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Nielsen theory, braids and fixed points of surface homeomorphisms

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Abstract

We study two problems in Nielsen fixed point theory using Artin's braid groups and the Nielsen–Thurston classification of surface homeomorphisms up to isotopy. The first is that of distinguishing Reidemeister classes of free group automorphisms realized by a braid (and thus induced by homeomorphisms of the 2-disc relative to a finite invariant set), for which we give a necessary and sufficient condition in terms of a conjugacy problem in the braid group. Consequently, one may use any braid conjugacy invariant (those of Garside's algorithm, linking numbers, topological entropy, etc.) and any link invariant (Alexander polynomial, splittability, etc.) to distinguish Reidemeister classes, giving much stronger criteria than those already known.

The second problem is that of deciding when two fixed points of a surface homeomorphism belong to the same Nielsen fixed point class. We give two criteria, the first in terms of certain reducing curves which can be checked using the Bestvina–Handel algorithm, the second using the multi-variable Alexander polynomial of a link associated with the suspension of the homeomorphism.

Finally we consider generalizations of Sharkovskii's theorem on the coexistence of periodic orbits of interval maps to homeomorphisms of the 2-disc. We show that for each $n \geqslant 5$ there exists a pseudo-Anosov orientation-preserving homeomorphism of the 2-disc relative to a periodic orbit of period n that does not have periodic orbits of all periods, with an analogous result for the 2-sphere. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $f : X \to X$ be a continuous self-map of a compact, connected polyhedron X, and let $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$ denote the set of *fixed points* of f. The Lefschetz–Hopf fixed point theorem states that if the *Lefschetz number* $L(f)$ of f is non-zero then every map

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homotopic to f has a fixed point [9]. As well as the existence of fixed points, one is also interested in finding a lower bound for the cardinality of $\mathrm{Fix}(f)$. Such a lower bound is given by $\mathrm{MF}[f]$, the minimum number of fixed points amongst all maps homotopic to f, but in general it is difficult to calculate explicitly.

Given a fixed point of f, a second interesting problem is (loosely speaking) to understand how it "interacts" with the topology of X and with other fixed points. We shall explain presently what we mean by this, but the reader should have in mind that within the framework of this paper, a fixed point will be interpreted as a component of a link in a 3-dimensional manifold formed by the suspension of a surface homeomorphism. Apart from its intrinsic interest, this interaction may be used to understand coexistence properties and forcing relations of periodic orbits of f.

These two problems are related to Nielsen fixed point theory [27]. Two fixed points belong to the same *fixed point class* of f if there exists an arc α joining them such that $f(\alpha)$ is homotopic to α keeping endpoints fixed during the homotopy. Combining this with the notion of fixed point index leads to the definition of the *Nielsen number* $N(f)$ of f, a homotopy invariant which indeed gives a lower bound for $\mathrm{MF}[f]$.

In order to compute $N(f)$, one must be able to decide whether two fixed points belong to the same fixed point class. Reidemeister showed that this geometric problem could be recast in an algebraic context. Let $\varphi : G \to G$ be an endomorphism of a group G. Then $u, v \in G$ are said to be *Reidemeister equivalent* or *φ-conjugate*, written $u \sim_{\mathrm{R}} v$, if there exists $w \in G$ such that $v = \varphi(w) \cdot u \cdot w^{-1}$. Let $[u]$ denote the *Reidemeister* or *φ-conjugacy class* of u. This leads to the problem of *distinguishing Reidemeister classes*, i.e., deciding whether two elements of G belong to the same Reidemeister class or not. Since two Reidemeister-equivalent elements of G abelianize (see Section 2.3 for a precise definition) to the same element, *abelianization* gives a simple criterion to distinguish Reidemeister classes. However, a measure of the difficulty of the problem is that this is one of the few effective criteria. We will describe the relationship between fixed point classes and Reidemeister classes in Section 2.2, but briefly, one interprets φ as the endomorphism f_π induced by f on the fundamental group $\pi_1(X)$ of X. One associates a Reidemeister class, or 'coordinate' to each fixed point of f. Then two fixed points belong to the same fixed point class if and only if their coordinates coincide.

Reidemeister defined a homotopy invariant $L_R(f)$ of f, the *Reidemeister trace*, which is an element of the Abelian group freely generated by the f_π-conjugacy classes. It is a powerful generalization of the Lefschetz number: it can be calculated as an alternating sum of traces on the cellular chain level, and it contains enough topological information to be able (up to distinguishing Reidemeister classes) to calculate $N(f)$, and to extract fixed point linking information. When X is a compact surface (which is the main subject of interest in this paper), Fadell and Husseini showed that $L_R(f)$ can be computed using Fox's free differential calculus (again up to the problem of distinguishing Reidemeister classes) just from f_π [11].

From the point of view of topological dynamics, much attention has been focussed in recent years on surface homeomorphisms [6,7,17,18,20,23,24,31]. The central result in this area is that of their classification up to isotopy, due to Nielsen and Thurston [12,35]. Given a homeomorphism f (relative to some given finite f-invariant subset A) of a compact surface M, perhaps with boundary ∂M, there exists a canonical or *Thurston representative* g isotopic to f that is one of three types: finite order (so $g^n = \mathrm{Id}_M$ for some $n \in \mathbb{N}$), pseudo-Anosov, or reducible. In the third case, the surface may be cut up into subsurfaces along a tubular neighbourhood of a finite, g-invariant set of mutually-disjoint curves (*reducing curves*), and the restriction of an appropriate iterate of g to each subsurface is either finite order or pseudo-Anosov. Given the action of f on the fundamental group $\pi_1(M \setminus A)$, one can effectively decide its Thurston type using an algorithm due to Bestvina and Handel [2], for which there exists an implementation [22]. Pseudo-Anosov homeomorphisms have many interesting dynamical properties: positive topological entropy (which is minimized within the isotopy class), and infinitely many periodic orbits that are isotopy stable [1,21]. Further, they have 'good' Nielsen-theoretic properties: every fixed point not in $A \cup \partial M$ has non-zero fixed point index, and every fixed point in the interior $\mathrm{Int}(M)$ of M is unique in its fixed point class. An important consequence of this is that if f is pseudo-Anosov then its Reidemeister trace contains precisely all of the fixed point linking information.

This paper will be devoted to the study of two problems: that of distinguishing Reidemeister classes for automorphisms of the free group $\mathbb{F}_n$ of rank n that are induced by orientation-preserving homeomorphisms of the 2-disc $\mathbb{D}^2$ relative to some n-point invariant set A, and that of deciding when two fixed points of a surface homeomorphism belong to the same fixed point class. As well as the Nielsen–Thurston classification, the main tool that we shall use is *Artin's theory of braids* [3]. Given a homeomorphism $f : (\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$, one can associate a braid β (an element of Artin's braid group B_n on n strings) with A by fixing an isotopy between the identity and f; the strings of β appear naturally by following A under the isotopy. This braid characterizes topologically the isotopy class of f relative to A. Such a braid induces an automorphism of $\mathbb{F}_n \cong \pi_1(\mathbb{D}^2 \setminus A)$ and of the corresponding symmetric group $\mathfrak{S}_n$. The subgroup B_{n+1}^n of B_{n+1} of elements whose induced permutation stabilizes $(n+1)$ will play an important rôle. Every fixed point $y \notin A$ of f defines an element of B_{n+1}^n (that associated with $A \cup \{y\}$) which may be considered as being obtained from β by adding an $(n+1)$st string. This string encodes precisely the topological interaction (its *linking information*) of the fixed point with A. We define U_{n+1} to be the kernel of the homomorphism $B_{n+1}^n \to B_n$ defined geometrically by removing the $(n+1)$st string. In fact, U_{n+1} is isomorphic to $\mathbb{F}_n$, and B_{n+1}^n may be decomposed as a semi-direct product $\mathbb{F}_n \rtimes B_n$. This latter fact can be interpreted geometrically using Artin's 'combing' operation (see Section 2.1).

Let φ be an automorphism of $\mathbb{F}_n$ induced by a braid $\beta \in B_n$. For each word $u \in \mathbb{F}_n$, we give an explicit construction of a braid $\beta_u \in B_{n+1}^n$, the *u-extension of β*, which is precisely $(u, \beta) \in \mathbb{F}_n \rtimes B_n$ (see Section 3.1). Then:

Theorem 1. *Let φ be an automorphism of $\mathbb{F}_n$ induced by a braid $\beta \in B_n$. Then $u, v \in \mathbb{F}_n$ are φ-conjugate if and only if β_u and β_v are conjugate in B_{n+1}^n via an element of U_{n+1}.*

Consequently, *any* braid conjugacy invariant such as those obtained by applying Garside's algorithm (and its improvements), linking number properties, Thurston type, etc., and *any* link invariant (because if two braids are conjugate then their closures represent the same link in $\mathbb{S}^3$) such as the Alexander polynomial or splittability, may be used to show that β_u and β_v are not conjugate in B_{n+1}, and thus that two words $u, v \in \mathbb{F}_n$ are not φ-conjugate. This gives much stronger criteria to distinguish Reidemeister classes than those already known. Notice that Garside's algorithm, which in any case is exponential in the number of strings, tells us whether the given braids are conjugate in B_{n+1}; it says nothing about whether the conjugacy is via an element of U_{n+1}. Indeed, in Section 3.4 we will give an explicit example of two braids β_u and β_v which are conjugate in B_{n+1}^n but not via an element of U_{n+1}. The proof of this fact is in itself interesting because it uses dynamical methods. In our setting, one could thus solve completely the problem of distinguishing Reidemeister classes if one were able to decide whether two braids are conjugate via an element of U_{n+1}.

Let $f : (\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ be an orientation-preserving homeomorphism, where A is an n-point, f-invariant set. Let $\beta \in B_n$ be the associated braid, and let φ be the induced free group automorphism. Given $u \in \mathbb{F}_n$, one constructs a homeomorphism $g_u : (\mathbb{D}^2, A \cup \{y_u\}) \to (\mathbb{D}^2, A \cup \{y_u\})$, the *topological u-extension of f* (see Section 3.2). By construction, g_u, considered as a homeomorphism $g_u : (\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$, is isotopic to f (and the induced automorphism of $\pi_1(\mathbb{D}^2 \setminus A)$ is also φ), $y_u \in \mathrm{Fix}(g_u) \setminus A$, and the u-extension $\beta_u \in B_{n+1}^n$ of β is the braid associated with $A \cup \{y_u\}$. Further, if we consider g_u as a homeomorphism relative to A (and not $A \cup \{y_u\}$) then the coordinate of y_u is equal to the φ-conjugacy class $[u]$ of u (see Proposition 12). By isotoping relative to A and $A \cup \{y_u\}$ respectively, we may suppose that f and g_u are the Thurston representatives within their isotopy classes. Their topological entropies can be calculated directly from β and β_u using the Bestvina–Handel algorithm, without knowing f and g_u explicitly. In Section 3.2 we prove the following result, which gives another criterion to distinguish Reidemeister classes:

Corollary 2. *Let φ, u and v be as in Theorem* 1, *and let g_u* (*respectively, g_v*) *be a topological u-* (*respectively, v-*) *extension of f. If u and v are φ-conjugate then g_u and g_v are topologically conjugate. In particular, they have the same topological entropy.*

If f is finite order then it is conjugate to a rigid rotation, and it has a simple fixed point structure. Thus the dynamical analysis of a general Thurston representative may be reduced to understanding what happens when f is pseudo-Anosov relative to A. Given $u \in \mathbb{F}_n$, let $g_u : (\mathbb{D}^2, A \cup \{y_u\}) \to (\mathbb{D}^2, A \cup \{y_u\})$ be a topological u-extension of f. If $[u]$ does not appear in the final expression for the Reidemeister trace of f then it follows from a result of Smillie [7,24] that the topological entropy of g_u will be strictly greater than that of f.

Consider the following extension to invariant sets of the definition of Nielsen equivalence. Given a continuous self-map f of a compact, connected polyhedron X, a finite f-invariant subset $A \subseteq X$ and an f-invariant subset $\mathcal{C} \subseteq X$, we say that $x \in \mathrm{Fix}(f)$ (or that the pair (x, f)) is *Nielsen equivalent to $\mathcal{C}$ relative to A* if there exists an arc $\alpha : [0, 1] \to X$ with $\alpha(0) = x$, $\alpha(1) \in \mathcal{C}$, and such that $f(\alpha)$ is homotopic to α relative to A by a homotopy $\{F_t\}_{t \in [0,1]}$ satisfying $F_t(0) = x$ and $F_t(1) \in \mathcal{C}$ for all $t \in [0, 1]$. We also allow for the possibility that $A \cap (\mathcal{C} \cup \{x\}) \neq \emptyset$. In this case, we require moreover that $F_t(s) \notin A$ for all $t \in [0,1]$ and all $s \in (0,1)$, and if for $i \in \{0,1\}$ there exists $\tau \in [0,1]$ such that $F_\tau(i) \in A$ then $F_t(i) \in A$ for all $t \in [0,1]$. In Section 3.3 we prove the following result:

Theorem 3. *Let $f : (\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ be an orientation-preserving, pseudo-Anosov homeomorphism. Given $u \in \mathbb{F}_n$, let $g_u : (\mathbb{D}^2, A \cup \{y_u\}) \to (\mathbb{D}^2, A \cup \{y_u\})$ be a topological u-extension of f. Then, relative to $A \cup \{y_u\}$, g_u is either pseudo-Anosov or reducible. Further*:

(a) *g_u is reducible relative to $A \cup \{y_u\}$ if and only if, relative to A, y_u is Nielsen equivalent either to the boundary $\partial\mathbb{D}^2$ of $\mathbb{D}^2$, or to A*;
(b) *let $v \in \mathbb{F}_n$, and let $g_v : (\mathbb{D}^2, A \cup \{y_v\}) \to (\mathbb{D}^2, A \cup \{y_v\})$ be a topological v-extension of f. If the pairs (y_u, g_u) and (y_v, g_v) are both Nielsen equivalent to $\partial\mathbb{D}^2$ relative to A then u and v are φ-conjugate if and only if the lengths of their abelianizations are equal*;
(c) *if g_u is pseudo-Anosov relative to $A \cup \{y_u\}$ then f has a fixed point whose coordinate is $[u]$ and whose fixed point class is of non-zero index if and only if the topological entropies of f and g_u are equal.*

In terms of the initial expression for the Reidemeister trace of f obtained by applying Fadell and Husseini's formula, we may determine which Reidemeister classes correspond to fixed point classes Nielsen equivalent to the boundary; there is at most one which is non-empty, and we can find explicitly its coordinate and determine its index by abelianization. Similarly, we may determine which Reidemeister classes correspond to fixed point classes Nielsen equivalent to A. For the remaining classes (those which are neither Nielsen equivalent to the boundary nor to A), we can decide effectively which of them are realized by fixed points of f. This does not mean though that we can determine exactly the final expression for the Reidemeister trace of f: the problem that we come up against is that of determining the indices of the fixed point classes realized by f. One can in fact determine the Reidemeister trace of f completely by looking at the train track given by the Bestvina–Handel algorithm.

Using the theory of generalized braid groups, it seems possible that results analogous to those of Theorems 1 and 3, and Corollary 2 may hold in the case of surfaces of higher genus. This is the subject of work in progress.

Let φ be an automorphism of $\mathbb{F}_n$ induced by an orientation-preserving homeomorphism $f : (\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ belonging to an irreducible (i.e., not reducible) isotopy class. In Corollary 16, we characterize those words that are φ-conjugate to 1. For example, if $\beta \in B_n$ realizes φ then w is φ-conjugate to 1 if and only if the link $\widehat{\beta_w}$ (the closure of β_w in $\mathbb{S}^3$) is split. We also give an effective criterion that may be tested using the Bestvina–Handel algorithm.

Now let M be a compact, connected surface, and let $f : (M, A) \to (M, A)$ be a homeomorphism, where $A \subseteq \mathrm{Int}(M)$ is a finite subset. In Sections 4 and 5, we consider the geometric problem of deciding whether two fixed points $y_1, y_2 \in \mathrm{Fix}(f) \setminus A$ belong to the same fixed point class for f. We study this in relation to Nielsen–Thurston theory. A simple closed curve $\mathcal{C} \subseteq \mathrm{Int}(M) \setminus \{y_1, y_2\}$ will be said to be *Y-reducing* if:

(a) $Y = \{y_1, y_2\}$, and $\mathcal{C}$ bounds a topological closed disc $\mathcal{D}$ (which we shall call a *Y-reducing disc*) such that $\mathcal{D} \cap (A \cup Y) = Y$, and

(b) $f(\mathcal{C})$ is homotopic to $\mathcal{C}$ relative to $A \cup Y$.

If it exists, one may exhibit such a curve by applying the Bestvina–Handel algorithm, for any Y-reducing curve is also a reducing curve for $f:(M, A \cup Y) \to (M, A \cup Y)$. This being the case, the two fixed points belong to the same fixed point class.

Theorem 4. *Let $f:(M, A) \to (M, A)$ be a surface homeomorphism, and let $Y = \{y_1, y_2\} \subseteq \mathrm{Fix}(f) \setminus A$.*

(a) *If there exists a Y-reducing curve then y_1 and y_2 belong to the same fixed point class.*

(b) *There exists a Y-reducing curve if and only if there exists an arc γ joining y_1 to y_2 such that $f(\gamma)$ is homotopic to γ relative to $A \cup Y$. This being the case, one may choose γ to be a simple arc.*

The condition in (b) is stronger than the condition needed for Nielsen equivalence (where the homotopy is just carried out relative to A). Although the converse to (a) is in general false, there are certain interesting cases where it holds (and which can be checked using the Bestvina–Handel algorithm):

Theorem 5. *Let $f:(M, A) \to (M, A)$ be a surface homeomorphism, and let $Y = \{y_1, y_2\} \subseteq \mathrm{Fix}(f) \setminus A$. Suppose that the isotopy class of $f:(M, A \cup Y) \to (M, A \cup Y)$ is finite order or reducible. Then y_1 and y_2 belong to the same fixed point class if and only if there exists a Y-reducing curve.*

It would be interesting to have a necessary and sufficient condition in the pseudo-Anosov case.

In Section 5, we give necessary conditions on the Alexander polynomial of certain links for the existence of a Y-reducing curve for a orientation-preserving disc homeomorphism $f:(\mathbb{D}^2, A \cup Y) \to (\mathbb{D}^2, A \cup Y)$, and for the two given fixed points to belong to the same fixed point class. Given an isotopy $\{f_t\}_{t\in[0,1]}$ between the identity and f, we construct three braids: one, $\beta \in B_{n+2}$ associated with $A \cup \{y_1, y_2\}$; and then for $i = 1, 2$, $\alpha_i \in B_{n+1}^n$ associated with $A \cup \{y_i\}$. We denote their closures by $\widehat{\beta}$, $\widehat{\alpha_1}$ and $\widehat{\alpha_2}$, respectively. Let $\Delta_{\widehat{\beta}}(t, s, u)$ denote the 3-variable Alexander polynomial of the closed braid $\widehat{\beta}$, where the indeterminates t, s and u correspond to the components of the link $\widehat{\beta}$ formed by A, y_1 and y_2, respectively, and let $\Delta_{\widehat{\alpha_1}}(t, s)$ and $\Delta_{\widehat{\alpha_2}}(t, u)$ denote the 2-variable Alexander polynomials of the closed braids $\widehat{\alpha_1}$ and $\widehat{\alpha_2}$, respectively, with the same convention for t, s and u. Given two finite f-invariant sets $\mathcal{P}$, $\mathcal{Q}$, let $\mathrm{Lk}(\mathcal{P}, \mathcal{Q}; \{f_t\}_{t\in[0,1]})$ denote the linking number of $\mathcal{P}$ about $\mathcal{Q}$ relative to the isotopy $\{f_t\}_{t\in[0,1]}$.

Theorem 6. *With the above notation*:

(a) *suppose that* y_1 *and* y_2 *belong to the same fixed point class for* $f:(\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$. *Then* $\Delta_{\widehat{\beta}}(t,s,1) = \Delta_{\widehat{\beta}}(t,1,s)$ *and* $\Delta_{\widehat{\alpha_1}}(t,s) = \Delta_{\widehat{\alpha_2}}(t,s)$.

(b) *suppose that there exists a* Y*-reducing curve. Let* $m = \mathrm{Lk}(y_1, y_2; \{f_t\}_{t\in[0,1]})$ *and* $l = \mathrm{Lk}(A, y_1; \{f_t\}_{t\in[0,1]}) = \mathrm{Lk}(A, y_2; \{f_t\}_{t\in[0,1]})$.
Then

$$\Delta_{\widehat{\beta}}(t,s,u) = \left(t^l (su)^m - 1\right) \cdot \Delta_{\widehat{\alpha_1}}(t, su) = \left(t^l (su)^m - 1\right) \cdot \Delta_{\widehat{\alpha_2}}(t, su).$$

These criteria may be used to show that two fixed points do not belong to the same fixed point class, and to prove the nonexistence of Y-reducing curves, and they are obviously stronger than just abelianizing the coordinates of the given fixed points. They may be verified using a formula due to Burau (see Section 5), just by computing matrix determinants, so are in general quick to check. The generalization of this result to the characterization of Alexander polynomials associated with reducible isotopy classes is also the subject of work in progress.

Finally in Section 6, we give an application of the fact that the Reidemeister trace of f contains all of the fixed point information if f is pseudo-Anosov. It is motivated by the following beautiful result in one-dimensional dynamics. Given a continuous map h of the interval, a consequence of Sharkovskii's theorem [33] is that if h has a periodic orbit of period 3 then it has periodic orbits of all periods, i.e., $\mathrm{Per}(h) = \mathbb{N}$ ($\mathrm{Per}(h)$ denotes the set of periods of periodic orbits of h). One may ask whether an analogous result is true for surface homeomorphisms. The specification of the period alone of a periodic orbit no longer suffices to elicit a positive response to this question. For example, for each $n \in \mathbb{N}$, the set of periods of a rigid rotation of $\mathbb{D}^2$ by $2\pi/n$ about its centre is $\{1, n\}$. We conclude that we have to place some topological restrictions on the given periodic orbit. A suitable restriction is that the isotopy class of the homeomorphism relative to the periodic orbit A is pseudo-Anosov. If the genus of the surface M is zero and the homeomorphism preserves orientation then the following results are known:

(i) If $M = \mathbb{D}^2$ and if A is a periodic orbit of period 3 then $\mathrm{Per}(f) = \mathbb{N}$ [17,30].

(ii) If $M = \mathbb{D}^2$ and if $\mathrm{Card}(A) = 3$ or 4 then $\mathrm{Per}(f) = \mathbb{N}$ [18] ($\mathrm{Card}(A)$ denotes the cardinality of the set A).

(iii) If M is the 2-sphere $\mathbb{S}^2$ and if $\mathrm{Card}(A) = 4$ then $\mathrm{Per}(f) = \mathbb{N}$ [18,31]. The case where f is orientation-reversing was also treated in [31].

(iv) If $M = \mathbb{S}^2$ and if $\mathrm{Card}(A) = 5$ then $\mathrm{Per}(f) = \mathbb{N}$ [18].

But if $M = \mathbb{D}^2$ or $\mathbb{S}^2$ then for each $n \geqslant 7$, $n \in \mathbb{N}$, there exist an n-point set $A \subseteq \mathrm{Int}(M)$ and a pseudo-Anosov homeomorphism f relative to A such that $\mathrm{Per}(f) \neq \mathbb{N}$ [31]. We resolve the outstanding cases, answering questions posed in [18,31].

Theorem 7.

(a) *For each* $n \geqslant 5$, $n \in \mathbb{N}$, *there exists a pseudo-Anosov homeomorphism* $f:(\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$, *where* $A \subseteq \mathrm{Int}(\mathbb{D}^2)$ *is a periodic orbit of period* n, *such that* $\mathrm{Per}(f) \neq \mathbb{N}$.

(b) *For each* $n \geqslant 6$, $n \in \mathbb{N}$, *there exist an* n*-point set* $B \subseteq \mathbb{S}^2$ *and a pseudo-Anosov homeomorphism* $g:(\mathbb{S}^2, B) \to (\mathbb{S}^2, B)$ *such that* $\mathrm{Per}(g) \neq \mathbb{N}$.

Conclusion (b) follows easily from (a) by collapsing down the boundary of $\mathbb{D}^2$ to a point $z \in \mathrm{Fix}(g)$, defining $g|_{\mathbb{S}^2\setminus\{z\}} = f|_{\mathrm{Int}(\mathbb{D}^2)}$, and taking B to be $A \cup \{z\}$. Then g is pseudo-Anosov relative to B because f is pseudo-Anosov relative to A, and $\mathrm{Per}(g) \subseteq \mathrm{Per}(f)$ because $\mathrm{Fix}(f) \cap \mathrm{Int}(\mathbb{D}^2) \neq \emptyset$ (by the Lefschetz–Hopf fixed point theorem, and taking into account the possible fixed point indices of fixed points of pseudo-Anosov homeomorphisms).

2. Preliminaries

In this section, we recall some facts that will be used in what follows concerning Artin's braid groups, Nielsen fixed point theory, and the Nielsen–Thurston classification of surface homeomorphisms up to isotopy.

2.1. Artin's braid groups and the combing operation

The basic reference for this section is [3]. For each $n \in \mathbb{N}$, *Artin's braid group B_n on n strings* admits a presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$, and with the following defining relations:

$$\begin{aligned} &\sigma_i\sigma_j = \sigma_j\sigma_i, && \text{where } |i-j| \geqslant 2 \text{ and } 1 \leqslant i, j \leqslant n-1,\\ &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, && \text{where } 1 \leqslant i \leqslant n-2. \end{aligned}$$

The elements of B_n are called *braids*. Every braid may be represented as a collection of n strings, or *geometric braid*, in $\mathbb{R}^2 \times [0,1]$. We define the *full twist braid* in B_n to be the braid $(\sigma_1 \cdots \sigma_{n-1})^n$; for $n \geqslant 3$, it generates the centre of B_n. Given a braid $\beta = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_m}^{\varepsilon_m} \in B_n$, where $1 \leqslant i_j \leqslant n-1$ for $1 \leqslant j \leqslant m$, let $\mathrm{es}(\beta) = \sum_{j=1}^m \varepsilon_j$ denote the *exponent sum* of β. The *closure* $\widehat{\beta}$ of β is the link in $\mathbb{S}^3$ obtained by identifying the initial points and end points of each of the braid strings.

Let $\mathbb{F}_n$ be a free group of rank n, with generators $x_1, \ldots, x_n$. Let $\mathrm{Aut}(\mathbb{F}_n)$ denote the group of (right) automorphisms of $\mathbb{F}_n$. Artin showed that B_n has a faithful representation as a subgroup of $\mathrm{Aut}(\mathbb{F}_n)$. The representation is induced by the group homomorphism $\xi : B_n \to \mathrm{Aut}(\mathbb{F}_n)$ given by:

$$(\sigma_i)\xi\colon \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \\ x_j \mapsto x_j, \quad \text{if } j \neq i, i+1. \end{cases} \tag{1}$$

From now on, we will identify B_n with its realization as a group of right automorphisms of $\mathbb{F}_n$, in particular we shall write $(x_j)\alpha$ for $(x_j)((\alpha)\xi)$ for all $\alpha \in B_n$. If β is an endomorphism of $\mathbb{F}_n$ then $\beta \in B_n \subseteq \mathrm{Aut}(\mathbb{F}_n)$ if and only if the following two conditions are satisfied:

$$(x_i)\beta = A_i x_{\rho(i)} A_i^{-1} \quad \text{for all } 1 \leqslant i \leqslant n, \text{ and} \tag{2}$$

$$(x_1 \cdots x_n)\beta = x_1 \cdots x_n, \tag{3}$$

where $\rho \in \mathfrak{S}_n$, $\mathfrak{S}_n$ being the symmetric group on n elements, and $A_i \in \mathbb{F}_n$ for all $1 \leqslant i \leqslant n$.

For each pair $1 \leqslant i < j \leqslant n+1$, we define a braid $T_{ij} \in B_{n+1}$ as follows:

$$T_{ij} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}.$$

Each T_{ij} may be interpreted geometrically as a positive twist of the jth string about the ith string. For each $2 \leqslant j \leqslant n+1$, $U_j = \langle T_{ij} \mid 1 \leqslant i < j \rangle$ is a free subgroup of rank $j-1$ of B_{n+1}. In particular, U_{n+1} is isomorphic to $\mathbb{F}_n$ via the isomorphism which for each $1 \leqslant i \leqslant n$ associates $T_{i,n+1}$ with $x_i \in \mathbb{F}_n$. A result of Artin states that every $\beta \in B_{n+1}$ may be written uniquely in the form $\beta = \pi_\beta \beta_2 \cdots \beta_{n+1}$, where π_β is a permutation braid, and $\beta_j \in U_j$ for all $2 \leqslant j \leqslant n+1$. This decomposition is known as *combing* the braid. The fact that it is unique may be used, for example, to solve the word problem in the braid group.

For $k \geqslant 1$, consider the subgroup B_{n+k}^n of B_{n+k} consisting of those elements for which the induced automorphism of the symmetric group $\mathfrak{S}_{n+k}$ fixes $(n+1), \ldots, (n+k)$ pointwise. Now B_n embeds naturally in B_{n+k}^n via the injective group homomorphism $\iota_k : B_n \hookrightarrow B_{n+k}^n$ which geometrically consists of adding k extra vertical strings. Conversely, B_{n+k}^n projects into B_n via the surjective group homomorphism $p_k : B_{n+k}^n \to B_n$ which geometrically consists of 'forgetting' the last k strings. Clearly, $p_k \circ \iota_k = \mathrm{Id}_{B_n}$.

Now take $k = 1$, and set $p = p_1$ and $\iota = \iota_1$. Then $\mathrm{Ker}(p) = U_{n+1} \cong \mathbb{F}_n$. In the group-theoretical sense, B_{n+1}^n is the extension of $\mathbb{F}_n$ by B_n. The combing operation may be interpreted as the decomposition of B_{n+1}^n as the semi-direct product of U_{n+1} (or isomorphically, $\mathbb{F}_n$) with B_n (with action ξ). In particular:

Lemma 8. *Let $\beta \in B_{n+1}^n$. Then there exist $\alpha \in B_n$ and $T \in U_{n+1}$, both unique, such that $\beta = \iota(\alpha) \cdot T$. Moreover, $\alpha = p(\beta)$.*

Proof. The uniqueness and existence of α and T follow from that of the combing operation. For the second part, observe that $\iota(\alpha)$ and T belong to B_{n+1}^n, and $p(T) = \mathrm{Id}_{B_n}$. Applying p to β, we have that $p(\beta) = (p \circ \iota)(\alpha) = \alpha$. $\square$

2.2. Nielsen equivalence and surface maps

The references for this section are [11,26,27]. Let X be a compact, connected polyhedron, and let $f : X \to X$ be a continuous self-map. The notion of Nielsen equivalence (as defined in the introduction) is an equivalence relation on $\mathrm{Fix}(f)$. The equivalence classes under this relation will be called *fixed point classes*. If x is an isolated fixed point of f then let $\mathrm{Ind}(x, f)$ denote its fixed point index. If $x, y \in \mathrm{Fix}(f)$ are Nielsen equivalent then we write $(x, f) \overset{\mathrm{N}}{\sim} (y, f)$. Each fixed point class F is an isolated subset of $\mathrm{Fix}(f)$, hence its *fixed point index* $\mathrm{Ind}(\mathsf{F}, f) \in \mathbb{Z}$ is well defined. A fixed point class will be called *essential* if its index is non-zero, and *inessential* otherwise. The (finite) number of essential fixed point classes is a homotopy invariant, the *Nielsen number* $N(f)$ of f. Every map homotopic to f has at least $N(f)$ fixed points. Since every map homotopic to f has at least $N(f)$ fixed points, the Nielsen number plays an important rôle in the study of fixed point theory. The problem is that it is difficult to calculate in general.

Nielsen equivalence may be characterized in other ways. One is that of lifting classes. Another, due to Jiang, is in terms of homotopy classes in the suspension. Let $\mathbb{T}_f$ be the mapping torus of f obtained from $X \times [0,1]$ by identifying $(x,1)$ with $(f(x),0)$ for all $x \in X$. The map f induces a natural semi-flow ψ on $\mathbb{T}_f$. There is a one-to-one correspondence between the periodic orbits of f and closed ψ-orbits. Then two fixed points of f belong to the same fixed point class if and only if their corresponding closed orbits of ψ are homotopic as closed curves in $\mathbb{T}_f$ [28].

As we indicated in the introduction, the (difficult) geometric problem of deciding whether two fixed points are Nielsen equivalent can be transformed into an (admittedly difficult) algebraic problem, that of distinguishing Reidemeister classes. In X, choose a basepoint x_0 and a path w from x_0 to $f(x_0)$. Set $\pi = \pi_1(X, x_0)$, and denote the composition $\pi \xrightarrow{f_\#} \pi_1(X, f(x_0)) \xrightarrow{w_\#} \pi$ by $f_\pi : \pi \to \pi$. So π is partitioned into f_π-conjugacy classes. We denote the set of f_π-conjugacy classes by π_R, and the Abelian group freely generated by π_R by $\mathbb{Z}\pi_R$. Both projections $\pi \to \pi_R$ and $\mathbb{Z}\pi \to \mathbb{Z}\pi_R$ will be denoted by the notation $u \mapsto [u]$. Notice that if $f \simeq g$ via a homotopy $\{f_t\}_{t\in[0,1]}$: $f \simeq g$ satisfying $x_0 \in \mathrm{Fix}(f_t)$ for all $t \in [0,1]$ then $f_\pi = g_\pi$ for any choice of path w.

Let $x \in \mathrm{Fix}(f)$. Choose a path c from x_0 to x. The f_π-conjugacy class of $\langle w(f \circ c)c^{-1}\rangle \in \pi$ is independent of the choice of c. We call this class the (f_π-) *coordinate* of x, denoted by $\mathrm{coord}(x, f)$. The choice of basepoint x_0 and path w from x_0 to $f(x_0)$ serve as the *reference frame* (x_0, w) for the coordinate. The relation between the notions of Reidemeister and Nielsen equivalence is that two fixed points of f belong to the same fixed point class if and only if they have the same f_π-coordinate in π_R. This being the case, we denote the coordinate of the fixed point class F by $\mathrm{coord}(\mathsf{F}, f)$. We say that an f_π-conjugacy class is *realized by* f if there exists a fixed point of f whose f_π-coordinate is that class, and that the class is *realized essentially* if the index of the corresponding fixed point class is non-zero.

There are two well-known methods which may help to distinguish Reidemeister classes, neither being algorithmic. The first is to apply directly the definition of Reidemeister equivalence. In particular, notice that:

$$\begin{aligned} f_\pi(v)\cdot u &\sim_{\mathrm{R}} u \cdot v, \\ f_\pi(u) &\sim_{\mathrm{R}} u, \end{aligned} \tag{4}$$

for all $u, v \in \pi$. The second method is that of *abelianization*. Consider the following composition $\eta \circ \theta$:

$$\pi \xrightarrow{\theta} H_1(X) \xrightarrow{\eta} \mathrm{Coker}\big(1 - f_{*1} : H_1(X) \to H_1(X)\big), \tag{5}$$

where θ is abelianization and η is the natural projection. Every f_π-conjugacy class is sent to a single element; we call the image of a coordinate under the map $\eta \circ \theta$ the *abelianized coordinate*. Thus if two elements of π have different abelianizations then they are in different f_π-conjugacy classes. The converse of this statement is false (see Example 2 of Section 3.4 for a counter-example). Nevertheless, these two methods often suffice to determine exactly $L_R(f)$, as we shall see in Section 6. A third partial solution was given by Ferrario [13].

Define the *Reidemeister trace* (also known as the generalized Lefschetz number) of f, denoted by $L_R(f)$, to be the element of $\mathbb{Z}\pi_R$ for which the coefficient of $[u] \in \pi_R$ is equal to the index of the fixed point class with coordinate $[u]$. In other words,

$$L_R(f) = \sum \mathrm{Ind}(\mathsf{F}, f) \cdot \mathrm{coord}(\mathsf{F}, f) \in \mathbb{Z}\pi_R, \tag{6}$$

where the sum is over the fixed point classes of f. This sum makes sense because there are only a finite number of essential fixed point classes. We call this formal sum the *reduced form* of $L_R(f)$, in the sense that each f_π-conjugacy class appears at most once. We define the *abelianized Reidemeister trace*, denoted by $L_H(f)$, to be the abelianization of $L_R(f)$. Both $L_R(f)$ and $L_H(f)$ are powerful homotopy invariants of f which can be used to obtain fixed point linking information (see Section 6 and [19] for some direct applications). The number of terms appearing in Eq. (6) is precisely $N(f)$.

Now suppose that X is a compact, connected surface with boundary. Let $\{x_1, \ldots, x_n\}$ be a free basis for $\pi = \pi_1(X, x_0)$. Fadell and Husseini gave the following formula for $L_R(f)$ [11]:

$$L_R(f) = [1] - \big[\mathrm{Tr}\big(J(f_\pi)\big)\big] \in \mathbb{Z}\pi_R, \tag{7}$$

where $J(f_\pi) = (\partial f_\pi(x_i)/\partial x_j)_{1 \leqslant i,j \leqslant n}$ is the $n \times n$ Jacobian with entries in $\mathbb{Z}\pi$, and the derivatives $\partial/\partial x_j$ being with respect to the Fox calculus. They showed that a similar result is true when $\partial X = \emptyset$, there being an extra term which corresponds to the action on the 2-chains. Given f_π, the trace of the Jacobian is in itself very simple to calculate. However, it is not known in general how to determine the reduced form of $L_R(f)$ from Eq. (7), the problem being that of distinguishing the Reidemeister classes that appear. This step is necessary in order to extract all of the homotopy-invariant fixed point linking information and to calculate $N(f)$.

2.3. Surface homeomorphisms

In what follows, M will be a compact, connected, orientable surface, perhaps with boundary, and $A \subseteq \mathrm{Int}(M)$ will be a finite n-point subset. If $M = \mathbb{D}^2$, let $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$ denote the class of orientation-preserving homeomorphisms of $\mathbb{D}^2$ which fix $\partial\mathbb{D}^2$ pointwise, and let $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A)$ denote the subset of those elements of $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$ which leave A invariant. By the Alexander trick, every element of $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$ is isotopic to the identity via an isotopy fixing the boundary pointwise.

In order to compare fixed point classes of different surface homeomorphisms (via their coordinates), we recall several definitions [7,8,21]. For $i = 0, 1$, let $f_i : (M, A) \to (M, A)$ be homeomorphisms, and let $y_i \in \mathrm{Fix}(f_i)$. Then (y_0, f_0) and (y_1, f_1) are *connected by isotopy* if there exist an isotopy $\{f_t\}_{t \in [0,1]}$: $f_0 \simeq f_1$ relative to A and an arc $\alpha : [0, 1] \to M$ such that $\alpha(0) = y_0$, $\alpha(1) = y_1$, and $\alpha(t) \in \mathrm{Fix}(f_t)$ for all $t \in [0, 1]$. Given a homeomorphism $f : (M, A) \to (M, A)$, $x \in \mathrm{Fix}(f)$ is said to be *unremovable* if for any homeomorphism g isotopic to f relative to A, there exists $y \in \mathrm{Fix}(g)$ such that (x, f) and

(y, g) are connected by isotopy. The pair (x, f) is *connected to A* if there exist $a \in A$ and a homeomorphism g such that (x, f) and (a, g) are connected by isotopy, and *separated from A* otherwise.

Given $x, y \in \mathrm{Fix}(f)$, x is *strong Nielsen equivalent* to y (written $(x, f) \overset{\mathrm{SN}}{\sim} (y, f)$) if (x, f) and (y, f) are connected by a contractible isotopy. We write $\mathrm{snc}(x, f)$ for the strong Nielsen class of x. Two strong Nielsen classes are said to be *connected by isotopy* if elements from each class are. Just as for Nielsen equivalence, the notion of strong Nielsen equivalence can be expressed in terms of the suspension manifold: two fixed points of f are strong Nielsen equivalent if and only if the corresponding simple closed curves are isotopic. In particular, if $x, y \in \mathrm{Fix}(f)$ then $(x, f) \overset{\mathrm{SN}}{\sim} (y, f)$ if and only if $(x, f) \overset{\mathrm{N}}{\sim} (y, f)$. Thus x and y belong to the same fixed point class for f if and only they are strong Nielsen equivalent. Also, (x, f) is connected to A if and only if it is (strong) Nielsen equivalent to a point of A.

If $M = \mathbb{D}^2$ then we shall restrict our attention to the class $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$. Otherwise, we shall suppose that the Euler characteristic of M is negative. In both of these situations, all self-isotopies of M are contractible [12, p. 22], and so $(x, f) \overset{\mathrm{SN}}{\sim} (y, f)$ if and only if (x, f) and (y, f) are connected by isotopy.

Let $B \subseteq \mathrm{Int}(M) \setminus A$ be a finite f-invariant set. In what follows, we shall often consider f in two ways: as a map $f : (M, A) \to (M, A)$, and as a map $f : (M, A \cup B) \to (M, A \cup B)$. If this is the case, and if we are considering the coordinate of x for $f_\pi : \pi_1(M \setminus A) \to \pi_1(M \setminus A)$ then we shall refer to the *coordinate of x relative to A* (relative to some given reference frame) if there is a risk of confusion.

Let us recall briefly the Nielsen–Thurston classification of surface homeomorphisms up to isotopy [12,23,35]. If $f : (M, A) \to (M, A)$ is a homeomorphism then there exists a canonical homeomorphism g isotopic to f relative to A, the *Thurston representative* of the isotopy class, that satisfies one of the following:

(i) g is *finite order* (there exists $m \in \mathbb{N}$ such that $g^m = \mathrm{Id}$).

(ii) g is *pseudo-Anosov*. This means that it preserves a transverse pair of measured singular foliations, expanding the measure uniformly along the leaves of one foliation and contracting it uniformly (by the same factor) along the leaves of the other. All of its fixed points lying in $\mathrm{Int}(M)$ are unremovable and unique in their (strong) Nielsen class.

(iii) g is *reducible*. There exists a finite g-invariant set of simple closed curves in $M \setminus A$ which are mutually disjoint, non-homotopic, and neither parallel to a single point of A, nor to the boundary. These curves are called *reducing curves* for g. By cutting M along g-invariant tubular neighbourhoods of these curves, we obtain a finite number of subsurfaces or *reducing components*, and the restriction of an appropriate iterate of g to each reducing component is either finite order or pseudo-Anosov. As defined, the set of reducing curves is not in general unique, but we shall always choose a canonical (minimal) set, which is unique up to isotopy [4,38].

We shall say that the *Thurston type* of g (and of the isotopy class of f) is finite order, pseudo-Anosov or reducible respectively. The Thurston type of an isotopy class may be determined from the induced action on the fundamental group using the *Bestvina–Handel* *algorithm* [2] (see [22] for UNIX- and DOS-executable implementations). The algorithm also finds reducing curves if there are any, and determines the *topological entropy* $h(g)$ of g and the associated train track in the pseudo-Anosov case. For the case $M = \mathbb{D}^2$, versions of the algorithm were also given independently by Los [32], and Franks and Misiurewicz [15].

Given a surface homeomorphism $f:(M, A) \to (M, A)$, we shall be interested in its isotopy-invariant fixed-point structure. For many purposes, such as the application of the Bestvina–Handel algorithm, it is sufficient to consider the induced action f_π of f on the (non-compact) complement $M \setminus A$. On the other hand, much of the Nielsen fixed point theory and the Reidemeister trace can only be applied directly to compact spaces. Although one could try and do relative Nielsen theory, it will be convenient for us instead to blow up the points of A: we recompactify $M \setminus A$ to a surface M_A by adding a boundary circle $\mathcal{C}_a$ for each point $a \in A$. If, further, f is the Thurston representative in its isotopy class then it may be extended to a homeomorphism $\overline{f}: M_A \to M_A$, called the *blow up of* f, by considering the induced action of f on the circle of unit vectors at each point of A [5]. Identifying the fundamental groups $\pi_1(M \setminus A)$ and $\pi_1(M_A)$ in the obvious way, it follows that the induced automorphisms $f_\pi : \pi_1(M \setminus A) \to \pi_1(M \setminus A)$ and $\overline{f}_\pi : \pi_1(M_A) \to \pi_1(M_A)$ are equal. Considering $M \setminus A$ to be a subset of M_A on which f and $\overline{f}$ coincide (so $\mathrm{Fix}(f) \cap (M \setminus A) = \mathrm{Fix}(\overline{f}) \cap (M_A \setminus \bigcup_{a \in A} \mathcal{C}_a)$), it is clear that $\mathrm{Ind}(x, f) = \mathrm{Ind}(x, \overline{f})$ for all $x \in \mathrm{Fix}(f) \cap (M \setminus A)$; and if $x, y \in \mathrm{Fix}(f) \cap (M \setminus A)$ then $(x, f) \overset{\mathrm{N}}{\sim} (y, f)$ (relative to A) if and only if $(x, \overline{f}) \overset{\mathrm{N}}{\sim} (y, \overline{f})$.

To understand the fixed point structure of $f_\pi : \pi_1(M \setminus A) \to \pi_1(M \setminus A)$, one may compute $L_R(\overline{f})$: in light of the previous sentence, the only difference between the fixed point structures of f and $\overline{f}$ appears at the points of $A \cap \mathrm{Fix}(f)$. Let a be such a point. If $\mathrm{Ind}(a, f) = +1$ then the restriction of $\overline{f}$ to $\mathcal{C}_a$ is a non-trivial rotation, so $\mathrm{Fix}(\overline{f}) \cap \mathcal{C}_a = \emptyset$, and there will be no terms in $L_R(\overline{f})$ corresponding to the blow-up of a. On the other hand, if $\mathrm{Ind}(a, f) \leqslant 0$ then $\mathrm{Ind}(\mathcal{C}_a, \overline{f}) = \mathrm{Ind}(a, f) - 1$. In particular, $\mathrm{Fix}(\overline{f}) \cap \mathcal{C}_a \neq \emptyset$, and any element of this set will be a fixed point Nielsen equivalent to $\mathcal{C}_a$. As we shall see in Section 3.3, we are able to detect such fixed points, and consequently determine exactly the fixed point linking information of f on $M \setminus A$. Thus it suffices to consider the fixed point structure of $\overline{f}$ (in particular $L_R(\overline{f})$) in order to determine that of f.

In defining the blow up of f, we supposed that f was the Thurston representative in its isotopy class; but the blow-up construction is also valid for homeomorphisms which are differentiable on A. If, however, f is not differentiable on A then we may carry out an isotopy of f relative to A whose support is an arbitrarily small neighbourhood of A so that the homeomorphism f' thus obtained is differentiable on A (a local smoothing) [10]. Since we are interested in the isotopy-invariant fixed point structure of f which is the same as that of f' (because f and f' are isotopic relative to A), we may (and in what follows shall) assume by considering f' rather than f if necessary, that f may be blown up at A, so that $L_R(\overline{f})$ is defined.

One may compare fixed point classes of isotopic homeomorphisms using isotopy connection, although this is difficult to verify in practice. A more practical method is to compare coordinates, but one needs to take care with the reference frames. Let $M = \mathbb{D}^2$. As we remarked previously, we shall restrict our attention to the class $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$.

Notice that any homeomorphism f of $\mathbb{D}^2$ may be extended to an element g in this class by gluing an exterior collar $\mathcal{C} \cong (0,1] \times \mathbb{S}^1$ to $\partial\mathbb{D}^2$ to obtain a new topological disc $\overline{\mathbb{D}}$ in such a way that $\mathrm{Fix}(g) \cap \mathcal{C} = \partial\overline{\mathbb{D}}$. Considering $\mathbb{D}^2$ to be a subset of $\overline{\mathbb{D}}$, any isotopy $\{f_t\}_{t\in[0,1]}$ of f extends to an isotopy $\{g_t\}_{t\in[0,1]}$ of g such that for all $t \in [0,1]$, f_t and g_t coincide on $\mathbb{D}^2$, and $\mathrm{Fix}(g_t) \cap \mathcal{C} = \partial\overline{\mathbb{D}}$. Two fixed points of f are Nielsen equivalent for f if and only if they are Nielsen equivalent for g. Let $x_0 \in \partial\mathbb{D}^2$ and $z_0 \in \partial\overline{\mathbb{D}}$ be basepoints, and let $\alpha \subseteq \overline{\mathbb{D}} \setminus \mathrm{Int}(\mathbb{D}^2)$ be a path from z_0 to x_0. Let $w \subseteq \partial\mathbb{D}^2$ be a path from x_0 to $f(x_0)$ homotopic to $\alpha^{-1} \cdot g(\alpha)$ keeping endpoints fixed. Taking $f_\pi : \pi_1(\mathbb{D}^2 \setminus A) \to \pi_1(\mathbb{D}^2 \setminus A)$ (respectively, $g_\pi : \pi_1(\overline{\mathbb{D}} \setminus A) \to \pi_1(\overline{\mathbb{D}} \setminus A)$) with respect to the frame (x_0, w) (respectively, $(z_0, *_{z_0})$, $*_{z_0}$ being the constant path at z_0), it follows that f_π and g_π are equal (up to identification of $\pi_1(\mathbb{D}^2 \setminus A, x_0)$ with $\pi_1(\overline{\mathbb{D}} \setminus A, z_0)$ via α). With respect to these frames, $\mathrm{coord}(x, f) = \mathrm{coord}(x, g)$ for each $x \in \mathrm{Fix}(f)$. Since $\mathrm{Fix}(g) \setminus \mathrm{Fix}(f) = \partial\overline{\mathbb{D}}$ and $\mathrm{Ind}(\partial\overline{\mathbb{D}}, g) = 0$, it follows that $L_R(\overline{g}) = L_R(\overline{f})$, and thus we may restrict our attention to elements of $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$, and take a reference frame of the form $(z_0, *_{z_0})$.

In general, Thurston representatives do not belong to $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$. An element f of $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A)$ will be called a *fixed-boundary Thurston representative* if it is a Thurston representative in its isotopy class up to collaring. By this, we mean that there exists an f-invariant open tubular neighbourhood $\mathcal{N}$ of a simple closed curve contained in $\mathrm{Int}(\mathbb{D}^2)$ such that $\mathbb{D}^2 \setminus \mathcal{N}$ has two connected components, one an annulus $\mathbb{A}$ containing $\partial\mathbb{D}^2$, and the other a topological closed disc which contains A, such that the restriction of f to this disc is a Thurston representative. As in the previous paragraph, we may suppose that $\mathrm{Fix}(f) \cap \mathrm{Int}(\mathbb{A}) = \emptyset$, and further that $h(f|_{\mathbb{A}}) = 0$. By extension, if the isotopy class in question is pseudo-Anosov then we shall refer to the *fixed-boundary pseudo-Anosov homeomorphism.*

The isotopy class of an element f of $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A)$ may be represented by a (non-unique) braid. Let $\{f_t\}_{t\in[0,1]}$: $\mathrm{Id} \simeq f$ be an isotopy such that $f_t \in \mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2)$ for all $t \in [0,1]$. The subset $(A, \{f_t\}_{t\in[0,1]}) = \{(f_t(A), t)\}_{t\in[0,1]} \subseteq \mathbb{D}^2 \times [0,1]$ is a *geometric braid* on n strings which may be identified with an element of B_n. If $x_0 \in \partial\mathbb{D}^2$, the group of automorphisms of $\pi_1(\mathbb{D}^2 \setminus A, x_0) \cong \mathbb{F}_n$ induced by elements of $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A)$ [3, p. 33] may be identified naturally with B_n. Considering β as an element of $\mathrm{Aut}(\mathbb{F}_n)$, it follows from Eq. (7) that

$$L_R(\overline{f}) = [1] - \big[\mathrm{Tr}\big(J(\beta)\big)\big] \in \mathbb{Z}\pi_R, \tag{8}$$

where $J(\beta) = (\partial((x_i)\beta)/\partial x_j)_{1\leqslant i,j\leqslant n}$, and $\overline{f}$ is the blow up of f. We write the right-hand side of Eq. (8) as follows:

$$\sum_{i=1}^{K} \mu_i[w_i] \in \mathbb{Z}\pi_R, \tag{9}$$

where $\mu_i \in \{\pm1\}$. One of our aims is to be able to decide which of the Reidemeister classes in this sum are realized essentially by $\overline{f}$ and by f.

3. Braid realizations of Reidemeister classes

Let $n \geqslant 1$, and let $\beta \in B_n$ be a braid. In Section 3.1, given a word $w \in \mathbb{F}_n$, we will construct a braid belonging to B^n_{n+1}. As we shall see in Section 3.2, this braid will represent the isotopy class of a disc homeomorphism f relative to a given finite invariant set, and the point corresponding to the added string will represent a fixed point whose f_π-conjugacy class is $[w]$. In Section 3.3, we will give criteria for the Reidemeister class $[w]$ to be realized essentially by f, and in Section 3.4, we shall give some examples.

3.1. Construction of a braid extension

With the notation of Section 2.1, for each $1 \leqslant j \leqslant n$, let $T_j = T_{j,n+1} \in B_{n+1}$. Then T_j has the following effect on the elements of $\mathbb{F}_{n+1}$:

$$(x_k)T_j = \begin{cases} x_k, & \text{if } 1 \leqslant k \leqslant j-1,\\ x_j x_{n+1} x_j x_{n+1}^{-1} x_j^{-1}, & \text{if } k = j,\\ x_j x_{n+1} x_j^{-1} x_{n+1}^{-1} x_k x_{n+1} x_j x_{n+1}^{-1} x_j^{-1}, & \text{if } j+1 \leqslant k \leqslant n,\\ x_j x_{n+1} x_j^{-1}, & \text{if } k = n+1, \end{cases} \tag{10}$$

with similar expressions for the $(x_k)T_j^{-1}$. Let $\beta \in B_n$, and let $w = x_{k_1}^{\varepsilon_1} \cdots x_{k_l}^{\varepsilon_l} \in \mathbb{F}_n$. Then we set:

$$T_w = T_{k_1}^{\varepsilon_1} \cdots T_{k_l}^{\varepsilon_l} \in U_{n+1} \leqslant B^n_{n+1}, \quad \text{and}$$
$$\beta_w = \iota(\beta) \cdot T_w \in B^n_{n+1}.$$

Thus β_w may be considered as the embedding of β in B^n_{n+1}, followed by a number of twists of the $(n+1)$st string. We shall also refer to β_w as the *w-extension of β*. In terms of the semi-direct product $B^n_{n+1} \cong \mathbb{F}_n \rtimes B_n$, β_w is nothing other than the element (w, β). We have the following split short exact sequence:

$$1 \to \mathbb{F}_n \to B^n_{n+1} \xrightarrow{p} B_n \to 1,$$

where the second map is the monomorphism $w \mapsto T_w$.

Proposition 9. *Given $\beta \in B_n$ and $w \in \mathbb{F}_n$, let $\beta_w = \iota(\beta) \cdot T_w \in B^n_{n+1}$. Let $(x_{n+1})\beta_w = A_{n+1} x_{n+1} A_{n+1}^{-1} \in \mathbb{F}_{n+1}$, written as a reduced word, and let $\rho : \mathbb{F}_{n+1} \to \mathbb{F}_n$ be the projection $x_i \mapsto x_i$ for $1 \leqslant i \leqslant n$, and $x_{n+1} \mapsto 1$. Then $\rho(A_{n+1}) = w$.*

Proof. By induction on the length l of w. We may suppose that $\varepsilon_j = \pm 1$ for all j. The result is clear if $l = 0$, so suppose that $w \in \mathbb{F}_n$ has length $l \geqslant 0$. Let $\eta = T_w \in B^n_{n+1}$. Then $\rho(A_{n+1}) = w$ by the induction hypothesis, where $(x_{n+1})\eta = A_{n+1} x_{n+1} A_{n+1}^{-1}$ is written as a reduced word. Now consider $w' = w x_k^{\varepsilon_k}$, where $1 \leqslant k \leqslant n$, and $\varepsilon_k = \pm 1$. Set $\eta' = T_{w'} \in B^n_{n+1}$. We shall suppose that $\varepsilon_k = +1$, the case that $\varepsilon_k = -1$ being similar. Since $\eta' = \eta \cdot T_k$, it follows from Eq. (10) that:

$$(x_{n+1})\eta' = \big(A_{n+1} x_{n+1} A_{n+1}^{-1}\big) T_k = \alpha x_k x_{n+1} x_k^{-1} \alpha^{-1}, \tag{11}$$

where $\alpha \in \mathbb{F}_{n+1} = (A_{n+1})T_k$ is a reduced word. So $\rho((A_{n+1})T_k) = \rho(\alpha)$. On the other hand, from Eq. (10), we see that $\rho((x_j)T_i) = \rho(x_j)$ for all $1 \leqslant j \leqslant n+1$ and for all $1 \leqslant i \leqslant n$, and so $\rho((u)T_i) = \rho(u)$ for all $u \in \mathbb{F}_{n+1}$ and for all $1 \leqslant i \leqslant n$. In particular, $\rho((A_{n+1})T_k) = \rho(A_{n+1}) = w$. If we write $(x_{n+1})\eta'$ in reduced form $A'_{n+1}x_{n+1}{A'}^{-1}_{n+1}$, then we have to show that $\rho(A'_{n+1}) = w'$. In other words, we need to look for cancellation in Eq. (11).

If there is no cancellation then $A'_{n+1} = \alpha x_k$, and we obtain the required result. So suppose that there is cancellation. Then α must be of the (reduced) form $\alpha = \alpha' x_{n+1}^m x_k^{-1}$, where $m \in \mathbb{Z}$ is chosen so that $|m|$ is maximal. Then $(x_{n+1})\eta' = \alpha' x_{n+1} {\alpha'}^{-1}$, as a reduced word, in other words $A'_{n+1} = \alpha'$. But $\rho(A'_{n+1}) = \rho(\alpha') = \rho(\alpha x_k) = w x_k = w'$. $\square$

The following result will be very useful.

Proposition 10. *If $w \in \mathbb{F}_n$ and $\beta \in B_n$ then $T_w \cdot \iota(\beta) = \iota(\beta) \cdot T_{(w)\beta}$.*

Proof. By induction on the length of w. In fact, it suffices to check that for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n-1$ and $\varepsilon \in \{\pm 1\}$, $T_{x_i^\varepsilon} \cdot \sigma_j^{\pm 1} = \sigma_j^{\pm 1} \cdot T_{(x_i^\varepsilon)\sigma_j^{\pm 1}}$. This follows from a straightforward calculation using the relations (1). $\square$

Theorem 11. *Let $\beta \in B_n$ be a braid, and let $\varphi \in \mathrm{Aut}(\mathbb{F}_n)$ be the associated free group automorphism. Let $v, w \in \mathbb{F}_n$. Then*:

(a) *v and w are φ-conjugate if and only if β_v and β_w are conjugate in B_{n+1}^n via an element of U_{n+1};*

(b) *β_v and β_w are conjugate via an element of B_{n+1}^n if and only if there exists $\delta \in B_n$ that commutes with β, and such that $(w)\delta$ and v are φ-conjugate.*

Remarks.

(1) Part (a) of this theorem is the statement of Theorem 1. It gives a necessary and sufficient condition to decide when two elements of $\mathbb{F}_n$ are Reidemeister equivalent. In particular, any function of the braid group invariant under conjugation may be used to show that two elements of $\mathbb{F}_n$ are not Reidemeister equivalent.

(2) Since $U_{n+1} \leqslant B_{n+1}^n$, condition (a) of the theorem implies condition (b). The converse is false: see the remarks at the end of Section 3.4 for a counter-example.

Proof of Theorem 11. (a) Suppose that $v, w \in \mathbb{F}_n$ are φ-conjugate. Then there exists $\gamma \in \mathbb{F}_n$ such that $v = \varphi(\gamma) \cdot w \cdot \gamma^{-1}$. So $T_\gamma \in U_{n+1}$, and:

$$\begin{aligned} T_\gamma \cdot \beta_w \cdot T_\gamma^{-1} &= T_\gamma \cdot \iota(\beta) \cdot T_w \cdot T_{\gamma^{-1}} \\ &= \iota(\beta) \cdot T_{\varphi(\gamma)} \cdot T_w \cdot T_{\gamma^{-1}} \\ &= \iota(\beta) \cdot T_{\varphi(\gamma)\cdot w \cdot \gamma^{-1}} = \iota(\beta) \cdot T_v = \beta_v, \end{aligned}$$

by Proposition 10.

Conversely, suppose that β_v and β_w are conjugate in B_{n+1}^n via an element of U_{n+1}. Then there exists an element $T \in U_{n+1}$ such that $T\beta_w T^{-1} = \beta_v$. Now T is of the form T_γ for some $\gamma \in \mathbb{F}_n$. So by Proposition 10,

$$T_\gamma \cdot \beta_w \cdot T_\gamma^{-1} = \beta_v,$$
$$T_\gamma \cdot \iota(\beta) \cdot T_w \cdot T_{\gamma^{-1}} = \iota(\beta) \cdot T_v,$$
$$\iota(\beta) \cdot T_{\varphi(\gamma)\cdot w\cdot\gamma^{-1}} = \iota(\beta) \cdot T_v, \quad \text{and thus}$$
$$T_{\varphi(\gamma)\cdot w\cdot\gamma^{-1}} = T_v \in U_{n+1}.$$

We conclude that $\varphi(\gamma) \cdot w \cdot \gamma^{-1} = v$ (U_{n+1} is isomorphic to $\mathbb{F}_n$), and thus v and w are φ-conjugate.

(b) This follows in a similar way: if $\alpha\beta_v = \beta_w\alpha$, where $\alpha = \iota(\delta) \cdot T_u \in B_{n+1}^n$, $\delta \in B_n$ and $u \in \mathbb{F}_n$, then $\iota(\delta\beta) \cdot T_{\varphi(u)\cdot v} = \iota(\beta\delta) \cdot T_{(w)\delta\cdot u}$. By uniqueness of the combing operation, $\delta\beta = \beta\delta$, and $\varphi(u) \cdot v = (w)\delta \cdot u$. Thus $(w)\delta = \varphi(u) \cdot v \cdot u^{-1}$, and $(w)\delta$ and v are φ-conjugate. The converse is clearly true also. □

3.2. *Topological braid extensions for disc homeomorphisms*

Let $A \subseteq \mathrm{Int}(\mathbb{D}^2)$ be an n-point set, let $x_0 \in \partial\mathbb{D}^2$, and let $f \in \mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A)$. Given $w \in \mathbb{F}_n$, we construct an element g_w of $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A \cup \{y_w\})$ satisfying the following conditions:

- (BE1) g_w is isotopic to f relative to A (in particular, $g_w|_A = f|_A$), and the isotopy is chosen so that $\partial\mathbb{D}^2$ is fixed pointwise during the isotopy.
- (BE2) $y_w \in (\mathrm{Fix}(g_w) \cap \mathrm{Int}(\mathbb{D}^2)) \setminus A$.
- (BE3) $\mathrm{coord}(y_w, g_w) = [w]$ relative to A.

Remark. The coordinates for f and g_w are taken in the same reference frame $(x_0, *_{x_0})$. Condition (BE1) implies that $f_\pi = (g_w)_\pi$, so the f_π- and $(g_w)_\pi$-conjugacy classes coincide.

We add a fourth condition which, by isotoping g_w relative to $A \cup \{y_w\}$ if necessary, we may assume to be satisfied:

- (BE4) g_w is the fixed-boundary Thurston representative in its isotopy class relative to $A \cup \{y_w\}$.

A homeomorphism g_w satisfying conditions (BE1)–(BE4) will be called a *topological w-extension of f*. To see how to construct g_w (at least up to isotopy relative to $A \cup \{y_w\}$), one may consider a braid realization. Let $\{f_t\}_{t\in[0,1]} : \mathrm{Id} \simeq f$ be an isotopy (fixing $\partial\mathbb{D}^2$ pointwise during the isotopy), and let $\beta \in B_n$ represent the geometric braid (A, f_t). Pick a point $y_w \in \mathrm{Int}(\mathbb{D}^2) \setminus A$, and let $g_w : (\mathbb{D}^2, A \cup \{y_w\}) \to (\mathbb{D}^2, A \cup \{y_w\})$ be an element of $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A \cup \{y_w\})$ that realizes the braid $\beta_w = \iota(\beta) \cdot T_w \in B_{n+1}^n$. We consider g_w to be obtained by the composition of two isotopies: during the first, y_w is fixed and the braid realized by A is β; the second isotopy realizes T_w, y_w corresponding to the $(n+1)$st string. The first two of properties (BE1)–(BE3) may be satisfied easily; if we forget the $(n+1)$st string of β_w then we recover β. The following proposition shows that property (BE3) is also satisfied:

Proposition 12. *With the notation of the above construction,* $\mathrm{coord}(y_w, g_w) = [w]$ *relative to A.*

Proof. Consider $\pi_1(\mathbb{D}^2 \setminus (A \cup \{y_w\}), x_0) \cong \langle x_1, \ldots, x_{n+1} \rangle$ (y_w corresponds to x_{n+1}). If $(x_{n+1})\beta_w = A_{n+1} x_{n+1} A_{n+1}^{-1}$, written as a reduced word, then $\rho(A_{n+1}) = w$ by Proposition 9.

Pick a loop representing $x_{n+1} \in \mathbb{F}_{n+1}$. In $\mathbb{D}^2 \setminus A$, collapse it down to an arc c joining x_0 to y_w. One checks that in $\pi_1(\mathbb{D}^2 \setminus A, x_0)$, $\langle g_w(c) \cdot c^{-1} \rangle = \rho(A_{n+1})$, and so $\operatorname{coord}(y_w, g_w) = [w]$ relative to A. $\square$

Remark. Consider the isotopy $\{f_t\}_{t \in [0,1]}$: $\mathrm{Id} \simeq f$. Let $x \in \operatorname{Fix}(f) \setminus A$, and let $\alpha \in B_{n+1}^n$ be the braid realized by $(A \cup \{x\}, \{f_t\}_{t \in [0,1]})$. It follows from Lemma 8 that $\alpha = \iota \circ p(\alpha) \cdot T$ as elements of B_{n+1}^n, for some $T \in U_{n+1}$. The isotopy $\{f_t\}_{t \in [0,1]}$ may be modified to reflect this decomposition.

The notion of topological w-extension may be used to decide whether the Reidemeister class $[w]$ is realized essentially by f. Let $w \in \mathbb{F}_n$, and let $g_w : (\mathbb{D}^2, A \cup \{y_w\}) \to (\mathbb{D}^2, A \cup \{y_w\})$ be a topological w-extension of f. If $z \in \operatorname{Fix}(f) \setminus A$ then consider the two homeomorphisms $f : (\mathbb{D}^2, A \cup \{z\}) \to (\mathbb{D}^2, A \cup \{z\})$ and g_w. Their geometric braids are represented by braids $\beta_u \in B_{n+1}^n$ (by the above remark with $T = T_u$ for some $u \in \mathbb{F}_n$) and $\beta_w \in B_{n+1}^n$ (by construction), respectively. Then:

Theorem 13. *With the above notation*:
- (a) $\operatorname{coord}(z, f) = \operatorname{coord}(y_w, g_w)$ *relative to A if and only if β_u and β_w are conjugate in B_{n+1}^n via an element of U_{n+1}*;
- (b) *let $v \in \mathbb{F}_n$, and let $g_v : (\mathbb{D}^2, A \cup \{y_v\}) \to (\mathbb{D}^2, A \cup \{y_v\})$ be a topological v-extension of f. Then* $\operatorname{coord}(y_v, g_v) = \operatorname{coord}(y_w, g_w)$ *if and only if* $\operatorname{snc}(y_v, g_v)$ *is connected by isotopy to* $\operatorname{snc}(y_w, g_w)$.

Proof. (a) This follows from the fact that $f_\pi = (g_w)_\pi$ (considered as automorphisms of $\pi_1(\mathbb{D}^2 \setminus A, x_0)$), Proposition 12 and Theorem 11.

(b) Since y_v and y_w are chosen arbitrarily, we may suppose that $y_v \neq y_w$. Let $\mathfrak{i} : B_{n+1} \hookrightarrow B_{n+2}^{n+1}$ denote the inclusion homomorphism. As for the topological w-extensions, we construct a homeomorphism $g_{vw} \in \operatorname{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A \cup \{y_v, y_w\})$ satisfying:

1. (1) g_{vw} is isotopic to f relative to A (in particular, $g_{vw}|_A = f|_A$), and the isotopy is chosen so that $\partial\mathbb{D}^2$ is fixed pointwise during the isotopy.
2. (2) $\{y_v, y_w\} \subseteq \operatorname{Fix}(g_{vw}) \setminus A$.
3. (3) $\operatorname{coord}(y_v, g_v) = \operatorname{coord}(y_v, g_{vw})$ and $\operatorname{coord}(y_w, g_w) = \operatorname{coord}(y_w, g_{vw})$, where coordinates are taken relative to A in a frame $(x_0, *_{x_0})$, with $x_0 \in \partial\mathbb{D}^2$.

This may be achieved by taking an element g_{vw} of $\operatorname{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A \cup \{y_v, y_w\})$ that realizes the braid $\mathfrak{i}(\iota(\beta)) \cdot \mathfrak{i}(T_v) \cdot \sigma_{n+1} \cdot \mathfrak{i}(T_w) \cdot \sigma_{n+1}^{-1} \in B_{n+2}^n$. If we forget the $(n+2)$nd (respectively, $(n+1)$st) string then we obtain the braid $\iota(\beta) \cdot T_v = \beta_v \in B_{n+1}^n$ (respectively, $\iota(\beta) \cdot T_w = \beta_w \in B_{n+1}^n$), while if we forget both of these strings, we obtain β. Thus g_{vw} is isotopic to g_v (respectively, g_w) relative to $A \cup \{y_v\}$ (respectively, $A \cup \{y_w\}$), and the isotopy may be chosen to lie within $\operatorname{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A \cup \{y_v\})$ (respectively $\operatorname{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A \cup \{y_w\})$). So relative to A, $\operatorname{snc}(y_v, g_v)$ is connected by isotopy to $\operatorname{snc}(y_v, g_{vw})$, $\operatorname{snc}(y_w, g_w)$ is connected by isotopy to $\operatorname{snc}(y_w, g_{vw})$, $\operatorname{coord}(y_v, g_v) =$

$\mathrm{coord}(y_v, g_{vw})$, and $\mathrm{coord}(y_w, g_w) = \mathrm{coord}(y_w, g_{vw})$. We consider f, g_v, g_w and g_{vw} to be elements of $\mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A)$. Relative to the frame $(x_0, *_{x_0})$, it follows that the induced automorphisms f_π, $(g_v)_\pi$, $(g_w)_\pi$ and $(g_{vw})_\pi$ of $\pi_1(\mathbb{D}^2 \setminus A, x_0)$ are all equal. The corresponding Reidemeister classes are thus all in terms of the action of the same free group automorphism.

So $\mathrm{coord}(y_v, g_v) = \mathrm{coord}(y_w, g_w)$ if and only if $\mathrm{coord}(y_v, g_{vw}) = \mathrm{coord}(y_w, g_{vw})$, which is in turn equivalent to $\mathrm{snc}(y_v, g_{vw}) = \mathrm{snc}(y_w, g_{vw})$, which is equivalent to the fact that $\mathrm{snc}(y_v, g_{vw})$ is connected by isotopy to $\mathrm{snc}(y_w, g_{vw})$. □

Proof of Corollary 2. Let $f \in \mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A)$ be such that $f_\pi = \varphi$, and let $\beta \in B_n$ be the braid that realizes φ. Given $u, v \in \mathbb{F}_n$, let $g_u : (\mathbb{D}^2, A \cup \{y_u\}) \to (\mathbb{D}^2, A \cup \{y_u\})$ and $g_v : (\mathbb{D}^2, A \cup \{y_v\}) \to (\mathbb{D}^2, A \cup \{y_v\})$ be topological u- and v-extensions of f, respectively. Since u and v are f_π-conjugate, it follows from Theorem 11 that β_u and β_v are conjugate in B_{n+1}^n via an element $T \in U_{n+1}$, and so are conjugate in B_{n+1}. These braids are realized by g_u and g_v, respectively. Identifying $\pi_1(\mathbb{D}^2 \setminus (A \cup \{y_u\}), x_0)$ with $\pi_1(\mathbb{D}^2 \setminus (A \cup \{y_v\}), x_0)$, we see that $(g_u)_\pi$ and $(g_v)_\pi$ are conjugate via an automorphism induced by T. Since T may be realized by a homeomorphism, it follows that g_u and g_v are topologically conjugate up to isotopy. By construction, g_u and g_v are the Thurston representatives in their respective isotopy classes. But the topological conjugate of a Thurston representative is also a Thurston representative in its isotopy class. Moreover, a Thurston representative is unique up to topological conjugacy in its isotopy class (recall that we take a canonical set of reducing curves), modulo the behaviour on the tubular neighbourhood of any reducing curves, which in any case we can suppose to be of some standard form depending essentially on the behaviour of the homeomorphism on the complement of the neighbourhood. Hence g_u and g_v are topologically conjugate, and so they have the same topological entropy. □

Remark. It follows from the above proof that any two topological w-extensions of a given homeomorphism are topologically conjugate.

3.3. A criterion for the realization of Reidemeister classes

In this section, we suppose that $A \subseteq \mathrm{Int}(\mathbb{D}^2)$ is an n-point subset, where $n \geqslant 3$, and that $f \in \mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A)$ is a fixed-boundary pseudo-Anosov homeomorphism.

Given $w \in \mathbb{F}_n$, let $g_w : (\mathbb{D}^2, A \cup \{y_w\}) \to (\mathbb{D}^2, A \cup \{y_w\})$ be a topological w-extension of f. By construction, g_w is isotopic to f relative to A. In its isotopy class relative to $A \cup \{y_w\}$, g_w is the fixed-boundary Thurston representative, so it is either reducible or pseudo-Anosov. We analyse these two cases separately.

3.3.1. The reducible case

Part (a) of Theorem 3 will follow by taking $g = g_w$ and $y = y_w$ in the statement of the following proposition:

Proposition 14. *Let $A \subseteq \mathrm{Int}(\mathbb{D}^2)$ be an n-point subset, and let $f:(\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ be an orientation-preserving homeomorphism whose isotopy class is pseudo-Anosov. Let g be isotopic to f relative to A, and suppose that $y \in \mathrm{Fix}(g) \setminus A$.*

(a) *The isotopy class of $g:(\mathbb{D}^2, A \cup \{y\}) \to (\mathbb{D}^2, A \cup \{y\})$ is reducible if and only if, relative to A, y is Nielsen equivalent either to $\partial\mathbb{D}^2$, or to A.*

(b) *Let (x_0, c) be a reference frame, where $x_0 \in \partial\mathbb{D}^2$ and $c \subseteq \partial\mathbb{D}^2$ is a path from x_0 to $g(x_0)$. Then, relative to A, y is Nielsen equivalent to $\partial\mathbb{D}^2$ if and only if $\mathrm{coord}(y, g) = [(x_1 \cdots x_n)^m]$ for some $m \in \mathbb{Z}$.*

Remarks.

(1) From this, one may use the Bestvina–Handel algorithm to decide whether a Reidemeister class represents a fixed point class Nielsen equivalent to the boundary, Nielsen equivalent to A, or Nielsen equivalent to neither.

(2) The result holds in particular if f is the Thurston or fixed-boundary Thurston representative in its isotopy class relative to A.

Proof of Proposition 14. We will prove part (a) of the proposition; part (b) will follow directly from the proof. If $y \in \partial\mathbb{D}^2$ then the isotopy class of $g:(\mathbb{D}^2, A \cup \{y\}) \to (\mathbb{D}^2, A \cup \{y\})$ is reducible, and y is Nielsen equivalent to $\partial\mathbb{D}^2$ relative to A. So let us suppose that $y \in \mathrm{Int}(\mathbb{D}^2)$. By isotoping relative to $A \cup \{y\}$ if necessary, we may further suppose that g is the Thurston representative in its isotopy class relative to $A \cup \{y\}$.

First, suppose that $g:(\mathbb{D}^2, A \cup \{y\}) \to (\mathbb{D}^2, A \cup \{y\})$ is reducible. Since g is isotopic to f relative to A, there are exactly two reducing components, one of which, $\mathcal{D}$, say, contains y. So g fixes each component setwise. There are two possibilities:

(i) $\mathcal{D}$ is a topological annulus, one of whose boundary components is $\partial\mathbb{D}^2$, and $\mathcal{D} \cap A = \emptyset$, or

(ii) $\mathcal{D}$ is a topological disc containing exactly one point a of A.

In both cases, $g|_{\mathcal{D}}:(\mathcal{D}, (\mathcal{D} \cap A) \cup \{y\}) \to (\mathcal{D}, (\mathcal{D} \cap A) \cup \{y\})$ is finite order. It is thus conjugate to rigid rotation and so must be the identity. If $\eta \subseteq \mathcal{D}$ is any arc joining y to $\partial\mathbb{D}^2$ in Case (i), or to a in Case (ii), then $g(\eta) = \eta$, and so relative to A, y is Nielsen equivalent to $\partial\mathbb{D}^2$ in Case (i), and to A in Case (ii).

To prove part (b) and the converse of part (a), first suppose that y is Nielsen equivalent to $\partial\mathbb{D}^2$ for g relative to A. Let (x_0, c) be a reference frame as in part (b). We first show that $\mathrm{coord}(y, g) = [(x_1 \cdots x_n)^m]$ for some $m \in \mathbb{Z}$, from which we shall conclude that g is reducible relative to $A \cup \{y\}$. As usual, we identify $\pi_1(\mathbb{D}^2 \setminus A, x_0)$ with $\mathbb{F}_n = \langle x_1, \ldots, x_n \rangle$. Since y is Nielsen equivalent to $\partial\mathbb{D}^2$ relative to A, there exist an arc $\alpha:[0,1] \to \mathbb{D}^2 \setminus A$ such that $\alpha(0) \in \partial\mathbb{D}^2$ and $\alpha(1) = y$, and a homotopy $\{\alpha_t\}_{t \in [0,1]}:[0,1] \to \mathbb{D}^2 \setminus A$ satisfying $\alpha_0 = g(\alpha)$, $\alpha_1 = \alpha$, and for all $t \in [0,1]$, $\alpha_t(0) \in \partial\mathbb{D}^2$ and $\alpha_t(1) = y$. Homotoping α if necessary, we may suppose that $\alpha(0) = x_0$. For each $t \in [0,1]$, the arc $\lambda_t = \{\alpha_s(0)\}_{0 \leqslant s \leqslant t} \cdot \alpha_t$ joins $\alpha(0) = g(x_0)$ to $\alpha(1) = y$, and relative to these two endpoints, it is homotopic to $g(\alpha) = \lambda_0$. Further, $\lambda_1 = \{\alpha_s(0)\}_{0 \leqslant s \leqslant 1} \cdot \alpha$ whose first segment $\{\alpha_s(0)\}_{0 \leqslant s \leqslant 1} \subseteq \partial\mathbb{D}^2$ is an arc joining $g(x_0)$ to x_0. Hence $\langle c \cdot g(\alpha) \cdot \alpha^{-1} \rangle = \langle c \cdot \{\alpha_s(0)\}_{0 \leqslant s \leqslant 1} \rangle \in \pi_1(\partial\mathbb{D}^2, x_0)$. Interpreting $\pi_1(\partial\mathbb{D}^2, x_0)$ as the infinite cyclic subgroup of $\pi_1(\mathbb{D}^2 \setminus A, x_0)$ generated by $\xi =$

$(x_1 \cdots x_n)$, it follows that $\langle c \cdot g(\alpha) \cdot \alpha^{-1} \rangle = \xi^m$, for some $m \in \mathbb{Z}$, and $\mathrm{coord}(y, g) = [\xi^m]$ relative to A. This proves the necessity of part (b).

Now suppose that $\mathrm{coord}(y, g) = [\xi^m]$ relative to A. As in Section 3.2, the braid associated with the isotopy class of $g : (\mathbb{D}^2, A \cup \{y\}) \to (\mathbb{D}^2, A \cup \{y\})$ may be written in the form $\beta_w = \iota(\beta) \cdot T_w \in B^n_{n+1}$ for some $w \in \mathbb{F}_n$, and so $\mathrm{coord}(y, g) = [w]$ relative to A by Proposition 12. Thus w and ξ^m are g_π-conjugate (where we consider g to be a homeomorphism relative to A), so there exists $\gamma \in \mathbb{F}_n$ such that $\xi^m = g_\pi(\gamma) \cdot w \cdot \gamma^{-1}$. Let $h : (\mathbb{D}^2, A \cup \{y\}) \to (\mathbb{D}^2, A \cup \{y\})$ be the Thurston representative of the isotopy class represented by the braid $\sigma = \iota(\beta) \cdot T_{\xi^m} \in B^n_{n+1}$. By Theorem 1 it follows that β_w and σ are conjugate in B^n_{n+1} (in fact, $T_\gamma \cdot \beta_w \cdot T_\gamma^{-1} = \sigma$). So as in the proof of Corollary 2, g and h are topologically conjugate. It is clear from the form of σ that h is reducible: there exists an h-invariant simple closed curve whose isotopy class is represented by the word $\xi \in \mathbb{F}_{n+1}$. Thurston type is a conjugacy invariant, so g is also reducible relative to $A \cup \{y\}$ (there exists a g-invariant simple closed curve whose isotopy class is represented by the word $(\xi)T_\gamma^{-1} \in \mathbb{F}_{n+1}$). The sufficiency of part (b) also follows, and this indeed completes the proof of part (b) and the first case of the converse of part (a).

Finally, the second case of the converse of part (a) may be deduced from the first case as follows. Suppose that y is Nielsen equivalent to A. There exists $a \in A$ such that y is Nielsen equivalent to a relative to A. Collapse down $\partial\mathbb{D}^2$ to a point z, and blow up a to a boundary circle to give a new topological disc $\mathbb{D}$. Then g induces a homeomorphism g' of $\mathbb{D}$. Set $A' = (A \setminus \{a\}) \cup \{z\}$. Then $y \in \mathrm{Fix}(g')$ is Nielsen equivalent to $\partial\mathbb{D}$ for g' relative to A'. By a similar argument to that of the first case, we conclude that g' is reducible relative to $A' \cup \{y\}$, and that g is reducible relative to $A \cup \{y\}$. □

Given a word $w = x_{i_1}^{\varepsilon_1} \cdots x_{i_r}^{\varepsilon_r}$ in the generators $x_1, \ldots, x_n$ of $\mathbb{F}_n$, define the *abelianized length of* w to be $\tau(w)$, where $\tau : \mathbb{F}_n \to \mathbb{Z}$ is the group homomorphism defined by $w \mapsto \sum_{j=1}^r \varepsilon_{i_j}$. If $f \in \mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A)$ then we shall say that w *is connected to* $\partial\mathbb{D}^2$ if y_w is Nielsen equivalent to $\partial\mathbb{D}^2$ relative to A for some (and hence any) topological w-extension $g_w : (\mathbb{D}^2, A \cup \{y_w\}) \to (\mathbb{D}^2, A \cup \{y_w\})$ of f.

Corollary 15. *For $w_i \in \mathbb{F}_n$, $i = 1, 2$, let $g_i : (\mathbb{D}^2, A \cup \{y_i\}) \to (\mathbb{D}^2, A \cup \{y_i\})$ be a topological w_i-extension of f. Suppose that the pairs (y_1, g_1) and (y_2, g_2) are both Nielsen equivalent to $\partial\mathbb{D}^2$ relative to A. Then*:

(a) *w_1 and w_2 are f_π-conjugate if and only if their abelianized lengths are equal*;

(b) $\mathrm{snc}(y_1, g_1)$ *and* $\mathrm{snc}(y_2, g_2)$ *are connected by isotopy if and only if the abelianizations of* $\mathrm{coord}(y_1, g_1)$ *and* $\mathrm{coord}(y_2, g_2)$ *are equal.*

The proof of the corollary follows from Theorem 13 and Proposition 14. This also proves part (b) of Theorem 3. One can thus decide effectively which of the Reidemeister classes appearing in Eq. (9) correspond to the Nielsen class of $\partial\mathbb{D}^2$, and among them, which are Reidemeister equivalent. One can also prove the following result which characterizes those words of $\mathbb{F}_n$ that are Reidemeister equivalent to 1:

Corollary 16. *Let φ be an automorphism of $\mathbb{F}_n$ that is induced by a braid $\beta \in B_n$. Suppose further that β is realized by an orientation-preserving homeomorphism $f : (\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ whose isotopy class relative to A is irreducible. Then*:

$$\begin{aligned}[1] &= \{w \in \mathbb{F}_n \mid \beta_w \text{ is conjugate to } \iota(\beta) \text{ in } B^n_{n+1} \text{ via an element of } U_{n+1}\} \\ &= \{w \in \mathbb{F}_n \mid \widehat{\beta_w} \text{ is a split link}\} \\ &= \{w \in \mathbb{F}_n \mid w \text{ is connected to } \partial\mathbb{D}^2\} \cap \mathrm{Ker}(\tau).\end{aligned}$$

Once again, this gives a criterion that may be verified using the Bestvina–Handel algorithm.

3.3.2. The pseudo-Anosov case

Given $w \in \mathbb{F}_n$, suppose that g_w is the fixed-boundary pseudo-Anosov homeomorphism relative to $A \cup \{y_w\}$. By condition (BE1) and [24], we have that $h(g_w) \geqslant h(f)$. The following result is part (c) of Theorem 3.

Theorem 17. *The Reidemeister class $[w]$ is realized essentially by f if and only if $h(f) = h(g_w)$.*

Remark. The topological entropy of a Thurston representative is equal to that of a fixed-boundary Thurston representative belonging to the same isotopy class. One can hence calculate the topological entropies of f and g_w using the Bestvina–Handel algorithm. Theorem 17 thus gives an effective criterion for the realization of the Reidemeister class $[w]$ by f.

Proof of Theorem 17. Let us first show that f realizes $[w]$ if and only if $h(f) = h(g_w)$. Suppose that $[w]$ is not realized by f. Since g_w is pseudo-Anosov relative to $A \cup \{y_w\}$, it follows from Proposition 14 that relative to A, y_w is neither Nielsen equivalent to $\partial\mathbb{D}^2$ nor to A. So y_w is separated from A. Further, g_w is isotopic to f relative to A, $y_w \in \mathrm{Fix}(g_w)$, and by Theorem 13, $\mathrm{snc}(y_w, g_w)$ is not connected by isotopy to any fixed point of f. It follows from a result of Smillie [7,24] that $h(g_w) > h(f)$.

Conversely, if $[w]$ is realized by $z \in \mathrm{Fix}(f) \setminus A$ then it follows from an argument similar to that of Corollary 2 that f and g_w are topologically conjugate, and so $h(f) = h(g_w)$.

Finally, if f realizes $[w]$ essentially then it realizes $[w]$. The converse is also true: if $[w]$ is realized by $z \in \mathrm{Fix}(f) \setminus A$ then it is separated from A because y_w is. Taking into account the possible fixed point indices of fixed point classes of pseudo-Anosov homeomorphisms, it follows that if z belongs to the fixed point class F then $\mathrm{Ind}(\mathsf{F}, f) \neq 0$. □

3.4. Comments and examples

With $f \in \mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A)$ as in Section 3.3 (pseudo-Anosov relative to A), one may determine which terms appearing in Eq. (9) are the coordinates of fixed point classes realized essentially by f. In particular:

(a) One may determine those terms that correspond to fixed point classes that are Nielsen equivalent to the boundary, and among them, those that are Reidemeister equivalent (Proposition 14). From this, the indices of the corresponding fixed point classes may be computed. In fact, at most one of these classes has non-zero (negative) index μ, and any remaining classes are empty.

(b) In a similar way, one may determine those terms that correspond to fixed point classes Nielsen equivalent to A. Again, there is at most one non-empty fixed point class Nielsen equivalent to each fixed point in A. Set $\mathcal{F}_A$ to be the sum of such terms: it may be zero, for example if $A \cap \mathrm{Fix}(f) = \emptyset$. As we indicated in Section 2.3, the structure of $\mathrm{Fix}(\overline{f}) \setminus \mathrm{Fix}(f)$ is encapsulated in $\mathcal{F}_A$ ($\overline{f}$ is the blow-up of f at A).

(c) The remaining terms in Eq. (9) correspond to fixed point classes that are Nielsen equivalent neither to the boundary nor to A. If w_i is such a term, one may decide whether or not $[w_i]$ corresponds to a fixed point class realized essentially by f by comparing the topological entropy of f with that of its topological w_i-extension (Theorem 17). We thus obtain a sum of the form:

$$L_R(\overline{f}) = \mu \cdot \left[(x_1 \cdots x_n)^m\right] + \mathcal{F}_A + \sum_{i=1}^{l} \mu_i \cdot [w_i], \tag{12}$$

where $\mu_i \in \{\pm 1\}$, $m \in \mathbb{Z}$, and $l \in \mathbb{N}$, and for $1 \leqslant i \leqslant l$, $\overline{f}$ and thus f realize essentially the fixed point class whose coordinate is $[w_i]$.

We now illustrate our results with some examples.

Example 1. Consider $\beta = \sigma_1\sigma_2^{-1} \in B_3$. It is well known that this braid represents a pseudo-Anosov isotopy class. Now x_1 and x_2 are Reidemeister equivalent because $(x_2)\beta = x_1$. We can also see this by applying Theorem 1 to the braid extensions β_{x_1} and β_{x_2}, since $T_{x_2^{-1}} \cdot \beta_{x_1} \cdot T_{x_2} = \beta_{x_2}$.

Example 2. Consider the words x_1 and $x_3x_2^{-1}x_1$ for the same braid as in Example 1. They cannot be distinguished by abelianization. The associated braid extensions are not conjugate because the topological entropies of the corresponding topological extensions are different. Moreover, the 2-variable Alexander polynomials of the associated closed braids are different (see also Theorem 6 and Section 5). Similarly, x_2 and x_3 are not Reidemeister equivalent.

Returning to Eq. (12), there is still an (open) problem: with these methods, it is not clear how one might determine the indices of the non-empty fixed point classes corresponding to the remaining terms (this could in fact be done by determining the train track of f using the Bestvina–Handel algorithm). All of these terms correspond to essential fixed point classes of f, but in general, there will be pairwise cancellation of terms whose indices are of opposite sign. The problem comes down to that of comparing strong Nielsen classes of different pairs (y_i, g_i). One idea is to construct an extension of an extension and use the notion of reducibility. Given $v, w \in \mathbb{F}_n$, consider the homeomorphism $g_{vw} : (\mathbb{D}^2, A \cup \{y_v, y_w\}) \to (\mathbb{D}^2, A \cup \{y_v, y_w\})$ constructed in the proof of part (b) of Theorem 13. If it is reducible (relative to $A \cup \{y_v, y_w\}$) then, relative to A, y_w and y_v belong to the same fixed point class for g_{vw}, and so (y_w, g_w) and (y_v, g_v) are connected by isotopy.

Example 3. Again take $\beta = \sigma_1\sigma_2^{-1} \in B_3$. We already know that $v = x_2$ and $w = x_1$ are Reidemeister equivalent. We can also see that corresponding fixed points belong to the same fixed point class for the homeomorphism g_{vw}. Consider the associated braid $\beta_{vw} = \sigma_1\sigma_2^{-1}\sigma_3\sigma_2^2\sigma_3^{-1}\sigma_4\sigma_3\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1} \in B_5$. One observes that there is a reducing curve whose isotopy class is of the form $x_2x_4x_2^{-1}x_5$ containing the strings corresponding to y_1 and y_2.

We shall come back to this type of reducibility in Section 4. The existence of such a reduction can also be indicated by comparing the Alexander polynomial of various closed braids (see Theorem 6).

However, the converse of the above observation does not hold: the fact that the two fixed points belong to the same fixed point class does not imply that there exists such a reduction. The reason is clear: in this setting, reducibility is relative to $A \cup \{y_v, y_w\}$, while belonging to the same fixed point class is just relative to A. As an example, consider the braid $\sigma_1^2\sigma_2^2\sigma_1^{-2}\sigma_2^{-2} \in B_3$ whose closure is the Borromean rings. This braid is realized by a pseudo-Anosov homeomorphism of the disc. But relative to the second fixed point, the first and third fixed points belong to the same fixed point class. For another example which is realized by a pseudo-Anosov homeomorphism of the form g_{vw}, it suffices to take $v = x_1$ and $w = x_2$ for the braid $\beta = \sigma_1\sigma_2^{-1} \in B_3$. Notice that this is similar to Example 3 above, except that we have exchanged v and w. With this in mind, one may ask the following question.

Question. Let φ be a free group automorphism φ realized by an n-braid. Given $v, w \in \mathbb{F}_n$ does there exist some explicit construction of another braid (perhaps similar to that of g_{vw}, but taking into account the twisting of the two added strings) such that this braid is reducible if and only if v and w are Reidemeister equivalent for φ?

One can sometimes distinguish certain Reidemeister classes in a simple way using abelianization *and* dynamical properties, as in the following example.

Example 4. Consider the free group automorphism induced by the braid $\beta = (\sigma_1\sigma_2^{-1})^2 \in B_3$. Then:

$$\beta\colon \begin{cases} x_1 \mapsto x_1x_3x_1^{-1}x_3^{-1}x_2x_3x_1x_3^{-1}x_1^{-1}, \\ x_2 \mapsto x_1x_3x_1^{-1}, \\ x_3 \mapsto x_3^{-1}x_2^{-1}x_3x_1x_3^{-1}x_2x_3. \end{cases}$$

Let $f : (\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ be the Thurston representative in an isotopy class which relative to A realizes the braid $\sigma_1\sigma_2^{-1}$. So $g = f^2 : (\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ is pseudo-Anosov relative to A because f is, and it realizes the braid β. Are x_1 and x_2 Reidemeister equivalent for

$g_\pi = \beta$? Abelianization alone does not help to decide, and there does not seem to be any simple way of using the relations (4) either. Moreover, the entropies of the associated braid extensions are equal.

In fact, x_1 and x_2 are not g_π-conjugate. Let $\overline{g}$ be the blow up of g at A. One can show easily that $L_R(\overline{g}) = [x_2] - [x_1x_3] + [x_1] + [x_3^{-1}] - [x_3^{-1}x_2^{-1}] + [x_3^{-1}x_2^{-1}x_3] - [1]$. These seven classes abelianize to $L_H(\overline{g}) = -t^2 + 2t - 1 + 2t^{-1} - t^{-2}$. Thus $\overline{g}$ has at least one fixed point of positive index corresponding to the abelianized coordinate t. It follows from Section 2.3 that the same is true for g. The maximal index of a fixed point of a pseudo-Anosov homeomorphism is $+1$, so g has at least two such fixed points of index 1, and since g is pseudo-Anosov relative to A, they must belong to different fixed point classes. But $[x_1]$ and $[x_2]$ are the only Reidemeister classes in $L_R(\overline{g})$ abelianizing to t, they are the coordinates of these two fixed points, and they are thus distinct as elements of π_R. So x_1 and x_2 are not g_π-conjugate.

Remarks.

(1) The preceding example also proves the assertion made in Remark (2) following Theorem 11: there exist $\beta \in B_n$, $v \in \mathbb{F}_n$, and $\delta \in B_n$ which commutes with β such that β_v and $\beta_{(v)\delta}$ are conjugate via an element of B_{n+1}^n but not necessarily via an element of U_{n+1}. For take $n = 3$, $\beta = (\sigma_1\sigma_2^{-1})^2 \in B_3$, $v = x_2$, $\delta = \sigma_1\sigma_2^{-1}$, and $w = (v)\delta = x_1$. We have just seen that v and $(v)\delta$ are not Reidemeister equivalent for β. So β_v and $\beta_{(v)\delta}$ are not conjugate via an element of U_4. However, δ commutes with β, and $(w)\delta = (x_2)\beta$ is Reidemeister equivalent to $v = x_2$ for β. It follows from part (b) of Theorem 11 that β_v and $\beta_{(v)\delta}$ are conjugate via an element of B_4^3 (they are in fact conjugate via $\iota(\delta)$).

(2) Let $h : (M, A) \to (M, A)$ be a pseudo-Anosov homeomorphism of a compact, connected surface M. For each $n \in \mathbb{N}$, the fixed points of h^n lying in $\mathrm{Int}(M) \setminus A$ belong to distinct fixed point classes. One might ask whether the braid types [6,7,18,21], considered relative to A, of two periodic orbits of h of the same period are distinct. The answer is no. For consider the pseudo-Anosov homeomorphism $g : (\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ in the previous example. For $i = 1, 2$, let $y_i \in \mathrm{Fix}(g) \setminus A$ be a fixed point which realizes the Reidemeister class $[x_i]$ for g. The preceding remark shows that, relative to A, y_1 and y_2 have the same braid type.

There are several possibilities for extensions of these ideas. One would be to generalize the extension construction to surfaces of higher genus. This could be undertaken using generalized braid groups. Another is to consider the case of periodic orbits, in particular, to find analogous criteria to decide when two periodic orbits are strong Nielsen equivalent in the sense of Asimov and Franks [1]. These two generalizations are the subject of work in progress.

4. Reidemeister classes and reducibility

In this section, we give a criterion in terms of reducibility to decide whether two fixed points of a surface homeomorphism belong to the same fixed point class.

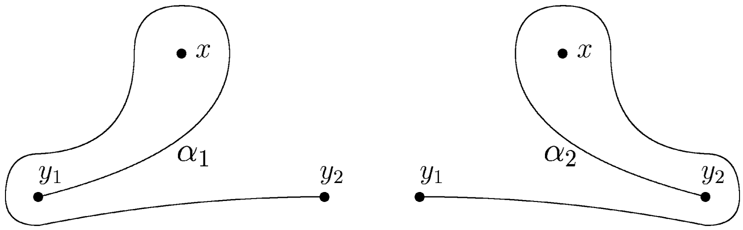

Fig. 1. The arcs α_1 and α_2.

Let M be a compact, connected, orientable surface, and let $A \subseteq \mathrm{Int}(M)$ be a finite subset. Let $f:(M, A) \to (M, A)$ be an orientation-preserving homeomorphism, and let $\{y_1, y_2\} = Y \subseteq \mathrm{Fix}(f) \setminus A$, where $y_1 \neq y_2$. By adding a collar containing no new fixed points to each boundary component of M, we may suppose without loss of generality that $y_1, y_2 \in \mathrm{Int}(M)$.

Suppose that y_1 and y_2 lie in the same fixed point class for $f:(M, A) \to (M, A)$. Then there exists an arc $c:[0,1] \to M \setminus A$ joining them such that $f(c)$ is homotopic to c relative to A, keeping endpoints fixed. A priori, c is not an embedding, but it can be made to be so, by 'pushing off' any self-intersections. But as we have already seen in Section 3.4, $f(c)$ is not necessarily homotopic to c relative to $A \cup Y$. As another example, consider the two arcs α_1 and α_2 shown in Fig. 1. They are homotopic relative to $A = \{x\}$, but are not homotopic relative to $A \cup Y$, and it is easy to construct a homeomorphism of the disc $\mathbb{D}^2$ that fixes pointwise x, y_1 and y_2, and that sends α_1 onto α_2.

With this in mind, we shall say that a simple closed curve $\mathcal{C} \subseteq \mathrm{Int}(M) \setminus Y$ is *Y-reducing* if:

(YR1) $\mathcal{C}$ bounds a topological closed disc $\mathcal{D}$ (which we shall call a *Y-reducing disc*) such that $\mathcal{D} \cap (A \cup Y) = Y$, and

(YR2) $f(\mathcal{C})$ is homotopic to $\mathcal{C}$ relative to $A \cup Y$.

This definition may be extended to larger finite subsets $Y \subseteq \mathrm{Fix}(f) \setminus A$. It follows from a theorem of Baer [10] that if condition (YR2) is satisfied then $f(\mathcal{C})$ is in fact isotopic to $\mathcal{C}$ relative to $A \cup Y$. This means that any Y-reducing curve is also (up to isotopy) a reducing curve (in the sense of Nielsen–Thurston theory) for the homeomorphism $f:(M, A \cup Y) \to (M, A \cup Y)$. By applying the Bestvina–Handel algorithm, one can thus decide effectively whether such a curve exists, and if so, the algorithm will exhibit a Y-reducing curve (which may not be unique). In particular, the following result, which is part (a) of Theorem 4, gives a criterion (stronger than abelianization of their coordinates) to decide whether two fixed points belong to the same fixed point class.

Proposition 18. *Suppose that $f:(M, A) \to (M, A)$ and Y are as above, and suppose further that there exists a Y-reducing curve. Then y_1 and y_2 belong to the same fixed point class.*

For the proof, it suffices to take an arc contained in the corresponding Y-reducing disc joining y_1 and y_2. In general, the converse of the proposition is false. As in Section 3.4, take

$M = \mathbb{D}^2$ and a homeomorphism whose suspension realizes the braid $\sigma_1^2\sigma_2^2\sigma_1^{-2}\sigma_2^{-2} \in B_3$. Its isotopy class is pseudo-Anosov, so there are no essential reducing curves, but relative to any one of the three points associated to the braid strings, the other two points belong to the same fixed point class.

There are however certain interesting cases, where with extra hypotheses, the converse of Proposition 18 is true. This being the case, the fact that two fixed points y_1 and y_2 of a surface homeomorphism f belong to the same fixed point class means that there exists a Y-reducing curve, and we can thus distinguish the Reidemeister classes of fixed points of f. This is indeed the case in the situations described in part (b) of Theorem 4, and in Theorem 5.

Proof of part (b) of Theorem 4. To prove the necessity of the condition, it suffices to take γ to be any simple arc joining y_1 to y_2 contained within the Y-reducing disc whose boundary is the given Y-reducing curve.

To prove sufficiency, isotope f relative to $A \cup Y$ to the 'standard form' $\varphi : (M, A \cup Y) \to (M, A \cup Y)$ of [29]. It follows from that paper (blowing up the points of Y to boundary circles if necessary—the corresponding boundary components are then φ-related) that y_1 and y_2 must both lie in the same finite order component M_0 which is fixed pointwise by φ, and that γ may be chosen to lie entirely within M_0. Now take $\mathcal{C} \subseteq \mathrm{Int}(M_0)$ to be any simple closed curve bounding a disc satisfying condition (YR1), then $\varphi(\mathcal{C}) = \mathcal{C}$, and so condition (YR2) is satisfied for f. $\square$

A similar argument proves Theorem 5. It would be interesting to have an analogous characterization of fixed point classes for the case where the isotopy class of $f : (M, A \cup Y) \to (M, A \cup Y)$ is pseudo-Anosov. Of course, the Borromean rings example shows again that equivalence in Theorem 5 does not hold in this case. We can interpret topologically the negative result of this example. From Jiang's characterization of fixed point classes in terms of curves in the suspension (see Section 2.2), it follows that two fixed points y_1, y_2 of a surface homeomorphism f belong to the same fixed point class if and only if the corresponding simple closed curves $\mathcal{C}_1$, $\mathcal{C}_2$ are freely isotopic in the mapping torus minus the image of A under the suspension flow. Since there is no Y-reducing curve, this means that there is no embedded annulus whose boundary components are $\mathcal{C}_1$ and $\mathcal{C}_2$, and whose interior avoids $\mathcal{C}_1 \cup \mathcal{C}_2$.

5. Reducibility and the Alexander polynomial

Let $\beta \in B_n$ be a braid, and let $\mathcal{L}_1, \ldots, \mathcal{L}_\mu$ denote the $\mu \geqslant 1$ components of the link $\widehat{\beta}$. The permutation ρ induced by β consists of μ disjoint cycles $\rho_1, \ldots, \rho_\mu$, where ρ_i corresponds to the component $\mathcal{L}_i$ of $\widehat{\beta}$. With the notation of Section 2.1, the link group $G = \pi_1(\mathbb{S}^3 \setminus \widehat{\beta})$ of the complement of $\widehat{\beta}$ in $\mathbb{S}^3$ admits the presentation $\langle x_1, \ldots, x_n \mid R_1, \ldots, R_n \rangle$, where R_i is the relation $A_i x_{\rho(i)} A_i^{-1} x_i^{-1}$ for $i = 1, \ldots, n$ [3]. Let ψ denote simultaneously the canonical group homomorphism $\mathbb{F}_n = \langle x_1, \ldots, x_n \rangle \to G$ and its extension $\mathbb{Z}\mathbb{F}_n \to \mathbb{Z}G$ to the group rings. Let $H = \langle t_1 \rangle \times \cdots \times \langle t_\mu \rangle$ denote the free abelian group of rank μ, and let φ denote simultaneously the homomorphism $G \to H$ which maps the generator x_i to the indeterminate t_j, where i belongs to the support of ρ_j, and its extension $\mathbb{Z}G \to \mathbb{Z}H$ to the group rings, where $\mathbb{Z}H$ is the ring of polynomials in $t_1, t_1^{-1}, \dots, t_\mu, t_\mu^{-1}$ with integer coefficients. For each $j = 1, \dots, \mu$, we may consider t_j to be at once a free generator of H and an oriented meridian of $\mathcal{L}_j$. There is a matrix representation $r : B_n \to \mathrm{GL}(n-1, \mathbb{Z}H)$, which we call the *link representation* of B_n [19]. One obtains the *reduced Burau representation* by identifying $t_1, \dots, t_\mu$ in $r(\beta)$ to a single symbol t.

Let $A \subseteq \mathrm{Int}(\mathbb{D}^2)$ be an n-point subset, and let $f \in \mathrm{Homeo}(\mathbb{D}^2, \partial\mathbb{D}^2, A)$ be such that A consists of μ distinct periodic orbits of f. Let $\{f_t\}_{t\in[0,1]} : \mathrm{Id} \simeq f$ be an isotopy that is fixed on $\partial\mathbb{D}^2$, and let $\beta \in B_n$ represent the geometric braid $(A, \{f_t\}_{t\in[0,1]})$. The link representation may be interpreted as a signed linking transition matrix for an Axiom A representative that realizes the braid β [14]. It is also strongly related to the abelianized Reidemeister trace: more precisely, $L_H(\overline{f}) = -\mathrm{Tr}(r(\beta))$ [16], and $\mathrm{Coker}(1 - \overline{f}_{*1}) = H$ [26,27]. Further, $L(\overline{f}) = L_H(\overline{f})|_{t_1=\cdots=t_\mu=1}$ is the usual Lefschetz number [16].

Burau showed that $\det(r(\beta) - \mathrm{Id}) = (1 + t + \cdots + t^{n-1}) \cdot \Delta_{\widehat{\beta}}(t)$ if $\mu = 1$, and $\det(r(\beta) - \mathrm{Id}) = (\varphi\psi(x_1 \cdots x_n) - 1) \cdot \Delta_{\widehat{\beta}}(t_1, \dots, t_\mu)$ if $\mu \geqslant 2$ [36]. This gives a simple method of computing the Alexander polynomials of closed braids, such as those appearing in Theorem 6, which is a stronger criterion than that of abelianization. Theorem 6 may be used to detect certain reducible isotopy classes, notably the Y-reducible ones of Section 4. The generalization of this is the subject of work in progress.

Part (a) of Theorem 6 follows from Theorem 1 and the fact that the closures of conjugate braids have the same Alexander polynomial. Part (b) may be proved by looking at a Jacobian matrix with respect to the Fox calculus of the action of β on a suitable set of generators for $\pi_1(\mathbb{D}^2 \setminus A)$; by conjugation, one may suppose that β is adapted to the reduction. The result also follows directly from [34,37].

The converse to Theorem 6 is false. Consider the following example (suggested by Jonathan Hillman). Take

$$\beta = \sigma_2\sigma_3\sigma_1^{-3}\sigma_2\sigma_3\sigma_4\sigma_3\sigma_2^{-1}\sigma_1^2\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1} \in B_6.$$

Let $f : (\mathbb{D}^2, A \cup \{y_1, y_2\}) \to (\mathbb{D}^2, A \cup \{y_1, y_2\})$ be an orientation-preserving homeomorphism that realizes β (via an isotopy $\{f_t\}_{t\in[0,1]}$), where A is a periodic orbit of period 4 corresponding to the first four strings, and y_1 and y_2 are fixed points of f corresponding to the 5th and 6th strings respectively. With the notation introduced just before the statement of Theorem 6, $\Delta_{\widehat{\beta}} = \Delta_{\widehat{\alpha_2}} = 0$ because the links $\widehat{\beta}$ and $\widehat{\alpha_2}$ are both split. Further, $\Delta_{\widehat{\alpha_1}} = 0$; this can be checked using the above formulae for the Alexander polynomial, or by observing that $\widehat{\alpha_1}$ may be deformed into the unsplittable link on p. 56 of [25]. So all the polynomials in Theorem 6 are identically zero (as are the linking numbers l and m). Suppose that there were to exist a Y-reducing curve. It follows from Theorem 4 that y_1 and y_2 belong to the same fixed point class for $f : (\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$. Since $\alpha_1 = \delta_u$ and $\alpha_2 = \delta_v$, where $\delta = \sigma_2\sigma_3\sigma_1^{-3}\sigma_2\sigma_3 \in B_4$, $u = x_2^{-1}x_1$ and $v = 1$ in $\mathbb{F}_4$, it follows from Theorem 1 that α_1 and α_2 must be conjugate via an element of U_5 (and thus conjugate in B_5). But $\widehat{\alpha_1}$ is an unsplittable link, while $\widehat{\alpha_2}$ is a split link, so α_1 and α_2 cannot be conjugate. One concludes that y_1 and y_2 must belong to distinct fixed point classes, and that there is no Y-reducing curve.

6. Periods for disc homeomorphisms

In this section we prove part (a) of Theorem 7 by constructing an explicit example for each $n \geqslant 5$.

Let $f:(\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ be an orientation-preserving homeomorphism, where $A \subseteq \mathrm{Int}(\mathbb{D}^2)$ is a periodic orbit of f, and the isotopy class of f relative to A is pseudo-Anosov. Let $g:(\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ be the pseudo-Anosov homeomorphism in this isotopy class. Two fixed points of g belong to the same fixed point class if and only if they both belong to $\partial\mathbb{D}^2$. If $z \in \mathrm{Fix}(g)$ were to belong to a non-essential fixed point class then it follows by studying the local foliation structure of pseudo-Anosov homeomorphisms that z would have to be a 1-pronged singularity of the foliations (e.g., [18,30]), and so $z \in A$. Since $A \cap \mathrm{Fix}(g) = \emptyset$, we conclude that z must lie in an essential fixed point class. So the element $L_R(\overline{f})$ describes exactly the linking information of the fixed points of the pseudo-Anosov homeomorphism g. In particular, the projected coordinate $\eta\theta(\mathrm{coord}(z, f)) = t^l$, where l is the algebraic linking number of z about A in some given suspension.

Proof of Theorem 7. Take $n \geqslant 5$. Let $\beta = \sigma_1\sigma_2\cdots\sigma_{n-3}\sigma_{n-2}^{-1}\sigma_{n-1}^{-1} \in B_n$. Let $f:(\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ be the Thurston representative of the isotopy class relative to A such that f_π may be identified with β. We will show that f is pseudo-Anosov and that it has no points of period 2.

We calculate the action of β (and hence f_π) on $\pi_1(\mathbb{D}^2 \setminus A)$ using Eq. (1):

$$\beta: \begin{cases} x_1 \mapsto x_1x_2\cdots x_{n-3}x_nx_{n-3}^{-1}\cdots x_2^{-1}x_1^{-1}, \\ x_i \mapsto x_{i-1}, \quad \text{for } 2 \leqslant i \leqslant n-2, \\ x_{n-1} \mapsto x_n^{-1}x_{n-2}x_n, \\ x_n \mapsto x_n^{-1}x_{n-1}x_n. \end{cases}$$

Applying Eqs. (4) and (8), we see that

$$L_R(\overline{f}) = [1] - \left[\sum_{i=1}^{n} \frac{\partial((x_i)\beta)}{\partial x_i}\right] = [x_1] + \left[x_n^{-1}\right] - [1]. \tag{13}$$

We claim that $f:(\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ is pseudo-Anosov. It cannot be finite order since the exponent sum $\mathrm{es}(\beta)$ of β would have to be a non-zero multiple of $n-1$ [6], and we see that $\mathrm{es}(\beta) = n-5$ (which is not divisible by $n-1$ if $n \geqslant 4$). On the other hand, f cannot be reducible. One can prove this using the Bestvina–Handel algorithm. We give an alternative short proof using linking number properties. Suppose that f were reducible relative to A. Let $\{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ be a set of reducing curves such that $f(\mathcal{C}_i) = \mathcal{C}_{i+1}$ for $1 \leqslant i \leqslant k-1$ and $f(\mathcal{C}_k) = \mathcal{C}_1$. Then $1 < k < n$ and $k \mid n$. Let $\mathcal{D}_0$ be the reducing component that contains

$\partial\mathbb{D}^2$; it is a k-holed disc. Each one of the other reducing components $\mathcal{D}_1, \ldots, \mathcal{D}_k$ contains $n/k > 1$ points of A. Suppose that $z \in \mathrm{Fix}(f) = \mathrm{Fix}(\overline{f})$. Since $\mathrm{Fix}(f) \cap (\bigcup_{i=1}^k \mathcal{D}_i) = \emptyset$ then $z \in \mathcal{D}_0$. Let $[\alpha] \in \mathbb{Z}\pi_R$ be the coordinate of z. Then the abelianized coordinate is $\eta\theta([\alpha]) = t^l \in \mathrm{Coker}(1 - \overline{f}_{*1})$, where l is the linking number of z about A. By reducibility, l must be a multiple of n/k. But from Eq. (13), there exist two fixed points whose abelianized coordinates are $t^{\pm 1}$, which implies that $n = k$. This contradicts our assumption that $n/k > 1$. Hence f must be pseudo-Anosov relative to A.

We now show that f has no points of period 2. It suffices to prove that $\mathrm{Fix}(f) = \mathrm{Fix}(f^2)$. By considering the abelianization of the three fixed point classes in Eq. (13), we conclude that they are distinct. On $\partial\mathbb{D}^2$, each of the two invariant foliations associated with the pseudo-Anosov homeomorphism has at least one singularity, the singularities of the two foliations alternate, the singularities are permuted by f, and they all have the same period. Remembering that two fixed points of f belong to the same fixed point class if and only if they lie in $\partial\mathbb{D}^2$, and by considering the index of a fixed point in terms of the local foliation structure [18,30], we conclude from Eq. (13) that:

(i) f has exactly two (interior) fixed points of positive index, corresponding to the f_π-conjugacy classes $[x_1]$ and $[x_n^{-1}]$;

(ii) f either has exactly one interior fixed point of negative index, or it has exactly two fixed points on $\partial\mathbb{D}^2$ (which correspond to singularities of the foliations). This corresponds to the f_π-conjugacy class $[1]$.

Similarly, β^2 has the following action on $\mathbb{F}_n$:

$$\beta^2\colon \begin{cases} x_1 \mapsto x_1 x_2 \cdots x_{n-3} x_n x_{n-3}^{-1} x_n^{-1} x_{n-1} x_n x_{n-3} x_n^{-1} x_{n-3}^{-1} \cdots x_2^{-1} x_1^{-1}, \\ x_2 \mapsto x_1 x_2 \cdots x_{n-3} x_n x_{n-3}^{-1} \cdots x_2^{-1} x_1^{-1}, \\ x_i \mapsto x_{i-2}, \quad \text{for } 3 \leqslant i \leqslant n-2, \\ x_{n-1} \mapsto x_n^{-1} x_{n-1}^{-1} x_n x_{n-3} x_n^{-1} x_{n-1} x_n, \\ x_n \mapsto x_n^{-1} x_{n-1}^{-1} x_{n-2} x_{n-1} x_n. \end{cases}$$

Applying Eqs. (4) and (8) (for f_π^2-conjugacy classes), we see that:

$$\begin{aligned} L_R(\overline{f}^2) &= [x_1 x_2] + [x_n^{-1} x_{n-1}^{-1}] - [1] \\ &= [(x_1)\beta \cdot x_1] + [(x_n^{-1})\beta \cdot x_n^{-1}] - [1]. \end{aligned}$$

So f^2 has exactly three fixed point classes, which are distinct (as classes of f^2). Since $f^2 : (\mathbb{D}^2, A) \to (\mathbb{D}^2, A)$ is also pseudo-Anosov relative to A, we conclude as before that:

(i) f^2 has exactly two (interior) fixed points of positive index, corresponding to the f_π^2-conjugacy classes $[(x_1)\beta \cdot x_1]$ and $[(x_n^{-1})\beta \cdot x_n^{-1}]$;

(ii) f^2 either has exactly one interior fixed point of negative index, or it has exactly two fixed points on $\partial\mathbb{D}^2$ (which correspond to singularities of the foliations). This corresponds to the f_π^2-conjugacy class $[1]$.

But $\mathrm{Fix}(f) \subseteq \mathrm{Fix}(f^2)$, hence it follows that the fixed points of f^2 are precisely the fixed points of f. So f has no periodic points of period 2. In particular, $\mathrm{Per}(f) \neq \mathbb{N}$. $\square$

Remark. Applying the Bestvina–Handel algorithm to these examples, and constructing a Markov partition, one may in fact show that $\mathrm{Per}(f) = \mathbb{N} \setminus \{2\}$.

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