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Slowly oscillating wave solutions of a single species reaction–diffusion equation with delay

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Dedicated to Professor Anatoliy Samoilenko on the occasion of his 70th birthday

Abstract

We study positive bounded wave solutions $u(t,x) = \phi(\nu \cdot x + ct)$, $\phi(-\infty) = 0$, of equation $u_t(t,x) = \Delta u(t,x) - u(t,x) + g(u(t-h,x))$, $x \in \mathbb{R}^m$ $(*)$. This equation is assumed to have two non-negative equilibria: $u_1 \equiv 0$ and $u_2 \equiv \kappa > 0$. The birth function $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ is unimodal and differentiable at 0 and κ. Some results also require the feedback condition $(g(s) - \kappa)(s - \kappa) < 0$, with $s \in [g(\max g), \max g] \setminus \{\kappa\}$. If additionally $\phi(+\infty) = \kappa$, the above wave solution $u(t,x)$ is called a travelling front. We prove that every wave $\phi(\nu \cdot x + ct)$ is eventually monotone or slowly oscillating about κ. Furthermore, we indicate $c^* \in \mathbb{R}_+ \cup \{+\infty\}$ such that Eq. $(*)$ does not have any travelling front (neither monotone nor non-monotone) propagating at velocity $c > c^*$. Our results are based on a detailed geometric description of the wave profile ϕ. In particular, the monotonicity of its leading edge is established. We also discuss the uniqueness problem indicating a subclass $\mathcal{G}$ of '*asymmetric*' tent maps such that given $g \in \mathcal{G}$, there exists exactly one positive travelling front for each fixed admissible speed.

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Keywords: Time-delayed reaction–diffusion equation; Slow oscillations; Small solution; Semi-wavefront; Travelling front; Single species population model

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1. Introduction and main results

We study travelling wave solutions of the delayed reaction–diffusion equation

$$u_t(t,x) = \Delta u(t,x) - u(t,x) + g\big(u(t-h,x)\big), \quad u(t,x) \geqslant 0,\ x \in \mathbb{R}^m, \tag{1}$$

which has exactly two non-negative equilibria $u_1 \equiv 0$, $u_2 \equiv \kappa > 0$. The nonlinear g is called *the birth function* and assumed to be non-negative. Throughout the paper we assume that g satisfies the following unimodality condition:

(UM) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and has only one positive local extremum point $s = s_M$ (global maximum point). Furthermore, $g(0) = 0$, $g(\kappa) = \kappa$, and $g'(0)$ and $g'(\kappa)$ exist with $g'(0) > 1$.

(UM) implies that g is strictly increasing on $[0, s_M]$ and decreasing on $[s_M, +\infty)$. For example, (UM) is satisfied for the diffusive Nicholson's blowflies equation. In a biological context, u is the size of an adult population, so we will consider only positive bounded wave solutions $u(x,t) = \phi(\nu \cdot x + ct)$, $\|\nu\| = 1$, with the wave speed $c > 0$.

Before going further, let us fix some terminology. We say that the wave solution $u(x,t) = \phi(\nu \cdot x + ct)$, $\|\nu\| = 1$, is a wavefront (or a travelling front), if the profile function ϕ satisfies $\phi(-\infty) = 0$ and $\phi(+\infty) = \kappa$. In the sequel, it will be convenient to work with the scaled function $\varphi(s) = \phi(cs)$. If $\phi(\nu \cdot x + ct) = \varphi(c^{-1}\nu \cdot x + t)$ is a wavefront of Eq. (1), then φ is a positive heteroclinic solution of the delay differential equation

$$\epsilon\varphi''(t) - \varphi'(t) - \varphi(t) + g\big(\varphi(t-h)\big) = 0, \quad t \in \mathbb{R}, \tag{2}$$

where $\epsilon = c^{-2} > 0$. Note that ϕ may be non-monotone. Following [8], we call positive bounded solutions $\phi(\nu \cdot x + ct)$ of Eq. (1) satisfying $\phi(-\infty) = 0$ semi-wavefronts. We will also say that φ is the wavefront (or semi-wavefront) for Eq. (2).

If we take $h = 0$ in Eq. (1), we get a *monostable* reaction–diffusion equations without delay. The problem of existence of travelling fronts for this equation is quite well understood. In particular, for each such equation we can indicate a positive real number c_* such that, for every $c \geqslant c_*$, it has exactly one travelling front $u(x,t) = \phi(\nu \cdot x + ct)$, see [8, Theorems 8.3(ii) and 8.7]. To find c_* we can use one of the variational principles for the front speeds, e.g., see [4,8,10]. Furthermore, Eq. (1) does not have any travelling front propagating at the velocity $c < c_*$. The profile ϕ is necessarily strictly increasing function, e.g. see [8, Theorem 2.39].

However, the situation changes drastically if we take $h > 0$. In fact, current research seems a long way from proving similar results concerning the existence, uniqueness and geometric properties of wavefronts for delayed equation (1). This is despite the fact that the existence of travelling fronts for Eq. (1) has been studied extensively in recent years (e.g. see [6,7,9,15,16,21,22,24,25]) for some specific subclasses of birth functions. Clearly, most of the available information is for the so-called monotone case, where g is monotone on $[0, \kappa]$. However, little is known so far about the number of *positive* wavefronts (modulo translation) for an arbitrary fixed $c \geqslant c_*$ even for equations with monotone birth functions. Moreover, we do not know how to determine c_* for a monotone g which does not meet the sublinearity condition

$$g(s) \leqslant g'(0)s, \quad s \geqslant 0. \tag{3}$$

The situation when g is not monotone on $[0, \kappa]$ is much more complicated. For example, it is not clear whether there exists Eq. (1) which does not have any travelling front.

The main results of this paper answer some questions raised above:

Theorem 1 *(Monotonicity of the leading edge of semi-wavefronts). Consider semi-wavefront* $u(x,t) = \phi(\nu \cdot x + ct)$, $\|\nu\| = 1$, *of Eq.* (1). *Then there exist some* $\tau_3 \geqslant \tau_2 \geqslant \tau_1 \in \mathbb{R} \cup \{+\infty\}$ *such that* $\phi'(s) > 0$ *on* $(-\infty, \tau_1) \cup (\tau_2, \tau_3)$ *and* $\phi'(s) < 0$ *on* (τ_1, τ_2). *Furthermore,* τ_1 *is finite if and only if* $\phi(\tau_1) > \kappa$. *Similarly,* τ_2 *is finite if and only if* $\phi(\tau_2) < \kappa$.

It is worth to mention that $\liminf_{s \to +\infty} \phi(s) \geqslant d > 0$, where d can be chosen to be independent of c, ϕ. More precisely, consider the following assumption (which is weaker than (UM)):

(B) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and such that, for some $0 < \zeta_1 < \zeta_2$,
1. $g([\zeta_1, \zeta_2]) \subseteq [\zeta_1, \zeta_2]$ and $g([0, \zeta_1]) \subseteq [0, \zeta_2]$;
2. $\min_{s \in [\zeta_1, \zeta_2]} g(s) = g(\zeta_1)$;
3. $g(x) > x$ for $x \in (0, \zeta_1]$ and g is differentiable at 0, with $g'(0) > 1$;
4. in $[0, \zeta_2]$, the equation $g(x) = x$ has exactly two solutions, 0 and κ.

The following result was proved in [24, Lemma 4.3]:

Proposition 2. *Assume* (B) *and suppose that* $\sup_{s \geqslant 0} g(s) \leqslant \zeta_2$. *Let* $\varphi \not\equiv 0$ *be a non-negative solution of Eq.* (2). *Then* φ *has the following uniform permanence property*: $\zeta_1 \leqslant \liminf_{t \to +\infty} \varphi(t) \leqslant \limsup_{t \to +\infty} \varphi(t) \leqslant \zeta_2$.

To state other theorems, we will introduce the concept of slowly oscillating solutions of Eq. (2). Below we follow closely the definition of slow oscillation proposed in [20, Section 8] and [19]. Here it is adapted for the case when the birth function g satisfies the following feedback condition:

$$\bigl(g(s) - \kappa\bigr)(s - \kappa) < 0, \quad s \in \bigl[g(\max g), \max g\bigr] \setminus \{\kappa\}. \tag{4}$$

Definition 1. Let $\psi : [\theta, +\infty) \to \mathbb{R}$ be a continuous function. We say that ψ is oscillatory if there exist sequences $\{t_n\}_{n \geqslant 1}$ and $\{t'_n\}_{n \geqslant 1}$ such that $t_n, t'_n \to +\infty$ and $\psi(t_n) < 0 < \psi(t'_n)$, $n \geqslant 1$.

Definition 2. Set $\mathbb{K} = [-h, 0] \cup \{1\}$. For any $v \in C(\mathbb{K}) \setminus \{0\}$ we define the number of sign changes by

$$\mathrm{sc}(v) = \sup\bigl\{k \geqslant 1\colon \text{there are } t_0 < \cdots < t_k \text{ such that } v(t_{i-1})v(t_i) < 0 \text{ for } i \geqslant 1\bigr\}.$$

We set $\mathrm{sc}(v) = 0$ if $v(s) \geqslant 0$ or $v(s) \leqslant 0$ for $s \in \mathbb{K}$. If $\varphi : [a - h, +\infty) \to \mathbb{R}$ is a solution of Eq. (2), we set $(\bar{\varphi}_t)(s) = \varphi(t+s) - \kappa$ if $s \in [-h, 0]$, and $(\bar{\varphi}_t)(1) = \varphi'(t)$. We will say that $\varphi(t)$ is slowly oscillating about κ if $\varphi(t) - \kappa$ is oscillatory and for each $t \geqslant a$, we have either $\mathrm{sc}(\bar{\varphi}_t) = 1$ or $\mathrm{sc}(\bar{\varphi}_t) = 2$.

The critical speeds c_*, c^* are defined as follows:

Definition 3. (a) Suppose that $g'(\kappa) \leqslant 0$. Let $c^* \in (0, +\infty]$ be the largest extended real number such that the equation

$$(c^*)^{-2} z^2 - z - 1 + g'(\kappa)\exp(-zh) = 0$$

does not have roots in the half-plane $\{\Re z > 0\}$ other than a positive real root.

(b) $c_* > 0$ is the smallest real number such that the equation

$$(c_*)^{-2} z^2 - z - 1 + g'(0)\exp(-zh) = 0$$

has at least one real root in the half-plane $\{\Re z > 0\}$.

It should be noted here that Corollary 18 below guarantees that c^* is well defined. The definition of c_* is fairly standard.

Theorem 3 *(Semi-wavefronts are either monotone or slowly oscillating). Assume that g satisfies the feedback condition* (4) *and that $g'(\kappa) < 0$, $g'(0) > 1$. If $u(x,t) = \phi(\nu \cdot x + ct)$, $\|\nu\| = 1$, is a semi-wavefront to Eq.* (1)*, then $\varphi(s) = \phi(cs)$ is eventually either monotone or slowly oscillating around κ. Furthermore, if $c > c^*$ then the profile φ has to develop non-decaying slow oscillations around κ.*

It follows from [20] that these non-decaying slow oscillations are asymptotically periodic if $g : [g(\max g), \max g] \to \mathbb{R}_+$ is decreasing.

Corollary 4 *(Admissible wavefront speeds and non-existence of fronts). If all the conditions of Theorem* 3 *are satisfied then Eq.* (1) *does not have any travelling front* (*neither monotone nor non-monotone*) *propagating at the velocity $c > c^*$ or $c < c_*$. In consequence, if c^* is less than c_*, then Eq.* (1) *does not possess any travelling front.*

If we denote by $\mathcal{A}$ the set of all admissible wavefront speeds for Eq. (1), then Corollary 4 says that $\mathcal{A} \subseteq [c_*, c^*]$ once the conditions of Theorem 3 have been met. Here we set $[c_*, c^*] = \emptyset$ if $c_* > c^*$. Note that, for an arbitrary g, it may happen that $\mathcal{A} \neq [c_*, c^*]$, e.g. see [10]. An interesting open problem is to find conditions on the birth function guaranteeing that $\mathcal{A} = [c_*, c^*]$. It was conjectured in [24] that condition (3) and the negativity of the Schwarz derivative of g are sufficient to have $\mathcal{A} = [c_*, c^*]$. Theorem 1.1 in [24] supports this conjecture and agrees with Corollary 4 by establishing that $\mathcal{A} \supseteq [c_*, c^\#]$ for some $c^\# \in [c_*, c^*]$ "reasonable close" to c^*. Observe here that the sublinearity condition (3) implies that c_* is the minimal speed of propagation of semi-wavefronts, see e.g. [24]. The importance of the negativity of the Schwarz derivative of g in our studies was showed in [14,24].

Finally, we discuss the uniqueness of positive wavefront (up to translations) for a given admissible speed c. Very few theoretical studies are devoted to the uniqueness problem for Eq. (1) and its non-local extensions. To the best of our knowledge, the first uniqueness result for a non-local version of Eq. (1) is due to Thieme and Zhao [23], who extended an integral-equations approach to scalar non-local reaction–diffusion equations with delay. Besides this work, it appears that uniqueness has been established for small delays in [1] and for sufficiently fast speeds in [2]. Here, we indicate a family $\mathcal{G}$ of unimodal piece-wise linear functions (i.e. tent maps) for which the problem of existence and uniqueness of travelling fronts can be solved in the closed form. The elements of $\mathcal{G}$ are defined as follows:

Let $d > 1$, $\theta > 0$, $a \in [-1, 1)$ be given and satisfy $a\theta + b = d\theta$, $a\kappa + b = \kappa$ for some b, κ. Then we have $b > 0$, $\kappa > \theta$. It is easy to verify that the piece-wise linear function

$$g(s) = g(s, a, d, \theta) := \begin{cases} ds, & \text{for } s \in [0, \theta], \\ as + b, & \text{if } s \in [\theta, \max\{\kappa, d\theta\}], \end{cases}$$

is continuous and satisfies $g(0) = 0$, $g(\kappa) = \kappa$.

Theorem 5 *(On the uniqueness of the travelling front). Set* $\mathcal{G} = \{g(s, a, d, \theta)\colon a \in [-1, 1),\ d > 1,\ \theta > 0\}$. *For every* $g \in \mathcal{G}$, *there exists exactly one wavefront for each fixed admissible speed.*

Since '*asymmetric*' tent maps mimic the main features of general unimodal birth functions, we hope that Theorem 5 can be extended to all unimodal smooth nonlinearities g. This would extend the uniqueness result from [8, Theorem 8.7] to equations with delay.

The structure of this paper is as follows. In Section 2, we prove the monotonicity of the leading edge of semi-wavefronts. We also establish that $\mathrm{sc}(\bar{\varphi}_{t'}) \in \{1, 2\}$ for an initial segment $\varphi_{t'}$ of the semi-wavefront considered within some g-invariant domain. In the third section, we study the dependence of roots of the characteristic equation at the positive steady state of Eq. (2) on the parameter $\epsilon = c^{-2}$. In Section 4, under the feedback condition, we establish that semi-wavefronts are (eventually) either monotone or slowly oscillating. This section also contains the most difficult part of the proof of Theorem 3: *if* $c > c^*$ *then the profile* φ *has to develop non-decaying slow oscillations around* κ. Finally, in Section 5 we show how the problem of existence travelling wavefronts can be solved in the closed form for the birth functions in $\mathcal{G}$. As a consequence, we establish that given $g \in \mathcal{G}$, there exists exactly one wavefront for each fixed admissible speed.

2. Monotonicity of the leading edge of semi-wavefronts

For given $\epsilon > 0$ we will denote by $\lambda < 0 < \mu$ the roots of $\epsilon z^2 - z - 1 = 0$. Also, we set $\epsilon' := \epsilon(\mu - \lambda)$. In this section, always assuming (UM), we study the monotonicity properties of semi-wavefronts to the equation

$$\epsilon\varphi''(t) - \varphi'(t) - \varphi(t) + g\big(\varphi(t-h)\big) = 0, \quad t \in \mathbb{R}. \tag{5}$$

Lemma 6. *Let* φ *be a semi-wavefront to Eq.* (5). *Then* $\epsilon \leqslant c_*^{-2}$ *and* $\varphi'(t) > 0$ *on some maximal interval* $(-\infty, \sigma)$.

Proof. Looking for a contradiction, we assume that there exists a sequence $t_n \to -\infty$ such that $\varphi'(t_n) = 0$ for every n. Set $\xi(t) = g(\varphi(t-h))/\varphi(t-h)$, $y_n(t) = \varphi(t + t_n)/\varphi(t_n)$. Since $\varphi(-\infty) = 0$, without the loss of generality we can suppose that $\varphi(t) \leqslant \varphi(t_n)$, $\xi(s + t_n) < 2g'(0)$ for all $t \leqslant t_n$, $s \leqslant 0$. It is clear that $y_n(0) = 1 = \max_{t \leqslant 0} y_n(t)$, $y_n'(0) = 0$, and that $y_n(t) > 0$ satisfies

$$\epsilon y''(t) - y'(t) - y(t) + \xi(t + t_n)y(t-h) = 0. \tag{6}$$

A partial integration of (6) yields

$$y_n'(t) = \frac{1}{\epsilon}\int_0^t e^{(t-s)/\epsilon}\big(y_n(s) - \xi(s+t_n)y_n(s-h)\big)\,ds, \tag{7}$$

from which we conclude the uniform boundedness of the sequence $\{y_n'(t)\}$:

$$\left|y_n'(t)\right| \leqslant 1 + 2g'(0), \quad t \leqslant 0,\ n \in \mathbb{N}. \tag{8}$$

Together with $0 < y_n(t) \leqslant 1$, $t \leqslant 0$, inequality (8) implies the pre-compactness of $\{y_n(t),\ n \in \mathbb{N}\}$ in the compact open topology of $C(\mathbb{R}_-, \mathbb{R}_+)$. Therefore, by the Arzelà–Ascoli theorem combined with the diagonal method, there is a subsequence $y_{n_j}(t)$ converging uniformly on bounded subsets of $\mathbb{R}_-$ to a continuous function $y(t), t \in \mathbb{R}_-$. Integrating (7) between t and 0 and then taking the limit as $n_j \to \infty$ in the obtained expression, we establish that $y(t)$, $t \leqslant 0$, satisfies

$$\epsilon y''(t) - y'(t) - y(t) + g'(0)y(t-h) = 0. \tag{9}$$

Additionally, $y'(0) = 0$ and $0 \leqslant y(t) \leqslant 1 = y(0)$, $t \leqslant 0$. Now, if $y(s) = 0$ for some $s < 0$ then $y'(s) = 0$, $y''(s) \geqslant 0$, so that (9) yields $y''(s) = 0$ and $y(s-h) = 0$. Hence, y solves the boundary value problem $y(s-h) = 0$, $y'(s) = 0$ for (9). As a consequence, after applying the variation of constants formula to Eq. (9) (or using Lemma 4.2 in [24]), we get that

$$0 = y(s) = \frac{g'(0)}{\epsilon(\mu e^{-\lambda h} - \lambda e^{-\mu h})}\int_{s-h}^{s}\big(e^{\lambda(s-h-u)} - e^{\mu(s-h-u)}\big)y(u-h)\,du.$$

This implies immediately that $y(t) = 0$ for all $t \in [s-2h, s-h]$. Therefore $y(t) \equiv 0$ for all $t \geqslant s - 2h$ which is not possible because of $y(0) = 1$. In this way, we have proved that Eq. (9) has a bounded *positive* solution on $\mathbb{R}_-$. As it was established in [12, Lemma A.1], this solution does not decay superexponentially. By [18, Proposition 7.2] (see also [12, Proposition 2.2]), this implies the existence of $b \geqslant 0$, $\delta > 0$ and a nontrivial eigensolution $y_1(t)$ of Eq. (9) on the generalized eigenspace associated with the (nonempty) set Λ of eigenvalues with $\Re\lambda = b$, such that $y(t) = y_1(t) + O(\exp((b+\delta)t))$, $t \to -\infty$. Now, Definition 3(b) assures that, for every $\epsilon > c_*^{-2}$ there are no eigensolutions of (9) associated to non-negative eigenvalues: hence $\Im\lambda \neq 0$ for all $\lambda \in \Lambda$. From [12, Lemma 2.3], we conclude that $y(t)$ is oscillatory, a contradiction.

Hence, $\epsilon \leqslant c_*^{-2}$ and thus Eq. (9) has exactly two real positive eigenvalues (counting multiplicity) $0 < \lambda_2(\epsilon) \leqslant \lambda_1(\epsilon)$ while other eigenvalues satisfy $\Re\lambda_j(\epsilon) < \lambda_2(\epsilon)$, e.g. see [24, Lemma 2.3]. Therefore, for every $b > \lambda_1(\epsilon)$, it holds that

$$y(t) = w(t) + \exp(bt)o(1), \quad t \to -\infty,$$

where $w(t)$ is a finite sum of eigensolutions of (9) associated to the eigenvalues λ_j with $\Re\lambda_j \geqslant 0$. Furthermore, the positivity of y implies that $w(t)$ incorporates only those eigensolutions of (9) that are associated to the *real* eigenvalues, i.e.

$$y(t) = \begin{cases} A_2 \exp(\lambda_2(\epsilon)t) + A_1 \exp(\lambda_1(\epsilon)t) + \zeta(t), & \text{if } \lambda_2(\epsilon) < \lambda_1(\epsilon), \\ \exp(\lambda_1(\epsilon)t)(A_2 + A_1 t) + \zeta(t), & \text{if } \lambda_2(\epsilon) = \lambda_1(\epsilon), \end{cases}$$

where ζ is a *small* solution of (9) at $-\infty$ in the sense that $\lim_{t\to-\infty} \zeta(t)\exp(bt) = 0$ for every $b \in \mathbb{R}$.

We claim that $\zeta(t) = 0$ for all $t \leqslant 0$. Indeed, suppose that $\zeta(q) \neq 0$ for some $q \leqslant 0$ and consider another small solution $v(t) = \zeta(q+t)$, $t \leqslant 0$, $v(0) \neq 0$, of (9). Multiplying this equation by $\exp(-zt)$ and then integrating obtained expression on $(-\infty, 0]$, we get that

$$\hat{v}(z) = \Phi(z)/\Delta(z),$$

where

$$\hat{v}(z) = \int_{-\infty}^{0} e^{-zs} v(s)\, ds, \qquad \Delta(z) = \epsilon z^2 - z - 1 + g'(0)e^{-zh},$$

$$\Phi(z) = \epsilon(zv(0) + v'(0)) - v(0) - g'(0) \int_0^h e^{-zs} v(s-h)\, ds.$$

Since v is a small solution, we find that $\hat{v}$ is an entire function. Furthermore, since $g'(0)v(0) \neq 0$ the entire functions $\Phi(z)$, $\Delta(z)$ are of the same exponential type h (see [5, Theorem 2.1, p. 137]). On the other hand, $\Phi(z)$, $\Delta(z)$ are polynomially bounded in the closed right half-plane. Thus, by [5, Corollary 2.3, p. 138], we get that $\hat{v}(z)$ is an entire function of exponential type 0. It is easy to see that $z\hat{v}(z)$ is uniformly bounded in $\Re z \geqslant 0$. Hence, an application of the Paley–Wiener theorem (see [5, Theorem 2.1]) yields $\hat{v}(z) = 0$, $z \in \mathbb{C}$. We can use now an inversion formula (e.g. see [11, p. 19]) to obtain that $v(t) = 0$ for all $t \leqslant 0$ contradicting to $v(0) \neq 0$.

In consequence, $\lim_{j\to\infty} y_{n_j}(t) = y(t)$, $t \leqslant 0$, where the convergence is uniform on each bounded subset of $\mathbb{R}_-$ and

$$y(t) = \begin{cases} A_2 \exp(\lambda_2(\epsilon)t) + A_1 \exp(\lambda_1(\epsilon)t), & \text{if } \lambda_2(\epsilon) < \lambda_1(\epsilon), \\ \exp(\lambda_1(\epsilon)t)(A_2 + A_1 t), & \text{if } \lambda_2(\epsilon) = \lambda_1(\epsilon). \end{cases} \tag{10}$$

Next, observe that $y_{n_j}(t)$ satisfies $y_{n_j}(0) = 1$, $y'_{n_j}(0) = 0$ and

$$y_{n_j}(t) = \frac{\mu e^{\lambda t} - \lambda e^{\mu t}}{\mu - \lambda} + \frac{1}{\epsilon'} \int_0^t \left(e^{\lambda(t-s)} - e^{\mu(t-s)}\right) \xi(s + t_{n_j}) y_{n_j}(s-h)\, ds \tag{11}$$

for all $t \in \mathbb{R}$. Taking limit, as $j \to \infty$, in (11) with $t \in [0, h]$, we see that $y_{n_j}(t)$ converges to $y(t)$ uniformly on $[0, h]$. Repeating the above procedure consecutively on the intervals

$[0,2h], [0,3h], \ldots,$ we establish that, in fact, $y_{n_j}(t)$ converges to $y(t)$ uniformly on every bounded subset of $\mathbb{R}$. Therefore $y(t)$, $t \in \mathbb{R}$, given by (10) must take only the non-negative values. It is easy to see that this requirement is incompatible with $y(0)=1$, $y'(0)=0$. □

Remark 7. Under additional conditions of C^2-smoothness of g at 0 and the hyperbolicity of Eq. (9), Lemma 6 was proved in [24, Remark 5.5 and Theorem 5.4].

Fix some semi-wavefront φ of (5) and set $\Gamma(t) := g(\varphi(t-h))$. Applying the variation of constants formula to (5), we obtain that

$$\varphi(t) = A'e^{\lambda t} + B'e^{\mu t} + \frac{1}{\epsilon'}\left\{\int_a^t e^{\lambda(t-s)}\Gamma(s)\,ds + \int_t^b e^{\mu(t-s)}\Gamma(s)\,ds\right\}. \tag{12}$$

Suppose for a moment that Γ is of bounded variation on $[a,b]$. Differentiating (12) and then integrating by parts Riemann–Stieltjes integrals [3, Theorem 7.6], we find that, for some $A, B \in \mathbb{R}$, the derivative $z(t) = \varphi'(t)$, $t \in [a,b]$, satisfies

$$z(t) = Ae^{\lambda t} + Be^{\mu t} + \frac{1}{\epsilon'}\left\{\int_a^t e^{\lambda(t-s)}\,d\Gamma(s) + \int_t^b e^{\mu(t-s)}\,d\Gamma(s)\right\}. \tag{13}$$

Lemma 8. *If z meets the boundary conditions $z(a) = z_0$, $z(0) = 0$, then*

$$\begin{aligned} z(t) &= \frac{e^{\lambda t} - e^{\mu t}}{e^{\lambda a} - e^{\mu a}}\left\{z_0 + \frac{1}{\epsilon'}\int_a^t \left(e^{\lambda(a-u)} - e^{\mu(a-u)}\right)d\Gamma(u)\right\} \\ &\quad + \frac{e^{\mu(t-a)} - e^{\lambda(t-a)}}{\epsilon'}\int_t^0 \frac{e^{-\mu u} - e^{-\lambda u}}{e^{-\mu a} - e^{-\lambda a}}\,d\Gamma(u), \end{aligned} \tag{14}$$

$$z'(0) = \frac{\lambda - \mu}{e^{\lambda a} - e^{\mu a}}\left\{z_0 + \frac{1}{\epsilon'}\int_a^0 \left(e^{\lambda(a-u)} - e^{\mu(a-u)}\right)d\Gamma(u)\right\},$$

$$z'(a) = \frac{\lambda e^{\lambda a} - \mu e^{\mu a}}{e^{\lambda a} - e^{\mu a}}z_0 + \frac{\mu - \lambda}{\epsilon'}\int_a^0 \frac{e^{-\mu u} - e^{-\lambda u}}{e^{-\mu a} - e^{-\lambda a}}\,d\Gamma(u).$$

Proof. Formula (14) follows from (13) after taking into consideration the boundary conditions. The representations for $z'(0)$, $z'(a)$ can be obtained in the following way: first, we integrate by parts both Riemann–Stieltjes integrals in (14). Then we find $z'(t)$ differentiating the obtained expression with respect to t. To get the above formulae for $z'(0)$, $z'(a)$, we need once more to integrate by parts. Observe here that, in general, we cannot differentiate Riemann–Stieltjes integrals in (14). □

Remark 9 *(Critical points of φ are isolated).* (1) Lemma 6 does not allow to have $\Gamma'(t)=0$ on any interval (p,q). Indeed, otherwise $\Gamma(t)\equiv \text{const}$, $t\in(p,q)$, and thus (UM) guarantees that $\varphi(t)\equiv\text{const}$, $t\in(p-h,q-h)$. Next, by considering Eq. (5), we get that $\Gamma'(t)=0$ on the interval $(p-h,q-h)$. Therefore, $\varphi(t)\equiv\text{const}$, $t\in(p-2h,q-2h)$. By applying repeatedly the above argument, we prove that $\varphi'(t)=0$ for all $t\in\bigcup_{j\geqslant 1}(p-jh,q-jh)$, in contradiction with Lemma 6.

(2) Another consequence of Lemmas 6, 8 is that the closed set $K=\{s\colon\ \varphi'(s)=0\}$ does not have finite limit points. Indeed, let $s_1\in\mathbb{R}$ be the first limit point of K. Since function $g(\varphi(t-h))$ is strictly monotone in both small one-sided neighborhoods $\mathcal{O}_l,\mathcal{O}_r$ of s_1, we see that $\Gamma(t)$ is of bounded variation on $\mathcal{O}_l\cup\mathcal{O}_r$. In consequence, Lemma 8 (where we assume that $s_1:=0$ and that $a\neq 0$ is some critical point of φ sufficiently close to $s_1=0$) can be used near s_1 to find that $\varphi''(s_1)\neq 0$. Therefore s_1 must be isolated in K.

Remark 9 implies that, for a semi-wavefront $\varphi(t)$, function $\Gamma(t)=g(\varphi(t-h))$ is piece-wise monotone, with finite number of local extrema on every compact subinterval of $\mathbb{R}$. In this way, Γ is locally of bounded variation, this justifies the use of the Riemann–Stieltjes integration in Lemmas 8, 10.

Lemma 10. *If $z(t)=\varphi'(t)$ satisfies $z(-\infty)=0$, $z(0)=0$ then*

$$z(t)=\frac{1}{\epsilon'}\left\{\left(e^{\lambda t}-e^{\mu t}\right)\int_{-\infty}^{t}e^{-\lambda s}\,d\Gamma(s)+e^{\mu t}\int_{t}^{0}\left(e^{-\mu s}-e^{-\lambda s}\right)d\Gamma(s)\right\}$$

$$=\frac{1}{\epsilon'}\left\{\left(e^{\lambda t}-e^{\mu t}\right)\int_{-\infty}^{0}e^{-\lambda s}\,d\Gamma(s)+\int_{0}^{t}\left(e^{\lambda(t-s)}-e^{\mu(t-s)}\right)d\Gamma(s)\right\}. \tag{15}$$

Proof. Formula (15) follows from (13) after taking into consideration the boundary conditions. To show the convergence of the improper Riemann–Stieltjes integrals, it suffices to integrate them by parts. □

Theorem 11. *Let φ be a semi-wavefront to Eq.* (5). *If $\tau\in\mathbb{R}$ is the leftmost point where $\varphi(\tau)=\kappa$ then $\varphi'(t)>0$, $t\in(-\infty,\tau]$.*

Proof. Take σ as in Lemma 6. Since $\sigma=+\infty$ implies that $\varphi(+\infty)=\kappa$ and $\varphi(t)<\kappa$, $t\in\mathbb{R}$, we may assume that $\sigma=0$ and $z(0)=\varphi'(0)=0$. Thus $z(t)=\varphi'(t)>0$ for all $t<0$. Next, arguing as in (7), (8), we find that $\varphi'(t)\leqslant\varphi(0)(1+2g'(0))$ for $t\leqslant 0$. Due to (5), this yields the uniform boundedness of $|\varphi''(t)|$ on $\mathbb{R}_-$. Therefore $\varphi'(t)$ is uniformly continuous on $\mathbb{R}_-$. An application of the Barbalat lemma (e.g. see [25, Lemma 2.3]) gives $\varphi'(-\infty)=0$.

In order to prove Theorem 11, it is sufficient to show that the inequality $\varphi(0)\leqslant\kappa$ cannot hold. Below, we consider two possible mutual positions of the points $\varphi(0)$ and s_M.

First, we consider the case when $\varphi(0)\leqslant s_M$ and $\varphi(0)\leqslant\kappa$. Then $\Gamma(t)=g(\varphi(t-h))$ is strictly increasing on $(-\infty,h)$. Since $z(t)=\varphi'(t)$ satisfies boundary conditions $z(-\infty)=0$, $z(0)=0$, we get from (15) that $z(t)<0$ for all $t\in(0,h]$. Thus $z(t)<0$ on some maximal interval $(0,\sigma_1)$. Note that σ_1 must be a finite real number since otherwise $\varphi'(t)<0$ on $(0,+\infty)$ implying $\varphi(+\infty)=0$. However, this contradicts the uniform persistence of semi-wavefronts established in Proposition 2. In consequence, $\sigma_1 > h$ is finite so that $\varphi'(\sigma_1) = z(\sigma_1) = 0$, $\varphi''(\sigma_1) \geqslant 0$ and $\varphi(\sigma_1) < \varphi(\sigma_1 - h)$. On the other hand, we see that (5) implies

$$\epsilon\varphi''(\sigma_1) - \varphi(\sigma_1) + g\big(\varphi(\sigma_1 - h)\big) = 0,$$

from which we obtain $\kappa > \varphi(\sigma_1 - h) > \varphi(\sigma_1) \geqslant g(\varphi(\sigma_1 - h))$, a contradiction.

Second, we suppose that $s_M < \kappa$ and $\varphi(0) \in (s_M, \kappa]$. Then $\varphi(t_*) = s_M$ for a unique $t_* < 0$.

Subcase I. If $t_* + h > 0$, then $\Gamma(t) = g(\varphi(t - h))$ is strictly increasing on $(-\infty, t_* + h]$ and we can use Lemma 10 to find that $z(t) < 0$, $t \in (0, t_* + h]$. Moreover, $\varphi''(0) = z'(0) < 0$ in view of Lemma 8 (where we take $a < 0$ so that $z_0 > 0$). Therefore, if $\varphi(0) \leqslant \kappa$ and if $\sigma_2 > 0$ denotes the leftmost positive point where $\varphi'(\sigma_2) = 0$, then $\sigma_2 > t_* + h$, $\varphi''(\sigma_2) \geqslant 0$ and $\varphi(\sigma_2) < \kappa$.

Subcase II. Now, assume that $t_* + h \leqslant 0$. Then $\varphi''(0) < 0$, since $\varphi''(0) = 0$ implies a contradiction: $\kappa \geqslant \varphi(0) = g(\varphi(-h)) > \kappa$. Suppose that $\varphi'(a) = 0$ for some $a \in (0, h]$. Since $\Gamma(t) = g(\varphi(t - h))$, $t \in [0, a]$, is strictly decreasing, an application of Lemma 8 yields $\varphi'(t) = z(t) < 0$, $t \in (0, a)$, and $\varphi''(a) = z'(a) > 0$. Hence, we can find at most one critical point $a \in (0, h]$. In any case, we see that if $t_* + h \leqslant 0$ then $\varphi'(t) < 0$ on $(0, a)$.

The above considerations show that if $\sigma_2 > 0$ denotes the leftmost positive point where $\varphi'(\sigma_2) = 0$, then $\sigma_2 > t_* + h$ and $\varphi''(\sigma_2) \geqslant 0$, $\varphi(\sigma_2) < \kappa$.

Next, let us suppose for a moment that $\varphi(\sigma_2) < s_M$. Then $\sigma_2 > t_* + h$ implies that $\varphi(\sigma_2) < \varphi(\sigma_2 - h) \leqslant \kappa$, a contradiction in view of

$$\varphi(\sigma_2) = \epsilon\varphi''(\sigma_2) + g\big(\varphi(\sigma_2 - h)\big) \geqslant g\big(\varphi(\sigma_2 - h)\big) > \varphi(\sigma_2 - h).$$

Therefore we have to suppose that $\varphi(\sigma_2) \geqslant s_M$. But then $\sigma_2 > t_* + h$ implies that $\kappa \geqslant \varphi(\sigma_2 - h) \geqslant s_M$ so that $\varphi(\sigma_2) \geqslant g(\varphi(\sigma_2 - h)) \geqslant \kappa$. This is again a contradiction.

The above said shows that $\varphi(t)$ is strictly increasing with $\varphi'(t) > 0$, at least until its first intersection with the positive equilibrium κ. □

Arguments used in the proof of Theorem 11 allow us to establish the strict monotonicity of *all* semi-wavefronts of Eq. (5) once g is monotone on $[0, \kappa]$:

Corollary 12. *Assume that continuous $g : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing on $[0, \kappa]$, there exists $g'(0) > 1$ and the equation $g(s) - s = 0$ has only two roots, 0 and κ. Then every semi-wavefront φ of Eq. (5) in fact is a travelling front. Moreover, $\varphi'(t) > 0$ for all $t \in \mathbb{R}$.*

Proof. We have that $s_M \geqslant \kappa$ while $\varphi'(t) > 0$ on some maximal semi-infinite interval $(-\infty, \sigma)$. If $\sigma = +\infty$, Corollary 12 is proved. If σ is finite, then Theorem 11 says that $\varphi(\sigma) > \kappa$, $\varphi'(\sigma) = 0$. To simplify the notations we put $\sigma := 0$. Therefore, by Lemma 8 (where we take $a < \sigma = 0$ so that $z_0 > 0$), we obtain that $\varphi''(\sigma) < 0$. This leads to the following contradiction:

$$\varphi(\sigma) = \epsilon\varphi''(\sigma) + g\big(\varphi(\sigma - h)\big) < g\big(\varphi(\sigma - h)\big) \leqslant \max\big\{\kappa, \varphi(\sigma - h)\big\}. \qquad \square$$

Theorem 13. *Let φ be a non-monotone semi-wavefront to Eq. (5). Then there exist $\tau_3 \geqslant \tau_2 > \tau_1$, $\tau_1 \in \mathbb{R}$, $\tau_2, \tau_3 \in \mathbb{R} \cup \{+\infty\}$, such that $\varphi'(t) > 0$ on $(-\infty, \tau_1) \cup (\tau_2, \tau_3)$, $\varphi(\tau_1) > \kappa$, and $\varphi'(t) < 0$ on (τ_1, τ_2). If τ_2 is finite then $\varphi(\tau_2) < \kappa < \varphi(\tau_2 - h)$ and $\tau_3 > \tau_2$. Finally, if $\tau_2 \in (\tau_1, \tau_1 + h]$ then $\varphi''(\tau_2) > 0$ and $\varphi'(t) > 0$ on $(\tau_2, \tau_1 + h]$.*

It is clear that Theorem 1 is essentially same as Theorem 13. Observe also that if $\tau_2 = +\infty$, then $\varphi(+\infty) = \kappa$ and wavefront φ can have only one extremum (global maximum) at τ_1. However, we do not know whether this can happen under our assumption (UM).

Proof of Theorem 13. From Corollary 12, we know that $s_M < \kappa$. Set $\tau_1 = \sup\{t\colon \varphi'(s) > 0,\ s \in (-\infty, t)\}$. This number is finite since φ is not monotone. Theorem 11 implies that $\varphi(\tau_1) > \kappa$, $\varphi'(\tau_1) = 0$. Next, let t_* be the unique point on $(-\infty, \tau_1)$ where $\varphi(t_*) = s_M$. We will consider several cases depending on possible mutual positions of the points $t_* + h$ and τ_1. In the sequel, to simplify the notations we set $\tau_1 = 0$.

Case A. First, suppose that $\varphi'(t) < 0$ on $(0, h]$ and set $\tau_2 = \sup\{t\colon \varphi'(s) < 0,\ s \in (0, t)\} > h$. When τ_2 is finite then $\varphi'(\tau_2) = 0$, $\varphi''(\tau_2) \geqslant 0$, and therefore $\varphi(\tau_2) \geqslant g(\varphi(\tau_2 - h))$. Since $\varphi(\tau_2) < \varphi(\tau_2 - h)$, this implies that $\kappa < \varphi(\tau_2 - h)$.

We *claim* that $\varphi'(t) > 0$ in some right neighborhood of $\tau_2 \in \mathbb{R}$. Indeed, otherwise $\varphi''(\tau_2) = 0$ and therefore $\varphi(\tau_2) = g(\varphi(\tau_2 - h))$. Since $\varphi(\tau_2) < \varphi(\tau_2 - h)$, this implies that $\varphi(\tau_2) < \kappa < \varphi(\tau_2 - h)$. Now invoking Proposition 2, we establish the existence of $\tau' > \tau_2$ such that $\varphi'(t) < 0$ on (τ_2, τ') and $\varphi'(\tau') = 0$. Observe that $\tau' - \tau_2 < h$. (Indeed, otherwise we have

$$\kappa > \varphi(\tau_2) \geqslant \varphi(\tau' - h) > \varphi(\tau') = \epsilon\varphi''(\tau') + g\bigl(\varphi(\tau' - h)\bigr) \geqslant g\bigl(\varphi(\tau' - h)\bigr),$$

a contradiction ($\kappa > \varphi(\tau' - h) > g(\varphi(\tau' - h))$ is incompatible with (UM)).) Furthermore, $\kappa < \varphi(\tau' - h)$ because of

$$\varphi(\tau' - h) > \varphi(\tau') = \epsilon\varphi''(\tau') + g\bigl(\varphi(\tau')\bigr) \geqslant g\bigl(\varphi(\tau' - h)\bigr).$$

Thus $\Gamma(t) = g(\varphi(t - h))$ strictly increases on (τ_2, τ'). Applying Lemma 8, we get a contradiction: $\varphi''(\tau_2) > 0$. The above claim is proved.

Now, since $\varphi'(t) > 0$ in some right neighborhood of $\tau_2 \in \mathbb{R}$, we can consider $b = \sup\{t\colon \varphi'(s) > 0,\ s \in (\tau_2, t)\} > \tau_2$. If $\varphi(\tau_2) \geqslant \kappa$ then b is finite, $b > h$ and $\varphi(b - h) \geqslant \kappa$, $\varphi'(b) = 0$, $\varphi''(b) \leqslant 0$. This gives $\kappa < \varphi(b) \leqslant g(\varphi(b - h)) \leqslant \kappa$, a contradiction. Hence, $\varphi(\tau_2) < \kappa$.

Case B. Next, assume that $t_* + h \leqslant 0$ and $\varphi'(a) = 0$ for some $a \in (0, h]$. Since $\Gamma(t) = g(\varphi(t - h))$, $t \in [0, a]$, is strictly decreasing, an application of Lemma 8 yields $\varphi'(t) = z(t) < 0$, $t \in (0, a)$, and $\varphi''(a) = z'(a) > 0$. Hence, we can find at most one critical point $a \in (0, h]$. Clearly, if such a point exists, we have that $\varphi'(t) < 0$ for all $t \in (0, a)$ and $\varphi'(t) > 0$, $t \in (a, h]$. The above said proves all conclusions of Theorem 13 (with $\tau_2 = a$) but the inequalities $\varphi(\tau_2) < \kappa$ and $\varphi(\tau_2 - h) > \kappa$.

Now, to see that the first inequality holds, let us consider $b := \sup\{t\colon \varphi'(s) > 0,\ s \in (\tau_2, t)\}$. If $\varphi(\tau_2) \geqslant \kappa$ then b is finite, $b > h$ and $\varphi(b - h) \geqslant \kappa$, $\varphi'(b) = 0$, $\varphi''(b) \leqslant 0$. This gives $\kappa < \varphi(b) \leqslant g(\varphi(b - h)) \leqslant \kappa$, a contradiction.

To prove the second inequality, we observe that $\varphi(\tau_2 - h) \geqslant s_M$. Then the relation $\varphi(\tau_2 - h) > \kappa$ follows from $\kappa > \varphi(\tau_2) \geqslant g(\varphi(\tau_2 - h))$.

Case C. Now, let us suppose, for an instance, that $t_* + h \leqslant 0$ and $\varphi'(t) > 0$ on the interval $(0, h]$. Set $c := \sup\{t\colon \varphi'(s) > 0,\ s \in (0, t)\}$. We see that c is finite, $\varphi'(c) = 0$, $\varphi''(c) \leqslant 0$, and $\Gamma(t) = g(\varphi(t - h))$ strictly decreases on $(0, c)$. So once more applying Lemma 8, we get a contradiction: $\varphi''(c) = z'(c) > 0$.

Note here that the situation when $t_* + h \leqslant 0$ and $\varphi'(t) < 0$, $t \in (0, h]$, was already considered in *Case* A.

Case D. Assume now that $t_* + h > 0$. Then $\Gamma(t) = g(\varphi(t-h))$ is strictly increasing on $(-\infty, t_* + h]$ and we can use Lemma 10 to find that $z(t) = \varphi'(t) < 0$, $t \in (0, t_* + h]$. Let us suppose for a moment that there exists $a \in (t_* + h, h]$ such that $\varphi'(a) = 0$. Then, by using the second formula of Lemma 8 on $[t_* + h, a]$, we obtain that $\varphi''(a) > 0$ (and thus we may set $\tau_2 = a$). This means that there is at most one critical point of φ on $(t_* + h, h]$ (and, in consequence, on $(0, h]$). Now, the proof of the inequality $\varphi(a) = \varphi(\tau_2) < \kappa$ is the same as in *Case* B. To prove that $\varphi(a-h) > \kappa$, it is sufficient to observe that $\varphi(a-h) \geqslant s_M$ and $\kappa > \varphi(a) \geqslant g(\varphi(a-h))$.

Finally, when $t_* + h > 0$ and $\varphi'(t) < 0$ on $(0, h]$, the proof is the same as in *Case* A. $\square$

Corollary 14. *Let φ be a non-monotone semi-wavefront to Eq.* (5). *Let τ_1 be the leftmost critical point of $\varphi(t)$. Then* $\mathrm{sc}(\bar{\varphi}_{\tau_1+h}) = 1$ *or* $\mathrm{sc}(\bar{\varphi}_{\tau_1+h}) = 2$ *and* $g(g(s_M)) \leqslant \varphi(t) \leqslant g(s_M) = \max_{s>0} g(s)$ *for all* $t \geqslant \tau_1$.

Proof. The property $\mathrm{sc}(\bar{\varphi}_{\tau_1+h}) \in \{1, 2\}$ is an immediate consequence of Theorem 13. Next, if φ has a local maximum at $\tau \geqslant \tau_1$ then

$$\varphi(\tau) = \epsilon\varphi''(\tau) + g\bigl(\varphi(\tau - h)\bigr) \leqslant g(s_M) = \max_{s>0} g(s).$$

Hence, we need only to prove that $\varphi(t) \geqslant g(g(s_M))$, $t \leqslant \tau_1$. Let τ_2 be as in Theorem 13. Then $g(s_M) \geqslant \varphi(\tau_2 - h) > \kappa$ and, in consequence

$$\varphi(\tau_2) = \epsilon\varphi''(\tau_2) + g\bigl(\varphi(\tau_2 - h)\bigr) \geqslant g\bigl(\varphi(\tau_2 - h)\bigr) \geqslant g\bigl(g(s_M)\bigr).$$

Now, suppose that $\hat{\tau}$ is the first critical point where $\varphi(\hat{\tau}) < g(g(s_M)) < \kappa$. We have that $\varphi(\hat{\tau} - h) > \varphi(\hat{\tau})$, $\kappa > \varphi(\hat{\tau})$, $\varphi'(\hat{\tau}) = 0$, $\varphi''(\hat{\tau}) \geqslant 0$ (and therefore $\varphi(\hat{\tau}) \geqslant g(\varphi(\hat{\tau} - h))$). It is easy to see that the above inequalities imply that $\varphi(\hat{\tau}) < \kappa < \varphi(\hat{\tau} - h) \leqslant g(s_M)$. Thus $\varphi(\hat{\tau}) \geqslant g(\varphi(\hat{\tau} - h)) \geqslant g(g(s_M))$, a contradiction. $\square$

Remark 15 *(Monotonicity without assuming the unimodality).* Some of the proofs given above do not use the full force of condition (UM). For example, as it can be easily checked, Lemma 6 holds true for all positive (including unbounded) solutions φ, $\varphi(-\infty) = 0$, if continuous g satisfies $g'(0) > 1$ and $g(s) > 0$, $s \in \mathbb{R}_+$. We can also repeat the first part of the proof of Theorem 11 to establish

Proposition 16. *Assume* (B) *with* $\sup_{s \geqslant 0} g(s) \leqslant \zeta_2$, *and suppose that* g *increases on* $[0, s_M]$, $s_M \in [\zeta_1, \kappa]$. *Let φ be a positive semi-wavefront of Eq.* (5). *Then there exists a unique* $\tau \in \mathbb{R} \cup \{+\infty\}$ *such that* $\varphi(\tau) = s_M$ *and* $\varphi'(s) > 0$ *for all* $s < \tau$.

3. Variational equation at the positive equilibrium

In this section, we study the zeros of the characteristic function

$$\chi(z, \epsilon) := \epsilon z^2 - z - 1 + a\exp(-zh), \quad a = g'(\kappa) < 0,$$

associated with the variational equation

$$\epsilon\varphi''(t) - \varphi'(t) - \varphi(t) + a\varphi(t - h) = 0 \tag{16}$$

about the equilibrium κ of Eq. (5). It is easy to check (e.g., see [24]) that all complex zeros of χ are simple and that, for some $\epsilon_0 > 0$, equation $\chi(z, \epsilon_0) = 0$ has a negative real root z_0 of the multiplicity 2. In fact, if $\epsilon \in (0, \epsilon_0)$, then χ has only one real (positive) root, and if $\epsilon > \epsilon_0$, then χ has exactly three real roots (one positive and two negative). So (z_0, ϵ_0) is a bifurcation point where two real roots merge and disappear as ϵ decreases through ϵ_0.

Lemma 17. *Fix $a < 0$; $\epsilon, h > 0$; $p \in [0, 1]$. Then function*

$$\chi_p(z, \epsilon) := \epsilon z^2 - p(z + 1) + a \exp(-zh)$$

(1) *has exactly two roots λ_0, λ_1, in the half-plane $\Re\lambda > -1$ for every fixed $\epsilon > \max\{2, -2ae^h\}$. Furthermore, these roots are real and $\lambda_1 < 0 < \lambda_0$;*
(2) *does not have any root in the semi-infinite horizontal strips $(-\infty, 0] \times (\pi(1 + 2k)/h, \pi(2 + 2k)/h)$, $(-\infty, 0] \times (-\pi(2 + 2k)/h, -\pi(1 + 2k)/h)$, $k \in \mathbb{N} \cup \{0\}$;*
(3) *has at most two roots (counting multiplicity) on the vertical line $\Re z = \alpha$, for every fixed $\alpha \in \mathbb{R}$.*

Proof. (1) Let $\mu = \mu(\epsilon, p) \leqslant 0$, $\nu = \nu(\epsilon, p) \geqslant 0$, be the roots of $\epsilon z^2 - pz - p = 0$. Since $\mu(\epsilon, p) > -1/2$ for $\epsilon > 2$, we have that

$$\left|\epsilon z^2 - pz - p\right| = \epsilon|z - \mu||z - \nu| \geqslant \epsilon/2 > |a|e^{-\Re zh}$$

for all z from the boundary of sufficiently large rectangles $[-1, A] \times [-B, B] \subset \mathbb{C}$. An application of the Rouché theorem completes the proof of (1).

(2) Indeed, if we take $z = x + iy$, $x \leqslant 0$, $\pm yh \in (\pi + 2\pi k, 2\pi + 2\pi k)$, then

$$\left|\Im\chi_p(z, \epsilon)\right| = 2\epsilon|xy| + p|y| - ae^{-xh}\left|\sin(yh)\right| > 0.$$

(3) Suppose that $z_1 = \alpha + iv \neq z_2 = \alpha + iu$, $|u| \neq |v|$, are two solutions of the equation $\chi_p(z, \epsilon) = 0$. Then

$$\left|\epsilon z_1^2 - pz_1 - p\right|^2 = |a|^2 \exp(-2\alpha h) = \left|\epsilon z_2^2 - pz_2 - p\right|^2,$$

which implies

$$\epsilon^2\left(u^2 + v^2\right) + 2(\epsilon\alpha - 0.5p)^2 + 2\epsilon p + p^2/2 = 0,$$

a contradiction. □

As it was observed in [24, Lemma 2.1 and Remark 2.2], if for fixed $a < 0$, $h, \epsilon_1 > 0$, the equation $\chi(z, \epsilon_1) = 0$ has a unique root in the half-plane $\{\Re z > 0\}$, then this property will be maintained for all $\epsilon > \epsilon_1$. In consequence, taking into account the remark before Lemma 17, we can deduce from the above lemma (with $p = 1$) the following

Corollary 18. *Fix $a < 0$, $h > 0$. Then there is $\epsilon^* \in [0, \max\{2, -2ae^h\}]$ such that $\chi(z, \epsilon)$ has only one zero in the half-plane $\{\Re z > 0\}$ if and only if $\epsilon \geqslant \epsilon^*$. In this way, $\epsilon^* = (c^*)^{-2}$. See Definition 3(a).*

Remarks 19, 20 below are motivated by [17, Section 6].

Remark 19. Fix $a < 0$, $h > 0$, $p \in [0, 1]$, and suppose that $\epsilon > \max\{2, -2ae^h\}$. Then Lemma 17 implies that each zero λ, $\Im\lambda \geqslant 0$, of $\chi_p(\lambda, \epsilon)$ belongs to the set $\{\lambda_0\} \bigcup_{k \geqslant 0} S_k$, where

$$S_0 = (-\infty, 0] \times [0, \pi/h], \qquad S_k = (-\infty, 0] \times \big[\pi 2k/h, \pi(1+2k)/h\big], \quad k \in \mathbb{N}.$$

Next, it is straightforward to see that $|a| \leqslant (\epsilon|z|^2 + |z| + 1)e^{\Re zh}$ for every zero z of $\chi_p(\lambda, \epsilon)$. In consequence, for each j we can indicate $x_j(|a|, \epsilon, h) < -1$ such that every zero $z \in S_j$ of $\chi_p(\lambda, \epsilon)$, $p \in [0, 1]$, satisfies $\Re z \in [x_j(|a|, \epsilon, h), -1]$. Hence, by the Rouché theorem, equations $\chi_1(z, \epsilon) = 0$ and $\chi_0(z, \epsilon) = 0$ have the same number of roots in each S_j. Note that $\chi_0(z, \epsilon) = 0$ can be written as

$$z^2 = \rho \exp(-zh), \quad \rho = |a|/\epsilon > 0. \tag{17}$$

Remark 20. Consider (17) for $\rho > 0$. It is a routine procedure to check that all complex roots of (17) are simple, there is a unique positive real root z_0, and the unique multiple (double) real root is $z = -2/h$. This root appears when $\rho_\sharp = 4/(he)^2$. As we have seen, if ρ is sufficiently small then all roots z, $\Im z \geqslant 0$, of this equation belong to $\{z_0\} \bigcup_{k \geqslant 0} S_k$. Now, take a simple root $z = z_j(\rho) \in S_j$ with $\Re z_j(\rho) < 0$. Observe that $z_j(\rho)$, $\rho \neq \rho_\sharp$, depends smoothly on ρ. If we let ρ increase, then $\exp(\Re z_j(\rho)h)|z_j(\rho)|^2 = \rho$ yields that $\Re z_j(\rho) > 0$ for sufficiently large ρ. If $z_j(\rho_j) = i\nu_j$, then, as it can be easily checked, we have that

$$\nu_j h = \pi(2j+1), \qquad \rho_j = \nu_j^2 = (\pi/h)^2(2j+1)^2, \qquad \rho_0 > \rho_\sharp,$$
$$\Re z_j'(\rho_j) = \frac{h}{4 + h^2\rho_j} > 0.$$

In consequence, every strip S_j, $j > 0$, possesses a unique root $z_j(\rho)$ for all $\rho \leqslant \rho_j$. When ρ increases trough ρ_j, this root crosses the imaginary axis from left to right. Hence, S_j does not contain any root of (17) for $\rho > \rho_j$. The same is true for the strip S_0, with the unique exception that S_0 contains two real roots $z_{01} \leqslant z_{02} < 0$ for $\rho \leqslant \rho_\sharp$. Furthermore, Lemma 17(3) implies that $\Re z_j(\rho) < \Re z_i(\rho)$, $\rho > 0$, if and only if $j > i$. If $\rho \leqslant \rho_\sharp$, then

$$\cdots < \Re z_2(\rho) < \Re z_1(\rho) < z_{01} \leqslant z_{02} < 0 < z_0.$$

If $\rho \in (\rho_\sharp, \rho_0]$, then

$$\cdots < \Re z_2(\rho) < \Re z_1(\rho) < \Re z_{01} = \Re z_{02} \leqslant 0 < z_0.$$

If $\rho > \rho_0$, then $\Re z_{01} = \Re z_{02} > 0$ and

$$\cdots < \Re z_2(\rho) < \Re z_1(\rho) < \Re z_{01} = \Re z_{02} < z_0.$$

Remarks 19, 20 imply the following

Lemma 21. *Take $a < 0$, $h > 0$, and $\epsilon \geqslant \epsilon^* = (c^*)^{-2}$. Then the set*

$$\Lambda = \{\lambda_j\}_{j>0} \cup \{\lambda_0, \lambda_{01}, \lambda_{02}\}$$

of all zeros λ_j, $\Im\lambda_j \geqslant 0$, of χ can be enumerated in such a way that either

$$\lambda_0 > 0 > \lambda_{01} \geqslant \lambda_{02} > \Re\lambda_1 > \Re\lambda_2 > \cdots$$

or

$$\lambda_0 > 0 \geqslant \Re\lambda_{01} = \Re\lambda_{02} > \Re\lambda_1 > \Re\lambda_2 > \cdots.$$

Furthermore, $\lambda_j \in S_j$ and $\lambda_{0k} \in S_0$.

The next result is a key to the proof of Theorem 3.

Theorem 22. *If $\epsilon^* = (c^*)^{-2} > 0$ is as in Corollary 18, then $\chi(z, \epsilon)$ does not have any zero in the strip $S_{00} := (-\infty, 0] \times [-2\pi/h, 2\pi/h]$ for every $\epsilon < \epsilon^*$.*

Proof. By Corollary 18 and Lemma 21, Theorem 22 holds if $\epsilon^* - \epsilon > 0$ is close to 0. Therefore, due to Lemma 17(2), if S_{00} contains zero $\lambda_j(\hat{\epsilon})$ of χ for some $\hat{\epsilon} < \epsilon^*$, it should enter the strip S_{00} crossing the interval $\mathcal{J} := [-2\pi i/h, 2\pi i/h]$ from the right to the left as ϵ is decreasing. This means that $\lambda_j(\epsilon)$ crosses $\mathcal{J}$ from the left to the right as ϵ increases from $\hat{\epsilon}$ to ϵ^*.

Now, the root $\lambda_j := \lambda_j(\hat{\epsilon}) \notin \mathbb{R}$ of $\chi(z, \hat{\epsilon}) = 0$ determines a unique smooth function $\lambda_j(\cdot): (\alpha_j, \beta_j) \to \mathbb{C}$ defined on some maximal open interval $(\alpha_j, \beta_j) \subseteq [0, +\infty)$ containing $\hat{\epsilon}$ and such that $\lambda_j(\hat{\epsilon}) = \lambda_j$, $\chi(\lambda_j(\epsilon), \epsilon) = 0$. We claim that the path $\lambda_j(\cdot): (\alpha_j, \beta_j) \to \mathbb{C}$ cannot cross the imaginary axis from the left to the right. Indeed, we have that

$$\lambda_j'(\epsilon) = -\frac{z^2}{2\epsilon z - 1 + h(\epsilon z^2 - z - 1)},$$

so that, at the moment $\tilde{\epsilon}$ of the eventual intersection we have $\lambda_j(\tilde{\epsilon}) = i\omega$ and

$$\Re\lambda_j'(\tilde{\epsilon}) = -\omega^2\bigl(1 + h + \epsilon h\omega^2\bigr)/\bigl(\bigl(1 + h\bigl(\epsilon\omega^2 + 1\bigr)\bigr)^2 + \omega^2(2\epsilon - h)^2\bigr) < 0,$$

a contradiction. Theorem 22 is completely proved. □

4. Proof of Theorem 3

In this section, in addition to (UM), we assume that g satisfies the feedback condition (4) and that $g'(\kappa) < 0$.

Let φ be a non-monotone semi-wavefront solution of Eq. (5). By Remark 9, all critical points of φ are isolated so that $\varphi(t) \not\equiv \text{const}$ on every open subinterval of $\mathbb{R}$. Let τ_1 be as in Theorem 13. Then Corollary 14 implies that $\varphi(t) \in [g(\max g), \max g]$ for every $t > \tau_1$ and that $\mathrm{sc}(\bar{\varphi}_{\tau_1+h}) = 1$ or $\mathrm{sc}(\bar{\varphi}_{\tau_1+h}) = 2$. Applying [19, Theorem 2.1], we find that $\mathrm{sc}(\bar{\varphi}_t) \in \{0, 1, 2\}$ for every $t > \tau_1 + h$.

It is easy to check that $\mathrm{sc}(\bar{\varphi}_s) > 0$ for all $s > \tau_1 + h$. Indeed, let us suppose, for an instant, that $\varphi(s+v) < \kappa$, $v \in [-h, 0]$, $\varphi'(s) < 0$, for some $s > \tau_1 + h$. Then Proposition 2 implies the existence of some $s_1 > s$ such that $\varphi(s_1 + v) < \kappa$, $v \in [-h, 0]$, $\varphi'(s_1) = 0$, $\varphi''(s_1) \geqslant 0$. Then we have

$$\kappa > \varphi(s_1) = \epsilon\varphi''(s_1) + g\big(\varphi(s_1 - h)\big) \geqslant g\big(\varphi(s_1 - h)\big) > \kappa,$$

a contradiction. The other case, when $\varphi(s+v) > \kappa$, $v \in [-h, 0]$, $\varphi'(s) > 0$, is similar.

Now, $\mathrm{sc}(\bar{\varphi}_t) = 1$ for all large t if and only if $\varphi(t)$ is eventually monotone. Furthermore, we claim that the situation when $\mathrm{sc}(\bar{\varphi}_t) = 2$, for all large t, is not possible. Indeed, to see this, it suffices to take some $s_2 > \tau_1 + h$ where $\varphi(t) - \kappa$ attains a positive local maximum. Since $\varphi'(s_2) = 0$ and $\mathrm{sc}(\bar{\varphi}_{s_2}) = 2$, we find that $\varphi(s_2 - h) > \kappa$ and $\varphi(s') < \kappa$ for some $s' \in (s_2 - h, s_2)$. Now, it is clear that every right neighborhood of s_2 should contain at least one point s_3 where $\varphi'(s_3) < 0$. But this means that $\mathrm{sc}(\bar{\varphi}_{s_3}) = 3$ if $s_3 > s_2$ is sufficiently close to s_2, a contradiction.

Hence, our previous analysis indicates the following two alternatives: either $\mathrm{sc}(\bar{\varphi}_t) = 1$ for all large t (so that $\varphi(t)$ is eventually monotone) or $\mathrm{sc}(\bar{\varphi}_t) : (\tau_1 + h, \infty) \to \{1, 2\}$ is not constant (so that φ is a slowly oscillating solution). However, in Lemma 25 below, we establish that the semi-wavefront φ cannot be eventually monotone if $c > c^*$. In order to prove this lemma, we need an auxiliary result:

Lemma 23. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $f(+\infty) = 0$. Given real numbers $d > 1$ and $\rho > 0$, let $\alpha = (\ln d)/\rho > 0$. Then either* (a) *$f(t) = O(e^{-\alpha t})$ at $+\infty$, or* (b) *there exists a sequence $t_j \to +\infty$ such that $f(t_j) = \max_{s \geqslant t_j} f(s)$ and $\max_{s \in [t_j - \rho, t_j]} f(s) \leqslant d f(t_j)$.*

Proof. Set

$$T = \Big\{t\colon\ f(t) = \max_{s \geqslant t} f(s) \text{ and } \max_{s \in [t-\rho, t]} f(s) \leqslant d f(t)\Big\}.$$

Then either (I) $T \neq \emptyset$ and $\sup T = +\infty$ and therefore the conclusion (b) of the lemma holds, or (II) T is a bounded set (without restricting the generality, in this case we may assume that $T = \emptyset$). Let us analyze more closely the second case (with $T = \emptyset$). Take an arbitrary $t > \rho$ and let $\hat{t} \geqslant t$ be defined as the leftmost point where $f(\hat{t}) = \max_{s \geqslant t} f(s)$. Since $t \notin T$, we have that $\hat{t} - t \leqslant \rho$. Let t_1 be defined by $f(t_1) = \max_{s \in [\hat{t} - \rho, \hat{t}]} f(s)$, our assumption about T implies that $\hat{t} - \rho \leqslant t_1 < t \leqslant \hat{t}$ and that $f(t_1) > d f(\hat{t}) \geqslant d f(t)$. Additionally, $f(t_1) = \max_{s \geqslant t_1} f(s)$. Next, we define t_2 as the leftmost point satisfying $f(t_2) = \max_{s \in [t_1 - \rho, t_1]} f(s)$. Note that $0 < t_1 - t_2 \leqslant \rho$, $f(t_2) = \max_{s \geqslant t_2} f(s)$ and $f(t_2) > d f(t_1)$. Proceeding in this way, we construct a decreasing sequence t_j such that $f(t_{j+1}) > d f(t_j)$ for every j. We claim that there exist an integer m such that $t_m \leqslant \rho$. Indeed, otherwise $t_j > \rho$ for all $j \in \mathbb{N}$ which implies the existence of $\lim t_j = t_*$ and $\lim f(t_j) = f(t_*)$. However, this is not possible because of $f(t_j) > d^j f(\hat{t}) > 0$. Hence, $t_m \in [0, \rho]$ for some integer m. Note that $t - t_m \leqslant \hat{t} - t_m \leqslant m\rho$ implying that $m \geqslant (t - t_m)/\rho \geqslant (t - \rho)/\rho$ and that

$$f(t) < d^{-m} f(t_m) \leqslant d^{-m} \max_{s \in [0,\rho]} f(s) \leqslant d e^{-\alpha t} \max_{s \in [0,\rho]} f(s). \qquad \square$$

Lemma 23 has the following immediate consequence.

Corollary 24. *Assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(+\infty) = 0$, does not decay superexponentially. Then, for every $\rho > 0$, there exist a sequence $t_j \to +\infty$ and a real $d > 1$ such that $f(t_j) = \max_{s \geqslant t_j} f(s)$ and $\max_{s\in[t_j-\rho,t_j]} f(s) \leqslant d f(t_j)$.*

Lemma 25. *The semi-wavefront φ cannot be eventually monotone if $c > c^*$.*

Proof. Suppose the theorem were false. Then there exists an eventually monotone travelling semi-wavefront $\varphi : \mathbb{R} \to \mathbb{R}_+$ moving with a positive velocity $c > c^*$. Now, the eventual monotonicity of φ implies that $\varphi(+\infty) \in \{0, \kappa\}$ while the uniform persistence of φ excludes the possibility of having $\varphi(+\infty) = 0$ (see Proposition 2). Thus $\varphi(+\infty) = \kappa$, so that actually φ is a travelling front.

Set $w(t) = \varphi(t) - \kappa$, then $w(t)$ is either decreasing and strictly positive or increasing and strictly negative, for all sufficiently large t. It is straightforward to verify that $w(t)$ satisfies

$$\epsilon w''(t) - w'(t) = w(t) + k(t)w(t-h), \quad k(t) := -\frac{g(\varphi(t-h)) - g(\kappa)}{\varphi(t-h) - \kappa}, \tag{18}$$

where, in view of $\varphi(+\infty) = \kappa$, it holds that $0 < k(t) < -2g'(\kappa)$, for all sufficiently large t. We can use now Lemma 3.1.1 from [13] to conclude that $w(t)$ cannot converge superexponentially to 0. This fact and Corollary 24 imply the existence of a sequence $t_j \to +\infty$ and a real number $d > 0$ such that $|w(t_j)| = \max_{s \geqslant t_j} |w(s)|$ and $\max_{s\in[t_j-3h,t_j]} |w(s)| \leqslant d|w(t_j)|$ for every j. Without the loss of generality we may assume that $w'(t_n) \leqslant 0$ and $0 < w(t) \leqslant w(t_n)$ for all $t \geqslant t_n$. Additionally, we can find a sequence $\{s_j\}$, $\lim(s_j - t_j) = +\infty$ such that $|w'(s_j)| \leqslant w(t_j)$. Now, since $w(t)$ satisfies (18), we conclude that every $y_j(t) = w(t + t_j)/w(t_j) > 0$ is a solution of

$$\epsilon y''(t) - y'(t) - y(t) - k(t + t_j)y(t - h) = 0, \quad t \in \mathbb{R}.$$

It is clear that $\lim_{j\to+\infty} k(t + t_j) = -g'(\kappa)$ uniformly on $\mathbb{R}_+$ and also that $0 < y_j(t) \leqslant d$ for all $t \geqslant -3h$, $j = 1, 2, 3, \ldots$.

We want to estimate $|y_j'(t)|$. Since $z_j(t) = y_j'(t)$ solves the initial value problem $z_j(s_j - t_j) = w'(s_j)/w(t_j) \in [-1, 0]$ for equation

$$\epsilon z'(t) - z(t) - y_j(t) - k(t + t_j)y_j(t - h) = 0, \quad t \in \mathbb{R},$$

we obtain that

$$y_j'(t) = e^{(t+t_j-s_j)/\epsilon} z_j(s_j - t_j) + \frac{1}{\epsilon}\int_{s_j-t_j}^{t} e^{(t-s)/\epsilon}\bigl(y_j(s) + k(s + t_j)y_j(s - h)\bigr)\, ds. \tag{19}$$

In consequence,

$$\bigl|y_j'(t)\bigr| \leqslant 1 + \bigl(2\bigl|g'(\kappa)\bigr| + 1\bigr)d, \quad t \in [-2h, s_j - t_j],\ j \in \mathbb{N}, \tag{20}$$

from which the uniform boundedness of the sequence $\{y_j'(t)\}$ on each compact interval $[-2h, \xi]$, $\xi > -2h$, follows. Together with $0 < y_j(t) \leqslant d$, $t \geqslant -3h$, inequality (20) implies the pre-compactness of the set $\{y_j(t),\ t \geqslant -2h,\ j \in \mathbb{N}\}$, in the compact open topology of $C([-2h,+\infty),\mathbb{R})$. Therefore, by the Arzelà–Ascoli theorem combined with the diagonal method, we can indicate a subsequence $y_{j_k}(t)$ converging to a continuous function $y(t)$, $t \in [-2h,+\infty)$. This convergence is uniform on every bounded subset of $[-2h,+\infty)$. Additionally we may assume that $\lim_{k\to\infty} y'_{j_k}(0) = y'_0$ exists.

Next, putting $s_j - t_j = 0$ in (19), we find that

$$y'_j(t) = e^{t/\epsilon} y'_j(0) + \frac{1}{\epsilon}\int_0^t e^{(t-s)/\epsilon}\big(y_j(s) + k(s+t_j)y_j(s-h)\big)\,ds, \quad t \geqslant -h.$$

Integrating this relation between 0 and t and then taking the limit as $j \to \infty$ in the obtained expression, we obtain that

$$y(t) = 1 + \epsilon\big(e^{t/\epsilon} - 1\big)y'_0 + \frac{1}{\epsilon}\int_0^t\int_0^\sigma e^{(\sigma-s)/\epsilon}\big(y(s) - g'(\kappa)y(s-h)\big)\,ds\,d\sigma, \quad t \geqslant -h.$$

Therefore $y(t)$ satisfies

$$\epsilon y''(t) - y'(t) - y(t) + g'(\kappa)y(t-h) = 0, \quad t \geqslant -h. \tag{21}$$

Additionally, $y(0) = 1$, $y'(0) = y'_0 \in [-1,0]$ and $0 \leqslant y(t) \leqslant d$, $t \geqslant -2h$. Clearly, $y \in C^2(\mathbb{R}_+)$ and we claim that $y(t) > 0$ for all $t \geqslant 0$. Observe here that $y(t)$, $t \geqslant -2h$, is non-increasing, and therefore $y(0) = 1$, $y(s) = 0$ imply $s > 0$. Let us suppose, for a moment, that $y(s) = 0$ and $y(r) > 0$, $r \in [-h,s)$. Then $y'(s) = 0$, $y(s-h) > 0$, so that (21) implies $y''(s) > 0$. Thus $y(t) > 0 = y(s)$ for all $t > s$ close to s which is not possible because y is non-increasing on $[-2h,+\infty)$.

Summing up, we have proved that Eq. (21) has a bounded positive solution on $\mathbb{R}_+$. As it was established in [13, Lemma 3.1.1], this solution does not decay superexponentially. From [18, Proposition 7.2] (see also [12, Proposition 2.2]), we conclude that there are $b \leqslant 0$, $\delta > 0$ and a nontrivial eigensolution $v(t)$ of Eq. (21) on the generalized eigenspace associated with the (nonempty) set Λ of eigenvalues with $\Re\lambda = b$, such that $y(t) = v(t) + O(\exp((b-\delta)t))$, $t \to +\infty$. On the other hand, since $c > c^*$, we know from Theorem 22 that there are no real negative eigenvalues of (21): hence $\Im\lambda \neq 0$ for all $\lambda \in \Lambda$. From [12, Lemma 2.3], we conclude that $y(t)$ is oscillatory, a contradiction. $\square$

Hence, if $c > c^*$, then φ is slowly oscillating around the positive steady state. In the remaining part of this section, we show that these oscillations are non-decaying. Arguing by contradiction, assume that $\varphi(+\infty) = \kappa$ for some $c > c^*$. Then $w(t) = \varphi(t) - \kappa$, $w(+\infty) = 0$, solves

$$\epsilon w''(t) - w'(t) - w(t) + g_1\big(w(t-h)\big) = 0, \quad t \in \mathbb{R}, \tag{22}$$

where $g_1(s) := g(s+\kappa) - \kappa$, $g_1(0) = 0$, $g'_1(0) = g'(\kappa)$, satisfies the feedback condition with respect to 0.

Since $w(+\infty) = 0$, there exists a sequence $t_n \to +\infty$ such that $|w(t_n)| = \max_{s \geqslant t_n}|w(s)|$. It follows from Remark 9 that the zeros of w are isolated, therefore $w(t_n) \neq 0$. Additionally, we can assume that w attains its local extremum at t_n so that $w'(t_n) = 0$, $w''(t_n)w(t_n) \leqslant 0$. Due to the feedback condition, these relations and (22) imply that $w(t_n)w(t_n - h) < 0$ and therefore $\mathrm{sc}(\bar{w}_{t_n})$

must be an odd integer. Hence, $\mathrm{sc}(\bar{w}_{t_n}) = 1$ so that there are a unique $z_n \in (t_n - h, t_n)$ and a finite set F_n such that $w(s) < 0$ for $s \in [t_n - h, z_n) \setminus F_n$ and $w(s) \geqslant 0$ for $s \in [z_n, t_n]$. Without restricting the generality, we can assume that $|w(t_n)| = \max\{|w(s)|\colon s \in [z_n, t_n]\}$, and that $\{r_n\}$, $r_n := t_n - z_n \in (0, h)$, is monotonically converging to $r_* \in [0, h]$.

Set $y_n(t) = w(t + z_n)/w(t_n)$, $t \in \mathbb{R}$, then $y_n(t)$ satisfies

$$\epsilon y''(t) - y'(t) - y(t) + p_n(t-h)y(t-h) = 0, \tag{23}$$

where

$$p_n(t) = \begin{cases} g_1(w(t+z_n))/w(t+z_n), & \text{if } w(t+z_n) \neq 0, \\ g'(\kappa), & \text{if } w(t+z_n) = 0. \end{cases}$$

Due to the above-mentioned properties of w, we find easily that $y_n(0) = 0$ and $|y_n(t)| \leqslant 1$, $t \geqslant 0$, and that $\lim_{n\to\infty} p_n(t) = g'(\kappa)$ uniformly in $t \in \mathbb{R}_+$. As a consequence, we may suppose that $p_n(t)/g'(\kappa) \in [0.9, 1.1]$ for all n and $t \geqslant 0$. We have also that $y_n(r_n) = 1$, $y_n(r_n - h) < 0$.

Next, we want to estimate $|y_n'(t)|$. Let $\{s_n\}$, $\lim(s_n - z_n) = +\infty$ be such that $w'(s_n) = 0$. Since $v_n(t) = y_n'(t)$ solves the initial value problem $v_n(s_n - z_n) = 0$ for

$$\epsilon v'(t) - v(t) - y_n(t) + p_n(t-h)y_n(t-h) = 0, \quad t \in \mathbb{R},$$

we obtain that

$$y_n'(t) = v_n(t) = \frac{1}{\epsilon}\int_{s_n - z_n}^{t} e^{(t-s)/\epsilon}\big(y_n(s) - p_n(s-h)y_n(s-h)\big)\,ds.$$

For all $t \in [h, s_n - z_n]$, we have that

$$\begin{aligned}
\left|y_n'(t)\right| &\leqslant \left|\frac{1}{\epsilon}\int_{s_n-z_n}^{t} e^{(t-s)/\epsilon}\left(\left|y_n(s)\right| + \sup_{x>0}\left|\frac{g(x)-\kappa}{x-\kappa}\right|\left|y_n(s-h)\right|\right)ds\right| \\
&\leqslant \left(\sup_{x>0}\left|\frac{g(x)-\kappa}{x-\kappa}\right| + 1\right)\frac{1}{\epsilon}\int_{t}^{s_n-z_n} e^{(t-s)/\epsilon}\,ds \leqslant \left(\sup_{x>0}\left|\frac{g(x)-\kappa}{x-\kappa}\right| + 1\right) := \rho, \\
\left|y_n''(t)\right| &\leqslant \epsilon^{-1}\left[\left|y_n'(t)\right| + \left|y_n(t)\right| + \left|p_n(t-h)\right|\left|y_n(t-h)\right|\right] \leqslant 2\epsilon^{-1}\rho.
\end{aligned}$$

Hence, the sequences $y_n(t)$, $y_n'(t)$ have subsequences which converge, uniformly on bounded subsets of $[h, +\infty)$, to continuous functions $y_*(t)$, $y_*'(t)$. Recalling the properties of y_n, we find that $\sup\{|y_*(s)|,\ s \geqslant h\} \leqslant 1$. Next, for all $t \in [2h, +\infty)$, it holds that

$$g_n(t) := p_n(t-h)y_n(t-h) \to g_*(t) := g'(\kappa)y_*(t-h),$$

uniformly on bounded subsets of $[2h, +\infty)$. We have that $0 \leqslant |g_*(t)| \leqslant |g'(\kappa)|$ for $t \geqslant 2h$.

In order to establish some further properties of $y_*(t)$, we indicate the family of all solutions of (23) which are bounded at $+\infty$:

$$y(t) = Ae^{\lambda t} + \frac{1}{\epsilon(\mu-\lambda)}\left\{\int_{2h}^{t} e^{\lambda(t-s)} g_n(s)\,ds + \int_{t}^{+\infty} e^{\mu(t-s)} g_n(s)\,ds\right\}, \quad t \geqslant 2h.$$

Replacing $y(t)$ with $y_n(t)$ in the latter formula, we find that

$$y_n(t) = A_n e^{\lambda t} + \frac{1}{\epsilon(\mu-\lambda)}\left\{\int_{2h}^{t} e^{\lambda(t-s)} g_n(s)\,ds + \int_{t}^{+\infty} e^{\mu(t-s)} g_n(s)\,ds\right\}, \quad t \geqslant 2h. \tag{24}$$

It is easy to prove now that the sequence $\{A_n\}$ is bounded: indeed, (24) implies that

$$|A_n| = \left| e^{-2h\lambda}\left(y_n(2h) - \frac{1}{\epsilon(\mu-\lambda)} \int_{2h}^{+\infty} e^{\mu(2h-s)} g_n(s)\,ds \right)\right| \leqslant \left| e^{-2h\lambda}\left(1 + \frac{\rho|\lambda|}{(\mu-\lambda)}\right)\right|.$$

Hence, taking the limit in (24) as $n \to +\infty$ (through considering a subsequence if necessary) we find that $y_*(t)$, $t \geqslant 2h$, satisfies

$$y_*(t) = Ae^{\lambda t} + \frac{1}{\epsilon(\mu-\lambda)}\left\{\int_{2h}^{t} e^{\lambda(t-s)} g_*(s)\,ds + \int_{t}^{+\infty} e^{\mu(t-s)} g_*(s)\,ds\right\}, \tag{25}$$

with some finite A. Next, (25) implies that $y_*(t)$ satisfies the linear equation

$$\epsilon y''(t) - y'(t) - y(t) + g'(\kappa) y(t-h) = 0, \quad t \geqslant 2h. \tag{26}$$

We claim that $y_*(t)$ is not a small solution. The proof of this claim, given below, is motivated by [17, Section 10].

So, on the contrary, let us suppose that $y_*(t)$ has superexponential decay. Since the characteristic function $\chi(z,\epsilon) = \epsilon z^2 - z - 1 + g'(\kappa)\exp(-zh)$ has the exponential type h, an application of [11, Theorem 3.1] assures that $y_*(t) = 0$ for all $t \geqslant 3h$. But then Eq. (26) implies that $y_*(t) = 0$, $y_*'(t) = 0$ for all $t \geqslant 2h$ and, in consequence, $y_*(t) = 0$, $y_*'(t) = 0$ all $t \geqslant h$.

Consider $y_{n,h}(s) := y_n(h+s)$, $s \in [-h, 0]$. We have that $y_{n,h} \in C[-h,0]$ and $|y_{n,h}| = 1$. Therefore, by the Banach–Alaoglu theorem, we can suppose that $y_{n,h} \rightharpoonup \gamma$ in $*$-weak topology of $L^\infty[-h,0]$. Integrating (23) between h and $t \geqslant h$, we get

$$\epsilon y_n'(t) - \epsilon y_n'(h) - \big(y_n(t) - y_n(h)\big) - \int_{h}^{t} \big(y_n(s) - p_n(s-h) y_n(s-h)\big)\,ds = 0. \tag{27}$$

We have already established that $y_n(t) \to y_*(t) = 0$, $y_n'(t) \to y_*'(t) = 0$, $p_n(t-h) \to g'(\kappa)$, uniformly on every compact subset of $[h, +\infty)$, and that $|y_n(t)| \leqslant 1$, $t \geqslant 0$. Therefore, we may take limit in (27) as $n \to \infty$ to find that $\int_h^t g'(\kappa)\gamma(s-2h)\,ds = 0$, $t \in [h, 2h]$. Hence, $\gamma = 0$ and therefore

$$\lim_{n\to\infty} \inf_{t\in[a,b]} \left|y_n(t)\right| = 0 \tag{28}$$

for every subinterval $[a, b] \subseteq [0, h]$. (Indeed, otherwise there exists $\varepsilon_0 > 0$ and a subsequence $\{y_{n_k}\}$ such that either $y_{n_k}(t) \geqslant \varepsilon_0$ or $y_{n_k}(t) \leqslant -\varepsilon_0$ for all $t \in [a, b]$. This means that $\int_a^b y_{n_k}(s)\,ds \nrightarrow 0$, contradicting to $y_{n,h} \rightharpoonup \gamma = 0$.)

We claim that there exists a sequence $\{s_{n_j}\}$, $s_{n_j} \in (r_{n_j}, h)$, such that $s_{n_j} \to r_*$ and $y'_{n_j}(s_{n_j}) < 2g'(\kappa) - 1$, $y''_{n_j}(s_{n_j}) = 0$. Below, we prove this statement considering three different possibilities (i)–(iii).

(i) If $r_* = h$, then we can define s_n by

$$y'_n(s_n) = \min_{s\in[r_n,h]} y'_n(s).$$

Recall that $y_n(h)$, $y'_n(h) \to 0$ and that $y_n(r_n) = 1$, $y'_n(r_n) = 0$. So actually $s_n \in (r_n, h)$ and $y''(s_n) = 0$.

(ii) Next, suppose that $r_* < h$, and that $\{r_n\}$ is increasing. For an arbitrary j satisfying $-(2^j - 2^{-j}) < 2g'(\kappa) - 1$, $r_* + 3 \cdot 2^{-j-1} < h$, we will fix two disjoint intervals

$$I_1 = \left[r_*, r_* + 2^{-j-1}\right], \qquad I_2 = \left[r_* + 2^{-j}, r_* + 3 \cdot 2^{-j-1}\right].$$

In view of (28), we can find $d_k \in I_k$ and integer n_j such that $r_* - r_{n_j} \leqslant 2^{-j-1}$; $|y_{n_j}(d_k)| \leqslant 4^{-j}$. But then $1 - 4^{-j} \leqslant y_{n_j}(r_{n_j}) - y_{n_j}(d_1) = y'_{n_j}(\theta_{n_j})(r_{n_j} - d_1)$ for some $\theta_{n_j} \in (r_{n_j}, d_1)$ so that

$$y'_{n_j}(\theta_{n_j}) \leqslant \frac{1 - 4^{-j}}{r_{n_j} - d_1} \leqslant -\left(2^j - 2^{-j}\right) < 2g'(\kappa) - 1.$$

Similarly,

$$-4^{-j} - 4^{-j} \leqslant y_{n_j}(d_2) - y_{n_j}(d_1) = y'_{n_j}(\xi_{n_j})(d_2 - d_1),$$

for some $\xi_{n_j} \in (d_1, d_2)$ so that, for $j \geqslant 2$,

$$y'_{n_j}(\xi_{n_j}) \geqslant \frac{-2 \cdot 4^{-j}}{d_2 - d_1} \geqslant \frac{-2 \cdot 4^{-j}}{2^{-j-1}} = -2^{-j+2} \geqslant y'_{n_j}(\theta_{n_j}),$$

$$r_{n_j} \leqslant \theta_{n_j} < \xi_{n_j} \leqslant r_* + 3 \cdot 2^{-j-1}.$$

Accordingly, if we set

$$y'_{n_j}(s_{n_j}) = \min_{s\in[r_{n_j}, r_* + 3\cdot 2^{-j-1}]} y'_{n_j}(s),$$

then

$$y''_{n_j}(s_{n_j}) = 0, \qquad y'_{n_j}(s_{n_j}) < 2g'(\kappa) - 1, \qquad s_{n_j} - r_{n_j} \leqslant 2^{-j+1}.$$

(iii) Finally, if $r_* < h$, and $\{r_n\}$ is decreasing, we may apply an argument similar to that used in (ii). Indeed, define $I_1 = [r_* + 2^{-j-2}, r_* + 2^{-j-1}]$, $I_2 = [r_* + 2^{-j}, r_* + 3 \cdot 2^{-j-1}]$ and choose a subsequence $\{n_j\}$ in such a way that $r_{n_j} \in [r_*, r_* + 2^{-j-3})$ and $|y_{n_j}(d_k)| \leqslant 4^{-j}$ for some $d_k \in I_k$. The rest of the proof is identical to that of (ii).

In this way, the above claim and (23) imply that

$$y_{n_j}(s_{n_j} - h) = \frac{y'_{n_j}(s_{n_j}) + y_{n_j}(s_{n_j})}{p_{n_j}(s_{n_j} - h)} \geqslant \frac{2}{p_{n_j}(s_{n_j} - h)/g'(\kappa)} \geqslant 2/1.1 > 1,$$

a contradiction, since $-h < r_{n_j} - h < s_{n_j} - h < r_{n_j}$ and

$$y_{n_j}(s) \leqslant 0, \quad s \in [r_{n_j} - h, 0), \qquad 0 \leqslant y_{n_j}(s) < 1, \quad s \in [0, r_{n_j}).$$

Therefore $y_*(t)$ is not a small solution.

Hence, by [18, Proposition 7.2], for every sufficiently large $|\nu|$, $\nu < 0$, we have that

$$y_*(t) = v(t) + O\big(\exp(\nu t)\big), \quad t \to +\infty,$$

where v is a *nonempty* finite sum of eigensolutions of (26) associated to the eigenvalues $\lambda_j \in F = \{\nu < \Re\lambda_j \leqslant 0\}$. Now, Theorem 22 says that, for every $\epsilon \in (0, \epsilon^*)$, $\epsilon^* = (c^*)^{-2}$,

$$F \cap (-\infty, 0] \times [-2\pi/h, 2\pi/h] = \emptyset.$$

In consequence, there exist $A > 0$, $\beta > 2\pi/h$, $\alpha \geqslant 0$, $\zeta \in \mathbb{R}$, such that

$$y_*(t) = \big(A\cos(\beta t + \zeta) + o(1)\big)e^{-\alpha t}, \quad t \geqslant 2h.$$

This implies the existence of an interval $(a, a+h)$, $a > 3h$, such that $y_*(t)$ changes its sign on $(a, a+h)$ at least three times. Since $(y_{n_j}(t), y'_{n_j}(t)) \to (y_*(t), y'_*(t))$ uniformly on $[a, a+h]$, we can conclude that $\varphi(t + z_{n_j}) - \kappa = w(t_{n_j})y_{n_j}(t)$ changes its sign on $(a, a+h)$ at least three times, for all large j. This contradicts to the assumption that φ is slowly oscillating around the positive steady state. In consequence, the equality $\varphi(+\infty) = \kappa$ cannot hold for $c > c^*$.

5. Uniqueness in the case of piece-wise linear birth function

Let $d > 1$, $\theta > 0$, κ, b, and $a \in [-1, 1)$ satisfy the relations $a\theta + b = d\theta$, $a\kappa + b = \kappa$. Then $\theta < \kappa$, $b > 0$, and the piece-wise linear function

$$g(s) = \begin{cases} ds, & \text{for } s \in [0, \theta], \\ as + b, & \text{if } s \in [\theta, \max\{\kappa, d\theta\}], \end{cases}$$

is continuous and satisfies $g(0) = 0$ and $g(\kappa) = \kappa$. Moreover, if $a \in [-1, 0)$ then $g : [g(\max g), \max g] \to \mathbb{R}_+$ is decreasing so that the feedback condition (4) is satisfied automatically.

In this section, we show how all the heteroclinic solutions of the corresponding equation

$$\epsilon\varphi''(t) - \varphi'(t) - \varphi(t) + g\big(\varphi(t-h)\big) = 0 \tag{29}$$

can be found in the closed form. It should be noted here that Eq. (29) has at least one heteroclinic solution (say φ) for every $\epsilon \in (0, 1/c_*^2]$ independently on the value of delay h, see [16,24].

Now, Theorem 13 (or Corollary 12) assures the existence of t_0 such that $\varphi'(t) > 0$, $t \leqslant t_0 - h$, and $\varphi(t_0 - h) = \theta$. Set $t_0 = 0$. Then, for all $t \leqslant 0$, such φ is a positive solution of the linear equation

$$\epsilon\varphi''(t) - \varphi'(t) - \varphi(t) + d\varphi(t-h) = 0. \tag{30}$$

The characteristic equation for (30) is

$$\epsilon\lambda^2 - \lambda - 1 + de^{-h\lambda} = 0, \tag{31}$$

and it has two positive real roots $0 < \lambda_1 \leqslant \lambda_2$ which dominate all complex roots λ_j's of (31) in the sense that $\Re\lambda_j < \lambda_1$, e.g. see [24, Lemma 2.3].

Case of the simple positive roots. At first we assume that $\lambda_1 < \lambda_2$. Then we obtain, for some $p \geqslant 0$, $p + q > 0$, that

$$\varphi(t) = pe^{\lambda_1(t+h)} + qe^{\lambda_2(t+h)}, \quad t \leqslant 0. \tag{32}$$

From (32) we get $\varphi(-h) = p + q = \theta$, so that $p = \theta - q$.

By Corollary 14, we have that $\varphi(t) \geqslant \theta$ for all $t \geqslant -h$. Hence, if $t > 0$, then

$$\epsilon\varphi''(t) - \varphi'(t) - \varphi(t) + a\varphi(t-h) + b = 0.$$

The change of variables $\varphi = y + \kappa$ transforms this equation into

$$\epsilon y''(t) - y'(t) - y(t) + ay(t-h) = 0. \tag{33}$$

Set $\psi(s) = \varphi(s) - \kappa$, $s \geqslant -h$. Then

$$\psi(s) = (\theta - q)e^{\lambda_1(h+s)} + qe^{\lambda_2(h+s)} - \kappa, \quad s \in [-h, 0],$$

$$\psi(0) = (\theta - q)e^{\lambda_1 h} + qe^{\lambda_2 h} - \kappa, \qquad \psi'(0) = \lambda_1(\theta - q)e^{\lambda_1 h} + q\lambda_2 e^{\lambda_2 h}.$$

Applying the Laplace transform $(\mathcal{L}y)(z) = \int_0^\infty e^{-zs} y(s)\,ds$ to Eq. (33), we get

$$\chi(z)(\mathcal{L}\psi)(z) = \epsilon\big(\psi'(0) + z\psi(0)\big) - \psi(0) - ae^{-zh}\int_{-h}^{0} \psi(s)e^{-zs}\,ds. \tag{34}$$

Here $\chi(z) = \epsilon z^2 - z - 1 + ae^{-hz}$. Since $|a| \leqslant 1$, characteristic function χ has a unique positive root ν while other characteristic values have negative real parts, see [24]. Therefore $\lim \psi(t) = 0$, $t \to \infty$, implies $\chi(\nu)(\mathcal{L}\psi)(\nu) = 0$. The last equation has the form $P(\nu, \lambda_1, \lambda_2)q + Q(\nu, \lambda_1, \lambda_2) = 0$, where

$$P(\nu,\lambda_1,\lambda_2)=\epsilon\lambda_2 e^{\lambda_2 h}-\epsilon\lambda_1 e^{\lambda_1 h}+\big(e^{\lambda_2 h}-e^{\lambda_1 h}\big)(\epsilon\nu-1)$$
$$-ae^{-h\nu}\int_{-h}^{0} e^{-\nu s}\big(e^{\lambda_2(h+s)}-e^{\lambda_1(h+s)}\big)\,ds,$$
$$Q(\nu,\lambda_1,\lambda_2)=\epsilon\big(\theta\lambda_1 e^{\lambda_1 h}+\nu\theta e^{\lambda_1 h}-\nu\kappa\big)+\kappa-\theta e^{\lambda_1 h}$$
$$-ae^{-h\nu}\int_{-h}^{0} e^{-\nu s}\big(\theta e^{\lambda_1(h+s)}-\kappa\big)\,ds.$$

Next, we establish that $P(\nu,\lambda_1,\lambda_2)>0$, proving that the partial derivative $P_{\lambda_2}(\nu,\lambda_1,\lambda_2)>0$. Observe here that $\lambda_2>\lambda_1$ and $P(\nu,\lambda_1,\lambda_1)=0$. Since $a<1$, $\lambda_2>0$, we have

$$P_{\lambda_2}(\nu,\lambda_1,\lambda_2)=e^{\lambda_2 h}\big(\epsilon+\epsilon\lambda_2 h+h(\epsilon\nu-1)\big)-a\int_{-h}^{0} e^{-\nu(s+h)}e^{\lambda_2(h+s)}(h+s)\,ds$$
$$\geqslant e^{\lambda_2 h}\big(\epsilon+\epsilon\lambda_2 h+h(\epsilon\nu-1)\big)-he^{\lambda_2 h}\int_{-h}^{0} e^{-\nu(s+h)}\,ds$$
$$=e^{\lambda_2 h}\left(\epsilon+\epsilon\lambda_2 h+h(\epsilon\nu-1)+h\frac{e^{-\nu h}-1}{\nu}\right)\geqslant e^{\lambda_2 h}(\epsilon+\epsilon\lambda_2 h)>0.$$

Note that

$$h\frac{e^{-\nu h}-1}{\nu}\geqslant h\frac{-\epsilon\nu^2+\nu}{\nu}=(-\epsilon\nu+1)h$$

due to relations $0=\epsilon\nu^2-\nu-1+ae^{-\nu h}\leqslant\epsilon\nu^2-\nu-1+e^{-\nu h}$.

Hence $q=-Q(\nu)/P(\nu)$ is determined uniquely and we can find a representation for $\psi(t)$ from (34) using the inverse Laplace transform:

$$\psi(t)=\mathcal{L}^{-1}\left[\frac{\epsilon(\psi'(0)+z\psi(0))-\psi(0)-ae^{-zh}\int_{-h}^{0}\psi(s)e^{-zs}\,ds}{\chi(z)}\right](t). \tag{35}$$

We observe here that [24, Lemma 3.1] implies that $p>0$ and $q<0$ in (32).

Case of the multiple positive roots. Now, let us consider the case when $\lambda_1=\lambda_2$. Then, for some $p\geqslant 0$, $p+q>0$, we have

$$\varphi(t)=e^{\lambda_1(t+h)}\big(p+q(t+h)\big),\quad t\leqslant 0. \tag{36}$$

From (36) we get $\varphi(-h) = p = \theta$, so that $p = \theta$. Therefore $\psi(s) = \varphi(s) - \kappa$, $s \in [-h, 0]$, satisfies:

$$\psi(s) = e^{\lambda_1(h+s)}\big(\theta + q(s+h)\big) - \kappa,$$
$$\psi(0) = e^{\lambda_1 h}(\theta + qh) - \kappa, \qquad \psi'(0) = e^{\lambda_1 h}(\lambda_1\theta + \lambda_1 qh + q). \tag{37}$$

We next apply the Laplace transform to Eq. (33) subject to initial conditions (37). Analyzing the equation $\chi(\nu)(\mathcal{L}\psi)(\nu) = 0$ (which is necessary to have $\psi(+\infty) = 0$), we conclude that it can be written as $P(\nu, \lambda_1)q + Q(\nu, \lambda_1) = 0$, where $Q(\nu, \lambda_1) = Q(\nu, \lambda_1, \lambda_1)$ (and $Q(\nu, \lambda_1, \lambda_2)$ is as above) and

$$P(\nu, \lambda_1) = e^{\lambda_1 h}(\epsilon\lambda_1 h + \epsilon + \epsilon\nu h - h) - ae^{-h\nu}\int_{-h}^{0} e^{-\nu s + \lambda_1(h+s)}(h+s)\,ds.$$

Since $a < 1$, $\lambda_1 > 0$, we have

$$P(\nu, \lambda_1) \geqslant e^{\lambda_1 h}\left(\epsilon\lambda_1 h + \epsilon + (\epsilon\nu - 1)h - h\int_{-h}^{0} e^{-\nu(h+s)}\,ds\right)$$
$$= e^{\lambda_1 h}\left(\epsilon\lambda_1 h + \epsilon + (\epsilon\nu - 1)h + \frac{h}{\nu}\big(e^{-\nu h} - 1\big)\right) \geqslant e^{\lambda_1 h}(\epsilon\lambda_1 h + \epsilon) > 0.$$

Hence $q = -Q(\nu)/P(\nu)$ is determined uniquely and $\psi(t)$ is given by (35).

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