $y'' = 0$).

E. Cartan [C2] and S. S. Chern [Ch] proved that each n-dimensional SNPC $\mathscr{X}^n$ can be immersed into the projective space $\mathbb{R}\mathbf{P}^N$. S. S. Chern [Ch] found that $N = n(n+1)/2 + [n/2]$, i.e., for $n = 2$ we have $N = 4$. If the invariant $v_5 = 0$ it is possible to immerse $\mathscr{X}^2$ into $\mathbb{R}\mathbf{P}^3$ and the image of $\mathscr{X}^2$ is a developable surface. For instance, the surface corresponding to the equation $y'' = 0$ is the projective plane (see E. Cartan [C2] and V. Prokofjev [Pr]). In all the other cases the immersion is possible into $\mathbb{R}\mathbf{P}^4$ only.

R. B. Gardner [Gr] presented the Cartan theory of SNPC in modern language and suggested an algorithm to construct a complete system of invariants under the general point transformation (8) for Eq. (10).

Using the method of R. B. Gardner, N. Kamran [K] found the complete set of invariants for the PI and PII Painlevé equations.

Our work is divided in two parts. In the first we investigate the case incorrectly considered by R. Liouville [L1], missed by A. Tresse [T2] and excluded from the consideration by G. Thomsen [Th] ($v_5 = w_1 = 0$). We prove that this case exactly corresponds to the family of equations with non-trivial holonomy group. This group is the subgroup of $PGL(2, \mathbb{R})$ that leaves some point of $\mathbb{R}\mathbf{P}^2$ fixed.

The characteristic property of above equations is that they can be transformed into the form $y'' = f(x, y)$. This form is very special (we call it a canonical form), the only point transformations preserving it are linear in y. As a result the equivalence problem for the equations in the canonical form become trivial.

One of the results of the presented theory is the Theorem 2.1, it states that all the Painlevé equations are in different equivalence classes with respect to the general point transformations (except for the several pointed out cases).

The problem of transforming the given equation to the canonical form or not can be solved constructively.

The second part of the article is devoted to the problem of immersion of SNPC into $\mathbb{R}\mathbf{P}^3$. We define all the necessary objects in detail and prove that any immersion of the SNPC from our class can be determined by three surfaces in $\mathbb{R}\mathbf{P}^3$. Two of them are arbitrary cones with common vertex. They can be considered as the parameters of the immersion. The third surface, we called it $\mathscr{P}_3$, carrys all the information about the function $f(x, y)$—the right-hand side of the equation in the canonical form. We construct such surfaces for all the Painlevé equations and show that some external restrictions on these surfaces ensures the uniqueness of immersions corresponding to each of the Painlevé equations. The constructed surfaces are the geometrical invariants uniquely characterising the equivalence classes corresponding to the Painlevé equations.

Part I: The Theory of Invariants and the Painlevé Equations

Usually we have to investigate an equation that is not in the canonical form. Is it possible to give some algorithmic procedure to find a point transformation reducing the equation to the canonical form or to prove that the equations are not equivalent?

We will give such an algorithmic procedure for any second order differential equation of the type (1) and Painlevé equations. In other words we will give the conditions under which some equation of the type (1) is equivalent to one of the Painlevé equations.

LEMMA 1. *Each equation of the type*

$$y'' = F(x, y, y') \tag{22}$$

can be transformed into the canonical form

$$y'' = f(x, y)$$

if and only if

$$\frac{\partial^4 F(x, y, y')}{\partial y'^4} = 0, \qquad v_5 = 0, \quad w_1 = 0. \tag{23}$$

Proof. The first condition in (23) means that Eq. (1) can be reduced to the form (10). Let now $v_5 = 0$, this means that v_5 is now an absolute invariant. Let us investigate the three possible cases.

(1) Both values L_1, L_2 are equal to zero. This case is trivial as the equation is equivalent to $y'' = 0$ [L1].

(2) Both values $L_1, L_2 \neq 0$. The values $-L_1, -L_2$ are the components of the tensor of the projective torsion and curvature (21). Under general point transformations $x = u(t, z)$, $y = v(t, z)$ these components transform in the following way

$$\tilde{L}_1 = \frac{\partial(v, u)}{\partial(t, z)} (L_1 u_t + L_2 v_t),$$

$$\tilde{L}_2 = \frac{\partial(v, u)}{\partial(t, z)} (L_1 u_z + L_2 v_z).$$

$\partial(v, u)/\partial(t, z)$ is the Jacobi determinant. By our assumptions it is possible to find new variables t, z such that $\tilde{L}_2 \equiv 0$. Note that any transformation of the type

$$x = u(t), \qquad y = v(t, z)$$

preserves this property. Thus this case can be reduced to the case considered below.

(3) Exactly one of the values L_1 or L_2 is equal to zero. If $L_1 = 0$ then from $v_5 = 0$ it follows $a_4(x, y) = 0$ and when $L_2 = 0$ then $a_1(x, y)$ must be equal to zero. After the substitution $x \to y$, $y \to x$ we get $L_1 \to L_2$, $L_2 \to L_1$, $a_1(x, y) \to a_4(x, y)$, $a_4(x, y) \to a_1(x, y)$. This means that we have to investigate only one of these cases. We take the second one: $L_2 = 0$, $a_1(x, y) = 0$. Consequently, the equation takes the form

$$y'' + 3a_2(x, y)\, y'^2 + 3a_3(x, y)\, y' + a_4(x, y) = 0. \tag{24}$$

The condition $w_1 = 0$ for the above equation means that

$$\frac{\partial a_3(x, y)}{\partial y} - 2\, \frac{\partial a_2(x, y)}{\partial x} = 0. \tag{25}$$

It can be proved through straightforward calculation that after the substitution

$$t = \int \exp\left(\int\left(-3a_3(x, y) + 6 \int \frac{\partial a_2(x, y)}{\partial x}\, dy\right) dx\right) dx,$$

$$z = \int \exp\left(3 \int a_2(x, y)\, dy\right) dy,$$

Equation (24) takes the canonical form

$$z'' = f(t, z). \tag{26}$$

COROLLARY 2. *The PI–PVI equations can be reduced to the canonical form* (26).

Proof. The first two equations, PI and PII, already have this form. The PIII equation after the substitution

$$t = \ln x, \qquad z = \ln y$$

will be

$$z'' = \alpha e^{t+z} + \beta e^{t-z} + \gamma e^{2(t+z)} + \delta e^{2(t-z)}. \tag{27}$$

The transformation

$$t = x, \qquad z = \sqrt{y}$$

puts the PIV equation into the form

$$z'' = \frac{3}{4} z^5 + 2tz^3 + (t^2 - \alpha) z + \frac{\beta}{2z^3}. \tag{28}$$

The following substitution

$$t = \ln x, \qquad z = \ln \frac{\sqrt{y}+1}{\sqrt{y}-1}$$

can be used for the PV equation. After it the PV equation takes the form

$$z'' = 4\alpha e^z \frac{1+e^z}{(1-e^z)^3} + 4\beta e^z \frac{1-e^z}{(1+e^z)^3} + \frac{\gamma}{4}(e^{t-z} - e^t + z)$$
$$+ \frac{\delta}{8}(e^{2(t-z)} - e^{2(t+z)}), \tag{29}$$

or equivalently

$$z'' = -\frac{2}{\sinh z}\left(\alpha \coth^2 \frac{z}{2} + \beta \tanh^2 \frac{z}{2}\right) - \frac{\gamma}{2} e^t \sinh z - \frac{\delta}{4} e^{2t} \sinh 2z. \tag{30}$$

Finally the PVI equation after the substitution (for more details see Appendix)

$$t = \frac{K(\sqrt{1-1/x})}{K(1/\sqrt{x})}, \qquad z = \frac{1}{2K(1/\sqrt{x})} \int_0^{\sqrt{y}} \frac{dt}{\sqrt{(1-t^2)(1-t^2/x)}},$$

will have the canonical form

$$4\pi^2 \frac{d^2 z}{dt^2} = \beta \wp'(z \mid 1/2, it/2) + \gamma \wp'_1 + \alpha \wp'_2 + \delta \wp'_3. \tag{31}$$

Here K is the complete elliptic integral of the first kind, $\wp'(z \mid \frac{1}{2}, i(t/2))$ is the derivative of the Weierstrass $\wp$-function with the periods 1 and it, and $\wp'_1$, $\wp'_2$, $\wp'_3$ are its shifts on the half-periods $1/2$, $it/2$, and $(1+it)/2$ correspondingly. This completes the proof.

Remark. The mentioned substitutions involving only transformations of the dependend variable for PIII and PV were used in [IKSY] in relation with the investigation of solutions near some fixed singular points of regular type. The authors reduced the equations PIII and PV to the so-called normal form with a main differential part of the type $x(xz'(x))'$. The elliptic form of PVI was obtained for the first time by other methods in

[M] and later independently in [BB]. As far as we know the reduction of all the Painlevé equations to the type (26) was not done elsewhere.

From now on we shall assume that all the equations are in canonical form. For such type of equations all formulae are much shorter and simpler. Now we list briefly the important properties and formulae for the equations in canonical form.

Main Properties of the Equations of the type $y'' = f(x, y)$.

(1) $v_5 = 0$, $w_1 = 0$ and $i_2 = f_{yyy}(x, y)$. If $i_2 \neq 0$ then i_2 generates some series of semi-invariants $i_{2m+2} = L_1(\partial i_{2m}/\partial y) - 2m i_{2m}(\partial L_1/\partial y)$ with weights equal $2m+2$, respectively.

(2) $L_1 = -f_{yy}(x, y)$, $L_2 = 0$.

We will consider now only point transformations that preserve the canonical form (26). The most general form of a point transformation (8) preserving the form (26) is given in the Lemma 2.

LEMMA 2. *Consider the equation of the type* $y'' = f(x, y)$. *Then the most general form of a point transformation* (8) *preserving the form* (26) *is*

$$x = c \int m^2(t)\, dt + c_0, \qquad y = m(t)\, z + n(t). \tag{32}$$

where c, c_0 *are constants* ($c \neq 0$) *and* $m(t)$, $n(t)$, $m(t) \neq 0$ *functions of the independent variable* t.

Proof. The proof of Lemma 2 follows from straightforward calculations by expressing d^2y/dx^2 in terms of the new variables. After substitution (32) we get $z'' = f^*(t, z)$ with f^* defined as

$$f^*(t, z) = -z m(t) \frac{d^2}{dt^2}\left(\frac{1}{m(t)}\right) - m(t) \frac{d}{dt}\left(\frac{1}{m^2(t)} \frac{dn(t)}{dt}\right) + c^2 m^3(t)\, f\left(c \int m^2(t)\, dt + c_0, m(t)\, z + n(t)\right). \tag{33}$$

Note that in this case the Jacobi determinant is equal to $cm^3(t)$.

We have seen that every equation of the type (10) with invariants $v_5 = 0$, $w_1 = 0$ can be reduced to the canonical form (26). Lemma 2 gives us now all possible substitutions that can be used to transform one of the equations in canonical form into another. If it is impossible to find such a substitution (32) then both equations are not equivalent to each other under general point transformations. The transformations of type (32) split all the equations in the canonical form into equivalence classes with respect to general point transformations. Each of the Painlevé equations defines its own series of equivalence classes which we will investigate in detail in the next section. The simplest of all the Painlevé equations is the first one PI (2). Its equivalence class can be described also very elementary. The class consists of the equations for which $v_5 = 0$, $w_1 = 0$, $i_2 = 0$ and the function f in (26) has the form $f(t, z) = g_2(t)\, z^2 + g_1(t)\, z + g_0(t)$, with an additional condition on the functions g_2, g_1, g_0. Let us formulate this condition. If the equation $z'' = g_2(t)\, z^2 + g_1(t)\, z + g_0(t)$ can be reduced to the first Painlevé equation (2) then there must exist a substitution of the type (32) with

$$m(t) = \left(\frac{g_2(t)}{6c^2}\right)^{1/5}, \qquad n(t) = \frac{c^2}{12}\left(g_1(t) + m(t)\left(\frac{m'(t)}{m^2(t)}\right)'\right),$$

and

$$g_0(t) = 6c^2 m^3(t)\, n^2(t) - \frac{m(t)}{c^2}\left(\frac{n'(t)}{m^2(t)}\right)' + c^2 m^3(t)\left(c\int m^2(t)\, dt + c_0\right). \qquad (34)$$

Only if the functions g_2, g_1, g_0 satisfy the condition (34) the equation $z'' = g_2(t)\, z^2 + g_1(t)\, z + g_0(t)$ is in the same equivalence class as Eq. (2).

2. THE EQUIVALENCE CLASSES GENERATED BY THE PAINLEVÉ EQUATIONS

In this section we point out the cases when some different Painlevé equations can be transformed one to another by some general point transformation.

Let us denote by $\{Pj(\alpha, \beta, \gamma, \delta)\}$ the set of equations equivalent to the jth Painlevé equation with parameters $\alpha, \beta, \gamma, \delta$.

Consider the sets

$$\{\mathrm{PI}\},$$

$$\{\mathrm{PII}\} = \bigcup_{\alpha} \{\mathrm{PII}(\alpha)\},$$

$$\{\mathrm{PIII}\} = \bigcup_{\alpha, \beta, \gamma, \delta} \{\mathrm{PIII}(\alpha, \beta, \gamma, \delta)\},$$

$$\{\mathrm{PIV}\} = \bigcup_{\alpha, \beta} \{\mathrm{PIV}(\alpha, \beta)\},$$

$$\{\mathrm{PV}\} = \bigcup_{\alpha, \beta, \gamma, \delta} \{\mathrm{PV}(\alpha, \beta, \gamma, \delta)\},$$

$$\{\mathrm{PVI}\} = \bigcup_{\alpha, \beta, \gamma, \delta} \{\mathrm{PVI}(\alpha, \beta, \gamma, \delta)\}.$$

From Lemma 2 and the canonical forms of the Painlevé equations the following theorems result

THEOREM 2.1. *$\{Pj\} \cap \{Pk\} = \emptyset, k < j, k, j \in \{I, II, III, IV, V, VI\}$, except for the case $k =$ III, $j =$ V,*

$$\{\mathrm{PIII}\} \cap \{\mathrm{PV}\} = \bigcup_{\gamma,\delta} \{\mathrm{PIII}(-\gamma, \gamma, -\delta, \delta)\} = \bigcup_{\gamma,\delta} \{\mathrm{PV}(0, 0, \gamma, \delta)\},$$

$$\{\mathrm{PIII}(-\gamma, \gamma, -\delta, \delta)\} = \{\mathrm{PV}(0, 0, 4\gamma, 8\delta)\}.$$

THEOREM 2.2. *Inside of the classes corresponding to each of the six Painlevé equations there are following relations hold.*

(1) *For the second Painlevé equation* (PII)

$$\{\mathrm{PII}(\alpha)\} \cap \{\mathrm{PII}(\tilde{\alpha})\} = \emptyset, \qquad \text{if} \quad \alpha^2 \neq \tilde{\alpha}^2,$$
$$\{\mathrm{PII}(\alpha)\} = \{\mathrm{PII}(-\alpha)\}.$$

(2) *For the third Painlevé equation* (PIII) *we consider the group of transformations of the parameters $(\alpha, \beta, \gamma, \delta)$ defined by generators*

$$(\alpha, \beta, \gamma, \delta) \simeq (rs\alpha, r/s\beta, (rs)^2\gamma, (r/s)^2\, \delta),$$
$$(\alpha, \beta, \gamma, \delta) \simeq (-\beta, -\alpha, -\delta, -\gamma),$$
$$(\alpha, \beta, 0, 0) \simeq (0, 0, 2\alpha, 2\beta),$$
$$(\alpha, 0, \gamma, 0) \simeq (r\alpha, 0, r\gamma, 0),$$

where $r, s > 0$. We shall write $(\alpha, \beta, \gamma, \delta) \simeq (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ if $(\alpha, \beta, \gamma, \delta)$ and $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ belong to the same orbit with respect to the action of this group. Then

$$\{\mathrm{PIII}(\alpha, \beta, \gamma, \delta)\} \cap \{\mathrm{PIII}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})\} = \emptyset$$

except for the cases when $(\alpha, \beta, \gamma, \delta) \simeq (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$.

(3) *For the fourth Painlevé equation* (PIV)

$$\{\mathrm{PIV}(\alpha, \beta)\} \cap \{\mathrm{PIV}(\tilde{\alpha}, \tilde{\beta})\} = \emptyset, \qquad \textit{if} \quad (\alpha, \beta) \neq (\tilde{\alpha}, \tilde{\beta}).$$

(4) *For the fifth Painlevé equation* (PV)

$$\{\mathrm{PV}(\alpha, \beta, \gamma, \delta)\} \cap \{\mathrm{PV}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})\} = \emptyset$$

except for the cases when PV is equivalent to PIII (*see previous theorem*) *and*

$$\{\mathrm{PV}(\alpha, \beta, \gamma, \delta)\} = \{\mathrm{PV}(\alpha, \beta, r\gamma, r^2\delta)\},$$

$$\{\mathrm{PV}(\alpha, -\alpha, 0, \delta)\} = \{\mathrm{PV}(4\alpha, 0, \delta/4, 0)\}.$$

(5) *For the sixth Painlevé equation* (PVI)

$$\{\mathrm{PVI}(\alpha, \beta, \gamma, \delta)\} \cap \{\mathrm{PVI}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})\} = \varnothing$$

except for the following cases

$$\{\mathrm{PVI}(\alpha, \beta, \gamma, \delta)\} = \{\mathrm{PVI}(\delta, \gamma, \beta, \alpha)\},$$

$$\{\mathrm{PVI}(\alpha, \beta, \beta, \alpha)\} = \{\mathrm{PVI}(4\alpha, 4\beta, 0, 0)\},$$

$$\{\mathrm{PVI}(\alpha, \alpha, \gamma, \gamma)\} = \{\mathrm{PVI}(0, 4\alpha, 4\gamma, 0)\},$$

$$\{\mathrm{PVI}(\beta, \beta, \beta, \beta)\} = \{\mathrm{PVI}(0, 16\beta, 0, 0)\}.$$

Remark. Note that we are considering real transformations only. If we consider complex point transformations, we must add to the mentioned equivalences the following ones,

$$\{\mathrm{PVI}(\alpha, \beta, \gamma, \delta)\} = \{\mathrm{PVI}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})\},$$

where $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ is an arbitrary permutation of $(\alpha, \beta, \gamma, \delta)$ and

$$\{\mathrm{PV}(\alpha, \beta, \gamma, \delta)\} = \{\mathrm{PV}(-\beta, -\alpha, -\gamma, \delta)\}.$$

The group of transformations for solutions related to different values of parameters for the third and fifth Painlevé equations in usual form was found and classified in [BK].

Part II: The geometry of the Equations $y'' = f(x, y)$

3. SPACES OF PROJECTIVE CONNECTION

We will explain the concept of the space of projective connection (SPC). First of all let us introduce the necessary definitions and notations.

Let $\mathscr{X}$ be a two-dimensional manifold. Consider a product bundle with a structure group $PGL(2, \mathbb{R})$: $\mathscr{X} \times \mathbb{R}\mathbf{P}^2 \xrightarrow{p} \mathscr{X}$, and some section $\hat{P}_0$ of this bundle:

$$\hat{P}_0 : \mathscr{X} \to \mathscr{X} \times \mathbb{R}\mathbf{P}^2, \qquad p(\hat{P}_0(X)) = X, \qquad \forall X \in \mathscr{X}.$$

By $\mathscr{P}_{aths}(\mathscr{X})$ we denote the set of all smooth oriented paths on $\mathscr{X}$.

Consider a smooth projective connection $\hat{\omega}$ on $\mathscr{X}$. The horizontal lift of $\mathscr{L} \in \mathscr{P}_{aths}(\mathscr{X})$ on the principal bundle as is well known defines the projective mapping of the fibre $p^{-1}(\mathscr{L}(0))$ over the starting point $\mathscr{L}(0)$ to the fibre $p^{-1}(\mathscr{L}(1))$ over the endpoint of the path $\mathscr{L}$. We denote this mapping by $\phi_\omega(\mathscr{L})$. The image of the marked point $\hat{P}_0(\mathscr{L}(0)) \subset p^{-1}(\mathscr{L}(0))$ under this mapping is some point of the fibre $p^{-1}(\mathscr{L}(1))$.

Note that any path $\mathscr{L} \in \mathscr{P}_{aths}(\mathscr{X})$ generates an ordered one-parametric family of paths $\bigcup_\lambda \mathscr{L}^\lambda \subset \mathscr{P}_{aths}(\mathscr{X})$. All $\mathscr{L}^\lambda$ are the parts of the $\mathscr{L}$ with starting points in the different points of $\mathscr{L}$ and fixed endpoint $\mathscr{L}(1)$.

The horizontal lift of each $\mathscr{L}$ defines some point $P^\lambda_{\mathscr{L}} \in p^{-1}(\mathscr{L}(1))$—the image of the marked point $\hat{P}_0$ in the fibre over the starting point of $\mathscr{L}^\lambda$. The ordered one-parametric set of points $P^\lambda_{\mathscr{L}}$ forms a path on the fibre over the endpoint of $\mathscr{L}$.

As a result the connection $\hat{\omega}$ and the section $\hat{P}_0$ determine the mapping

$$\Gamma_\omega : \mathscr{P}_{aths}(\mathscr{X}) \to \mathscr{P}_{aths}(\mathscr{X} \times \mathbb{R}\mathbf{P}^2), \qquad \Gamma_\omega \mathscr{L} \subset p^{-1}(\mathscr{L}(1)).$$

The mapping Γ_ω is called the developing of $\mathscr{L}$ on the fibre over the endpoint $\mathscr{L}(1)$.

The above construction gives a correspondence between the points $X^\lambda \in \mathscr{L}$ of $\mathscr{L} \in \mathscr{P}_{aths}(\mathscr{X})$ and the points $P^\lambda_{\mathscr{L}} \in \Gamma_\omega \mathscr{L} : X_\lambda \to P^\lambda_{\mathscr{L}}$.

Let the point $X^\lambda \in \mathscr{L} \subset \mathscr{X}$ move along $\mathscr{L}$ with non-zero velocity. Consider the corresponding point $P^\lambda_{\mathscr{L}}$.

Definition 3.1. The pair $\hat{\omega}, \hat{P}_0$ is called non-degenerate if the velocity of the point $P^\lambda_{\mathscr{L}} \in \Gamma_\omega \mathscr{L} \subset p^{-1}(\mathscr{L}(1))$ is non-zero for all $\mathscr{L} \in \mathscr{P}_{aths}(\mathscr{X})$ at all points $X^\lambda \in \mathscr{L}$.

Definition 3.2. A line $\mathscr{L} \in \mathscr{P}_{aths}(\mathscr{X})$ is called a geodesic for the SPC if its developing $\Gamma_\omega \mathscr{L}$ is a part of a straight line.

Let us choose the bundle coordinates in such a way that the first row of the connection matrix form $\hat{\omega}$, which defines the projective connection, corresponds to the $\hat{P}_0$.

Remark. It is easy to see that the non-degeneracy of the connection implies that the Pfaff forms $\hat{\omega}^1_0(X)$ and $\hat{\omega}^2_0(X)$ are forming a basis of the cotangent space $T^*_X(\mathscr{X})$ for all $X \in \mathscr{X}$.

DEFINITION 3.3. Two projective connections $\hat{\omega}$ and $\hat{\hat{\omega}}$ are equivalent if their forms connected by the gauge transformation,

$$\hat{\omega}(X) = -\kappa^{-1}(X)\, d\kappa(X) + \kappa^{-1}(X)\, \hat{\hat{\omega}}(X)\, \kappa(X), \qquad \kappa(X) \in PGL(2, \mathbb{R}), \tag{35}$$

where the first row of the matrix $\kappa(X)$ is proportional to $(1, 0, 0)$.

Note that equivalent connections have the same sets of geodesics and are non-degenerate at the same time. So we can speak about the non-degenerate equivalence classes and about the geodesics of some equivalence class.

DEFINITION 3.4. The space with the projective connection SPC is the set $\{\mathscr{X} \times \mathbb{R}\mathbf{P}^2 \xrightarrow{p} \mathscr{X},\ \hat{P}_0,\ \hat{\omega}\}$, where

(1) $\mathscr{X} \times \mathbb{R}\mathbf{P}^2 \xrightarrow{p} \mathscr{X}$ is a trivial bundle with a structure group $PGL(2, \mathbb{R})$,

(2) $\hat{P}_0 : \mathscr{X} \to \mathscr{X} \times \mathbb{R}\mathbf{P}^2$ is a section of this bundle,

(3) $\hat{\omega}, \hat{P}_0$ is a non-degenerate equivalence class of the projective smooth connections.

Let us choose any coordinates (x, y) on $\mathscr{X}$ and frames $\hat{\mathbf{P}}(X) = (\hat{P}_0, \hat{P}_1, \hat{P}_2)^T (X)$ in all fibres in such a way that the first vertex $\hat{P}_0(X)$ is defined from the fixed section.

The connection form $\hat{\omega}$ can be interpreted as the projective correspondence between the close fibres $p^{-1}(X(x, y))$ and $p^{-1}(X(x+dx, y+dy))$: $\hat{\omega}(X)$ is such a matrix that for a point $\hat{P} = \vec{z}\hat{\mathbf{P}}(X) \in p^{-1}(X)$ the point $\phi_\omega(\mathscr{L}_{XX'})(\hat{P}) = \hat{P}' \in p^{-1}(X')$ has the coordinates $\vec{z}(\mathbf{I} - \hat{\omega}(X))$ in the frame $\hat{\mathbf{P}}(X')$

$$\phi_\omega(\mathscr{L}_{XX'})(\vec{z}\hat{\mathbf{P}}(X)) = \vec{z}(\mathbf{I} - \hat{\omega}(X))\, \hat{\mathbf{P}}(X'),$$

where we denote $X = X(x, y)$, $X' = X(x+dx, y+dy)$ and $\mathscr{L}_{XX'}$ is the path connected points X and X', i.e., $\mathscr{L}_{XX'}(t) = X(x+t dx, y+t dy)$, $t \in [0, 1]$.

Remark. We do not distinguish frames with proportional coordinates of vertexes, respectively matrices ω which differ one from another by matrices proportional to unit matrix 1-form.

Remark. For any projective connection $\hat{\omega}$ we can fix the frames in the fibres in such a way that $2\hat{\omega}_0^0 = \hat{\omega}_1^1 + \hat{\omega}_2^2$. We can obtain such frames by adding vectors, proportional to $\hat{P}_0(X) \in \mathbb{R}^3 \backslash \{0\}$ to the other vertices $P_1(X), P_2(X) \in \mathbb{R}^3 \backslash \{0\}$.

In accordance with the first remark we put $\hat{\omega}_0^0=0$ that gives the normalisation

$$\hat{\omega}_2^2=-\hat{\omega}_1^1, \qquad \hat{\omega}_0^0=0. \tag{36}$$

THEOREM 3.1 (E. Cartan). *Let $\{\mathscr{X}\times\mathbb{R}\mathbf{P}^2 \xrightarrow{p} \mathscr{X}, \hat{P}_0, \hat{\omega}\}$ be a SPC.*

Let $\hat{\omega}_0^1$, $\hat{\omega}_0^2$ be a basis of $\mathbf{T}_X^*(\mathscr{X})$ (for example $\hat{\omega}_0^1=dx$, $\hat{\omega}_0^2=dy$ in some coordinates x, y on $\mathscr{X}$) and let the normalisation (36) holds. Then the equality

$$\phi_\omega(\mathscr{L}_{XX'})(\hat{P}_0,\hat{P}_1,\hat{P}_2)^T(X)=(\mathbf{I}-\hat{\omega}(X))(\hat{P}_0,\hat{P}_1,\hat{P}_2)^T(X')+o(X,X'),$$

where by $o(X,X')$ we denote a term of higher order when $X'\to X$, determines $\hat{\omega}(X)$ uniquely and fixes the frames $(\hat{P}_0,\hat{P}_1,\hat{P}_2)(X)$ in the fibres up to a right constant factor $M\in PGL(2,\mathbb{R})$.

Consider the SPC defined by the matrix of 1-forms $\omega(X)$. According to Definition 3.2 the points $X(x,y(x))\in\mathscr{X}$ lie on some geodesic of this SPC if the equation

$$\frac{d^2y}{dx^2}=\Pi_{22}^1\left(\frac{dy}{dx}\right)^3+(2\Pi_{12}^1+\Pi_{21}^1)\left(\frac{dy}{dx}\right)^2+(2\Pi_{11}^1-\Pi_{12}^2)\frac{dy}{dx}-\Pi_{11}^2, \tag{37}$$

where $\omega_j^i=\Pi_{j1}^i\,dx+\Pi_{j2}^i\,dy$, holds.

We see that there are many SPC with the same geodesics.

According to E. Cartan let us consider the tensor $R_{\alpha,\,l,\,k}^{\beta}$ of curvature and torsion of a SPC (see formula (21)).

DEFINITION 3.5. The SPC is called a space of normal projective connection (SNPC) if all the components of the curvature and torsion tensor except of R_{112}^0 and R_{212}^0 vanish:

$$R_{0lk}^i=0, \qquad R_{jlk}^i=0, \quad i,j,l,k=1,2. \tag{38}$$

Remark. Consider an arbitrary small cycle $\mathscr{L}^0$ that begins and ends at $\hat{P}_0(X)$, i.e., $\mathscr{L}^0(0)=\mathscr{L}^0(1)=\hat{P}_0(X)$. The identities (38) imply that the infinitesimal projective transformation $\Gamma_\omega\mathscr{L}^0$ corresponding to $\mathscr{L}^0$ moves the point $\hat{P}_0(X)$ to itself and any straight line containing $\hat{P}_0$ into itself. This shows that this definition is correct.

THEOREM 3.2 (E. Cartan). *For any equation in the form*

$$y''+a_1(x,y)\,y'^3+3a_2(x,y)\,y'^2+3a_3(x,y)\,y'+a_4(x,y)=0$$

there exist exactly one SNPC whose equation of geodesics (37) *coincides with this equation. This SNPC is defined by the equalities* (20).

The proof of this important theorem can be found in [C2].

The calculation of the non-zero elements of the curvature tensor gives

$$R^0_{112} = -L_1, \qquad R^0_{212} = -L_2,$$

where L_1 and L_2 are the quantities (14), (15) introduced by R. Liouville (see [L1]).

COROLLARY (R. Liouville). *Equation* (10) *is reducible to* $y'' = 0$ *if and only if* $L_1 = L_2 = 0$.

We put aside for some time the analytical theory and introduce the special class of SPC—*the SPC, immersible into* $\mathbb{R}\mathbf{P}^3$.

Let us denote by $\mathbb{R}\mathbf{P}^3_2$ a set of (two-dimensional) planes in $\mathbb{R}\mathbf{P}^3$.

Now, let π_0, P_0, P_3 be any smooth maps,

$$\pi_0 \colon \mathscr{X} \to \mathbb{R}\mathbf{P}^3_2, \qquad P_0 \colon \mathscr{X} \to \mathbb{R}\mathbf{P}^3, \quad P_3 \colon \mathscr{X} \to \mathbb{R}\mathbf{P}^3. \tag{39}$$

and assume that the following conditions hold:

(1) for any $X \in \mathscr{X}$, $P_0(X) \in \pi_0(X)$,

(2) $P_3(X) \notin \pi_0(X)$,

(3) the straight line $(P_3(X), P_0(X))$ is not tangent to the surface $\mathscr{P}_0 = \bigcup_{X \in \mathscr{X}} P_0(X)$ at the point $P_0(X)$.

It can be easily verified that the projection from $P_3(X)$ of $\pi_0(X) \supset P_0(X)$ on $\pi_0(X') \supset P_0(X')$, where $X' \to X$ define some horizontal direction field on the principal bundle, i.e., the projective connection on $\mathscr{X}$.

DEFINITION 3.6 A space of projective connection is called immersible into $\mathbb{R}\mathbf{P}^3$ if it can be obtained by the construction described above.

Consider single-valued function $\mathbf{P}(X) \in PGL(3, \mathbb{R})$,

$$\mathbf{P}(X) = (P_0(X), P_1(X), P_2(X), P_3(X))^T,$$

where $P_k(X)$ are the verteces of a frame in $\mathbb{R}\mathbf{P}^3$, so that $P_1(X)$, $P_2(X)$ are arbitrary verteces on $\pi_0(X)$, and denote $d\mathbf{P}(X)\,\mathbf{P}^{-1}(X)$ by $\omega(X)$. The form ω is closed in according to its definition.

The definition of connection $\hat{\omega}$ by the projection from $P_3(X)$ and Definition 3.3 of the equivalent connections leads to the following theorem.

THEOREM 3.3. *A SPC* $\{\mathcal{X} \times \mathbb{R}\mathbf{P}^2 \xrightarrow{p} \mathcal{X}, \hat{P}_0, \hat{\omega}\}$ *is immersible into* $\mathbb{R}\mathbf{P}^3$ *if and only if there exists a* 4×4 *matrix* $\omega(X)$ *of* 1*-forms on* $\mathcal{X}$ *so that*

(a) *its first* 3×3 *block* $\hat{\omega}(X)$ *coincides with some of the matrices of* 1*-forms that defines this SPC,*

(b) *the form* $\omega(X)$ *is closed.*

Consider some set $\{\pi_0, P_0, P_3\}$. It defines three two-dimensional manifolds in $\mathbb{R}\mathbf{P}^3$

(1) the surface Π_0 enveloping the set of planes $\pi_0(X)$ (the plane $\pi_0(X)$ is the tangent plain to Π_0 at the point $\Pi(X)$),

(2) the surface $\mathcal{P}_0 = \bigcup_{X \in \mathcal{X}} P_0(X)$,

(3) the surface $\mathcal{P}_3 = \bigcup_{X \in \mathcal{X}} P_3(X)$,

and the correspondence

$$\pi_0(X) \leftrightarrow P_0(X) \leftrightarrow P_3(X).$$

It is not difficult to prove following statement.

PROPOSITION 3.1. *Let* $\Pi_0, \mathcal{P}_0, \mathcal{P}_3$ *be three two-dimensional sub-manifolds of* $\mathbb{R}\mathbf{P}^3$, *and let the one-to-one correspondence "*$\leftrightarrow$*" between their points* $\Pi \leftrightarrow P_0 \leftrightarrow P_3$ *be given. Further let the conditions of non-degeneracy hold*

(1) *the plane* π_0 *tangent to* Π_0 *at the point* Π *does not contain the point* $P_3 \in \mathcal{P}_3$ *that corresponds to* Π,

(2) *the straight line* (P_3, P_0) *passing through the points* $P_3 \leftrightarrow P_0$ *intersects the surface* $\mathcal{P}_0$ *transversally.*

Then $\Pi_0, \mathcal{P}_0, \mathcal{P}_3$ *and the correspondence* $\leftrightarrow$ *define a SPC, denoted by* $\{\Pi_0, \mathcal{P}_0, \mathcal{P}_3, \leftrightarrow\}$

Remark. A projective motion of $\{\Pi_0, \mathcal{P}_0, \mathcal{P}_3, \leftrightarrow\}$ in $\mathbb{R}\mathbf{P}^3$ does not change SPC.

4. THE ODE OF THE SECOND ORDER AND THE CARTAN THEORY OF SPC

Consider the equation in the canonical form, i.e., Eq. (10) with $a_1 = a_2 = a_3 = 0$. We denote $a_4 = -f$ and write the equation in the form $y'' = f(x, y)$.

As we can see (formula (20)), the SNPC for $y'' = f(x, y)$ can be assigned by the matrix

$$\hat{\omega} = \begin{pmatrix} 0 & dx & dy \\ -f_y\,dx & 0 & f\,dx \\ 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

THEOREM 4.1. *A SNPC, corresponding to the class of ODE, equivalent to* $y'' = f(x, y)$, *is immersible into* $\mathbb{R}\mathbf{P}^3$ *and all its immersions* $\{\Pi_0, \mathscr{P}_0, \mathscr{P}_3, \leftrightarrow\}$ *can be defined by the frames* $\mathbf{P}(X) = (P_0,\ P_1,\ P_2,\ P_3)^T$ $(X) \in PGL(3, \mathbb{R})$

$$\mathbf{P}(X) = \begin{pmatrix} 1 & 0 & y & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ f_y & 0 & yf_y - f & 1 \end{pmatrix} \begin{pmatrix} s \\ s_x \\ K \\ s_{xx} \end{pmatrix}, \tag{41}$$

where $s = (s^0(x), s^1(x), s^2(x), s^3(x))$ *is a function* $s: \mathbb{R} \to \in \mathbb{R}^4 \backslash \{0\}$ *so that the matrix* $(s, s_x, K, s_{xx})^T$ *is non degenerate,* $K \in \mathbb{R}^4 \backslash \{0\}$ *is some constant vector. The immersion is defined by* $\mathbf{P}(X)$, $\pi_0(X) = (P_0(X), P_1(X), P_2(X)) \in \Pi_0 \subset \mathbb{R}\mathbf{P}^3_2$, $X \in \mathscr{X} \subset \mathbb{R}^2$.

Proof. The proof is directly obtained solving the equation $d(\omega(X)) = 0$, where $\omega(X)$ is an arbitrary matrix with the prescribed first block (40).

The matrix $(s, s_x, K, s_{xx})^T$ is non-degenerate. Therefore in any point $x \in \mathbb{R}$ we can write

$$s_{xxx} = c_0 s + c_1 s_x + c_k K + c_2 s_{xx}, \tag{42}$$

where c_0, c_1, c_k, c_2 are some functions of x.

Let us calculate $\omega = d\mathbf{P}\mathbf{P}^{-1}$

$$\omega = \begin{pmatrix} 0 & dx & dy & 0 \\ -f_y\,dx & 0 & f\,dx & dx \\ 0 & 0 & 0 & 0 \\ df_y + (c_0 - c_2 f_y)\,dx & (f_y + c_1)\,dx & (-f_x - yc_0 + c_k + c_2 f)\,dx & c_2\,dx \end{pmatrix}.$$

We can see that $d\mathbf{P}\mathbf{P}^{-1}$ depends on $s(x)$ through the coefficients c_0, c_1, c_k, c_2 and as a result we have the following proposition.

PROPOSITION 4.1. *The immersions* (41) *corresponding to* s *and* $\tilde{s}$ *having the same coefficients* c_j *in the decomposition* (42) *can be moved one to another by some projective transformation of* $\mathbb{R}\mathbf{P}^3$.

DEFINITION 4.1. A SNPC defined by equations equivalent to some $y'' = f(x, y)$ we call $\mathscr{C}$-space and its immersions we call $\mathscr{C}$-immersions.

From Theorem 4.1 it follows that for all $\mathscr{C}$-immersions the plane $\pi_0(X)$ is the tangent plane to $\mathscr{P}_0$ at the point $P_0(X)$, consequently $\Pi_0 = \mathscr{P}_0$ holds and we can write $\{\mathscr{P}_3, \mathscr{P}_0, \leftrightarrow\}$ instead of $\{\Pi_0, \mathscr{P}_0, \mathscr{P}_3, \leftrightarrow\}$. The surface $\mathscr{P}_0$ is a cone (see Fig. 1) and the lines $x = \mathrm{const} \subset \mathscr{X}$ are the pre-images of the rulings of the cone.

For any $\mathscr{C}$-immersion $\{\mathscr{P}_3, \mathscr{P}_0, \leftrightarrow\}$ the curves $\mathscr{L}_3 \subset \mathbb{R}\mathbf{P}^3$

$$\mathscr{L}_3(x) = \bigcup_{y \in \mathbb{R}} P_3(x, y), \qquad x = \mathrm{const}$$

are well defined. It can be shown that they are plane curves.

Let us denote the plane containing $\mathscr{L}_3(x)$ by $\pi(x)$.

The ruling of the cone $\mathscr{P}_0$

$$l_0(x) = \bigcup_{y \in \mathbb{R}} P_0(x, y) \in \mathscr{P}_0, \qquad x = \mathrm{const},$$

corresponding to the points of $\mathscr{L}_3(x)$, belong to the same plane $\pi(x)$. The straight lines $(P_0, P_3)(x, y) = l(x, y) \subset \pi(x)$ are tangent to $\mathscr{P}_3$ at P_3 and transversally intersect $\mathscr{P}_0$. If the plane curves $\mathscr{L}_3(x) = \mathscr{P}_3 \cap \pi(x)$ are given, the lines $l(x, y)$ which determine the correspondence "$\leftrightarrow$" are given too, they are the tangents to $\mathscr{L}_3(x)$ at $P_3(x, y)$.

For any $\mathscr{C}$-space the cone $\mathscr{P}_0 = \bigcup_{x \in \mathbb{R}} l_0(x)$ and the one-parametric set of planes $\pi(x) \supset l_0(x)$ can be defined in an arbitrary way if the non degeneracy condition holds. It means that $l_0(x) = \pi_0(x) \cap \pi(x)$ are straight lines, i.e., $\pi_0(x)$ and $\pi(x)$ do not coincide.

We see that for the reconstruction of the $\mathscr{C}$-space $\{\mathscr{P}_3, \mathscr{P}_0, \leftrightarrow\}$ it is sufficient to have a cone $\mathscr{P}_0$, a set of planes $\pi(x)$ transversally intersecting $\mathscr{P}_0$ along the rulings $l_0(x)$ and some surface $\mathscr{P}_3$

$$\mathscr{P}_3 = \bigcup_{x \in \mathbb{R}} \mathscr{L}_3(x), \qquad \mathscr{L}_3(x) \subset \pi(x), \qquad l_0(x) = \pi_0(x) \cap \pi(x).$$

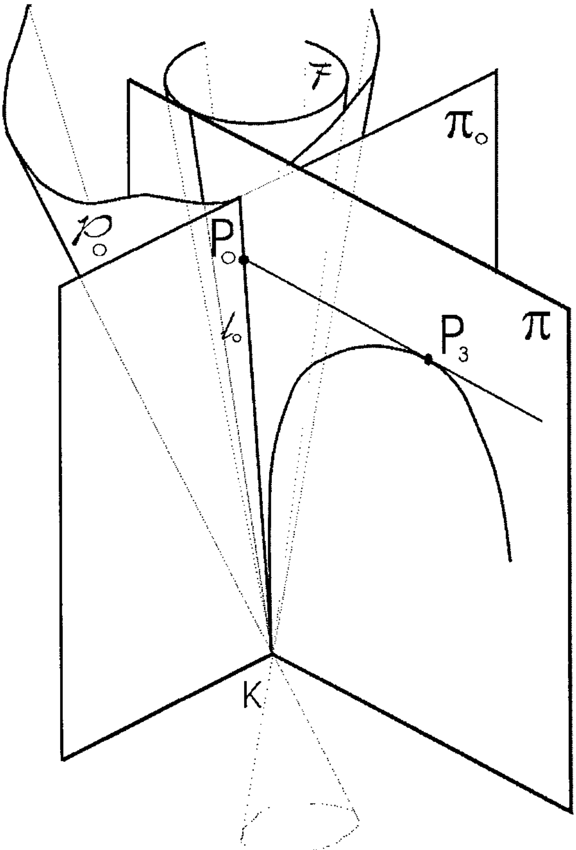

FIG. 1. The $\mathscr{C}$-immersion of SNPC.

Consider the planes $\pi(x) \in \mathbb{R}\mathbf{P}_2^3$. All these planes contain the vertex K of the cone $\mathscr{P}_0$, consequently $\pi(x)$ envelope some other cone with the same vertex K. We denote this cone by $\mathscr{F}$. Note that $\mathscr{F}$ may be degenerated to a straight line.

The set $\{\mathscr{P}_3, \mathscr{P}_0, \mathscr{F}\}$ defines the $\mathscr{C}$-space. All our considerations are local and we suppose that there is existing only one plane containing $l_0(x)$ and tangent to $\mathscr{F}$.

Assume that we have two immersions of the same $\mathscr{C}$-space, i.e., $\{\mathscr{P}_3, \mathscr{P}_0, \mathscr{F}\}$ and $\{\tilde{\mathscr{P}}_3, \tilde{\mathscr{P}}_0, \tilde{\mathscr{F}}\}$. Then for any $x \in \mathbb{R}$ the plane curves $\mathscr{L}_3(x)$ and $\tilde{\mathscr{L}}_3(x)$ are projectively equivalent. The projective motion that brings one into another moves $l_0(x)$ to $\tilde{l}_0(x)$, and $K \in l_0(x)$ to the $\tilde{K} \in l_0(x)$.

PROPOSITION 4.2. *Let two $\mathscr{C}$-immersions $\{\mathscr{P}_3, \mathscr{P}_0, \mathscr{F}\}$ and $\{\tilde{\mathscr{P}}_3, \mathscr{P}_0, \mathscr{F}\}$ with same $\mathscr{P}_0$ and $\mathscr{F}$ be given. Let $s\colon \mathbb{R} \to \mathbb{R}^4\backslash\{0\}$ be any directrix of $\mathscr{P}_0$ with such parametrisation $s(x)$ that $s_{xx}(x) \in \pi(x)$.*

Then the $\mathscr{C}$-spaces, determined by this immersions are equivalent if and only if there exist functions $m(x), n(x)\colon \mathbb{R} \to \mathbb{R}$ and a constant c so that the coordinates (l^0, l^k, l^2) of the corresponding points P_3 and $\tilde{P}_3$ in the frame (s, K, s_{xx}) on $\pi(x)$ are connected by

$$(r^0, r^k, r^2) = q(\tilde{r}^0, \tilde{r}^k, \tilde{r}^2)\begin{pmatrix} m & \mu & 0 \\ 0 & 1 & 0 \\ \dfrac{-c^2}{m^2}\dfrac{d^2}{dx^2}\dfrac{1}{m} & \dfrac{c^2}{m^2}\dfrac{d}{dx}\left(\dfrac{1}{m^2}\dfrac{d}{dx}\mu\right) & \dfrac{c^2}{m^3} \end{pmatrix}, \tag{43}$$

where $q = q(x, y)$ is some scalar factor whose concrete value doesnt play any role (the coordinates are homogeneous).

Let some $\mathscr{C}$-space be given, we can see that the possible transformations of $\mathscr{P}_3$ with fixed $\mathscr{P}_0$ and $\mathscr{F}$ are very similar to the transformations of the right-hand side of $y'' = f(x, y)$ under the substitutions preserving this form. Using this observation we will fix the $\mathscr{C}$-immersions (i.e., the sets $\{\mathscr{P}_3, \mathscr{P}_0, \mathscr{F}\}$) for the Painlevé equations.

5. QUADRIC IMMERSIONS. QUADRIC IMMERSIONS OF THE PAINLEVÉ EQUATIONS

For a given $\mathscr{C}$-space the immersion $\{\mathscr{P}_3, \mathscr{P}_0, \mathscr{F}\}$ can be chosen in a number of different ways (they are parametrised by $s(x)\colon \mathbb{R} \to \mathbb{R}^4\backslash\{0\}$). We will make now an appropriate choice.

DEFINITION 5.1. The $\mathscr{C}$-immersion $\{\mathscr{P}_3, \mathscr{P}_Q, \mathscr{F}\}$ we will call quadric immersion if $\mathscr{P}_Q$ is a cone with a quadric directrix and $\mathscr{F}$ is one of its rulings.

DEFINITION 5.2. We will call the coordinates $(x, y)\colon \mathbb{R}^2 \to \mathscr{P}_Q$ on the cone $\mathscr{P}_Q$ with the fixed ruling $\mathscr{F}$ canonical if a frame $A, B, S_2, K \subset \mathbb{R}^4\backslash\{0\}$ can be found so that a point $P_0(x, y) \in \mathscr{P}_Q$ with coordinates x, y has the representation

$$P_0(x, y) = A + Bx + S_2x^2 + Ky \subset \mathbb{R}\mathbf{P}^3, \qquad S_2 \in \mathscr{F}.$$

DEFINITION 5.3. We will say that the quadric immersion $\{\mathscr{P}_3, \mathscr{P}_Q, \mathscr{F}\}$ is connected with the equation $y'' = f(x, y)$ if a frame A, B, S_2, K can be found so that

$$\mathscr{P}_3 = \bigcup_{x,\, y \in \mathbb{R}} (A + Bx + S_2 x^2)\, f_y(x, y) + K(y f_y(x, y) - f(x, y)) + 2S_2,$$

$$\mathscr{P}_Q = \bigcup_{x,\, y \in \mathbb{R}} A + Bx + S_2 x^2 + Ky,$$

$$\mathscr{F} = (K, S_2).$$

This corresponds to the choice $c_0 = c_1 = c_2 = c_k = 0$ in (42).

An equation $y'' = f(x, y)$ uniquely determines the quadric immersion connected with it, i.e., any of two such immersions can be moved one to another by some projective transformation in $\mathbb{R}\mathbf{P}^3$. The quadric immersion determine f—the function connected with it up to the transformation $f \to \tilde{f}$,

$$\tilde{f}(x, y) = a^2 f(ax + b, \alpha y + \beta + \gamma x + \delta x^2)/\alpha - 2a^2\delta/\alpha, \tag{44}$$

where $a, b, \alpha, \beta, \gamma, \delta$ are some constants.

This can be formulated as

THEOREM 5.1. *Quadric immersions connected with the equations* $y'' = f(x, y)$ *and* $y'' = \tilde{f}(x, y)$ *can be moved one into another by some projective transformation of* $\mathbb{R}\mathbf{P}^3$ *if and only if* (44) *holds for some constants* $a, b, \alpha, \beta, \gamma, \delta$.

Proof. The proof of the theorem is based on the two following considerations.

(1) The transformations

$$\begin{aligned} x &\to ax + b \\ y &\to \alpha y \\ y &\to y + \beta + \gamma x + \delta x^2 \end{aligned} \tag{45}$$

of the variables in the $y'' = f(x, y)$ and the projective transformations

$$\begin{aligned} (A, B, S_2, K) &\to ((A + Bb + S_2 b^2)/a^2, (B + 2S_2 b)/a, S_2, K/a^2) \\ (A, B, S_2, K) &\to (A, B, S_2, \alpha K) \\ (A, B, S_2, K) &\to (A + \beta K, B + \gamma K, S_2 + \delta K, K) \end{aligned} \tag{46}$$

of the $\mathbb{R}\mathbf{P}^3 \supset \{\mathscr{P}_3, \mathscr{P}_Q, \mathscr{F}\}$, where $P_3(x, y) = (A + Bx + S_2 x^2)\, f_y + K(y f_y - f) + 2S_2$, $P_0(x, y) = A + Bx + S_2 x^2 + yK$, $\mathscr{F} = (K, S_2)$ leads to the same transformations of the quadric immersions.

(2) Any of two canonical coordinates on $\mathscr{P}_Q$ are related by some combination of the transformations (45).

Let us define some special types of projective transformations preserving the cone $\mathscr{P}_Q$ and its fixed ruling $\mathscr{F}$.

DEFINITION 5.4. Let $\rho \in \mathbb{R}\mathbf{P}^3_2$ be some plane, $K \notin \rho$. We define σ_ρ as the projective transformation (it is unique) with the properties

(1) σ_ρ maps any of the rulings of $\mathscr{P}_Q$ into itself,

(2) all the points of the plane ρ are fixed under σ_ρ,

(3) $\sigma_\rho^2 = \mathrm{id}$, $\sigma_\rho \neq \mathrm{id}$.

Let A, B, S_2, K be a frame so that $A, B, S_2 \in \rho$, K be the vertex of $\mathscr{P}_Q$. We can see that the transformation σ_ρ is given by

$$\sigma_\rho(x_1 A + x_2 B + x_3 S_2 + x_4 K) = x_1 A + x_2 B + x_3 S_2 - x_4 K.$$

DEFINITION 5.5. Let us define $T \subset PGL(3, \mathbb{R})$ as the one-parametric subgroup of $PGL(3, \mathbb{R})$ with the properties:

(1) any $T_\mu \in T$ maps any of the rulings of $\mathscr{P}_Q$ onto itself,

(2) all the points of the tangent plane to $\mathscr{P}_Q$ along $\mathscr{F}$ (we denote this plane by π_∞) are the fixed points for any $T_\mu \in T$, and T_μ has no more fixed points.

In any canonical coordinates the transformations from T are translations:

$$x \to x, \qquad y \to y + \text{const}.$$

DEFINITION 5.6. Let $l \subset \pi_\infty$ be a straight line lying in the tangent plane to $\mathscr{P}_Q$ along $\mathscr{F}$, let $l \neq \mathscr{F}$ and $K \notin l$.
We define $\Phi^l \subset PGL(3, \mathbb{R})$ as the one-parametric subgroup of $PGL(3, \mathbb{R})$ with the properties:

(1) any $\phi \in \Phi^l$ moves $\mathscr{P}_Q$ and $\mathscr{F}$ into itself,

(2) all the points of $\mathscr{F}$ are fixed for any $\phi \in \Phi^l$, and ϕ has no more fixed points on $\mathscr{P}_Q$,

(3) any $\phi \in \Phi^l$ moves all the quadrices that are tangent to the line l at the point $l \cap \mathscr{F}$ into themselves.

The class of such subgroups we will denote by $\Phi = \bigcup_{l \in \pi_\infty} \Phi^l$.

We can choose the canonical coordinates in such a way that the transformations from Φ^l are the shifts

$$x \to x + \text{const}, \qquad y \to y.$$

We will investigate $\{\mathcal{P}_3, \mathcal{P}_Q, \mathcal{F}\}$ near the vertex K. K is a singular point for our construction; we will give some special definitions.

DEFINITION 5.7. We say that $\{\mathcal{P}'_3, \mathcal{P}_Q, \mathcal{F}\}$ and $\{\mathcal{P}''_3, \mathcal{P}_Q, \mathcal{F}\}$ have at the vertex K of $\mathcal{P}_Q$ a contact of the nth order if the sections of $\mathcal{P}'_3$ and $\mathcal{P}''_3$ by the planes $\pi \supset \mathcal{F}$ have at K nth order contact for all planes π except for the plane π_∞, the tangent plane to $\mathcal{P}_Q$ along $\mathcal{F}$.

DEFINITION 5.8. We say that the quadric immersion $\{\mathcal{P}_3, \mathcal{P}_Q, \mathcal{F}\}$ has Φ-property of nth order at K if there is such a $\Phi^l \in \Phi$ that $\{\mathcal{P}_3, \mathcal{P}_Q, \mathcal{F}\}$ and $\{\phi\mathcal{P}_3, \mathcal{P}_Q, \mathcal{F}\}$ have nth order contact at K for all $\phi \in \Phi^l$.

THEOREM 5.2. *Let*

$$f(x, y) = y^n(1 + y^{-2}\mathrm{O}(1)), \qquad y \to \infty,$$

where $\mathrm{O}(1)$ *is an analytic function of the variable* y^{-1} *in some neighbourhood of* $y^{-1} = 0$ *for all* x, $n > 1$.
Then

(1) *the quadric immersion* $\{\mathcal{P}_3, \mathcal{P}_Q, \mathcal{F}\}$ *related to* $y'' = f(x, y)$ *has the* Φ*-property of the* $(n+1)$*st order, and*

(2) *any quadric immersion* $\{\tilde{\mathcal{P}}_3, \mathcal{P}_Q, \mathcal{F}\}$ *of the SNPC having the* Φ*-property of* $(n+1)$*st order and corresponding to the equivalence class of the equation* $y'' = f(x, y)$ *can be moved into* $\{\mathcal{P}_3, \mathcal{P}_Q, \mathcal{F}\}$ *by some projective transformation.*

Proof. It is not difficult to verify that the curves $\mathcal{L}_1$ and $\mathcal{L}_2$

$$\mathcal{L}_i = \bigcup_{y \in \mathbb{R}} (S f_{iy} + K(y f_{i_y} - f_i) + 2S_2),$$

$$f_i(y) = a_i y^n + b_i y^{n-1} + y^{n-2}\mathrm{O}(1), \qquad i = 1, 2$$

have $(n+1)st$ order contact at $y^{-1} = 0$ if and only if $a_1 = a_2$, $b_1 = b_2$.

(1) Let us choose a frame A, B, S_2, K and construct the quadric immersion related with $f(x, y)$:

$$\mathcal{P}_3 = \bigcup_{x, y \in \mathbb{R}} (A + Bx + S_2 x^2) f_y + K(y f_y - f) + 2S_2. \tag{47}$$

We can see that the transformations ϕ_μ from Φ^l, where $l=(S_2, B)$ are $(A, B, S_2, K) \to (A+B\mu+S_2\mu^2,\ B+2S_2\mu,\ S_2, K)$. Such transformations move the plane $\pi(x)=(K, S_2, A+Bx+S_2x^2)$ to the plane $\pi(x+\mu)$. The sections $\mathcal{L}(x)=\mathcal{P}_3\cap\pi(x)$ and $\mathcal{L}(x+\mu)=\mathcal{P}_3\cap\pi(x+\mu)$ have the same coefficients ($a=1$ and $b=0$) in the decomposition $f(x, y)=ay^n+by^{n-1}+\cdots$ so they have $(n+1)$st order contact for all x and μ.

(2) Consider $\{\tilde{\mathcal{P}}_3, \mathcal{P}_Q, \mathcal{F}\}$—some quadric immersion with the Φ-property of $(n+1)$st order, and let Φ^l be such subgroup that $\{\tilde{\mathcal{P}}_3, \mathcal{P}_Q, \mathcal{F}\}$ and $\{\phi\tilde{\mathcal{P}}_3, \mathcal{P}_Q, \mathcal{F}\}$ have $(n+1)$st order contact if $\phi\in\Phi^l$.

Let $S_2=l\cap\mathcal{F}$, $B\in l$, A be such points that the quadric $\bigcup_{x\in\mathbb{R}} A+Bx+S_2x^2$ lies on $\mathcal{P}_Q$.

Let the tangential equation of the section of $\tilde{\mathcal{P}}_3$ by $\pi(x)=(K, S_2, A+Bx+S_2x^2)$ be defined by $\tilde{f}(x, y)$:

$$\tilde{P}_3(x, y)=(A+Bx+S_2x^2)\,\tilde{f}_y+K(y\tilde{f}_y-\tilde{f})+2S_2, \qquad \tilde{P}_3\in\tilde{\mathcal{P}}_3\cap\pi(x).$$

The equation $y''=\tilde{f}(x, y)$, related to this quadric immersion lies in the same equivalence class as $y''=f(x, y)$ (they define the same SNPC), consequently (see (33)) there exists such $m(x)$, $n(x)$, c that

$$\tilde{f}=c^2m^3f\left(c\int m^2\,dx,\ my+n\right)+ym\frac{d^2}{dx^2}\frac{1}{m}-m\frac{d}{dx}\left(\frac{1}{m^2}\frac{d}{dx}n\right),$$

so $\tilde{f}=a(x)\,y^n+b(x)\,y^{n-1}+O(y^{n-2})$.

From the Φ-property of $(n+1)$st order it follows that $a=\text{const}$, $b=\text{const}$ which means that $m=\text{const}$, $n=\text{const}$. Theorem 5.1 states that the quadric immersions related to $f(x, y)$ and $a_1^2 f(a_1x+b_1, my+n)/m$ can be moved one into another in $\mathbb{R}\mathbf{P}^3$ (we put $a_1=cm^2$).

For the Painlevé equations the following theorem holds.

THEOREM 5.3. *Let A, B, S_2, K be a frame. Let us denote* $S(x)=A+Bx+S_2x^2$, $\mathcal{P}_Q=\bigcup_{x,\,y\in\mathbb{R}} S(x)+Ky$, $\mathcal{F}=(S_2, K)$,

$$\mathcal{P}^I=\bigcup_{x,\,y\in\mathbb{R}} S(x)\,12y+K(6y^2-x)+2S_2,$$

$$\mathcal{P}^{II}=\bigcup_{x,\,y\in\mathbb{R}} S(x)(6y^2+x)+K(4xy-\alpha)+2S_2,$$

$$\mathcal{P}^{IV}=\bigcup_{x,\,y\in\mathbb{R}} S(x)\left(\frac{15}{4}y^4+6xy^2+x^2-\alpha-\frac{3\beta}{2y^4}\right)$$
$$+K\left(3y^5+4xy^3-\frac{2\beta}{y^3}\right)+2S_2.$$

The immersions $\{\mathcal{P}^I, \mathcal{P}_Q, \mathcal{F}\}$, $\{\mathcal{P}^{II}, \mathcal{P}_Q, \mathcal{F}\}$, $\{\mathcal{P}^{IV}, \mathcal{P}_Q, \mathcal{F}\}$ *are the unique quadric immersions with the* Φ*-properties of 3rd, 4th, and 6th orders of the SNPC corresponding to the I, II, and IV Painlevé equations.*

Consider the symmetries T and σ_ρ (see Definitions 4.4 and 4.5). The invariance of $\{\mathcal{P}_3, \mathcal{P}_Q, \mathcal{F}\}$ with respect to some T_μ: $\{\mathcal{P}_3, \mathcal{P}_Q, \mathcal{F}\} = \{T_\mu \mathcal{P}_3, \mathcal{P}_Q, \mathcal{F}\}$ means periodicity with a constant period μ ($\mu_x = 0$) of the function $f(x, y)$, i.e., $f(x, y+\mu) = f(x, y)$, which defines $\mathcal{P}_3$ in some frame, see (47).

Let $\{\mathcal{P}_3, \mathcal{P}_Q, \mathcal{F}\} = \{\sigma_\rho \mathcal{P}_3, \mathcal{P}_Q, \mathcal{F}\}$. Let us choose A, B, S_2 in such a way that $\rho = (A, B, S_2)$, $\mathcal{P}_Q = \bigcup_{x \in \mathbb{R}} A + Bx + S_2 x^2$, $\mathcal{F} = (K, S_2)$. For this frame the function $f(x, y)$, defining $\mathcal{P}_3$ is odd, i.e., $f(x, y) = -f(x, -y)$.

From these two considerations it follows that if

(1) $\{\mathcal{P}_3, \mathcal{P}_Q, \mathcal{F}\}$ is invariant under some σ_ρ, T_μ,

(2) $\{\tilde{\mathcal{P}}_3, \mathcal{P}_Q, \mathcal{F}\}$ is invariant under some $\sigma_{\tilde\rho}$, $T_{\tilde\mu}$

and determine the same SNPC, they can be moved one into another by some projective transformation of $\mathbb{R}\mathbf{P}^3$.

For the equation PVI we have the following result.

THEOREM 5.4. *Let* A, B, S_2, K *be a frame and* $S(x)$, $\mathcal{P}_Q$, $\mathcal{F}$ *be as before*

$$S(x) = A + Bx + S_2 x^2, \qquad \mathcal{P}_Q = \bigcup_{x,\, y \in \mathbb{R}} S(x) + Ky, \qquad \mathcal{F} = (S_2, K).$$

Let

$$\mathcal{P}^{VI} = \bigcup_{x,\, y \in \mathbb{R}} S(x)\, f_y^{VI} + K(y f_y^{VI} - f^{VI}) + 2S_2,$$

where $f^{VI}(x, y) = \alpha \wp'(y \mid 1/2, ix/2) + \beta \wp'_1 + \gamma \wp'_2 + \delta \wp'_3$, *and the notations are the same as in* (31).

Then the immersion $\{\mathcal{P}^{VI}, \mathcal{P}_Q, \mathcal{F}\}$, *is the unique quadric immersion of the SNPC corresponding to the equation PVI and invariant under some* σ_ρ *and* $T_\mu \in T$.

The immersions related with PV and PIII can not be fixed by such simple symmetries as T_μ-invariance[1], Φ-property or σ_ρ-invariance.

We can uniquely fix the quadric immersion using the properties of the function $f(x, y)$. For PIII and PV we will proceed this way. By Theorem 5.1 the quadric immersion determines, the function $f(x, y)$ related to this immersion up to the transformations (44).

[1] We will not introduce T_μ-invariance for complex μ.

Consider the quadric immersions $\{\mathscr{P}^{III}, \mathscr{P}_Q, \mathscr{F}\}$ and $\{\mathscr{P}^{V}, \mathscr{P}_Q, \mathscr{F}\}$, where

$$\mathscr{P}^j = \bigcup_{x,\, y \in \mathbb{R}} S(x)\, f^j(x, y) + K(y f_y^j(x, y) - f(x, y)) + 2S_2, \qquad j = \mathrm{III}, \mathrm{V}$$

and

$$f^{\mathrm{III}}(x, y) = \alpha e^{x+y} + \beta e^{x-y} + \gamma e^{2(x+y)} + \delta e^{2(x-y)},$$

$$f^{\mathrm{V}}(x, y) = 4\alpha e^z \frac{1+e^z}{(1-e^z)^3} + 4\beta e^z \frac{1-e^z}{(1+e^z)^3} + \frac{\gamma}{4}(e^{t-z} - e^t + z)$$
$$+ \frac{\delta}{8}(e^{2(t-z)} - e^{2(t+z)}).$$

THEOREM 5.5. *The immersion* $\{\mathscr{P}^{III}, \mathscr{P}_Q, \mathscr{F}\}$ *is the unique quadric immersion of SNPC corresponding to the third Painlevé equation if the function* $f(x, y)$ *has the following properties* (*see Definition* 5.3)

(1) $f(x, y)$ *is a rational function of* e^{cy}, *where* c *is a constant, i.e.,* $f(x, y) = R(x, e^{cy}) = R(x, z)$, $z = e^{cy}$,

(2) *the term* $r_0(x)$ *in the decomposition* $R(x, z) = \sum_{k=-2}^{2} r_k(x)\, z^k$ *of* R *in Laurent series with respect to the second argument does not depend on* x, *i.e.,* $r_0(x) = r_0 = \mathrm{const}$.

THEOREM 5.6. *The immersion* $\{\mathscr{P}^{V}, \mathscr{P}_Q, \mathscr{F}\}$ *is the unique quadric immersion of SNPC corresponding to the fifth Painlevé equation if the function* $f(x, y)$ *has the following properties*

(1) $f(x, y)$ *is a rational function of* e^{cy}, *where* c *is a constant* $f(x, y) = R(x, e^{cy}) = R(x, z)$, $z = e^{cy}$,

(2) *the term* $r_0(x)$ *in the decomposition of*

$$R(x, z) = r_0(x) + \sum_{j,\, k \neq 0} r_{k,\, j}(x)(z - z_j)^k, \qquad z_j \in \{-1, 0, 1\}$$

of R *into partial fractions with respect to the second argument does not depend on* x, *i.e.,* $r_0(x) = r_0 = \mathrm{const}$.

APPENDIX

The Elliptic Form of the Sixth Painlevé Equation

The sixth Painleve equation (PVI) was obtained by B. Gambier [Gm] and it is the most fundamental equation of all the six Painlevé equations (see [I, IKSY]). It was named "the master equation" because it can be degenerated to all others Painlevé equations. The PVI is also the most voluminous one. For this reason there had been a lot of attempts to find more simple forms of this equation, comfortable for further investigations. The form presented in this paper is the simplest and the only one which is symmetric with respect to permutations of all constants $(\alpha, \beta, \gamma, \delta)$. Moreover, it is the first form of the PVI equation, where a deep connection with the theory of elliptic functions can be seen.

THEOREM 5.7. *Consider the equation PVI*

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx}$$
$$+\frac{y(1-y)(x-y)}{x^2(x-1)^2}\left(\alpha-\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\left(\frac{1}{2}-\delta\right)\frac{x(x-1)}{(y-x)^2}\right). \tag{48}$$

It takes the form

$$4\pi^2\frac{d^2}{d\tau^2}u=\beta\wp'(u\mid 1/2, i\tau/2)+\gamma\wp'_1+\alpha\wp'_2+\delta\wp'_3. \tag{49}$$

after the substitution

$$x=\frac{\theta_3^4(0\mid i\tau)}{\theta_2^4(0\mid i\tau)},\qquad y=\frac{\theta_3^2(0\mid i\tau)}{\theta_2^2(0\mid i\tau)}\frac{\theta_1^2(u\mid i\tau)}{\theta_0^2(u\mid i\tau)} \tag{50}$$

or

$$\tau=\frac{K(\sqrt{1-1/x})}{K(1/\sqrt{x})},\qquad u=\frac{1}{4K(1/\sqrt{x})}\int_0^y\frac{dt}{\sqrt{t(1-t)(1-t/x)}}. \tag{51}$$

Here $\wp'(u\mid\frac{1}{2}, i(\tau/2))$ is the derivative with respect to u of the Weierstrass $\wp$-function with the periods 1 and $i\tau$, and $\wp'_1$, $\wp'_2$, $\wp'_3$ are its shifts on the half-periods $1/2$, $i\tau/2$, and $(1+i\tau)/2$ correspondently.

Proof. Let us introduce the new variable $z=z(x)$:

$$z=\int_0^y\frac{dt}{\sqrt{t(1-t)(x-t)}},\qquad y=\operatorname{sn}^2\left(\frac{z}{2k};k\right),\quad k=\frac{1}{\sqrt{x}}.$$

We denote $Q=\sqrt{y(1-y)(x-y)}$,

$$I_1=\int_0^y\frac{dt}{(x-t)\sqrt{t(1-t)(x-t)}},\qquad I_2=\int_0^y\frac{dt}{(x-t)^2\sqrt{t(1-t)(x-t)}}$$

consequently

$$\frac{dy}{dx}=Q\left(\frac{dz}{dx}z+\frac{1}{2}I_1\right),\qquad \frac{dI_1}{dx}=-\frac{3}{2}I_2+\frac{dy}{dx}\frac{1}{(x-y)\,Q},$$

$$\frac{d^2y}{dx^2}=\frac{1}{Q}\left(\frac{\partial Q}{\partial x}+\frac{dy}{dx}\frac{\partial Q}{\partial y}\right)\frac{dy}{dx}+Q\left(\frac{d^2z}{dx^2}-\frac{3}{4}I_2+\frac{1}{2}\frac{dy}{dx}\frac{1}{(x-y)\,Q}\right)$$

$$=\frac{1}{2}\left(\frac{\partial}{\partial x}\log(y(1-y)(x-y))+\frac{dy}{dx}\frac{\partial}{\partial y}\log(y(1-y)(x-y))\right)\frac{dy}{dx}$$

$$+Q\frac{d^2z}{dx^2}-\frac{3}{4}QI_2+\frac{1}{2}\frac{dy}{dx}\frac{1}{x-y}$$

$$=Q\frac{d^2}{dx^2}z+\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2+\frac{dy}{dx}\frac{1}{x-y}-\frac{3}{4}QI_2,$$

and PVI can be rewritten in the form

$$\frac{d^2}{dx^2}z=-\frac{2x-1}{x(x-1)}\frac{d}{dx}z+\frac{3}{4}I_2-\frac{1}{2}\frac{2x-1}{x(x-1)}I_1+J_{\alpha\beta\gamma\delta},$$

where

$$J_{\alpha\beta\gamma\delta}=\frac{Q}{x^2(x-1)^2}\left(\alpha-\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}-\delta\frac{x(x-1)}{(x-y)^2}\right)$$

$$=\alpha\frac{k^7}{(1-k^2)^2}\operatorname{sn}\left(\frac{z}{2k};k\right)\operatorname{cn}\left(\frac{z}{2k};k\right)\operatorname{dn}\left(\frac{z}{2k};k\right)$$

$$-\beta\frac{k^5}{(1-k^2)^2}\frac{\operatorname{cn}(z/2k;k)\operatorname{dn}(z/2k;k)}{\operatorname{sn}^3(z/2k;k)}$$

$$+\gamma\frac{k^5}{1-k^2}\frac{\operatorname{sn}(z/2k;k)\operatorname{dn}(z/2k;k)}{\operatorname{cn}^3(z/2k;k)}-\delta\frac{k^7}{1-k^2}\frac{\operatorname{sn}(z/2k;k)\operatorname{cn}(z/2k;k)}{\operatorname{dn}^3(z/2k;k)}$$

$$=\beta f(z)+\gamma f(z+T_1/2)+\alpha f(z+T_2/2)+\delta f(z+(T_1+T_2)/2),$$

and

$$f(z)=-\frac{k^5}{(1-k^2)^2}\frac{\operatorname{cn}(z/2k;k)\operatorname{dn}(z/2k;k)}{\operatorname{sn}^3(z/2k;k)}.$$

We see that $f(z)$ is the double-periodic (elliptic) function of z with the periods $T_1=4kK$, $T_2=4ikK'$, $K=K(k)$, $K'=K(\sqrt{1-k^2})$, $k=1/\sqrt{x}$.

Using $K=\pi\theta_3^2(0\mid i\tau)/2$ and

$$\wp'(u\mid\omega,\omega')=-\frac{2}{(2\omega)^3}\frac{(\theta_1'(0\mid i\tau))^3}{\theta_0(0\mid i\tau)\,\theta_2(0\mid i\tau)\,\theta_3(0\mid i\tau)}\times\frac{\theta_0(u/2\omega\mid i\tau)\,\theta_2(u/2\omega\mid i\tau)\,\theta_3(u/2\omega\mid i\tau)}{(\theta_1(u/2\omega\mid i\tau))^3}$$

we get

$$f(z)=\frac{1}{2\pi^3\theta_3^6(0\mid i\tau)}\frac{k^5}{k'^4}\,\wp'\left(u\,\middle|\,\frac{1}{2},\,i\tau/2\right),$$

where $u=z/4kK$, $\tau=K'/K$.

It can be easily verified by differentiation that

$$\int_0^y\left(3\,\frac{x(x-1)}{(x-t)^2}-2\,\frac{2x-1}{x-t}+1\right)\frac{dt}{\sqrt{t(1-t)(x-t)}}=-2\,\frac{\sqrt{y(1-y)(x-y)}}{(x-y)^2}$$

so

$$\frac{3}{4}I_2-\frac{1}{2}\frac{2x-1}{x(x-1)}I_1=-\frac{Q}{2x(x-1)(x-y)^2}-\frac{z}{4x(x-1)}$$

gives

$$\frac{1}{x(x-1)}\frac{d}{dx}\left(x(x-1)\frac{d}{dx}z\right)=-\frac{z}{4x(x-1)}+J_{\alpha\beta\gamma\delta}. \tag{52}$$

Let us introduce the new variables (λ, u) instead of (x, z),

$$\lambda=\frac{1}{x}=k^2,\qquad z=4\sqrt{\lambda}m(\lambda)\,u,\quad \frac{d}{dx}=-\lambda^2\frac{d}{d\lambda},$$

where we choose the function $m(\lambda)$ in such a way that the linear term in the right-hand side of (52) cancels.

For m we get the hypergeometric equation,

$$\lambda(1-\lambda)\frac{d^2}{d\lambda^2}m-(2\lambda-1)\frac{d}{d\lambda}m-\frac{1}{4}m=0. \tag{53}$$

Its solutions $(\pi/2)\,F(\frac{1}{2},\frac{1}{2},1,\lambda)$ and $(\pi/2)\,F(\frac{1}{2},\frac{1}{2},1,1-\lambda)$ are the complete elliptic integrals K, K' if we consider them as the functions of k^2 (see [A, WW]).

Consider the new variable τ

$$c\frac{d}{d\tau}=\lambda(1-\lambda)\,m^2\frac{d}{d\lambda}, \qquad \text{where } c \text{ is a constant.}$$

From this condition it follows that τ must be the ratio of any two solutions of (53). We choose $m=K$, $\tau=K'/K=K(\sqrt{1-1/x})/K(1/\sqrt{x})$.

For the calculation of the constant c we put $k^2=k'^2=1/2$. For this value of k^2 we have

$$k^2k'^2K^2\frac{d}{dk^2}(K'/K)=-k'^2K'k\frac{d}{dk}K=-k'^2(2EK'-KK')=-\pi/4,$$

consequently, $c=-\pi/4$, and

$$\frac{d}{dx}=\frac{\pi}{4}\frac{1}{K^2}\frac{k^2}{k'^2}\frac{d}{d\tau}.$$

The substitution of this expression into (52) completes the proof.

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