The Torelli theorem for the moduli spaces of connections on a Riemann surface

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Abstract

Let $(X, x_0)$ be any one-pointed compact connected Riemann surface of genus $g$, with $g \geq 3$. Fix two mutually coprime integers $r > 1$ and $d$. Let $\mathcal{M}_X$ denote the moduli space parametrizing all logarithmic $\text{SL}(r, \mathbb{C})$-connections, singular over $x_0$, on vector bundles over $X$ of degree $d$. We prove that the isomorphism class of the variety $\mathcal{M}_X$ determines the Riemann surface $X$ uniquely up to an isomorphism, although the biholomorphism class of $\mathcal{M}_X$ is known to be independent of the complex structure of $X$. The isomorphism class of the variety $\mathcal{M}_X$ is independent of the point $x_0 \in X$. A similar result is proved for the moduli space parametrizing logarithmic $\text{GL}(r, \mathbb{C})$-connections, singular over $x_0$, on vector bundles over $X$ of degree $d$. The assumption $r > 1$ is necessary for the moduli space of logarithmic $\text{GL}(r, \mathbb{C})$-connections to determine the isomorphism class of $X$ uniquely.

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1. Introduction

Our aim here is to show that the algebraic isomorphism class of a smooth moduli space of $\text{SL}(r, \mathbb{C})$-connections on a compact Riemann surface determines uniquely the isomorphism class of the Riemann surface. This moduli space is canonically biholomorphic to a certain representation space...
of the fundamental group of the Riemann surface; the biholomorphism is constructed by sending a connection to its monodromy representation. Therefore, the biholomorphism class of the moduli space is independent of the complex structure of the Riemann surface. We note that any two smooth projective varieties that are biholomorphic are actually algebraically isomorphic. However, this is not true for quasi-projective varieties.

Let \( X \) be a compact connected Riemann surface of genus \( g \), with \( g \geq 3 \). Fix a base point \( x_0 \in X \). Take any integer \( r \geq 2 \). Fix an integer \( d \) which is coprime to \( r \). The de Rham differential \( f \mapsto df \) defines a logarithmic connection on the line bundle \( \mathcal{O}_X(dx_0) \) which is singular exactly over \( x_0 \).

Let \( \mathcal{M}_X \) denote the moduli space parametrizing logarithmic connections \( (E, D) \) on \( X \) singular exactly over \( x_0 \) and satisfying the following three conditions:

1. \( E \) is a holomorphic vector bundle over \( X \) of rank \( r \) with \( \bigwedge^r E \cong \mathcal{O}_X(dx_0) \),
2. the logarithmic connection on \( \bigwedge^r E \) induced by the logarithmic connection \( D \) on \( E \) coincides with the connection on \( \mathcal{O}_X(dx_0) \) given by the Rham differential, and
3. the residue \( \text{Res}(D, x_0) = -\frac{d}{r} \text{Id}_{E|_{x_0}} \).

This moduli space \( \mathcal{M}_X \) is an irreducible smooth quasi-projective variety defined over \( \mathbb{C} \) of dimension \( 2(r^2 - 1)(g - 1) \). The isomorphism class of the variety \( \mathcal{M}_X \) is independent of the choice of the point \( x_0 \in X \); see Remark 2.2 for the details.

We prove the following Torelli type theorem:

**Theorem 1.1.** The isomorphism class of the Riemann surface \( X \) is uniquely determined by the isomorphism class of the variety \( \mathcal{M}_X \). In other words, if \( (Y, y_0) \) is another one-pointed compact connected Riemann surface of genus \( g \) and \( \mathcal{M}_Y \) the moduli space of rank \( r \) logarithmic connections \( (E, D) \) over \( Y \) singular exactly over \( y_0 \) such that \( \bigwedge^r E \cong \mathcal{O}_X(dy_0) \), the connection on \( \bigwedge^r E \) induced by \( D \) is the de Rham connection on \( \mathcal{O}_X(dy_0) \), and \( \text{Res}(D, y_0) = -\frac{d}{r} \text{Id}_{E|_{y_0}} \), then the two varieties \( \mathcal{M}_X \) and \( \mathcal{M}_Y \) are isomorphic if and only if the two Riemann surfaces \( X \) and \( Y \) are isomorphic.

The proof of Theorem 1.1 involves investigations of the mixed Hodge structure of the moduli space \( \mathcal{M}_X \).

In the special case where \( r = 2 < g - 1 \), a similar result was proved in [3].

Fix a point \( x' \in X \setminus \{x_0\} \): \( X' \). Let

\[
\text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C})) \subset \text{Hom}(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))
\]

be the space of homomorphisms from the fundamental group \( \pi_1(X', x') \) to \( \text{SL}(r, \mathbb{C}) \) satisfying the condition that the image of the free homotopy class of oriented loops in \( X' \) around \( x_0 \) (with anticlockwise orientation) is \( \exp(2\pi \sqrt{-1}d/r) \cdot I_{r \times r} \). Any oriented loop in \( X' \) gives a conjugacy class in \( \pi_1(X', x') \). The above condition says that all elements in the conjugacy class of the oriented loops around \( x_0 \) are mapped to \( \exp(2\pi \sqrt{-1}d/r) \cdot I_{r \times r} \).

Let

\[
\mathcal{R}_g := \text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))/\text{SL}(r, \mathbb{C})
\]

be the quotient for the action of \( \text{SL}(r, \mathbb{C}) \) given by the conjugation action of \( \text{SL}(r, \mathbb{C}) \) on itself. The algebraic structure of \( \text{SL}(r, \mathbb{C}) \) induces an algebraic structure on \( \mathcal{R}_g \). It is known that \( \mathcal{R}_g \) is an irreducible smooth quasi-projective variety defined over \( \mathbb{C} \) of dimension \( 2(r^2 - 1)(g - 1) \). The variety \( \mathcal{R}_g \) does not depend on the choice of the point \( x' \) as the fundamental groups of a space for two different base points
are canonically identified up to an inner automorphism. The isomorphism class of the variety \( \mathcal{R}_g \) is independent of the complex structure of \( X \). The isomorphism class depends only on the genus of \( X \) for fixed \( r \) and \( d \).

Given any logarithmic connection \((E, D) \in \mathcal{M}_X\), consider the corresponding monodromy representation of \( \pi_1(X', x') \) on the fiber \( E_{x'} \). From the conditions on \( D \) it follows that this monodromy representation lies in \( \text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C})) \). Consequently, we get a map

\[
\sigma : \mathcal{M}_X \longrightarrow \mathcal{R}_g
\]

that sends any connection to its monodromy representation. It is known that \( \sigma \) is actually a biholomorphism \([31, \text{p. 26, Theorem 7.1}]\). This immediately implies that the complex manifold \( \mathcal{M}_X \) is biholomorphic to the complex manifold \( \mathcal{M}_Y \) defined in \( \text{Theorem 1.1} \).

Take \( r \) and \( d \) as above. Let \( \widehat{M}_X \) denote the moduli space parametrizing logarithmic connections \((E, D)\) on \( X \), singular exactly over \( x_0 \), such that

- \( \text{rank}(E) = r \), and
- \( \text{Res}(D, x_0) = -\frac{d}{r} \text{Id}_{E_{x_0}} \).

The second condition implies that \( \text{degree}(E) = d \). It is known that \( \widehat{M}_X \) is a smooth irreducible quasiprojective variety defined over \( \mathbb{C} \) of dimension \( 2(r^2(g - 1) + 1) \). The isomorphism class of the variety \( \widehat{M}_X \) is independent of the choice of the point \( x_0 \).

We prove the following analog of \( \text{Theorem 1.1} \):

**Theorem 1.2.** The isomorphism class of the Riemann surface \( X \) is uniquely determined by the isomorphism class of the variety \( \widehat{M}_X \).

Since \( \widehat{M}_X \) is biholomorphic to the representation space of the fundamental group obtained by replacing \( \text{SL}(n, \mathbb{C}) \) with \( \text{GL}(n, \mathbb{C}) \) in Eq. (1.1), the biholomorphism class of \( \widehat{M}_X \) is also independent of the complex structure of \( X \).

It should be mentioned that \( \text{Theorem 1.2} \) fails for \( r = 1 \); see \( \text{Remark 5.4} \) for the details.

We do not know if \( \text{Theorems 1.1 and 1.2} \) remain valid when \( g = 2 \). In our proof of \( \text{Theorems 1.1 and 1.2} \), we needed the codimension of the subvariety in \( \text{Lemma 3.1} \) to be at least two. We should mention that this is the only place where our assumption \( g > 2 \) is used.

2. Biholomorphism class of the moduli space

Let \( X \) be a compact connected Riemann surface. The genus of \( X \), which will be denoted by \( g \), is assumed to be at least three. The holomorphic cotangent bundle of \( X \) will be denoted by \( K_X \). The holomorphic tangent bundle of \( X \) will be denoted by \( TX \). Fix a base point \( x_0 \in X \).

Take a holomorphic vector bundle \( E \) over \( X \). A logarithmic connection on \( E \) singular over \( x_0 \) is a holomorphic differential operator

\[
D : E \longrightarrow E \otimes K_X \otimes \mathcal{O}_X(x_0)
\]  

(2.1)

satisfying the Leibniz identity which says that

\[
D(fs) = f \cdot D(s) + s \otimes df,
\]
where $f$ is any locally defined holomorphic function on $X$, and $s$ is any locally defined holomorphic section of $E$. The above Leibniz identity implies that the order of $D$ is exactly one. More precisely, a logarithmic connection on $E$ singular over $x_0$ is a holomorphic differential operator of order one

$$D \in H^0(X, \text{Diff}_X^1(E, E \otimes K_X \otimes \mathcal{O}_X(x_0)))$$

whose symbol, which is a holomorphic section of the vector bundle

$$\text{Hom}(E, E \otimes K_X \otimes \mathcal{O}_X(x_0)) \otimes TX = \text{End}(E) \otimes \mathcal{O}_X \mathcal{O}_X(x_0),$$

coincides with the section of $\text{End}(E) \otimes \mathcal{O}_X \mathcal{O}_X(x_0)$ given by the identity automorphism of the vector bundle $E$.

The fiber over $x_0$ of the line bundle $K_X \otimes \mathcal{O}_X \mathcal{O}_X(x_0)$ is canonically identified with $\mathbb{C}$. This identification is obtained by sending any holomorphic section of $K_X \otimes \mathcal{O}_X \mathcal{O}_X(x_0)$ defined around $x_0$, which is a meromorphic form with a pole at $x_0$ of order at most one, to its residue at $x_0$.

Let $D$ be a logarithmic connection on $E$ singular over $x_0$. Consider the composition

$$E \xrightarrow{D} E \otimes K_X \otimes \mathcal{O}_X(x_0) \rightarrow (E \otimes K_X \otimes \mathcal{O}_X(x_0))_{x_0} = E_{x_0},$$

where the last equality is obtained using the above mentioned identification of the line $(K_X \otimes \mathcal{O}_X(x_0))_{x_0}$ with $\mathbb{C}$, and the second homomorphism is the evaluation at $x_0$ of sections. This composite homomorphism is $\mathcal{O}_X$-linear. Hence it gives a linear endomorphism of the fiber $E_{x_0}$. This endomorphism is called the residue of $D$ at $x_0$, and it is denoted by $\text{Res}(D, x_0)$. (See [9, p. 53].) We have

$$\text{degree}(E) = -\text{trace}(\text{Res}(D, x_0))$$

(see the last sentence of Corollary B.3 in [12, p. 186]).

Fix an integer $r \geq 2$, and also fix an integer $d$ which is coprime to $r$. Let $D_0$ denote the logarithmic connection on the line bundle $\mathcal{O}_X(dx_0)$, singular over $x_0$, defined by the de Rham differential that sends any locally defined meromorphic function $f$ on $X$ to the form $df$; here we identify the sheaf of sections of the line bundle $\mathcal{O}_X(dx_0)$ with the sheaf of meromorphic functions on $X$ with a pole at $x_0$ of order at most $d$. It is straightforward to check that the residue of $D_0$ at $x_0$ is $-d$. Note that this also follows from Eq. (2.2).

Let $\mathcal{M}_X$ denote the moduli space parametrizing all pairs $(E, D)$ of the following type:

- $E$ is a holomorphic vector bundle of rank $r$ over $X$ with $\bigwedge^r E \cong \mathcal{O}_X(dx_0)$,
- $D$ is a logarithmic connection on $E$ singular over $x_0$ with

$$\text{Res}(D, x_0) = -\frac{d}{r} \text{Id}_{E_{x_0}}$$

- the connection on $\bigwedge^r E \cong \mathcal{O}_X(dx_0)$ induced by $D$ coincides, by some (hence any) isomorphism between $\bigwedge^r E$ and $\mathcal{O}_X(dx_0)$, with the de Rham connection $D_0$ on $\mathcal{O}_X(dx_0)$ defined above.

See [30,31,27] for the construction of this moduli space $\mathcal{M}_X$. The scheme $\mathcal{M}_X$ is a reduced and irreducible quasiprojective variety defined over $\mathbb{C}$, and its (complex) dimension is $2(r^2 - 1)(g - 1)$, where $g$ is the genus of $X$.

Take any logarithmic connection $(E, D) \in \mathcal{M}_X$. Since $d$ and $r$ are mutually coprime, the connection $D$ is irreducible in the sense that no nonzero proper holomorphic subbundle of $E$ is preserved by $D$ [6, p. 787, Lemma 2.3]. Using this it follows that the variety $\mathcal{M}_X$ is smooth. See [14] for some topological properties of $\mathcal{M}_X$. 
Let $X' := X \setminus \{x_0\}$ be the complement. Fix a point $x' \in X'$. The point $x_0$ gives a conjugacy class in the fundamental group $\pi_1(X', x')$ as follows: Let

$$f : \mathbb{D} \to X'$$

be an orientation preserving embedding of the closed unit disk $\mathbb{D} := \{z \in \mathbb{C} | \|z\|^2 \leq 1\}$ into the Riemann surface $X$ such that $f(0) = x_0$. The free homotopy class of the map $\partial \mathbb{D} = S^1 \to X'$ obtained by restricting $f$ to the boundary of $\mathbb{D}$ is independent of the choice of $f$. The orientation of $\partial \mathbb{D}$ coincides with the anti-clockwise rotation around $x_0$. Any free homotopy class of oriented loops in $X'$ gives a conjugacy class in $\pi_1(X', x')$. Let $\gamma$ denote the orbit in $\pi_1(X', x')$, for the conjugation action of $\pi_1(X', x')$ on itself, defined by the above free homotopy class of oriented loops associated to $x_0$.

Let

$$\text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C})) \subset \text{Hom}(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))$$

be the space of homomorphisms from $\pi_1(X', x')$ to $\text{SL}(r, \mathbb{C})$ satisfying the condition that the image of $\gamma$ (defined above) is $\exp(2\pi \sqrt{-1} d/r) \cdot I_{r \times r}$. It should be clarified that since $\exp(2\pi \sqrt{-1} d/r) \cdot I_{r \times r}$ is in the center of $\text{SL}(r, \mathbb{C})$, a homomorphism sends the orbit $\gamma$ in $\pi_1(X', x')$ (for the adjoint action of $\pi_1(X', x')$ on itself) to $\exp(2\pi \sqrt{-1} d/r) \cdot I_{r \times r}$ if and only if there is an element in the orbit which is mapped to $\exp(2\pi \sqrt{-1} d/r) \cdot I_{r \times r}$.

Take any homomorphism $\rho \in \text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))$. Let $(V, \nabla)$ be the flat vector bundle or rank $r$ over $X'$ given by $\rho$. The monodromy of $\nabla$ along the oriented loop $\gamma$ is $\exp(2\pi \sqrt{-1} d/r) \cdot I_{r \times r}$. Using the logarithm $2\pi \sqrt{-1} d/r \cdot I_{r \times r}$ of the monodromy, the vector bundle $V$ over $X'$ extends to a holomorphic vector bundle $\overline{V}$ over $X$, and furthermore, the connection $\nabla$ on $V$ extends to a logarithmic connection $\overline{\nabla}$ on the vector bundle $\overline{V}$ over $X$ such that $(\overline{V}, \overline{\nabla}) \in \mathcal{M}_X$ (see [21, p. 159, Theorem 4.4]).

The easiest way to construct the above mentioned extension $(\overline{V}, \overline{\nabla})$ is to fix a ramified Galois covering of the surface $X$

$$\phi : Y \to X$$

of degree $r$ which is totally ramified over $x_0$ (it may have other ramification points). Let $y_0 \in Y$ be the unique point such that

$$\phi(y_0) = x_0.$$ 

(2.4)

The connection $\phi^*\nabla$ on the vector bundle $\phi^*V$ over $\phi^{-1}(X') = Y \setminus \{y_0\} \subset Y$ has trivial monodromy around the point $y_0$. Indeed, this follows from the given condition that the map $\phi$ is totally ramified over $x_0$, combined with the fact that the order of the matrix $\exp(2\pi \sqrt{-1} d/r) \cdot I_{r \times r}$ is a divisor of the degree of $\phi$. Consequently, the flat vector bundle $(\phi^*V, \phi^*\nabla)$ has a canonical extension to $Y$ as a flat vector bundle (take the direct image of the local system by the inclusion map $\phi^{-1}(X') \hookrightarrow Y$). Let $(W, \nabla')$ denote this flat vector bundle over $Y$ obtained by extending $(\phi^*V, \phi^*\nabla)$. Let $\nabla_0$ denote the de Rham logarithmic connection on the line bundle $\mathcal{O}_Y(d_{y_0})$ that sends any locally defined meromorphic function $f$ to $df$. The vector bundle $W \otimes \mathcal{O}_Y(d_{y_0})$ is equipped with the logarithmic connection

$$\nabla'' := \nabla' \otimes \text{Id}_{\mathcal{O}_Y(d_{y_0})} + \text{Id}_W \otimes \nabla_0.$$ 

The direct image $\phi_*(W \otimes \mathcal{O}_Y(d_{y_0}))$ over $X$ is equipped with an action of the Galois group $\text{Gal}(\phi)$ for $\phi$, with $\text{Gal}(\phi)$ acting trivially on $X$. The above mentioned holomorphic vector bundle $\overline{V}$ over $X$ is identified with the invariant part $(\phi_*(W \otimes \mathcal{O}_Y(d_{y_0})))^{\text{Gal}(\phi)}$ for this action of $\text{Gal}(\phi)$ on
\(\phi_s(W \otimes \mathcal{O}_Y(\text{d}y_0))\). The logarithmic connection \(\nabla\) on \(\nabla\) coincides with the one induced by the logarithmic connection \(\nabla''\) on \(W \otimes \mathcal{O}_Y(\text{d}y_0)\).

Since \(\text{SL}(r, \mathbb{C})\) is an algebraic group defined over \(\mathbb{C}\), and \(\pi_1(X', x')\) is a finitely presented group, the representation space \(\text{Hom}(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))\) is a complex algebraic variety in a natural way. The closed subvariety \(\text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))\) defined in Eq. (2.3) is actually smooth. The smoothness of \(\text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))\) follows from the fact that any representation in \(\text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))\) is irreducible; the irreducibility of such a representation follows from [6, p. 787, Lemma 2.3]. The conjugation action of \(\text{SL}(r, \mathbb{C})\) on itself induces an action of \(\text{SL}(r, \mathbb{C})\) on \(\text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))\). The action of any \(T \in \text{SL}(r, \mathbb{C})\) on \(\text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))\) sends any homomorphism \(\rho\) to the homomorphism \(\pi_1(X', x') \rightarrow \text{SL}(r, \mathbb{C})\) defined by \(\beta \mapsto T^{-1} \rho(\beta) T\). Let

\[
\mathcal{R}_g := \text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))/\text{SL}(r, \mathbb{C})
\]  

be the quotient space for this action.

The algebraic structure of \(\text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))\) induces an algebraic structure on the quotient \(\mathcal{R}_g\). The scheme \(\mathcal{R}_g\) is an irreducible smooth quasiprojective variety of dimension \(2(r^2 - 1)(g - 1)\) defined over \(\mathbb{C}\). We recall that \(\mathcal{R}_g\) and \(\mathcal{M}_X\) are known as the \textit{Betti moduli space} and the \textit{de Rham moduli space} respectively (see [29,30,14]).

If we replace the base point \(x' \in X'\) by another point \(x'' \in X'\), then we can construct an isomorphism of \(\pi_1(X', x')\) with \(\pi_1(X', x'')\) by fixing a path connecting \(x'\) to \(x''\). Consequently, \(\pi_1(X', x')\) and \(\pi_1(X', x'')\) are identified up to an inner automorphism. This in turn implies that the variety \(\mathcal{R}_g\) is canonically identified with the variety obtained by replacing \(x'\) with \(x''\) in the construction of \(\mathcal{R}_g\). In other words, \(\mathcal{R}_g\) does not depend on the choice of the point \(x'\).

Given any two one-pointed compact connected oriented \(C^\infty\) surfaces of genus \(g\), there is an orientation preserving diffeomorphism between them that takes the marked point in one surface to the marked point in the other surface. Therefore, the isomorphism class of the variety \(\mathcal{R}_g\) depends only on the integers \(g, r\) and \(d\). In particular, the isomorphism class of this variety is independent of the complex structure of the topological surface \(X\).

Given any \((E, D) \in \mathcal{M}_X\), the monodromy representation of \(D\)

\[
\pi_1(X', x') \rightarrow \text{Aut}(E_{x'})
\]

gives an element of \(\mathcal{R}_g\). The inverse map is obtained from the earlier mentioned construction that associates an element of \(\mathcal{M}_X\) to each element of \(\text{Hom}^0(\pi_1(X', x'), \text{SL}(r, \mathbb{C}))\).

These two maps are inverses of each other [31, p. 26, Theorem 7.1]. In particular, the following proposition holds.

**Proposition 2.1.** The moduli space \(\mathcal{M}_X\) is biholomorphic to \(\mathcal{R}_g\). Therefore, if \((Y, y_0)\) is another one-pointed compact connected Riemann surface of genus \(g\) and \(\mathcal{M}_Y\) the corresponding moduli space of logarithmic connections, then the two complex manifolds \(\mathcal{M}_X\) and \(\mathcal{M}_Y\) are biholomorphic.

**Remark 2.2.** The isomorphism class of the variety \(\mathcal{M}_X\) is independent of the choice of the point \(x_0 \in X\). To prove this, take another point \(x_1 \in X\). Let \(\mathcal{M}'_X\) be the moduli space of logarithmic connections on \(X\), singular exactly over \(x_1\), obtained by replacing \(x_0\) with \(x_1\) in the construction of \(\mathcal{M}_X\).

Fix a holomorphic line bundle \(L\) over \(X\) such that \(\mathcal{O}_X(x_1 - x_0)\) is isomorphic to \(L^{\otimes d}\). Let \(D'_0\) denote the de Rham logarithmic connection on the line bundle \(\mathcal{O}_X(x_1 - x_0)\) that sends any locally defined
meromorphic function $f$ to $df$. We note that there is a unique logarithmic connection $D_0$ on $L$, singular over both $x_0$ and $x_1$, such that the logarithmic connection on the tensor product $L \otimes d \cong O_X(x_1 - x_0)$ induced by $D_0$ coincides with the de Rham connection $D'_0$ on $O_X(x_1 - x_0)$.

For any $(E, D) \in \mathcal{M}_X$, it is easy to see that $(E \otimes L, D \otimes \text{Id}_L + \text{Id}_E \otimes D_0) \in \mathcal{M}_X'^{1}$. The map

$$\mathcal{M}_X \rightarrow \mathcal{M}_X'^{1}$$

defined by

$$(E, D) \mapsto (E \otimes L, D \otimes \text{Id}_L + \text{Id}_E \otimes D_0)$$

is an algebraic isomorphism of varieties.

In the next section we will investigate the algebraic structure of $\mathcal{M}_X$.

3. The second intermediate Jacobian of the moduli space

Let $U \subset \mathcal{M}_X$ be the Zariski open subset parametrizing all $(E, D)$ such that the underlying vector bundle $E$ is stable. The openness of this subset follows from [28, p. 182, Proposition 10]. Let $\mathcal{N}_X$ denote the moduli space parametrizing all stable vector bundles $E$ over $X$ with rank$(E) = r$ and $\bigwedge^r E \cong O_X(dx_0)$. The moduli space $\mathcal{N}_X$ is an irreducible smooth projective variety of dimension $(r^2 - 1)(g - 1)$ defined over the field $\mathbb{C}$. Let

$$\Phi : U \rightarrow \mathcal{N}_X$$

denote the forgetful map that sends any $(E, D)$ to $E$.

It is know that any $E \in \mathcal{N}_X$ admits a logarithmic connection $D$ such that $(E, D) \in \mathcal{M}_X$ [6, p. 787, Lemma 2.3]. Therefore, the projection $\Phi$ in Eq. (3.2) is surjective. Furthermore, $\Phi$ makes $U$ an affine bundle over $\mathcal{N}_X$. More precisely, $U$ is a torsor over $\mathcal{N}_X$ for the holomorphic cotangent bundle $T^*\mathcal{N}_X$. This means that the fibers of the vector bundle $T^*\mathcal{N}_X$ act freely transitively on the fibers of $\Phi$ [6, p. 786]. Since $\mathcal{M}_X$ is irreducible, and $U$ is nonempty, the subset $U \subset \mathcal{M}_X$ is Zariski dense.

**Lemma 3.1.** Let $\mathcal{Z} := \mathcal{M}_X \setminus U$ be the complement of the Zariski open dense subset. The codimension of the Zariski closed subset $\mathcal{Z}$ in $\mathcal{M}_X$ is at least $(r - 1)(g - 2) + 1$.

**Proof.** Let $E$ be a holomorphic vector bundle over $X$ with rank$(E) = r$ and $\bigwedge^r E \cong O_X(dx_0)$. Assume that $E$ admits a logarithmic connection $D$ singular over $x_0$ such that $(E, D) \in \mathcal{M}_X$.

Fix a logarithmic connection $D$ on $E$ singular over $x_0$ such that $(E, D) \in \mathcal{M}_X$. Let $T \in H^0(X, \text{End}(E))$ be a holomorphic endomorphism of $E$ satisfying the identity

$$D \circ T = (T \otimes \text{Id}_{K_X \otimes O_X(x_0)}) \circ D.$$  

We will show that there is a complex number $\lambda$ such that $T = \lambda \cdot \text{Id}_E$, or in other words, any endomorphism of the pair $(E, D)$ is a scalar multiplication.
To prove this, first note that as $X$ is a compact connected Riemann surface, it does not admit any nonconstant holomorphic functions. In particular, $\text{trace}(T^j)$ is a constant function on $X$ for all $j \geq 1$. This implies that the eigenvalues, as well as their multiplicities, of $T(x) \in \text{End}(E_x)$ are independent of $x \in X$. Consequently, for any eigenvalue $\alpha$ of $T(x)$, where $x$ is some point of $X$, the generalized eigenspaces of the endomorphisms $T(y)$, $y \in X$, for the eigenvalue $\alpha$ patch together to define a holomorphic subbundle of $E$ of positive rank. Let $E^\alpha$ denote this holomorphic subbundle of $E$ defined by the generalized eigenspaces for the eigenvalue $\alpha$.

From the identity in Eq. (3.3) it follows immediately that $E^\alpha$ is preserved by the connection $D$. Since $D$ is irreducible [6, p. 787, Lemma 2.3], this implies that $E^\alpha = E$. In other words, for any $y \in X$, the endomorphism $T(y) \in \text{End}(E_y)$ has exactly one eigenvalue, namely $\alpha$.

Consider the endomorphism

$$T_0 = T - \alpha \cdot \text{Id}_E \in H^0(X, \text{End}(E)).$$

Since $\alpha$ is the only eigenvalue of $T(y)$, $y \in X$, it follows that $T_0$ is a nilpotent endomorphism of $E$. From Eq. (3.3) it follows immediately that

$$D \circ T_0 = (T_0 \otimes \text{Id}_{K_X \otimes \mathcal{O}_X(x_0)}) \circ D. \quad (3.4)$$

Let $E_0 \subset E$ be the coherent subsheaf defined by the kernel of the above endomorphism $T_0$. Note that as $T_0$ is nilpotent, the subsheaf $E_0$ is nonzero. From Eq. (3.4) it follows that $E_0$ is preserved by the connection $D$. Since $D$ is irreducible, this implies that $E_0 = E$. Consequently, $T_0 = 0$. In other words, we have $T = \alpha \cdot \text{Id}_E$.

Let $\mathcal{D}$ denote the space of all logarithmic connections $D$ on $E$, singular exactly over the point $x_0$, such that $(E, D) \in \mathcal{M}_X$. We note that $\mathcal{D}$ is an affine space for the vector space $H^0(X, \text{ad}(E) \otimes K_X)$, where $\text{ad}(E) \subset \text{End}(E)$ is the subbundle of rank $r^2 - 1$ defined by the sheaf of endomorphisms of $E$ of trace zero [6, p. 786]. We have shown above that for the natural action of the global automorphism group $\text{Aut}(E)$ on $\mathcal{D}$, the isotropy at any point is the subgroup defined by all automorphisms of the form $\lambda \cdot \text{Id}_E$ with $\lambda \in \mathbb{C}^\ast$.

Therefore, the space of all isomorphism classes of logarithmic connections $D$, on the given vector bundle $E$, such that $(E, D) \in \mathcal{M}_X$ is of dimension

$$h^0(X, \text{ad}(E) \otimes K_X) - (\dim \text{Aut}(E) - 1) = h^1(X, \text{ad}(E)) - h^0(X, \text{ad}(E)) = (r^2 - 1)(g - 1),$$

where the last equality is the Riemann–Roch formula, and the first equality follows from the Serre duality.

Therefore, if a holomorphic vector bundle $E'$ over $X$ admits a logarithmic connection $D'$ singular over $x_0$ such that $(E', D') \in \mathcal{M}_X$, then the dimension of the space of all isomorphism classes of such logarithmic connections on $E'$ is actually independent of the choice of $E'$.

We will now estimate the dimension of the isomorphism classes of non-semistable vector bundles over $X$ that arise in the family parametrized by $\mathcal{M}_X$.

Take any $(E, D) \in \mathcal{M}_X$ such that the underlying vector bundle $E$ is not stable. Since $r$ and $d$ are mutually coprime, this implies that $E$ is not semistable. Let

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E \quad (3.5)$$

be the Harder–Narasimhan filtration of $E$. We recall that the collection of pairs of integers $\{(\text{rank}(E_i), \text{degree}(E_i))\}_{i=1}^{\ell}$ is called the Harder–Narasimhan polygon of $E$ (see [28, p. 173]).
The space of all isomorphism classes of holomorphic vector bundles over $X$, whose Harder–Narasimhan polygon coincides with that of the given vector bundle $E$, is of dimension at most $r^2(g - 1) - (r - 1)(g - 2)$. This follows from [2, p. 247–248]; we will give below the details of the argument.

Since
\[
\frac{\text{degree}(E_i/E_{i-1})}{\text{rank}(E_i/E_{i-1})} > \frac{\text{degree}(E_{i+1}/E_i)}{\text{rank}(E_{i+1}/E_i)}
\]
for all $i \in [1, \ell - 1]$ in Eq. (3.5), the integer $c$ in the middle of page 247 of [2] can actually be taken to be zero. Indeed, this follows immediately from the inductive construction of $c$ in [2, p. 248] (see the comment in [2, p. 247] just after $c$ is introduced). In other words, $a_r$ defined in [2, p. 247] satisfies the inequality
\[
a_r \leq r^2(g - 1) - (r - 1)(g - 2) - g
\]
(see [2, p. 247]); note that the term $\sum_{1 \leq i < j \leq r} n_i n_j$ in [2, p. 247] is at least $(\sum_i n_i) - 1$ (the index $\ell$ here is $r$ in [2], and $\sum_i n_i$ in [2] is $r$ here). The dimension of the space of all isomorphism classes of vector bundles over $X$ whose Harder–Narasimhan polygon coincides with the Harder–Narasimhan polygon of the vector bundle $E$ is at most $a_r$ [2, p. 248].

Therefore, the space of all isomorphism classes of vector bundles $E'$ over $X$ whose Harder–Narasimhan polygon coincides with that of $E$, and $\bigwedge^r E' \cong \mathcal{O}_X(dx_0)$, is of dimension at most $r^2(g - 1) - (r - 1)(g - 2) - g$.

Since the dimension of the space of all isomorphism classes of logarithmic connections, lying in $\mathcal{M}_X$, on any given vector bundle $E'$ is $(r^2 - 1)(g - 1)$ (assuming that it admits a connection), the subvariety of $\mathcal{M}_X$ parametrizing all pairs of the form $(E', D') \in \mathcal{M}_X$ such that the Harder–Narasimhan polygon of $E'$ coincides with that of $E$ is at most $r^2(g - 1) - (r - 1)(g - 2) - g + (r^2 - 1)(g - 1)$; that this subset of $\mathcal{M}_X$ is algebraic follows from [28, p. 182, Proposition 10]. We also note that there are only finitely many Harder–Narasimhan polygons that occur for the vector bundles over $X$ in the family parametrized by $\mathcal{M}_X$ [28, p. 183, Proposition 11].

Since $\dim \mathcal{M}_X = 2(r^2 - 1)(g - 1)$, we have
\[
\dim \mathcal{M}_X - (r^2(g - 1) - (r - 1)(g - 2) - g + (r^2 - 1)(g - 1)) = (r - 1)(g - 2) + 1.
\]
This completes the proof of the lemma. $\square$

For any $i \geq 0$, the $i$-th cohomology of a complex variety with coefficients in $\mathbb{Z}$ is equipped with a mixed Hodge structure [10,11].

**Proposition 3.2.** The mixed Hodge structure $H^3(\mathcal{M}_X, \mathbb{Z})$ is pure of weight three. More precisely, the mixed Hodge structure $H^3(\mathcal{M}_X, \mathbb{Z})$ is isomorphic to the Hodge structure $H^3(\mathcal{N}_X, \mathbb{Z})$, where $\mathcal{N}_X$ is the moduli space of stable vector bundles introduced at the beginning of this section.

**Proof.** Consider the diagram of morphisms
\[
\mathcal{N}_X \leftarrow \Phi \mathcal{U} \rightarrow \mathcal{M}_X,
\]
where $\Phi$ is the projection in Eq. (3.2), and $\iota$ is the inclusion map.
Since $\mathcal{U} \xrightarrow{\phi} \mathcal{N}_X$ is a fiber bundle with contractible fibers, the induced homomorphism
\[ \phi^*: H^i(\mathcal{N}_X, \mathbb{Z}) \rightarrow H^i(\mathcal{U}, \mathbb{Z}) \] (3.7)
is an isomorphism for all $i \geq 0$. Therefore, $\phi$ induces an isomorphism of the two mixed Hodge structures $H^i(\mathcal{N}_X, \mathbb{Z})$ and $H^i(\mathcal{U}, \mathbb{Z})$. Since $\mathcal{N}_X$ is a smooth projective variety, we know that $H^i(\mathcal{N}_X, \mathbb{Z})$ is a pure Hodge structure of weight $i$. Thus $H^i(\mathcal{U}, \mathbb{Z})$ is a pure Hodge structure of weight $i$. In particular, $H^3(\mathcal{U}, \mathbb{Z})$ is a pure Hodge structure of weight three.

Let $H^3(\mathcal{M}_X, \mathcal{U}, \mathbb{Z}) \rightarrow H^3(\mathcal{M}_X, \mathbb{Z}) \xrightarrow{\psi} H^3(\mathcal{U}, \mathbb{Z}) \rightarrow H^4(\mathcal{M}_X, \mathcal{U}, \mathbb{Z})$ (3.8)
be the long exact sequence of relative cohomologies. We note that the above homomorphism $\psi$ is a morphism of mixed Hodge structures. In fact, Eq. (3.8) itself is an exact sequence of mixed Hodge structures [11, p. 43, Proposition (8.3.9)], but we will not need this here. Let
\[ Z := \mathcal{M}_X \setminus \mathcal{U} \]
be the complement, which is a closed subscheme. Since $g \geq 3$ and $r \geq 2$, from Lemma 3.1 we know that the (complex) codimension of $Z$ in $\mathcal{M}_X$ is at least two. Therefore, the relative cohomology has the following properties:
\[ H^i(\mathcal{M}_X, \mathcal{U}, \mathbb{Z}) = 0 \] (3.9)
for all $i < 4$, and $H^4(\mathcal{M}_X, \mathcal{U}, \mathbb{Z})$ is torsion-free. Consequently, Eq. (3.8) gives the exact sequence
\[ 0 \rightarrow H^3(\mathcal{M}_X, \mathbb{Z}) \xrightarrow{\psi} H^3(\mathcal{U}, \mathbb{Z}) \rightarrow H^4(\mathcal{M}_X, \mathcal{U}, \mathbb{Z}) \] (3.10)

We observed earlier that $H^3(\mathcal{U}, \mathbb{Z})$ is a pure Hodge structure of weight three. Therefore, for any smooth compactification $\overline{\mathcal{U}}$ of $\mathcal{U}$, the homomorphism of mixed Hodge structures
\[ H^3(\overline{\mathcal{U}}, \mathbb{Q}) \rightarrow H^3(\mathcal{U}, \mathbb{Q}), \]
induced by the inclusion map $\mathcal{U} \hookrightarrow \overline{\mathcal{U}}$, is surjective [10, p. 39, Corollaire (3.2.17)]. Consequently, choosing $\overline{\mathcal{U}}$ to be a smooth compactification of $\mathcal{M}_X$, from the surjectivity of the composition homomorphism of mixed Hodge structures
\[ H^3(\overline{\mathcal{U}}, \mathbb{Q}) \rightarrow H^3(\mathcal{M}_X, \mathbb{Q}) \xrightarrow{\psi} H^3(\mathcal{U}, \mathbb{Q}) \]
we conclude that the homomorphism
\[ \psi_\mathbb{Q}: H^3(\mathcal{M}_X, \mathbb{Q}) \rightarrow H^3(\mathcal{U}, \mathbb{Q}) \] (3.11)
induced by the map $\iota$ in Eq. (3.6) is surjective.

We noted earlier that $H^4(\mathcal{M}_X, \mathcal{U}, \mathbb{Z})$ is torsion-free. Therefore, from the surjectivity of $\psi_\mathbb{Q}$ in Eq. (3.11) we conclude that the homomorphism $\psi$ in Eq. (3.10) is surjective. Consequently, the mixed Hodge structure $H^3(\mathcal{M}_X, \mathbb{Z})$ is isomorphic to the pure Hodge structure $H^3(\mathcal{N}_X, \mathbb{Z})$ of weight three. This completes the proof of the proposition. □

Let
\[ J^2(\mathcal{M}_X) := H^3(\mathcal{M}_X, \mathbb{C})/(F^2 H^3(\mathcal{M}_X, \mathbb{C}) + H^3(\mathcal{M}_X, \mathbb{Z})) \] (3.12)
be the intermediate Jacobian of the mixed Hodge structure \( H^3(\mathcal{M}_X) \) (see [8, p. 110]). The intermediate Jacobian of any mixed Hodge structure is a generalized torus [8, p. 111]. Let

\[
J^2(N_X) := H^3(N_X, \mathbb{C}) / (F^2 H^3(N_X, \mathbb{C}) + H^3(N_X, \mathbb{Z}))
\]

be the intermediate Jacobian for \( H^3(N_X, \mathbb{Z}) \), which is a complex torus.

**Proposition 3.3.** The intermediate Jacobian \( J^2(M_X) \) is isomorphic to \( J^2(N_X) \), which is isomorphic to the Jacobian \( \text{Pic}^0(X) \) of the Riemann surface \( X \).

**Proof.** From **Proposition 3.2** it follows immediately that \( J^2(M_X) \) is isomorphic to the intermediate Jacobian \( J^2(N_X) \).

On the other hand, there is a natural isomorphism of \( J^2(N_X) \) with \( \text{Pic}^0(X) \) [24, p. 392, Theorem 3]; this was proved earlier in [23] for \( r = 2 \) (see [23, p. 1201, Theorem]). This completes the proof of the proposition. □

In the next section we will construct a natural polarization on \( J^2(M_X) \).

### 4. Polarization on the intermediate Jacobian

The aim in this section is to show that the Riemann surface \( X \) and the variety \( M_X \) determine each other uniquely up to isomorphisms. We start with a lemma.

**Lemma 4.1.** \( H^2(M_X, \mathbb{Z}) = \mathbb{Z} \).

**Proof.** Consider the long exact sequence of relative cohomologies

\[
H^2(M_X, U, \mathbb{Z}) \longrightarrow H^2(M_X, \mathbb{Z}) \longrightarrow H^2(U, \mathbb{Z}) \longrightarrow H^3(M_X, U, \mathbb{Z})
\]
given by \( \iota \) in Eq. (3.6). Using Eq. (3.9), from this exact sequence we conclude that \( H^2(M_X, \mathbb{Z}) = H^2(U, \mathbb{Z}) \). Therefore, from the isomorphism in Eq. (3.7) we have

\[
H^2(M_X, \mathbb{Z}) = H^2(N_X, \mathbb{Z}).
\]  

Finally, \( H^2(N_X, \mathbb{Z}) = \mathbb{Z} \) [1, p. 582, Proposition 9.13]. This completes the proof of the lemma. □

Set \( m := (r^2 - 1)(g - 1) - 3 \). Let

\[
F : \left( \bigwedge^2 H^3(M_X, \mathbb{Q}) \right) \otimes H^2(M_X, \mathbb{Q}) \otimes m \longrightarrow H^2(r^2 - 1)(g - 1)(M_X, \mathbb{Q})
\]

be the homomorphism defined by \( (\alpha \wedge \beta) \otimes \gamma^m \longmapsto \alpha \cup \beta \cup \gamma^m \).

**Proposition 4.2.** The dimension of the image of the homomorphism \( F \), defined in Eq. (4.2), is one.

**Proof.** We will use the properties of a certain moduli space of Higgs bundles over \( X \) which is naturally diffeomorphic to \( M_X \) and is known as the Dolbeault moduli space (see [29,14]).

Let \( \mathcal{H}_X \) denote the moduli space parametrizing all stable Higgs bundles \((E, \theta)\) over \( X \) of the following form:

- \( E \) is a holomorphic vector bundle of rank \( r \) with \( \bigwedge^r E \cong \mathcal{O}_X(dx_0) \), and
- \( \text{trace}(\theta) \in H^0(X, K_X) \) vanishes identically.
It is known that the moduli space $\mathcal{M}_X$ is naturally diffeomorphic to $\mathcal{M}_X$ (see [14]). A quick way to construct the diffeomorphism is the following.

Fix a ramified Galois covering $\phi$ as in Eq. (2.4). For any Higgs bundle $(E, \theta) \in \mathcal{H}_X$, consider the pulled back Higgs bundle $((\phi^* E) \otimes \mathcal{O}_Y)(-d\gamma_0), \phi^* \theta)$ over $Y$, where $\gamma_0$ is the point in Eq. (2.5). This Higgs bundle is polystable of degree zero, because the pull back of any polystable Higgs bundle by a Galois covering map remains polystable. Let $(W, \nabla)$ be the flat vector bundle over $Y$ corresponding to $((\phi^* E) \otimes \mathcal{O}_Y)(-d\gamma_0), \phi^* \theta)$ [17,29]. The flat connection $\nabla$ on $W$ and the de Rham logarithmic connection on the line bundle $\mathcal{O}_Y(d\gamma_0)$ together induce a logarithmic connection $\nabla'$ on the vector bundle $W \otimes \mathcal{O}_Y \mathcal{O}_Y(d\gamma_0)$. Let

$$V := (\phi_*(W \otimes \mathcal{O}_Y \mathcal{O}_Y(d\gamma_0)))^{Gal(\phi)}$$

be the invariant direct image on $X$ for the natural action of the Galois group $Gal(\phi)$ on the direct image. The logarithmic connection $\nabla'$ being invariant under the action of the Galois group, induces a logarithmic connection $\nabla'$ on $V$. It is easy to see that $(V, \nabla') \in \mathcal{M}_X$. Sending any $(E, \theta)$ to $(V, \nabla')$, a diffeomorphism of $\mathcal{H}_X$ with $\mathcal{M}_X$ is obtained. This diffeomorphism does not depend on the choice of the covering $\phi$.

Since $\mathcal{H}_X$ and $\mathcal{M}_X$ are diffeomorphic, from Lemma 4.1 we have $H^2(\mathcal{H}_X, \mathbb{Q}) = \mathbb{Q}$. Let

$$\Gamma : \left(\bigwedge^2 H^3(\mathcal{H}_X, \mathbb{Q})\right) \otimes H^2(\mathcal{H}_X, \mathbb{Q}) \otimes (r^2-1)(g-1)-3 \longrightarrow H^{2(r^2-1)(g-1)}(\mathcal{H}_X, \mathbb{Q}) \ (4.3)$$

be the homomorphism defined by $(\alpha \wedge \beta) \otimes \gamma \mapsto \alpha \cup \beta \cup \gamma$.

Comparing the above homomorphism $\Gamma$ with $F$ defined in Eq. (4.2) we conclude that the following lemma implies that $\dim \text{Image}(F) \leq 1$.

**Lemma 4.3.** The dimension of the image of the homomorphism $\Gamma$, defined in Eq. (4.3), is at most one.

**Proof.** To prove this lemma, we consider the Hitchin map

$$H : \mathcal{H}_X \longrightarrow \bigoplus_{i=2}^r H^0(X, K_X^{\otimes i}) \ (4.4)$$

defined by $(E, \theta) \longmapsto \sum_{i=2}^r \text{trace}(\theta^i)$ [17,18]. This map $H$ is algebraic and proper [18, [26, p. 291, Theorem 6.1]. The fiber of $H$ over $(0, \ldots, 0)$ is known as the nilpotent cone. The nilpotent cone is a finite union of complete subvarieties of $\mathcal{H}_X$ of complex dimension $(r^2 - 1)(g - 1)$.

Each component of the nilpotent cone defines an element of $H_{2(r^2-1)(g-1)}(\mathcal{H}_X, \mathbb{Q})$, and these elements together generate $H_{2(r^2-1)(g-1)}(\mathcal{H}_X, \mathbb{Q})$. This can be proved as follows.

The moduli space $\mathcal{H}_X$ has a natural structure of a noncompact Kähler manifold. It is equipped with the following holomorphic action of $\mathbb{C}^*$:

$$\lambda \cdot (E, \theta) = (E, \lambda \cdot \theta),$$

where $\lambda \in \mathbb{C}^*$ and $(E, \theta) \in \mathcal{H}_X$. Also, $\mathbb{C}^*$ acts on $\bigoplus_{i=2}^r H^0(X, K_X^{\otimes i})$ as

$$\lambda \cdot \sum_{i=2}^r \omega_i = \sum_{i=2}^r \lambda^i \cdot \omega_i,$$

where $\omega_i \in H^0(X, K_X^{\otimes i})$. The Hitchin map $H$ is equivariant with respect to these actions of $\mathbb{C}^*$. 

The action of the subgroup $U(1) \subset \mathbb{C}^*$ on $\mathcal{H}_X$ preserves the Kähler metric, and furthermore, this action is Hamiltonian. The moment map $\mu$ for this action is defined as follows:

$$\mu(E, \theta) = -\frac{1}{2} \|\theta\|^2 = -\sqrt{-1} \int_X \text{trace} \left( \theta \wedge \theta^* \right),$$

where the Hermitian metric on $E$ is the one that satisfies the Hermitian–Yang–Mills equation. Using Uhlenbeck’s compactness theorem, Hitchin proved that the moment map $\mu$ is proper [17]. Clearly $\mu$ is bounded above by 0. By a result due to Frankel [13], a proper moment map for a Hamiltonian action is a perfect Bott–Morse function (the result of [13] is stated for compact Kähler manifolds, but it works in the noncompact case as well). The critical submanifolds of $\mu$ are the fixed points of the $\mathbb{C}^*$-action on $\mathcal{H}_X$. Using the fact that the Hitchin map $H$ is equivariant for the actions of $\mathbb{C}^*$ it follows immediately that the fixed point set is contained in the nilpotent cone.

We will index the critical submanifolds by $C_\eta, \eta \in I$. For each $C_\eta$, let $D_\eta$ be the associated downward Morse flow submanifold, i.e., the collection of points $x \in \mathcal{H}_X$ whose flow under $-\mu$ converges to $C_\eta$ (using the Kähler form, any one-form gives a vector field). We will show that

$$\bigcup_{\eta \in I} D_\eta = H^{-1}(0, \ldots, 0), \quad (4.5)$$

which is the nilpotent cone. To prove the equality in Eq. (4.5), first note that given any $z \in \mathcal{H}_X$ with $H(x_0) \neq (0, \ldots, 0)$, the flow of $z$ diverges as the flow of $H(x_0)$ diverges (recall that the map $H$ is $\mathbb{C}^*$-equivariant). On the other hand, the nilpotent cone is compact, and it is preserved by the action of $\mathbb{C}^*$. Therefore, for any $z \in \mathcal{H}_X$ in the nilpotent cone, the flow of $z$ converges. This proves the equality in Eq. (4.5). (See also [15].)

Since the nilpotent cone has finitely many irreducible components, the index set $I$ is finite, i.e., the number of critical submanifolds is finite. Hence there is a real number $c < 0$ such that there are no critical points $(E, \theta)$ with $\mu((E, \theta)) < c$. Consequently, $\mathcal{H}_X$ retracts to $\mu^{-1}([c, 0])$, and hence it also retracts to the union $\bigcup_{\eta \in I} D_\eta$, which is the nilpotent cone.

The nilpotent cone is a Lagrangian subvariety of $\mathcal{H}_X$ [20, p. 648, Théorème (0.3)]. Hence each component of the nilpotent cone is a complete subvariety of (complex) dimension $(r^2 - 1)(g - 1)$. Therefore, each component of the nilpotent cone defines an element of $H_{2(r^2-1)(g-1)}(\mathcal{H}_X, \mathbb{Q})$, and these elements together generate $H_{2(r^2-1)(g-1)}(\mathcal{H}_X, \mathbb{Q})$.

As the next step in the proof of the lemma, we now consider the moduli space $\mathcal{M}_g^1$ parametrizing all isomorphism classes of one-pointed compact Riemann surfaces $(Y, y)$ of genus $g$ with $\text{Aut}(Y) = e$. This moduli space is a smooth irreducible quasi-projective variety of dimension $3g - 2$ defined over $\mathbb{C}$. There is a universal family of one-pointed Riemann surfaces

$$p : \mathcal{C}_g^1 \longrightarrow \mathcal{M}_g^1, \quad (4.6)$$

The marked points are given by a section of the projection $p$.

Let

$$P : \mathcal{H} \longrightarrow \mathcal{M}_g^1 \quad (4.7)$$

be the family of moduli spaces of stable Higgs bundles corresponding to the family of Riemann surfaces in Eq. (4.6). Therefore, for any one-pointed Riemann surface $x := (X, x_0) \in \mathcal{M}_g^1$, the fiber $P^{-1}(\overline{x})$ is
the moduli space \( \mathcal{H}_X \) parametrizing all stable Higgs bundles \((E, \theta)\) of rank \( r \) with \( \bigwedge^r E \cong \mathcal{O}_X(dx_0) \) and \( \text{trace}(\theta) = 0 \).

For any \( j \geq 1 \), consider the direct image on \( \mathcal{M}_g^1 \)

\[ \mathcal{V}_j := p_*K^{i\otimes i}_{\text{rel}}, \]

where \( K_{\text{rel}} \) is the relative cotangent bundle on \( \mathcal{C}_g^1 \) for the projection \( p \) in Eq. (4.6). We have the relative Hitchin map

\[ \tilde{H} : \tilde{\mathcal{H}} \rightarrow \bigoplus_{j=2}^r \mathcal{V}_j \]  

(4.8)

defined by \((E, \theta) \mapsto \sum_{i=2}^r \text{trace}(\theta^i)\) (see Eq. (4.4)). This \( \tilde{H} \) is clearly an algebraic morphism. The relative nilpotent cone is the inverse image \( \tilde{H}^{-1}(0_{\mathcal{M}_g^1}) \), where \( 0_{\mathcal{M}_g^1} \) is the zero section of the vector bundle in Eq. (4.8).

Consider the local system \( R^{2(r^2-1)(g-1)}P_*=\mathbb{C} \) on \( \mathcal{M}_g^1 \), where \( \mathbb{C} \) is the constant local system on \( \tilde{\mathcal{H}} \) (constructed in Eq. (4.7)) with stalk \( \mathbb{C} \), and \( P \) is the projection in Eq. (4.7). Take any \( \chi = (X, x) \) in \( \mathcal{M}_g^1 \) and any component \( Z \) of the nilpotent cone of \( \mathcal{H}_X \). The element in

\[ H_{2(r^2-1)(g-1)}(\mathcal{H}_X, \mathbb{Q}) = H^{2(r^2-1)(g-1)}(\mathcal{H}_X, \mathbb{Q})^* \]

given by \( Z \) extends uniquely as a section of \((R^{2(r^2-1)(g-1)}P_*=\mathbb{C})^* \), over any contractible analytic open subset \( U_\chi \subset \mathcal{M}_g^1 \) containing the point \( \chi \). Using the above construction of the relative nilpotent cone we conclude that this section satisfies the following condition: for each point \( y = (Y, y) \) in \( U_\chi \), the evaluation of the section at \( y \) corresponds to a component of the nilpotent cone of \( \mathcal{H}_Y \).

We noted earlier that each component of the nilpotent cone in \( \mathcal{H}_X \) (there are finitely many of them) defines an element of \( H_{2(r^2-1)(g-1)}(\mathcal{H}_X, \mathbb{Q}) \), and these elements together generate \( H_{2(r^2-1)(g-1)}(\mathcal{H}_X, \mathbb{Q}) \). This together with the above observation on relative nilpotent cone imply that the monodromy of the local system \( R^{2(r^2-1)(g-1)}P_*=\mathbb{C} \) is a quotient group of the permutation group of the components of the nilpotent cone. In particular, the monodromy of the local system \( R^{2(r^2-1)(g-1)}P_*=\mathbb{C} \) is a finite group.

Fix a base point \( x_0 := (X_0, x_0) \in \mathcal{M}_g^1 \). Let \( G_{\mathbb{Z}} \) (respectively, \( G_{\mathbb{C}} \)) denote the group of all automorphisms of \( H^1(X_0, \mathbb{Z}) \) (respectively, \( H^1(X_0, \mathbb{C}) \)) preserving the symplectic pairing given by the cup product. Choosing a symplectic basis of \( H^1(X_0, \mathbb{C}) \), the groups \( G_{\mathbb{Z}} \) and \( G_{\mathbb{C}} \) get identified with \( \text{Sp}(2g, \mathbb{Z}) \) and \( \text{Sp}(2g, \mathbb{C}) \) respectively.

Consider the local system \( R^1p_*\mathbb{Z} \) on \( \mathcal{M}_g^1 \), where \( p \) is the projection in Eq. (4.6), and \( \mathbb{Z} \) is the constant local system on \( \mathcal{C}_g^1 \) with stalk \( \mathbb{Z} \). Using its monodromy, the group \( G_{\mathbb{Z}} \) is a quotient of the fundamental group

\[ \Gamma_g^1 := \pi_1(\mathcal{M}_g^1, x_0). \]

This group \( \Gamma_g^1 \) is known as the mapping class group, and the kernel of the projection of \( \Gamma_g^1 \) to \( G_{\mathbb{Z}} \) is known as the Torelli group.

The cohomology algebra \( H^*(\mathcal{H}_X, \mathbb{C}) \) is generated by the Künneth components of a universal vector bundle over \( X \times \mathcal{H}_X \) [22, p. 73, Theorem 7]; this was proved earlier in [16] for \( r = 2 \) (see [16, p. 64],
(6.1)). From this it follows immediately that for any $i \geq 0$, the local system $R^i P_* \mathbb{C}$ is a quotient of some local system on $\mathcal{M}_g^1$ of the form

$$\mathcal{W} := \bigoplus_{j=1}^{\ell} \left( \left( \bigoplus_{i=0}^{2} R^i P_* \mathbb{C} \right)^{a_j} \otimes b_j \right),$$

where $a_j, b_j \in \mathbb{N}$, and $p$ is the projection in Eq. (4.6). We note that using the marked point, a universal Higgs bundle can be rigidified. Since both $R^0 P_* \mathbb{C}$ and $R^2 P_* \mathbb{C}$ are constant local systems, and $\text{Sp}(2g, \mathbb{C})$ is Zariski dense in $\text{Sp}(2g, \mathbb{C})$ [7, p. 179], we conclude the following:

(1) For all $i \geq 0$, the Torelli group is in the kernel of the monodromy representation

$$\Gamma_g^1 \longrightarrow \text{Aut}((R^i P_* \mathbb{C})_{\Sigma_0})$$

of the mapping class group for the local system $R^i P_* \mathbb{C}$, where $\Sigma_0 \in \mathcal{M}_g^1$ is the base point. Therefore, the homomorphism in Eq. (4.9) factors through the quotient $G_Z$ of $\Gamma_g^1$. 

(2) The homomorphism

$$G_Z \longrightarrow \text{Aut}((R^i P_* \mathbb{C})_{\Sigma_0})$$

obtained from Eq. (4.9) extends to a representation of $G_\mathbb{C}$.

To prove the second assertion, note that as $G_Z$ is Zariski dense in $G_\mathbb{C}$, the kernel of the surjective homomorphism of $G_Z$-modules

$$\mathcal{W}_{\Sigma_0} \longrightarrow (R^i P_* \mathbb{C})_{\Sigma_0}$$

is preserved by the action of $G_\mathbb{C}$ on $\mathcal{W}_{\Sigma_0}$, thus inducing an action of $G_\mathbb{C}$ on the quotient.

We noted earlier that the monodromy of the local system $R^{2(r^2-1)(g-1)} P_* \mathbb{C}$ is a finite group. In other words, the monodromy representation for $R^{2(r^2-1)(g-1)} P_* \mathbb{C}$ factors through a finite quotient of $G_Z \cong \text{Sp}(2g, \mathbb{Z})$. Any finite index subgroup of $\text{Sp}(2g, \mathbb{Z})$ is Zariski dense in $\text{Sp}(2g, \mathbb{C})$ [7, p. 179]. Since a Zariski dense subgroup is in the kernel of the homomorphism

$$G_\mathbb{C} \longrightarrow \text{Aut}((R^{2(r^2-1)(g-1)} P_* \mathbb{C})_{\Sigma_0})$$

obtained by extending the monodromy representation of $G_Z$ in Eq. (4.10) for the local system $R^{2(r^2-1)(g-1)} P_* \mathbb{C}$, we conclude that the above homomorphism is the trivial homomorphism. In other words, the monodromy of $R^{2(r^2-1)(g-1)} P_* \mathbb{C}$ is trivial. Consequently, the local system $R^{2(r^2-1)(g-1)} P_* \mathbb{C}$ is a constant one.

Consider the homomorphism $\tilde{\Gamma}$ constructed in Eq. (4.3) with coefficients in $\mathbb{C}$ instead of $\mathbb{Q}$. The pointwise construction of it (over the points of $\mathcal{M}_g^1$) yields a homomorphism of local systems

$$\tilde{\Gamma} : \left( \bigwedge^{2} R^3 P_* \mathbb{C} \right) \otimes (R^2 P_* \mathbb{C})^{\otimes (r^2-1)(g-1)-3} \longrightarrow R^{2(r^2-1)(g-1)} P_* \mathbb{C}.$$  

(4.11)

From Lemma 4.1 it follows that $R^2 P_* \mathbb{C}$ is a constant local system of rank one.

There is a canonical isomorphism $H^1(\mathcal{N}_X, \mathbb{Z}) = H^1(X, \mathbb{Z})$, where $\mathcal{N}_X$ is the moduli space of stable vector bundles over $X$ defined in Section 3 [24, p. 392, Theorem 3]. This isomorphism is constructed using the Künneth component of the second Chern class of a universal vector bundle over $X \times \mathcal{N}_X$. The
Therefore, using Proposition 3.2 it follows that
\[ R^3 p_* \underline{C} = R^1 p_* \underline{C}, \]
where \( p \) is the projection in Eq. (4.6).

If we identify \( G \) with \( \text{Sp}(2g, \mathbb{Z}) \) by choosing a symplectic basis of \( H^1(X_0, \mathbb{Z}) \), then \( R^1 p_* \underline{C} \) gets identified with the local system on \( \mathcal{M}_g^1 \) associated to the standard representation of \( \text{Sp}(2g, \mathbb{Z}) \). From this it follows that the local system \( \bigwedge^2 R^1 p_* \underline{C} \) decomposes as
\[ \bigwedge^2 R^1 p_* \underline{C} = L^0 \oplus L^1, \]
where \( L^0 \) is a constant local system of rank one and \( L^1 \) is an irreducible local system of rank \( g(2g-1)-1 \) on \( \mathcal{M}_g^1 \). This decomposition corresponds to the decomposition of the \( \text{Sp}(2g, \mathbb{C}) \)-module \( \bigwedge^2 \mathbb{C}^{2g} \) as a direct sum of the trivial \( \text{Sp}(2g, \mathbb{C}) \)-module of dimension one with an irreducible \( \text{Sp}(2g, \mathbb{C}) \)-module. Since \( L^1 \) is irreducible of rank at least two, and both \( R^2(p^2-1)(g-1) P_* \underline{C} \) and \( R^2 P_* \underline{C} \) are constant local systems, the restriction of the homomorphism \( \tilde{\Gamma} \) (constructed in Eq. (4.11)) to the sublocal system
\[ L^1 \otimes (R^2 P_* \underline{C})^\otimes((p^2-1)(g-1)-3) \subset \bigwedge^2 R^3 P_* \underline{C} \otimes (R^2 P_* \underline{C})^\otimes((p^2-1)(g-1)-3) \]
vanishes identically.

Therefore, the homomorphism \( \tilde{\Gamma} \) factors through the one dimensional quotient local system
\[ L^0 \otimes (R^2 P_* \underline{C})^\otimes((p^2-1)(g-1)-3). \] This immediately implies that the dimension of the image of the homomorphism \( \Gamma \) defined in Eq. (4.3) is at most one. This completes the proof of the lemma. \( \square \)

Continuing with the proof of Proposition 4.2, from Lemma 4.3 it follows that
\[ \dim \text{Image}(F) \leq 1. \]
We will complete the proof of the proposition by showing that \( \text{Image}(F) \neq 0 \).

Let
\[ \gamma \in H^2(\mathcal{N}_X, \mathbb{Q}) \]  \hspace{1cm} (4.12)
be an ample class of the smooth projective variety \( \mathcal{N}_X \) (defined in Section 3). The Hard Lefschetz theorem for \( \mathcal{N}_X \) says that there are
\[ \alpha, \beta \in H^3(\mathcal{N}_X, \mathbb{Q}) \]  \hspace{1cm} (4.13)
such that
\[ 0 \neq \alpha \cup \beta \cup \gamma^\otimes((p^2-1)(g-1)-3) \in H^2(p^2-1)(g-1)(\mathcal{N}_X, \mathbb{Q}) = \mathbb{Q} \]  \hspace{1cm} (4.14)
(recall that \( \dim_{\mathbb{C}} \mathcal{N}_X = (p^2 - 1)(g - 1) \)).

Let
\[ \tilde{\alpha}, \tilde{\beta} \in H^3(\mathcal{M}_X, \mathbb{Q}) \]  \hspace{1cm} (4.15)
be the elements given by \( \alpha \) and \( \beta \) respectively (in Eq. (4.13)) using the isomorphism \( H^3(\mathcal{N}_X, \mathbb{Q}) = H^3(\mathcal{M}_X, \mathbb{Q}) \) in Proposition 3.2. Similarly, let
\[ \tilde{\gamma} \in H^2(\mathcal{M}_X, \mathbb{Q}) \]  \hspace{1cm} (4.16)
be the element given by \( \gamma \) using the isomorphism in Eq. (4.1).
We will show that
\[
F((\tilde{\alpha} \wedge \tilde{\beta}) \otimes \nabla^\otimes((r^2-1)(g-1)-3)) \neq 0, \tag{4.17}
\]
where \( F \) is the homomorphism in Eq. (4.2).

The projection \( \Phi \) in Eq. (3.2) has a natural \( C^\infty \) section given the unique unitary logarithmic connection singular over \( x_0 \) on any stable vector bundle. We will briefly explain the construction of this section. Fix a ramified Galois covering \( \phi \) as in Eq. (2.4). Given any stable vector bundle \( E \in \mathcal{N}_X \), consider the vector bundle \( (\phi^*E) \otimes \mathcal{O}_Y(-dy_0) \) over \( Y \), where \( y_0 \) is the point in Eq. (2.5). This vector bundle is polystable of degree zero. Hence it admits a unique unitary flat connection \( \nabla \) [25]. Let \( \nabla' \) be the logarithmic connection on
\[
\phi^*E = ((\phi^*E) \otimes \mathcal{O}_Y(-dy_0)) \otimes \mathcal{O}_Y(dy_0)
\]
induced by \( \nabla \) and the de Rham logarithmic connection on \( \mathcal{O}_Y(dy_0) \). The above mentioned unique unitary connection on \( E \) is the unique logarithmic connection \( \nabla'' \) on \( E \) such that the pullback \( \phi^*\nabla'' \) coincides with \( \nabla' \). It is easy to see that \( (E, \nabla'') \in \mathcal{M}_X \).

Let
\[
\tau : \mathcal{N}_X \rightarrow \mathcal{U}
\]
be the smooth section of \( \Phi \) that associates to any stable vector bundle \( E \) the unique unitary logarithmic connection \( \nabla'' \) on it. Let
\[
\tilde{\tau} \in H_{2(r^2-1)(g-1)}(\mathcal{M}_X, \mathbb{Q})
\]
be the homology class defined by the image of \( \tau \) using the inclusion of \( \mathcal{U} \) in \( \mathcal{M}_X \).

Consider the natural duality pairing between the cohomology \( H^{2(r^2-1)(g-1)}(\mathcal{M}_X, \mathbb{Q}) \) and the homology \( H_{2(r^2-1)(g-1)}(\mathcal{M}_X, \mathbb{Q}) \). It is straightforward to check that
\[
F((\tilde{\alpha} \wedge \tilde{\beta}) \otimes \nabla^\otimes((r^2-1)(g-1)-3))(\tilde{\tau}) = (\alpha \cup \beta \cup \gamma^\otimes((r^2-1)(g-1)-3)) \cap [\mathcal{N}_X] \in \mathbb{Q}, \tag{4.18}
\]
where \( \alpha, \beta \) and \( \gamma \) are the cohomology classes in Eq. (4.13) and Eq. (4.12) respectively, and \( \tilde{\alpha}, \tilde{\beta} \) and \( \tilde{\alpha} \) are constructed in Eqs. (4.15) and (4.16) respectively. Using Eq. (4.18) we conclude that the assertion in Eq. (4.17) follows from Eq. (4.14). Consequently, \( \text{Image}(F) \neq 0 \). We have already shown that \( \dim \text{Image}(F) \leq 1 \). This completes the proof of the proposition. \( \square \)

Now we are in a position to prove our main result.

**Theorem 4.4.** Let \((X, x_0)\) and \((Y, y_0)\) be two compact connected one-pointed Riemann surfaces of genus \( g \), with \( g \geq 3 \). Let \( \mathcal{M}_X \) and \( \mathcal{M}_Y \) be the corresponding moduli spaces of connections defined as in Section 2. The two varieties \( \mathcal{M}_X \) and \( \mathcal{M}_Y \) are isomorphic if and only if the two Riemann surfaces \( X \) and \( Y \) are isomorphic.

**Proof.** From Proposition 4.2 we know that \( \dim \text{Image}(F) = 1 \). Consequently, fixing a generator of \( \text{Image}(F) \), and also fixing a generator of \( H^2(\mathcal{M}_X, \mathbb{Q}) \), the homomorphism \( F \) gives a cohomology class
\[
\theta \in \bigwedge^2 H^3(\mathcal{M}_X, \mathbb{Q})^* = H^2(J^2(\mathcal{M}_X), \mathbb{Q}). \tag{4.19}
\]
Using the diagram in Eq. (3.6), we have an isomorphism of $J^2(M_X)$ with $J^2(N_X)$ (Proposition 3.3). Consider the Néron–Severi group

$$\text{NS}(J^2(N_X)) := H^{1,1}(J^2(N_X)) \cap H^2(J^2(N_X), \mathbb{Z}).$$

There is a natural element

$$\theta \in \text{NS}(J^2(N_X))_{\mathbb{Q}} := \text{NS}(J^2(N_X)) \otimes_{\mathbb{Z}} \mathbb{Q} = H^{1,1}(J^2(N_X)) \cap H^2(J^2(N_X), \mathbb{Q})$$

deﬁned by

$$\alpha \wedge \beta \mapsto (\alpha \cup \beta \cup \gamma \otimes ((r^2-1)(g-1)-3)) \cap \left[ N_X \right] \in \mathbb{Q},$$

where $\alpha, \beta \in H^3(N_X, \mathbb{Q})$, and $\gamma$ is the ample generator of $H^2(N_X, \mathbb{Z})$. The Hard Lefschetz theorem for $N_X$ says that $\theta \neq 0$. We will show that $\theta$ is a scalar multiple of a principal polarization on $J^2(N_X)$.

Let

$$\overline{\varphi} : J \to M^1_g$$

be the universal Jacobian for the universal family of one-pointed Riemann surfaces in Eq. (4.6). It is known that any section of the local system $R^2\overline{\varphi}_* \mathbb{Q}$ which is a Hodge cycle over every point of $M^1_g$ (i.e., an element of NS(Pic$^0(Y))_{\mathbb{Q}$ for every Riemann surface $Y$), must be a scalar multiple of the section of $R^2\overline{\varphi}_* \mathbb{Q}$ given by the theta line bundle on the Jacobians; this is well known (see [4, p. 712]).

We have $J^2(N_X) \cong \text{Pic}^0(X)$ (see [24, p. 392, Theorem 3]). Therefore, in view of the above remark, the cohomology class $\theta$ is a rational multiple of a principal polarization on $J^2(N_X)$.

Given a nonzero rational multiple, say $\theta'$, of a principal polarization on an abelian variety $A$, there is a unique way to scale $\theta'$ to get back the principal polarization on $A$. Indeed, this is an immediately consequence of the following two properties of a principal polarization:

- a principal polarization on $A$ is an ample class on $A$, and
- a principal polarization is indivisible as an element of $H^2(A, \mathbb{Z})$.

Note that a free $\mathbb{Z}$-module of rank one has exactly two generators.

Let $\tilde{\theta}$ be the unique principal polarization on $J^2(N_X)$ which is a rational multiple of $\theta$. Therefore, the principally polarized abelian variety $(J^2(N_X), \tilde{\theta})$ is isomorphic to $\text{Pic}^0(X)$ equipped with the canonical principal polarization on it given by the theta line bundle.

Comparing $\theta$ with $\theta$ deﬁned in Eq. (4.19) we conclude that the above mentioned identiﬁcation of $J^2(N_X)$ with $J^2(M_X)$ takes $\tilde{\theta}$ to a nonzero multiple of $\theta$. Now using the earlier remark that a principal polarization can be recovered from any nonzero multiple of it we conclude that $\theta$ gives a principal polarization $\hat{\theta}$ on $J^2(M_X)$. Furthermore, the principally polarized abelian variety $(J^2(M_X), \hat{\theta})$ is isomorphic to the principally polarized abelian variety $(J^2(N_X), \tilde{\theta})$.

We have noted earlier that the principally polarized abelian variety $(J^2(N_X), \tilde{\theta})$ is isomorphic to $\text{Pic}^0(X)$ equipped with the canonical principal polarization on it given by the theta line bundle. Therefore, from the classical Torelli theorem, which says that the isomorphism class of the principally polarized abelian variety $\text{Pic}^0(X)$ equipped with the principal polarization given by the theta line bundle determines the Riemann surface $X$ uniquely up to an isomorphism, we conclude that the isomorphism class of the variety $M_X$ determines the Riemann surface $X$ uniquely up to an isomorphism.

We saw in Remark 2.2 that the isomorphism class of the variety $M_X$ is independent of the choice of the base point $x_0$ of $X$. This immediately implies that the isomorphism class of the variety $M_X$ is
determined uniquely by the isomorphism class of the Riemann surface $X$. Therefore, the proof of the theorem is complete. □

In the next section we will prove a similar result for the moduli space of $\text{GL}(n, \mathbb{C})$-connections.

5. Moduli of $\text{GL}(n, \mathbb{C})$-connections

As before, fix two mutually coprime integers $r$ and $d$, with $r \geq 2$. Let $(X, x_0)$ be a one-pointed compact Riemann surface of genus $g$, with $g \geq 3$.

Let $\widehat{M}_X$ be the moduli space parametrizing all logarithmic connections $(E, D)$ over $X$ singular exactly over $x_0$ and satisfying the following two conditions:

- $\text{rank}(E) = r$, and
- $\text{Res}(D, x_0) = -\frac{d}{r} \text{Id}_{E_{x_0}}$.

From Eq. (2.2) it follows that if $(E, D) \in \widehat{M}_X$, then degree$(E) = d$. The moduli space $\widehat{M}_X$ is an irreducible smooth quasiprojective variety defined over $\mathbb{C}$. Its dimension is $2r^2(g - 1) + 2$.

A surjective morphism $p_0 : \widehat{M}_X \rightarrow A_0$, where $A_0$ is an abelian variety defined over $\mathbb{C}$, will be called universal if for any morphism $p : \widehat{M}_X \rightarrow A$ as in Eq. (5.1), where $A$ is any abelian variety defined over $\mathbb{C}$, there is a unique morphism $\gamma : A_0 \rightarrow A$ such that $\gamma \circ p_0 = p$.

**Proposition 5.1.** The morphism

$$p_0 : \widehat{M}_X \rightarrow \text{Pic}^d(X)$$

defined by $(E, D) \mapsto \wedge^r E$ is universal.

**Proof.** From [6, p. 787, Lemma 2.3] it follows immediately that the morphism $p_0$ in Eq. (5.2) is surjective. To prove that it is universal, let $A$ be an abelian variety defined over $\mathbb{C}$ and $p$ a morphism to $A$ as in Eq. (5.1). Fix any closed point $z \in \text{Pic}^d(X)$. Let

$$p^z : p_0^{-1}(z) \rightarrow A$$

be the restriction of $p$ to the closed subvariety $p_0^{-1}(z) \subset \widehat{M}_X$.

Fix a holomorphic line bundle $L_0$ over $X$ of degree zero such that the line bundle $(L_0^*)^{\otimes r} \otimes_{\mathcal{O}_X} \mathcal{O}_X(dx_0)$ corresponds to the point $z \in \text{Pic}^d(X)$. Consider the moduli space $\mathcal{M}_X$ defined in Section 2. Given any $(E, D) \in p_0^{-1}(z)$, there is a unique holomorphic connection $\nabla$ on $L_0$ (the connection $\nabla$ depends on $D$) such that

$$(E \otimes L_0, D \otimes \text{Id}_{L_0} + \text{Id}_E \otimes \nabla) \in \mathcal{M}_X.$$
Let
\[ \gamma : p_0^{-1}(z) \rightarrow \mathcal{M}_X \] (5.5)
be the morphism that sends any \((E, D) \in p_0^{-1}(z)\) to \((E \otimes L_0, D \otimes \text{Id}_{L_0} + \text{Id}_E \otimes \nabla)\) in Eq. (5.4). This map \(\gamma\) is algebraic, and it makes \(p_0^{-1}(z)\) an affine bundle over \(\mathcal{M}_X\) for the trivial vector bundle over \(\mathcal{M}_X\) with fiber \(H^0(X, K_X)\). In other words, \(H^0(X, K_X)\) acts freely transitively on each of the fibers of \(\gamma\).

Any algebraic morphism \(f : \mathbb{C}P^1 \rightarrow A'\), where \(A'\) is a complex abelian variety, is a constant morphism. Indeed, the cotangent bundle of \(A'\) is generated by global sections, and \(H^0(\mathbb{C}P^1, K_{\mathbb{C}P^1}) = 0\), implying that the differential of \(f\) vanishes. Therefore, any algebraic morphism from a Zariski open subset of \(\mathbb{C}P^1\) to \(A'\) is a constant morphism.

From the above observation it follows that the restriction of the morphism \(p^\circ \) (defined in Eq. (5.3)) to any fiber of \(\gamma\) (defined in Eq. (5.5)) is a constant morphism. Consequently, there is unique morphism
\[ \gamma_0 : \mathcal{M}_X \rightarrow A \] (5.6)
such that \(\gamma_0 \circ \gamma = p^\circ\).

Let
\[ \gamma_1 : \mathcal{U} \rightarrow A \] (5.7)
be the restriction of \(\gamma_0\) to the open dense subset \(\mathcal{U} \subset \mathcal{M}_X\) defined in Eq. (3.1). Since any fiber of the morphism \(\Phi\) in Eq. (3.2) is an affine space, the restriction of \(\gamma_1\) to any fiber of \(\Phi\) is a constant morphism. Therefore, there is a unique morphism
\[ \gamma_2 : \mathcal{N}_X \rightarrow A \] (5.8)
such that \(\gamma_2 \circ \Phi = \gamma_1\), where \(\gamma_1\) is defined in Eq. (5.7).

The variety \(\mathcal{N}_X\) is known to be unirational, in fact, it is rational [19, p. 520, Theorem 1.2]. Hence a nonempty Zariski open subset of it is covered by the images of \(\mathbb{C}P^1\). This implies that the morphism \(\gamma_2\) in Eq. (5.8) is a constant one.

Therefore, \(\gamma_1\) in Eq. (5.7) is a constant morphism. Since \(\mathcal{U}\) is Zariski dense in \(\mathcal{M}_X\), the morphism \(\gamma_0\) in Eq. (5.6) is a constant one. Consequently, \(p^\circ\) in Eq. (5.3) is a constant morphism. This implies that the morphism \(p\) factors through a morphism \(\text{Pic}^d(X) \rightarrow A\). Hence the morphism \(p_0\) in Eq. (5.2) is universal. This completes the proof of the proposition. \(\square\)

From Proposition 5.1 it follows that the isomorphism class of the variety \(\widehat{\mathcal{M}}_X\) determines the pair \(\text{Pic}^d(X)\) and the projection \(p_0\) in Eq. (5.2) up to an automorphism of \(\text{Pic}^d(X)\). Therefore, the isomorphism class of the variety \(\mathcal{M}_X\) determines the isomorphism class of the variety \(p_0^{-1}(z)\) for some \(z \in \text{Pic}^0(X)\). More precisely, for any point \(w \in \widehat{\mathcal{M}}_X\), the isomorphism class of the variety \(p_0^{-1}(p_0(w))\) is determined by the isomorphism class of the variety \(\mathcal{M}_X\).

Since the morphism \(\gamma\) in Eq. (5.5) is an affine fibration, in particular, the fibers are contractible, the induced homomorphism
\[ \gamma^* : H^i(\mathcal{M}_X, \mathbb{Z}) \rightarrow H^i(p_0^{-1}(z), \mathbb{Z}) \]
is an isomorphism for all $i \geq 0$. Therefore, $J^2(\mathcal{M}_X) \cong J^2(p_0^{-1}(z))$. Furthermore, the cup product
\[
\left( \bigwedge^2 \mathcal{H}^3(p_0^{-1}(z), \mathbb{Q}) \right) \otimes \mathcal{H}^2(p_0^{-1}(z), \mathbb{Q}) \otimes (r^2-1)(s-1) \rightarrow \mathcal{H}^2(r^2-1)(s-1)(p_0^{-1}(z), \mathbb{Q})
\]
defined as in Eq. (4.2) gives, after rescaling, a principal polarization on $J^2(p_0^{-1}(z))$. The resulting principally polarized abelian variety is clearly isomorphic to $(J^2(\mathcal{M}_X), \tilde{\theta})$ constructed in the proof of Theorem 1.2.

Therefore, using Theorem 4.4, we have the following theorem:

**Theorem 5.2.** The isomorphism class of the variety $\widehat{\mathcal{M}}_X$ determines the Riemann surface $X$ uniquely up to an isomorphism.

**Remark 5.3.** The isomorphism class of the variety $\widehat{\mathcal{M}}_X$ is independent of the choice of the point $x_0$. This can be shown using the morphism defined in Remark 2.2. More precisely, if $\widehat{\mathcal{M}}_{X}^1$ is the moduli space of logarithmic connection obtained by replacing $x_0$ with a different point $x_1$ in the construction of $\widehat{\mathcal{M}}_X$, then the morphism
\[
\widehat{\mathcal{M}}_X \rightarrow \widehat{\mathcal{M}}_{X}^1
\]
defined by
\[
(E, D) \mapsto (E \otimes L, D \otimes \text{Id}_L + \text{Id}_E \otimes D_0)
\]
is an algebraic isomorphism of varieties, where $(L, D_0)$ is defined in Remark 2.2.

Also, the complex manifold $\widehat{\mathcal{M}}_X$ is naturally biholomorphic to the subset of $\text{Hom}(\pi_1(X \setminus \{x_0\}), \text{GL}(r, \mathbb{C}))$ parametrizing all representations that sends the anticlockwise oriented loop around $x_0$ to $\exp(2\pi \sqrt{-1}d/r) \cdot I_{r \times r} \in \text{GL}(r, \mathbb{C})$. As before, the biholomorphism is obtained by sending a connection to the corresponding monodromy representation.

**Remark 5.4.** It can be shown that Theorem 5.2 fails for rank one case. To prove this, let $\widehat{\mathcal{M}}^1$ denote the moduli space of rank one regular connections on $X$. So there is an algebraic morphism
\[
\psi : \widehat{\mathcal{M}}^1 \rightarrow \text{Pic}^0(X) =: T
\]
that sends any $(L, D) \in \widehat{\mathcal{M}}^1$ to $L \in \text{Pic}^0(X)$.

Let $\Omega_T^1$ denote the holomorphic cotangent bundle of the abelian variety $T = \text{Pic}^0(X)$. Since any two holomorphic connections on a given holomorphic line bundle over $X$ differ by a holomorphic one-form on $X$, the morphism $\psi$ makes $\widehat{\mathcal{M}}^1$ a $\Omega_T^1$-torsor. In other words, for each point $t \in T$, the fiber $\psi^{-1}(t)$ is an affine space for the vector space $(\Omega_T^1)_x = H^0(X, K_X)$. Isomorphism classes of $\Omega_T^1$-torsors are parametrized by $H^1(T, \Omega_T^1)$. It is known that the above $\Omega_T^1$-torsor $\widehat{\mathcal{M}}^1$ corresponds to $\Theta/2$, where $\Theta \in H^1(T, \Omega_T^1)$ is the class of a theta divisor on $T$ [5, p. 308, Theorem 2.11].

Elements of $H^1(T, \Omega_T^1)$ correspond to the translation invariant vector bundle homomorphisms from the smooth tangent bundle $T^{0,1}T$ of type $(0, 1)$ to the cotangent bundle $\Omega_T^1$; here translation invariant means invariant for the translation action of the torus $T$ on itself. The class $\Theta/2 \in H^1(T, \Omega_T^1)$ corresponds to an isomorphism of $\Omega_T^1$ with $T^{0,1}T$.

Let $\text{Aut}(\Omega_T^1)$ denote the group of all holomorphic automorphisms of the vector bundle $\Omega_T^1$. Consider the natural action of $\text{Aut}(\Omega_T^1)$ on the space of all translation invariant vector bundle homomorphisms
from $T^{0,1}T$ to $\Omega^1_I$: the action of any $\tau \in \text{Aut}(\Omega^1_I)$ sends a homomorphism $\beta$ to $\tau \circ \beta$. It is easy to see that the action of $\text{Aut}(\Omega^1_I)$ on the space of all translation invariant isomorphisms of $T^{0,1}T$ with $\Omega^1_I$ is transitive.

This implies that if

$$\psi_i : A_i \longrightarrow T,$$

$i = 1, 2$, are two algebraic $\Omega^1_I$-torsors such that the translation invariant homomorphisms, from $T^{0,1}T$ to $\Omega^1_I$, corresponding to $\psi_1$ and $\psi_2$ are both isomorphisms, then the variety $A_1$ is isomorphic to $A_2$.

Therefore, if $X$ and $Y$ are two compact connected Riemann surfaces such that $\text{Pic}^0(X)$ is isomorphic to $\text{Pic}^0(Y)$, then the moduli space of rank one regular connections on $X$ is isomorphic to the moduli space of rank one regular connections on $Y$. Since there are non-isomorphic compact Riemann surfaces with isomorphic Jacobians, we conclude that Theorem 5.2 fails for $r = 1$.

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References