Physics Letters B 751 (2015) 19–24

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

De Sitter vacua in ghost-free massive gravity theory

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ARTICLE INFO

Article history:
Received 11 July 2015
Received in revised form 25 September 2015
Accepted 6 October 2015
Available online 19 October 2015
Editor: M. Cvetič

Keywords:
Massive gravity
De Sitter space

ABSTRACT

We present a simple procedure to obtain a large class of different versions of the de Sitter solution in the ghost-free massive gravity theory via applying the Gordon ansatz. For these solutions the physical metric describes a hyperboloid in 5D Minkowski space, while the flat reference metric depends on the Stuckelberg field $T(t,r)$ subject to $(\partial_t T)^2 - (\partial_r T)^2 = 1$. This equation admits infinitely many solutions, hence there are infinitely many de Sitter vacua with different physical properties. Only the simplest solution with $T = t$ has previously been studied, as it is manifestly homogeneous and isotropic, but this solution turns out to be unstable. However, other solutions could be stable. We require the timelike isometry to be common for both metrics and this gives physically distinguished solutions since only for them the canonical Killing energy is time-independent. We conjecture that these solutions minimize the energy and are therefore stable. We also show that in some cases solutions can be homogeneous and isotropic in a non-manifest way such that their symmetries are not obvious. All of this suggests that the theory may admit physically interesting cosmologies.

1. Introduction

The discovery of the ghost-free massive gravity theory by de Rham, Gabadadze, and Tolley (dRGT) [1] (see [2,3] for a review) opens up the possibility to explain the dark energy and the cosmic acceleration [4,5] by a tiny mass of the gravitons. The dRGT field equations admit the de Sitter solution with the cosmological constant mimicked by the graviton mass. This solution can describe the late time cosmic acceleration, but a special analysis is needed to decide whether its other properties are physically acceptable.

A closer look reveals that the de Sitter solution in the dRGT theory is actually not unique, and a number of its versions have been found [6–14]. A special attention was received by one particular solution whose physical and reference metrics are both of the manifestly homogeneous and isotropic Freedman–Lemaître–Robertson–Walker (FLRW) form [10]. However, a detailed analysis reveals that this solution is unstable [15,16]. For other known solutions only the physical metric is manifestly FLRW while the reference metric looks inhomogeneous, for which reason they are considered to be less interesting [9]. All of this has reduced the interest towards the dRGT theory, the focus shifting towards its extensions, as for example the bigravity [17–25] and other generalizations admitting FLRW solutions [26–29].

However, we would like to argue in this paper that it may be premature to abandon the dRGT theory on the basis of negative evidence obtained from just one solution, because the theory actually admits infinitely many other solutions that could be physically interesting. They all have the same physical (de Sitter) metric but different values of the reference metric depending on the Stuckelberg field $T(t,r)$ subject to a complicated differential equation [9, 11–14]. Below we shall describe a simple way to obtain these solutions by applying the Gordon ansatz [30] and using the global embedding coordinates for the de Sitter space. The T-equation then assumes a simple form, $(\partial_t T)^2 - (\partial_r T)^2 = 1$, whose essentially general solution is known. The simplest solution $T = t$ is unstable [15,16] but other solutions could be stable. One can choose $T(t,r)$ in such a way that both metrics are invariant under the timelike isometry, which gives distinguished solutions since only for them the canonical Killing energy is time independent. We conjecture that their energy is minimal and hence these solutions are stable. We also give explicit examples where the reference metric looks inhomogeneous but shares with the physical metric the same translational and rotational isometries. Hence, solutions previously considered to be non-FLRW can actually be homogeneous and isotropic. All of this suggests that physically interesting dRGT cosmologies may exist.

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http://dx.doi.org/10.1016/j.physletb.2015.10.012

2. The dRGT massive gravity

The theory is defined on a four-dimensional spacetime manifold endowed with two Lorentzian metrics, the physical one $g_{\mu\nu}$ and the flat reference metric $f_{\mu\nu}=\eta_{AB}\partial_\mu\Phi^A\partial_\nu\Phi^B$ with $\eta_{AB}=\mathrm{diag}[-1,1,1,1]$. The scalars $\Phi^A(x)$ are sometimes called Stuckelberg fields. The theory is defined by the action

$$S=\frac{M_{\rm Pl}^2}{m^2}\int\left(\frac{1}{2}R(g)-\mathcal{U}\right)\sqrt{-g}\,d^4x\,, \tag{2.1}$$

where the metrics and all coordinates are assumed to be dimensionless, the length scale being the inverse graviton mass $1/m$. The interaction between the two metrics is determined by the tensor $\gamma^\mu_{\ \nu}$ subject to $(\gamma^2)^\mu_{\ \nu}\equiv\gamma^\mu_{\ \alpha}\gamma^\alpha_{\ \nu}=g^{\mu\alpha}f_{\alpha\nu}$, hence, using the hat to denote matrices, one has $\hat\gamma=\sqrt{\hat g^{-1}\hat f}$. If λ_A are the eigenvalues of $\hat\gamma$ then the interaction potential is

$$\mathcal{U}=b_0+\sum_{n=1}^{3}b_k\mathcal{U}_k \tag{2.2}$$

where b_0,b_k are parameters and $\mathcal{U}_k$ are defined by (with $[\gamma]\equiv\mathrm{tr}(\hat\gamma)$ and $[\gamma^k]\equiv\mathrm{tr}(\hat\gamma^k)$)

$$\mathcal{U}_1=\sum_A\lambda_A=[\gamma],\qquad \mathcal{U}_2=\sum_{A<B}\lambda_A\lambda_B=\frac{1}{2!}([\gamma]^2-[\gamma^2]),$$
$$\mathcal{U}_3=\sum_{A<B<C}\lambda_A\lambda_B\lambda_C=\frac{1}{3!}([\gamma]^3-3[\gamma][\gamma^2]+2[\gamma^3]). \tag{2.3}$$

The metric $g_{\mu\nu}$ and the scalars Φ^A are the variables of the theory. Varying the action with respect to $g_{\mu\nu}$ gives the Einstein equations $G_{\mu\nu}=T_{\mu\nu}$ with

$$T^\mu_{\ \nu}=\{b_1+b_2\mathcal{U}_1+b_3\mathcal{U}_2\}\gamma^\mu_{\ \nu}-\{b_2+b_3\mathcal{U}_1\}(\gamma^2)^\mu_{\ \nu}$$
$$+b_3(\gamma^3)^\mu_{\ \nu}-\mathcal{U}\,\delta^\mu_\nu\,. \tag{2.4}$$

Varying the action with respect to Φ^A gives the conservation conditions $\nabla_\mu T^\mu_\nu=0$, but these equations are not independent and follow from the Bianchi identities for the Einstein equations.

3. De Sitter space

The field equations $G_{\mu\nu}=T_{\mu\nu}$ admit solutions for which the physical metric is de Sitter. The de Sitter space can be globally visualized as the hyperboloid

$$-X_0^2+\sum_i X_i^2+X_4^2=\alpha^2 \tag{3.1}$$

in the 5D Minkowski space with the metric

$$ds^2=-dX_0^2+\sum_i dX_i^2+dX_4^2. \tag{3.2}$$

The 4D geometry induced on the hyperboloid fulfills the equations $G^\mu_\nu+\Lambda\delta^\mu_\nu=0$ with $\Lambda=3/\alpha^2$. Rescaling the coordinates, $X_0=\alpha t$, $X_i=\alpha x_i$, $X_4=\alpha r$ with $x_i\equiv(x,y,z)$, the metric reads

$$ds_g^2=\alpha^2\left\{-dt^2+dr^2+dx^2+dy^2+dz^2\right\}$$
$$=\alpha^2\left\{-dt^2+dr^2+dR^2+R^2\,d\Omega^2\right\} \tag{3.3}$$

where $d\Omega^2=d\vartheta^2+\sin^2\vartheta d\varphi^2$ and the following constraint is imposed

$$R^2\equiv x^2+y^2+z^2=1+t^2-r^2. \tag{3.4}$$

Let us choose the flat reference metric as

$$ds_f^2=\alpha^2u^2\left\{-dT^2+dX^2+dY^2+dZ^2\right\}, \tag{3.5}$$

where u is a constant and T, X, Y, Z are the Stuckelberg fields.

The constants α, u and the functions T, X, Y, Z in the above formulas can be chosen such that the two metrics fulfill the field equations. It turns out that it is sufficient to make sure that the following relation is fulfilled (the Gordon ansatz) [30],

$$f_{\mu\nu}=\omega^2\left(g_{\mu\nu}+(1-\zeta^2)V_\mu V_\nu\right), \tag{3.6}$$

where ω,ζ are functions and

$$g^{\mu\nu}V_\mu V_\nu\equiv V^\mu V_\mu=-1. \tag{3.7}$$

If Eq. (3.6) holds, then one can see at once that

$$\gamma^\mu_{\ \nu}=\omega\left(\delta^\mu_\nu+(1-\zeta)V^\mu V_\nu\right) \tag{3.8}$$

fulfills $\gamma^\mu_\alpha\gamma^\alpha_\nu=g^{\mu\alpha}f_{\alpha\nu}$. One has $(\gamma^n)^\mu_{\ \nu}=\omega^n\left(\delta^\mu_\nu+(1-\zeta^n)V^\mu V_\nu\right)$ and so the energy–momentum tensor (2.4) becomes

$$T^\mu_\nu=-\{P_0(\omega)-\zeta\omega P_1(\omega)\}\delta^\mu_\nu+\omega(\zeta-1)P_1(\omega)V^\mu V_\nu\,. \tag{3.9}$$

Here we have defined

$$P_m(\omega)\equiv b_m+2b_{m+1}\,\omega+b_{m+2}\,\omega^2\,, \tag{3.10}$$

where $m=0,1,2$ (assuming that $b_4=0$). Let us set $\omega=u$ where u is a constant chosen such that

$$P_1(u)=0. \tag{3.11}$$

Then the energy–momentum tensor (3.9) reduces to

$$T^\mu_\nu=-P_0(u)\delta^\mu_\nu \tag{3.12}$$

and the Einstein equations become $G^\mu_\nu+\Lambda\delta^\mu_\nu=0$ with $\Lambda=P_0(u)$, hence the de Sitter metric (3.3) will fulfill these equations if

$$\frac{3}{\alpha^2}=\Lambda=P_0(u). \tag{3.13}$$

Therefore, choosing u, α according to (3.11), (3.13), the metrics (3.3) and (3.5) will fulfill the field equations, if only the functions T, X, Y, Z can be adjusted such that the Gordon relation (3.6) holds.

Let us choose in (3.5) $T=T(t,r)$, $X=x$, $Y=y$, $Z=z$ so that the f-metric becomes

$$ds_f^2=\alpha^2u^2\left\{-dT^2+dx^2+dy^2+dz^2\right\}$$
$$=\alpha^2u^2\left\{-dT^2+dR^2+R^2d\Omega^2\right\}. \tag{3.14}$$

The two metrics (3.3) and (3.14) are related to each other as

$$ds_f^2=u^2\left(ds_g^2+dt^2-dr^2-dT^2\right). \tag{3.15}$$

This will be compatible with the Gordon relation (3.6) if

$$\partial_\mu t\partial_\nu t-\partial_\mu r\partial_\nu r-\partial_\mu T\partial_\nu T=(1-\zeta^2)V_\mu V_\nu\,. \tag{3.16}$$

Assuming that the indices μ,ν correspond to (t,r,ϑ,φ) yields $V_\vartheta=V_\varphi=0$ and

$$(\partial_t T)^2-1=(\zeta^2-1)V_t^2\,,$$
$$(\partial_r T)^2+1=(\zeta^2-1)V_r^2\,,$$
$$\partial_t T\,\partial_r T=(\zeta^2-1)V_tV_r\,. \tag{3.17}$$

From the first two of these relations one obtains

$$V_t^2 = \frac{(\partial_t T)^2 - 1}{\zeta^2 - 1}, \qquad V_r^2 = \frac{(\partial_r T)^2 + 1}{\zeta^2 - 1}, \tag{3.18}$$

inserting which into the normalization condition (3.7) determines ζ. Finally, inserting (3.18) into the third relation in (3.17) yields

$$(\partial_t T)^2 (\partial_r T)^2 = ((\partial_t T)^2 - 1)((\partial_r T)^2 + 1) \tag{3.19}$$

and therefore

$$(\partial_t T)^2 - (\partial_r T)^2 = 1. \tag{3.20}$$

This completes the procedure, because V_μ and ζ are determined by the above formulas and the Gordon relation is fulfilled.

Summarizing, the de Sitter solution in the theory is described by (3.3), (3.14) where u, α are defined by (3.11), (3.13) and T is a solution of the differential equation (3.20). Since there are infinitely many T's subject to (3.20), there are infinitely many versions of the de Sitter solution. They all have the same physical metric (3.3) but differ one from the other by the choice of T in the reference metric (3.14). The physical properties of solutions with different T's, as for example their stability, can be different.

These solutions were actually discussed previously [9,11–14], but within a different computation scheme yielding the T-equation in a form that gives little hope to solve it (see Eq. (6.7) below). Our procedure yields the simple equation (3.20) that can be solved. In addition, new solutions can be obtained by slightly modifying the procedure. Specifically, it was assumed in the above derivation that both metrics have the same spatial $SO(3)$ symmetry. However, let us choose instead

$$ds_f^2 = \alpha^2 u^2 \left\{ -dt^2 + dx^2 + dy^2 + dZ^2 \right\} \tag{3.21}$$

with $Z = Z(r, z)$, so that the two metrics share the same $SO(1, 2)$ symmetry in the t, x, y subspace. Repeating the above analysis one obtains

$$(\partial_r Z)^2 + (\partial_z Z)^2 = 1, \tag{3.22}$$

and this gives new solutions. When expressed in the standard spherical coordinates, the f-metric will not even look spherically symmetric since for generic Z it has no common with the g-metric $SO(3)$ symmetry, although it has its own $SO(3)$ in the x, y, Z space. Below we shall only be discussing equation (3.20) since the analysis of (3.22) is similar. We have also tried to make all Stuckelberg fields T, X, Y, Z non-trivial but could not obtain new solutions in this way.

4. The simplest solution

Even though there are infinitely many solutions of Eq. (3.20), almost all known dRGT cosmologies reported in the literature correspond just to the simplest choice,

$$T = t. \tag{4.1}$$

A slightly more general choice is

$$T = \cosh(\xi)\, t + \sinh(\xi)\, r \tag{4.2}$$

with a constant ξ. However, the value of ξ can be changed by boosts in the t, r plane of the ambient 5D Minkowski space, which does not affect the g-metric (3.3), hence one can set $\xi = 0$ without loss of generality. Rewriting (4.1) in different coordinates gives results which look very different, and it has not been recognized that they actually describe the same solution. Let us therefore see what happens when this solution is expressed in the standard spatially flat, closed, or open coordinate systems.

4.1. Flat slicing

Let us express t, r, R in (3.3) in terms of two new coordinates τ and ρ as

$$t = \sinh\tau + \frac{\rho^2}{2} e^\tau, \qquad r = \cosh\tau - \frac{\rho^2}{2} e^\tau, \qquad R = e^\tau \rho. \tag{4.3}$$

This solves the constraint (3.4) and transforms the de Sitter metric (3.3) to the standard FLRW form with flat spatial sections,

$$ds_g^2 = \alpha^2 \{ -d\tau^2 + a^2(\tau)(d\rho^2 + \rho^2 d\Omega^2) \}, \tag{4.4}$$

where $a(\tau) = e^\tau$. The function $T = t$ can be represented as

$$T = \frac{1}{2} \int \frac{d\tau}{\dot{a}(\tau)} + \frac{1}{2}\left(1 + \rho^2\right) a(\tau). \tag{4.5}$$

This is the well-known solution, found first for special [9] and then for arbitrary [12–14] values of b_k. Using (4.2) instead of (4.1), the solution can be generalized to include an integration constant. Although the g-metric (4.4) is manifestly homogeneous and isotropic, the f-metric (3.14), when expressed in the τ, ρ coordinates, becomes non-diagonal and ρ-dependent, which suggests that it is inhomogeneous. For this reason it is often said that the dRGT theory does not admit genuinely homogeneous and isotropic cosmologies with flat spatial sections [9]. However, we shall shortly comment on this.

4.2. Closed slicing

If one chooses in (3.3)

$$t = \sinh(\tau), \qquad r = \cosh(\tau)\cos(\rho),$$
$$R = \cosh(\tau)\sin(\rho), \tag{4.6}$$

this solves the constraint (3.4) and the de Sitter metric (3.3) assumes the FLRW form with closed spatial sections,

$$ds_g^2 = \alpha^2 \{ -d\tau^2 + a^2(\tau)(d\rho^2 + \sin^2(\rho) d\Omega^2) \}, \tag{4.7}$$

with $a(\tau) = \cosh(\tau)$. These coordinates cover the whole of de Sitter space. The Stuckelberg field is $T = \sinh(\tau)$ and the f-metric (3.14) expressed in the τ, ρ coordinates is again non-diagonal and ρ-dependent, which suggests that there are no genuinely homogeneous and isotropic cosmologies with closed spatial sections either.

4.3. Open slicing

For the open slicing one has

$$t = \sinh(\tau)\cosh(\rho), \qquad r = \cosh(\tau),$$
$$R = \sinh(\tau)\sinh(\rho), \tag{4.8}$$

and the g-metric becomes

$$ds_g^2 = \alpha^2 \{ -d\tau^2 + a^2(\tau)(d\rho^2 + \sinh^2(\rho) d\Omega^2) \}, \tag{4.9}$$

with $a(\tau) = \sinh(\tau)$. The Stuckelberg field is $T = \sinh(\tau)\cosh(\rho)$, and the specialty now is that the f-metric (3.14) becomes diagonal in the τ, ρ coordinates,

$$ds_f^2 = \alpha^2 u^2 \{ -\cosh(\tau)^2 d\tau^2 + a^2(\tau)(d\rho^2 + \sinh^2(\rho) d\Omega^2) \}. \tag{4.10}$$

This solution, discovered in [10], is broadly considered to be the only genuinely homogeneous and isotropic dRGT cosmology because both metrics are manifestly homogeneous and isotropic and

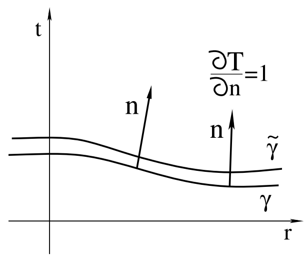

Fig. 1. The T-equation in the method of characteristics.

share the same rotational and translational Killing symmetries. However, this solution is equivalent to its flat and closed versions considered above. Therefore, the latter have the same rotational and translational isometries and hence they are also homogeneous and isotropic, although this fact is not manifest in their case. The conclusion is that solutions can be homogeneous and isotropic in a non-manifest way.

At the same time, being homogeneous and isotropic, the $T = t$ solution is not static, whereas the de Sitter space is static. This can be seen as follows.

4.4. Static slicing

Setting

$$t = \sqrt{1-\rho^2}\,\sinh(\tau), \qquad r = \sqrt{1-\rho^2}\,\cosh(\tau),$$
$$R = \rho \tag{4.11}$$

solves the condition (3.4) and reduces the de Sitter metric (3.3) to the static form

$$ds_g^2 = \alpha^2 \left\{ -\Sigma\, d\tau^2 + \frac{d\rho^2}{\Sigma} + \rho^2 d\Omega^2 \right\} \tag{4.12}$$

with $\Sigma = 1 - \rho^2$. The $T = t$ solution then becomes

$$T(\tau,\rho) = \sqrt{1-\rho^2}\,\sinh(\tau), \tag{4.13}$$

and it is non-static even in static coordinates. Therefore, the g-metric is invariant under the action of the timelike Killing vector $\partial/\partial\tau$, but the Stuckelberg field T and the f-metric are not invariant. As a result, the timelike isometry is not shared by both metrics.

As the solution $T = t$ is not static, it is unlikely to describe the "ground state" of the theory. This is probably the reason why this solution was found to be unstable [15,16]. Therefore, we need to consider other solutions for T.

5. Other solutions

Solutions of the T-equation $(\partial_t T)^2 - (\partial_r T)^2 = 1$ can be constructed in different ways. A fairly general solution containing an arbitrary function $W(\xi)$ is given by [31]

$$T = \cosh(\xi)\,t + \sinh(\xi)\,r + W(\xi),$$
$$0 = \sinh(\xi)\,t + \cosh(\xi)\,r + \frac{dW(\xi)}{d\xi}, \tag{5.1}$$

where the second line implicitly determines the dependence of ξ on t, r. Together with (4.2), this gives if not all but probably almost all solutions. However, this formula is difficult to use since one cannot explicitly determine $\xi(t,r)$ for a generic $W(\xi)$.

The T-equation can also be integrated by applying the method of characteristics [31], which has a simple geometric interpretation. Let us consider the 2D Minkowski space spanned by $x^a \equiv \{t, r\}$ with the metric $g_{ab} = \mathrm{diag}[1, -1]$. The T-equation reads $g^{ab}\partial_a T \partial_b T \equiv \langle \nabla T, \nabla T\rangle = 1$. Let $\gamma = x^a(s)$ be a spacelike curve and let T be constant along γ. At every point of γ there is a unit timelike normal n such that $\langle n, n\rangle = 1$ and $\langle n, \partial/\partial s\rangle = 0$. The T-equation is equivalent to $\partial T/\partial n = 1$ [31].

This allows one to pass from γ where $T = T(\gamma)$ to a neighboring curve $\tilde{\gamma}$ where $T = T(\tilde{\gamma})$ (see Fig. 1) and so on, thereby extending T to the whole of the space. The solution is therefore defined, up to an additive constant, by the choice of the initial curve γ. For example, the solution (4.2) can be obtained by choosing γ to be a straight line.

In practice, solutions of $(\partial_t T)^2 - (\partial_r T)^2 = 1$ can be obtained by changing the variables and then separating them [32]. Let us illustrate this by passing to the static coordinates (4.11), in which case the T-equation becomes

$$\frac{1}{\Sigma}\left(\frac{\partial T}{\partial \tau}\right)^2 - \frac{\Sigma}{1-\Sigma}\left(\frac{\partial T}{\partial \rho}\right)^2 = 1. \tag{5.2}$$

It is easy to see that $T(\tau,\rho)$ given by (4.13) fulfills this equation, but now we can obtain also other solutions, in particular those for which dT does not depend on time and the f-metric is static. The most general solution of this type is obtained by separating the variables,

$$T = \sqrt{1+q^2}\,\tau + \int \frac{\rho\, d\rho}{\Sigma}\sqrt{q^2+\rho^2}, \tag{5.3}$$

where q is an integration constant. If $q = 0$ then the solution becomes especially simple,

$$T = \tau + \int \frac{d\rho}{\Sigma} - \rho \equiv V - \rho, \tag{5.4}$$

and choosing V and ρ as coordinates, the two metrics become

$$ds_g^2 = \alpha^2\{-\Sigma\, dV^2 + 2dV d\rho + \rho^2 d\Omega^2\},$$
$$ds_f^2 = u^2\alpha^2\{-dV^2 + 2dV d\rho + \rho^2 d\Omega^2\}. \tag{5.5}$$

Solutions (5.3) are distinguished, since only for them the canonical Killing energy is time-independent.

6. Energy

We considered above the de Sitter solution (3.3), (3.14) expressed in several special coordinate systems. Let us now express it in arbitrary coordinates η, χ. The two metrics then read

$$ds_g^2 = \alpha^2\left\{-N^2 d\eta^2 + \frac{1}{\Delta^2}(d\chi + \beta\, d\eta)^2 + R^2\, d\Omega^2\right\},$$
$$ds_f^2 = \alpha^2 u^2\left\{-dT^2 + dR^2 + R^2\, d\Omega^2\right\} \tag{6.6}$$

where N, β, Δ, R, T depend on η, χ. Using this parametrization, one can directly analyse the field equation $G_{\mu\nu} = T_{\mu\nu}$ (without using the Gordon ansatz) and check [13,14] that they reduce to $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ with $\Lambda = P_0(u)$ and $P_1(u) = 0$ provided that $\alpha^2 = 3/\Lambda$ and

$$\left(\dot{T} - \beta T' + N\Delta R'\right)^2 - \left(\dot{R} - \beta R' + N\Delta T'\right)^2 - \left(\Delta(\dot{T}R' - \dot{R}T') + N\right)^2 = 0; \tag{6.7}$$

the dot and prime denoting, respectively, $\partial/\partial\eta$ and $\partial/\partial\chi$. This is the analogue of equation (3.20) expressed in arbitrary coordinates.

If the coordinates coincide with the coordinates of the ambient 5D Minkowski space used in (3.3), $\eta = t$, $\chi = r$, then one finds by comparing (6.6) with (3.3)

$$N = \frac{1}{\sqrt{1+\eta^2}}, \qquad R = \sqrt{1+\eta^2-\chi^2},$$
$$\Delta = NR, \qquad \beta = -\frac{\eta\chi}{1+\eta^2}. \tag{6.8}$$

Inserting this to (6.7) reduces the equation to (3.20). (Equation (6.7) can be avoided and T remains arbitrary if $b_2 + b_3 u = 0$ [8,35,36].)

The parametrization (6.6) can be used to compute the canonical energy. If the coordinates are chosen such that the unitary gauge condition is fulfilled, $\eta = T$, $\chi = R$, then the energy is given by expressions obtained in [33,34]. However, the energy can be computed also for an arbitrary choice of η, χ and for a non-trivial Stuckelberg field $T(\eta, \chi)$. This computation will be presented separately but its result is as follows: the energy on a hypersurface Σ_η of constant time η is

$$E[\eta, T] = u^2\alpha^4 P_2(u) \int R^2\,(\dot{T}R' - \dot{R}T')\,d\chi\,. \tag{6.9}$$

In this formula $T(\eta, \chi)$ is assumed to be a solution of (6.7) and the energy depends on choice of this solution. The energy depends also on choice of the spacelike hypersurface Σ_η over which the integration is carried out. Therefore, the energy $E[\eta, T]$ takes two arguments of which the first indicates the choice of the hypersurface and the second one refers to the solution chosen.

Let us go to the unitary gauge where $\eta = T$ and $\chi = R$. Then one will have the "unitary energy" evaluated on hypersurfaces Σ_T of constant T

$$E[T, T] = u^2\alpha^4 P_2(u) \int \chi^2 d\chi\,, \tag{6.10}$$

which depends neither on value of T nor on the functional form of $T(\eta, \chi)$. This may appear puzzling, but in fact this merely indicates that the unitary time is not always the best choice. Indeed, as different solutions differ from each other precisely by the form of $T(\eta, \chi)$, choosing T as time coordinate implies that every solution has its own time with its own hypersurfaces Σ_T and therefore with its own definition of the energy. It is then meaningless to compare energies of different solutions, since all of them are defined differently. Hence, the fact that the unitary energy (6.10) is the same for any $T(\eta, \chi)$ is probably not very relevant.

We therefore conclude that the time coordinate should be chosen differently, and the geometrically distinguished choice is the Killing time associated with the timelike de Sitter isometry. Choosing the static coordinates defined in (4.11) and setting $\eta = \tau$ and $\chi = \rho = R$ gives then the "Killing energy"

$$E[\tau, T] = u^2\alpha^4 P_2(u) \int \partial_\tau T\,\rho^2 d\rho \tag{6.11}$$

where $T(\tau, \rho)$ fulfills (5.2). However, we notice that, since for a generic solution $T(\tau, \rho)$ the derivative $\partial_\tau T$ depends on τ, the Killing energy is time-dependent. In particular, the energy is time-dependent for the $T = t$ solution expressed by (4.13).

One can wonder why the energy obtained from the same canonical Hamiltonian is conserved in the unitary gauge but becomes time-dependent when the Stuckelberg field T is non-trivial. The answer is that the energy depends on the choice of time. If the spacetime is foliated by hypersurfaces Σ_T of constants unitary time T then the energy is the same on any Σ_T. However, for other foliations it may become hypersurface dependent.

Since we adopt the viewpoint that the Killing time is physically distinguished, we wonder when the "Killing energy" (6.11) can be time-independent as the de Sitter space itself. The answer is that the energy will be time-independent if $\partial_\tau T$ is time independent, and then the timelike isometry is common for both metrics. However, all such solutions are given by (5.3), in which case

$$E = u^2\alpha^4 P_2(u)\sqrt{1+q^2} \int \rho^2 d\rho\,. \tag{6.12}$$

Depending on values of the theory parameters b_k, this energy can be positive, negative, or zero.

The actual value of the background energy is probably not so important, but it is important to know if it is minimal or not. We conjecture that the static solutions (5.3) correspond to the energy minima and are therefore stable. Therefore, they are candidates for describing the de Sitter ground state in the theory. To prove the conjecture will require to compute the energy for deformations of the de Sitter background, similarly to what was done for deformations of the Minkowski space [33,34]. We presently have partial results supporting our conjecture, but the corresponding analysis is complicated and will be presented separately.

7. Conclusions

We have shown that the dRGT theory admits infinitely many de Sitter vacua labeled by solutions of $(\partial_t T)^2 - (\partial_r T)^2 = 1$. The simplest solution $T = t$ is manifestly homogeneous and isotropic when written in the open slicing, but it is unstable. Therefore, one should study other solutions. One could worry that other solutions will not be homogeneous and isotropic, because their reference metric is inhomogeneous. However, as we have seen, this is not necessarily the case, as the reference metric can look inhomogeneous while sharing translational isometries with the physical metric.

The important issue is the number of common isometries of the two metrics. Since each of them describes a maximal symmetry space, each metric has ten isometries, some of which can be common, as for example the $SO(3)$ rotational isometries. The number of common isometries depends on choice of T, for example for $T = t$ this number is six, but the same could be true for other choices of T as well.

Requiring the timelike isometry to be common for both metrics reduces the set of solutions to a one-parameter family (5.3). These solutions are physically distinguished since only for them the Killing energy is time-independent. We conjecture that they are stable and describe the de Sitter ground state in the theory. Their stability will be demonstrated if one shows that the energy increases for any deformations of the de Sitter background. However, the corresponding analysis goes beyond the scope of the present paper and will be reported separately.

As a final remark, we notice that the unitary energy can also be used to study stability of the solutions. Although the unitary energy is not good to compare different de Sitter solutions, it can be used to compare a given background solution and its deformations. For example, for the $T = t$ solution the unitary time is just the global time of the ambient Minkowski space, and one should keep this gauge choice also for deformations of the background. One could then compute the energy of deformed configurations evaluated on the $t = const.$ hypersurface and compare it with the background value (6.10). This would provide a non-perturbative confirmation (or disproof) of the perturbative instability of the solution discovered in [15].

Acknowledgements

We are grateful to Gary Gibbons and Kei-ichi Maeda for constructive suggestions. The work of M.S.V. was partly supported by the Russian Government Program of Competitive Growth of the Kazan Federal University.

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