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On Combinatorics of Al-Salam Carlitz Polynomials

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A new combinatorial interpretation of the moments of Al-Salam Carlitz polynomials as 'striped' skew-shapes is used to explain the cancellation in the moments of Viennot theory for these polynomials.

1. INTRODUCTION

The Al-Salam Carlitz polynomials $U_n^{(a)}(x)$ (see [1] or [2, p. 195]), are a family of orthogonal polynomials defined by the following generating function

$$\frac{(w)_\infty(aw)_\infty}{(xw)} = \sum_{n=0}^{\infty} U_n^{(a)}(x)\frac{w^n}{(q)_n}, \tag{1}$$

where $(a)_\infty$ denotes the product $\Pi_{i=0}^{\infty}(1-aq^i)$ and $(a)_n = \Pi_{i=0}^{n-1}(1-aq^i)$.

These polynomials satisfy a recurrence relation

$$U_{n+1}^{(a)}(x) = (x-(1+a)q^n)U_n^{(a)}(x) + aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x), \qquad n \geqslant 1, \tag{2}$$

with initial conditions $U_{-1}^{(a)}(x) = 0$ and $U_0^{(a)}(x) = 1$.

Let $\left[{n \atop k}\right]_q$ denote the q-binomial number, i.e.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\Pi_{i=1}^{k}(1-q^{n-k+i})}{\Pi_{i=1}^{k}(1-q^i)} = \frac{(q)_n}{(q)_k(q)_{n-k}}.$$

The generating function (1) can be used to obtain an explicit expression for the polynomials, i.e.

$$U_n^{(a)}(x) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q (-1)^{n-i} q^{(n-i)(n-i-1)/2} \prod_{j=0}^{i-1}(x-aq^j),$$

which is equivalent to

$$U_n^{(a)}(x) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q (-1)^i x^{n-i} \left(\sum_{j=0}^{i} \begin{bmatrix} i \\ j \end{bmatrix}_q q^{(i-j)(i-j-1)/2 + j(j-1)/2} a^j \right). \tag{3}$$

Let L be the linear functional with respect to which $\{U_n^{(a)}(x)\}_{n\geqslant 0}$ are orthogonal. Then the nth moments have the following expressions:

$$L(x^n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k. \tag{4}$$

The orthogonality of $\{U_n^{(a)}(x)\}_{n\geqslant 0}$ is

$$L(U_m^{(a)}(x)U_n^{(a)}(x)) = (-a)^n q^{n(n-1)/2}(q)_n \delta_{mn}, \tag{5}$$

which will be proved combinatorially in Section 4.

Viennot gives a combinatorial model for orthogonal polynomials in the context of weighted paths and Motzkin paths. His model is based on three-term recurrence relations for orthogonal polynomials, [6, 7], and can be applied to any family of orthogonal polynomials. However, many known orthogonal polynomials have more

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structured combinatorial models, which are different from Viennot's general one (see [4, 5]).

Since the coefficient of $U_{n-1}^{(a)}(x)$ in the recurrence relation of (2) has mixed signs but the expressions for $U_n^{(a)}(x)$ in (3) and moments $L(x^n)$ in (4) have no terms to be cancelled, the $U_n^{(a)}(x)$'s and moments in Viennot's model contain some terms to be cancelled out.

In this paper we interpret $U_n^{(a)}(x)$ and the moments using the explicit expressions and find a weight-preserving bijection and several weight-preserving sign-reversing (*wpsr*) involutions to explain the cancellations in the Viennot's model of the polynomials $U_n^{(a)}(x)$ and the moments $L(x^n)$. The main results of the paper, which are in Section 3, are the encoding of Motzkin paths as striped skew-shapes and a *wpsr* involution on striped skew-shapes, achieving the desired cancellations.

2. Models of $U_n^{(a)}(x)$

According to Viennot [6], $U_n^{(a)}(x)$ is the generating function of the paths of length n, from 0 to n, with three different weighted steps: for some integer i,

(1) a step from i to $i+1$ of weight x,

(2) a step from i to $i+1$ of weight $-q^i - aq^i$, and

(3) a step from i to $i+2$ of weight $-aq^i(q^{i+1}-1)$.

We split a step of length 2 to a sequence of two steps of length 1 of weight $-aq^i$ and $q^{i+1}-1$ respectively; and replace a step of weight $-q^i - aq^i$ with two types of steps of weight $-q^i$, and $-aq^i$ respectively; and replace a step of $q^{i+1}-1$ with two types of steps of weight q^{i+1}, and -1 respectively. Then we can see that $U_n^{(a)}(x)$ is the generating function of the paths of length n with steps, for each $i \in \{0, 1, \ldots, n-1\}$,

(i) a step from i to $i+1$ of weight x, and

(ii) a step from i to $i+1$ of weight -1, q^i, $-q^i$ or $-aq^i$,

with a condition that in each path steps of weight -1 or q^i are preceded by a step the weight of which contains a factor $-a$. This condition comes from the fact that q^{i+1} is split from a step of length 2 of weight $-aq^i(q^{i+1}-1)$. There is a *wpsr* involution on this set, the fixed set of which consists of paths where, in terms of weight of steps, no $-aq^i$ is followed by q^{i+1} or $-q^{i+1}$. The involution is defined as follows: if a path contains a sequence $-aq^i$ and $\pm q^{i+1}$, then find the smallest such i and change the sequence to $-aq^i$ and $\mp q^{i+1}$. The fixed set of this *wpsr* involution consists of steps,

(i) a step from i to $i+1$ of weight x, and

(ii) a step from i to $i+1$ of weight -1, $-q^i$, $-aq^i$,

where each occurrence of a step of weight -1 is preceded by a step of weight $-aq^i$. This fixed set can be regarded as the weighted set, denoted (T_n, w), of all multi-permutations of length n with entries from $\{x, -1, -a\}$, where the weight w is defined as follows: for $\sigma = \sigma_0\sigma_1 \cdots \sigma_{n-1} \in (T_n, w)$, let $w(\sigma) = \Pi_{i=0}^{n-1} w(\sigma_i)$, where

$$w(\sigma_i) = \begin{cases} x, & \text{if } \sigma_i = x, \\ -aq^i, & \text{if } \sigma_i = -a, \\ -q^i, & \text{if } \sigma_i = -1,\ \sigma_{i-1} \neq -a, \\ -1, & \text{if } \sigma_i = -1,\ \sigma_{i-1} = -a. \end{cases}$$

However, there is another interpretation of $U_n^{(a)}(x)$, which can be read from the generating function of $U_n^{(a)}(x)$ or the explicit expression in equation (3). Let T_n denote the same set as before, i.e. the set of all multi-permutations of length n of $\{x, -a, -1\}$. We assume that $-a < -1 < x$ symbolically. We define a new weight w' on T_n as follows:

$$w'(\sigma_i) = \begin{cases} x, & \text{if } \sigma_i = x, \\ -aq^i, & \text{if } \sigma_i = -a, \\ -q^k, & \text{if } \sigma_i = -1 \quad \text{and} \quad k = |\{j : \sigma_j = -1 \text{ or } x,\ 0 \leqslant j < i\}|, \end{cases}$$

The exponent of q in $w'(\sigma)$ can be regarded as the number of inversions, where, in addition to usual inversions, each pair $(-1, -1)$ or (a, a) is counted as an inversion.

We claim that (T_n, w) and (T_n, w') have the same distribution. Note that if $q = 1$ then it is obvious. To handle the general case, we define a weight-preserving bijection Θ from (T_n, w) to (T_n, w'). Let $\sigma = \sigma_0\sigma_1 \cdots \sigma_{n-1}$ be an element of (T_n, w). Then $\Theta(\sigma) \in (T_n, w')$ is defined as follows. For each occurrence of -1 from the left in order:
(i) if it is preceded by either x or -1, then exchange each $-a$ to the left of it, one by one from right to left, with the next entry;
(ii) otherwise, exchange each of x or -1 to the left of it, one by one from right to left, with the next entry (of x or -1) (this is equivalent to moving the $-a$ just before the -1 to the beginning).
Let $\Theta(\sigma)$ be the final multi-permutation.

Since each step preserves the contribution of -1 to the weight, $\Theta(\sigma) \in (T_n, w')$ has the same weight as $\sigma \in (T_n, w)$. Moreover, this process is reversible. To go backward, we start from the rightmost -1. For each -1, if the starting entry is either x or -1, then exchange each $-a$ to the left of it, one by one from left to right, with the next entry, else exchange each occurrence of x or -1 to the left of it, one by one from left to right, with the next entry or, equivalently, move the starting $-a$ to the left of the -1, making it adjacent to the -1.

Hence $\Theta : (T_n, w) \to (T_n, w')$ is a weight-preserving bijection.

For instance, if $\sigma = (-1, -a, x, -1, x, -a, -1, -a) \in (T_n, w')$, then

$$\sigma = (\boxed{-1}, -a, x, -1, x, -a, -1, -a) \Rightarrow (-1, -a, x, \boxed{-1}, x, -a, -1, -a) \Rightarrow$$
$$(-1, x, -a, -1, x, -a, \boxed{-1}, -a) \Rightarrow (-a, -1, -a, x, -1, x, -1, -a) = \Theta(\sigma).$$

Note that we do nothing for the leftmost -1 in the first step. In each step, we rearrange the elements to the left of the boxed -1, according to the above rules.

3. Moments as 'Striped' Skew Shapes

In Viennot's theory [6], the nth moment $L(x^n)$ is the weight-generating function of the Motzkin paths on the plane of length n from $(0, 0)$ to $(n, 0)$ with the following steps:
(i) a step from (i, j) to $(i+1, j+1)$ of weight aq^i;
(ii) a step from (i, j) to $(i+1, j)$ of weight aq^i of q^i;
(iii) a step from (i, j) to $(i+1, j-1)$ of weight q^i or -1.
Since there are steps with a negative weight, $L(x^n)$ in Viennot's model is a sum involving some terms of negative coefficients. However, $L(x^n)$, in equation (4), has only positive terms. This suggests that there exist a *wpsr* involution, explaining the cancellation.

From the expression of $L(x^n)$ in equation (4), it is clear that $L(x^n)$ can be interpreted as a partition inside an $(n-k) \times k$ rectangle. A partition here is a finite weakly decreasing sequence of non-negative integers. Let P_n be the set of all multi-permutations of length n with entries $\{1, a\}$. Put a weight w on P_n as follows: for $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in P_n$, let $w(\sigma) = a^k q^l$, where k is the number of a's in σ and l is the number of pairs (i, j), $i < j$, such that $\sigma_i = a$ and $\sigma_j = 1$. An element σ in P_n with

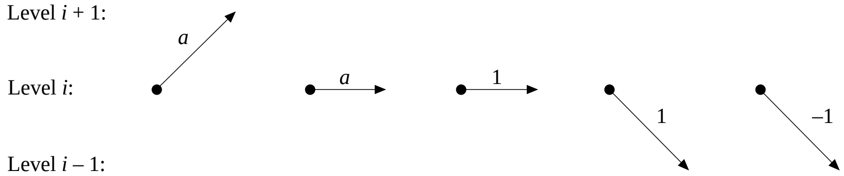

FIGURE 1. Five different kinds of step.

exactly k a's corresponds to a partition the Ferrers diagram of which fits inside an $(n-k)\times k$ rectangle, where a partition is represented as a lattice path from the lower left corner of the rectangle to the upper right corner, with a '*right*' step for a and an '*up*' step for 1. The power of q in $w(\sigma)$ corresponds to the integer of which σ is a partition.

It is not clear that the weight-generating function of P_n is equivalent to that of Motzkin paths for Al-Salam Carlitz polynomials given by Viennot [6]. We show combinatorially that the weight-generating function of P_n is the moment of x^n for Al-Salam Carlitz polynomials.

In this section, we encode Motzkin paths as '*striped*' skew-shapes and define a combinatorial *wpsr* involution on '*striped*' skew-shapes, the fixed point set of which is (P_n, w).

3.1. Encoding of a Motzkin path as a striped skew shape. We will call a step in a Motzkin path an a-step, if its weight contains a; a 1-step, if its weight is 1; or a (-1)-step, if its weight is -1. We will also use adjectives, *up, horizontal* and *down,* to describe steps in Motzkin paths. Note that there are five different kinds of step; namely, an *up* or *horizontal* a-step, a *horizontal* or *down* 1-step, and a *down* (-1)-step. One of each type of step shows in Figure 1.

We will represent a moment path of length n as a skew shape λ/μ, with some diagonal stripes, inside a rectangle of size $(n-k)\times k$ for some k. Given a lattice path from $(0,0)$ to $(k, n-k)$, there exists the unique partition λ the Ferrers diagram of which is enclosed by the lattice path and the line $x=0$ and the line $y=n-k$.

We describe how a skew shape is obtained from a moment path. We begin at the point $(0,0)$. The partition λ is determined by n steps from $(0,0)$ to $(k, n-k)$ for some k. If the ith step in the moment path is an a-step, then the ith step of λ is a horizontal unit step, called a 'right' step; otherwise, it is a vertical unit step, called an '*up*' step.

The partition μ is also determined by n steps from $(0,0)$ to $(k, n-k)$. If the ith step in the moment path is an up a-step or a horizontal 1-step, then the ith step of λ is an '*up*' step; otherwise, it is a '*right*' step.

Since the number of *up* a-steps in a Motzkin path is equal to the number of *down* steps in the path, both λ and μ have the same number of *right* steps and the same number of *up* steps.

We assume that a skew shape of shape λ/μ consists of $|\lambda|-|\mu|$ unit squares. If the ith path in the moment path is a (-1)-step, then we put a white circle inside each box in the diagonal starting from the box containing the ith step of μ and ending with the box containing the ith step of λ.

We put a black circle inside each box in μ. We will call a box with a black circle a *black box* and a box with a white circle a *white box.*

In Figure 2 it is shown how we obtain a striped skew shape from a Motzkin path.

Let M_n be the set of all objects that we can obtain from the above encoding of Motzkin paths. We can define M_n formally as follows. Note that a partition is a weakly decreasing finite sequence of non-negative integers. In particular, we allow 0 as a part of a partition.

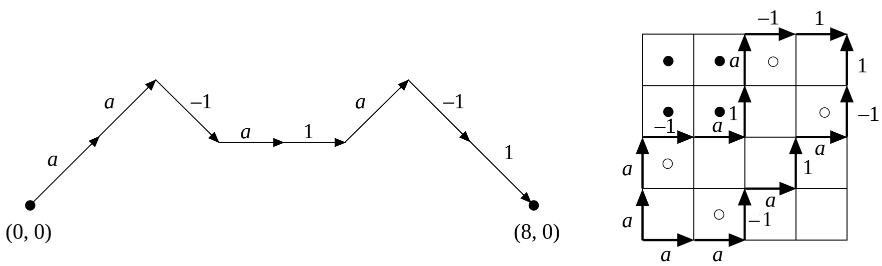

FIGURE 2. Encoding of a Motzkin path: $\mu = 2200$, $\lambda = 4432$.

DEFINITION 3.1. A striped skew shape of shape λ/μ inside a rectangle $k \times (n-k)$ is a region enclosed by two partitions $\lambda = \lambda_1\lambda_2 \cdots \lambda_k$ and $\mu = \mu_1\mu_2 \cdots \mu_k$ such that:
(i) for all i, $1 \leqslant i \leqslant k$, we have $0 \leqslant \mu_i \leqslant \lambda_i \leqslant n-k$;
(ii) some diagonals from north-west to south-east may become a *stripe* if the top leftmost box in the diagonal is the topmost box in the column of λ/μ in which the box belongs, and the bottom rightmost box in the diagonal is the rightmost box in the row of λ/μ in which the box belongs.
A stripe is denoted by putting a white circle inside each box in it. Let $M_{n,k}$ be the set of all striped skew shapes of shape λ/μ inside a rectangle $k \times (n-k)$ and let M_n be the disjoint union of $M_{n,0}$, $M_{n,1}$, $M_{n,2}, \ldots, M_{n,n}$. We can put a weight w on M_n. For $\pi \in M_{n,k}$, define the weight of π as

$$w(\pi) = (-1)^s a^k q^{|\lambda|-|\mu|-l},$$

where l is the number of boxes in stripes in π, and s is the number of stripes.

EXAMPLE. If π is the striped skew-shape in Figure 3, then $w(\pi) = (-1)^2 a^5 q^5$.

It is clear from the above description that the encoding of Motzkin paths described earlier defines a weight-preserving bijection between the set of Motzkin paths for $U_n^{(a)}(x)$ and the weighted set (M_n, w). We state this fact as a theorem without a proof.

THEOREM 3.1. *There is a weight-preserving bijection between Motzkin paths for the moments of* $U_n^{(a)}(x)$ *and the weighted set* (M_n, w).

3.2. An involution on striped skew shapes. We now want to define a *wpsr* involution the fixed set of which is P_n. Let π be a striped skew shape of shape λ/μ. The basic idea of the involution is that we change the boxes in a certain vertical strip of μ to white boxes, or a diagonal stripe of white boxes in λ/μ to black boxes. After changing the color of boxes, we arrange the colored boxes by '*floating*'. Black boxes float to the left, resulting in enlargement of μ, and white boxes float to the right, forming a diagonal stripe. If some sequence of boxes changes color and can be floated, then this sequence is called '*changeable*'.

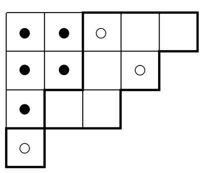

FIGURE 3. A striped skew shape of shape λ/μ: $\mu = 2210$, $\lambda = 5431$.

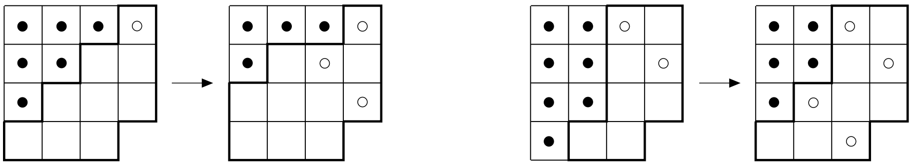

FIGURE 4. Examples of Cases 1.1 and 1.2.

We now want to define a *wpsr* involution Π on (M_n, w). Let σ be a striped skew-shape of shape λ/μ inside a $k \times (n-k)$ rectangle. If μ has any non-zero parts, then let k_1 be the number of positive parts in μ. Recall that, in Definition 3.1, a stripe is a diagonal sequence of boxes in λ/μ, such that the top leftmost box in the diagonal is the topmost box in the column of λ/μ in which the box belongs, and the bottom rightmost box in the diagonal is the rightmost box in the row of λ/μ in which the box belongs. If σ has any stripes, then let k_2 be the largest integer k_2 such that σ has a stripe ending in the k_2th row of λ; otherwise, set $k_2 = 0$. There are three cases to be considered.

CASE 1. Suppose that $k_1 > k_2$. Consider the diagonal in λ/μ ending at the last box in the k_1th row of λ. Let d_1 be the length of this diagonal. Note that a vertical strip is a skew shape which has exactly one box at each of its rows. There are two subcases, as follows.

Case 1.1. If the upper-leftmost box of the diagonal is the topmost box of a column of λ/μ, then we change the vertical strip of μ consisting of d_1 boxes contained in the last d_1 rows of μ to white boxes and float them, to form a stripe of length d_1.

Case 1.2. If the upper-leftmost box of the diagonal is not the topmost box of a column of λ/μ, then we change the vertical strip of μ consisting of $d_1 + 1$ boxes contained in the last $d_1 + 1$ rows of μ to white boxes and float them, to form a stripe of length $d_1 + 1$.

Examples for Cases 1.1 and 1.2 are shown in Figure 4.

CASE 2. Suppose that $k_1 \leqslant k_2$ and $k_2 > 0$. In this case, we change each box in the stripe ending at the k_2th row of λ to a black box, and float them to the left. If we reverse the arrows in Figure 4, we obtain examples of this case.

CASE 3. Suppose that $k_1 = k_2 = 0$. Then we do nothing.

We define $\Pi(\sigma)$ to be the resulting striped skew shape. It is clear that Π is a *wpsr* involution and Π fixes σ iff $k_1 = k_2 = 0$. In fact, Case 2 is the reverse operation of Case 1. We state this as a theorem.

THEOREM 3.2. *The map* Π *defined on* (M_n, w) *is a wpsr involution. Moreover, a striped skew shape of shape* λ/μ *is fixed by* Π *iff* $\mu = 00\cdots 0$ *and it has no stripes.*

4. ORTHOGONALITY OF $U_n^{(a)}(x)$

We first interpret $U_n^{(a)}(x)$ as (T_n, w) defined in Section 2. For each pair of integers (m, n), we define a set $O_{m,n}$ as the set of all pairs of sequences (σ, τ), where σ is a multi-permutation of length m of $1, -1, a, -a$ and τ is a multi-permutation of length n of $1, -1, a, -a$. We put a weight w on $O_{m,n}$. Let (σ, τ) be an element of $O_{m,n}$, where $\sigma = \sigma_0\sigma_1\cdots\sigma_{m-1}$, $\tau = \tau_0\tau_1\cdots\tau_{n-1}$. The weight of (σ, τ) is defined as

$$w(\sigma, \tau) = \prod_{i=0}^{m-1} w(\sigma_i) \prod_{j=0}^{n-1} w(\tau_j),$$

where $w(\sigma_i)$ and $w(\tau_j)$ are defined as follows:

$$w(\sigma_i)=\begin{cases} a, & \text{if } \sigma_i=a,\\ q^k, & \text{if } \sigma_i=1 \text{ and } k \text{ is the number of occurrences of } a \text{ before } \sigma_i,\\ -aq^i, & \text{if } \sigma_i=-a,\\ -q^i, & \text{if } \sigma_i=-1,\quad \sigma_{i-1}\neq -a,\\ -1, & \text{if } \sigma_i=-1,\quad \sigma_{i-1}=-a.\end{cases}$$

$$w(\tau_j)=\begin{cases} a, & \text{if } \tau_j=a,\\ q^k, & \text{if } \tau_j=1 \text{ and } k \text{ is the number of occurrences of } a \text{ before } \sigma \text{ and } \tau_j,\\ -aq^j, & \text{if } \tau_j=-a,\\ -q^j, & \text{if } \tau_j=-1,\quad \tau_{j-1}\neq -a,\\ -1, & \text{if } \tau_j=-1,\quad \tau_{j-1}=-a.\end{cases}$$

Note that the definition of $w(\sigma_i)$ and that of $w(\tau_j)$ are the same except for the second case.

From equation (4) and Theorem 3.2, we know that the nth moment

$$L(x^n)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q a^k$$

is interpreted as the weight of the sequences of $\{1, a\}$ of length n, where the weight contains an appropriate factor of q. So it is clear that

$$L(U_m^{(a)}(x)U_n^{(a)}(x))=\sum_{(\sigma,\tau)\in O_{m,n}} w(\sigma,\tau). \tag{6}$$

We now prove the orthogonality of $\{U_n^{(a)}(x)\}_{n\geqslant 0}$ by finding appropriate involutions in $O_{m,n}$. In fact, we will use two involutions Γ and Ψ defined below to accomplish it.

A *wpsr* involution Γ on the σ part:

Assume that $\sigma_i=1$ or -1 and $\sigma_j\neq\pm 1$ for all $j<i$. There are four cases:

(1) $\sigma_i=-1$ and $\sigma_{i-1}=-a$: set $\sigma_i=1$ and move each a to the left of σ_i to the right by 1 unit, by interchanging adjacent a and $-a$.

(2) $\sigma_i=-1$ and $\sigma_{i-1}=a$: set $\sigma_i=1$ and move each $-a$ to the left of σ_i to the right by 1 unit, by interchanging adjacent a and $-a$.

(3) $\sigma_i=1$ and $\sigma_1=-a$: set $\sigma_i=-1$ and move each a to the left of σ_i to the left by 1 unit, by interchanging adjacent a and $-a$.

(4) $\sigma_i=1$ and $\sigma_1=a$: set $\sigma_i=-1$ and move each $-a$ to the left of σ_i to the left by 1 unit, by interchanging adjacent a and $-a$.

It can be shown that these operations define a *wpsr* involution Γ. An element (σ,τ) is fixed under Γ if σ contains neither 1 nor -1.

Now we define another *wpsr* involution Ψ on the fixed set. Let k be the number of 1's in τ. If $k<m$, then changing the sign of σ_k is a *wpsr* involution. Therefore the final fixed set consists of (σ,τ), where σ consists of only a or $-a$ and τ contains at least m 1's. Hence, if $m>n$, then the fixed set is the empty set, which explains the orthogonality of $U_m^{(a)}(x)$ and $U_n^{(a)}(x)$ for $m\neq n$. For $m=n$, the fixed set consists of (σ,τ), where σ consists of only a or $-a$ and τ of only 1's. The weight of this set will be

$$\prod_{i=0}^{n-1}(aq^n-aq^i)=(-a)^n q^{n(n-1)/2}\prod_{i=1}^{n}(1-q^i).$$

We next interpret $U_n^{(a)}(x)$ as (T_n, w') defined in Section 2. In this case, the weight of the set $O_{m,n}$ defined at the beginning of this section should be changed. The change is made to only -1. As in (T_n, w), the weight of -1 will be $-q^k$ if k is the number of occurrences of either a or 1 or 1 or -1 to the left of it. The orthogonality is easier to explain. Changing the sign of first ± 1 in σ is a *wpsr* involution. The fixed set is the same as before and we can apply the *wpsr* involution Ψ on the fixed set to obtain the orthogonality for $m \neq n$.

5. Remarks

Recently, de Médicis, Stanton and White [3] interpreted q-Charlier polynomials combinatorially. Among other things, they interpreted the linearization coefficients

$$L(C_{n_1}(x, a; q)C_{n_2}(x, a; q)C_{n_3}(x, a; q))$$

combinatorially. Since it is known that

$$C_n(x, a; q) = a^n U_n^{(-1/a(1-q))}\left(\frac{x}{a} - \frac{1}{a(1-q)}\right),$$

we may expect that their approach will work for $U_n^{(a)}(x)$'s. But rescaling seems to make the problem very different.

The linearization coefficients $L(C_{n_1}(x, a; q)C_{n_2}(x, a; q)$ in [3] can be used to show that $L(U_{n_1}^{(a)}(x)U_{n_2}^{(a)}(x)U_{n_3}^{(a)}(x))$ is equal to

$$(q)_{n_1} \sum_{m=0}^{n_1+n_2-n_3} (-a)^{n_3+m} \begin{bmatrix} n_2 \\ n_1+n_2-n_3-m \end{bmatrix}_q \begin{bmatrix} n_3 \\ n_3-n_2+m \end{bmatrix}_q$$
$$\times \sum_{k=0}^{\min(m, n_1+n_2-n_3-m)} q^N (-1)^{n_1+n_2-n_3-m-k} (q)_{n_2-k} \begin{bmatrix} n_1+n_2-n_3-m \\ k \end{bmatrix}_q \begin{bmatrix} n_3-n_2+m \\ m-k \end{bmatrix}_q,$$

where

$$N = \binom{n_1}{2} + \binom{n_2}{2} + \binom{k+1}{2} + (n_3 - n_1 + m)(n_3 - n_2 + k) + m(m - n_1 - k).$$

Combinatorial interpretation of the above expression, similar to that in [3], will be interesting.

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