General solution of a kind of quantum coloured Yang–Baxter equation (I) ♠

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Received 9 November 2005
Available online 31 March 2006
Submitted by Steven G. Krantz

Abstract

In this paper, we give a new method to solve the quantum coloured Yang–Baxter matrix equation (QCYBE), and the general solution for a kind of QCYBE is given.
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Keywords: Yang–Baxter equation; General solution

1. Introduction

Let A(u), B(u) be m × m matrices with meromorphic function entries of u. By the so-called Yang–Baxter equation (YBE) one means the matrix equation

\[ A(u)B(u + v)A(v) = B(v)A(u + v)B(u). \] (1.1)

The YBE, which was proposed independently by Yang [17,18] and Baxter [3], applies in many branches of physics and mathematics. Let \( \tilde{R}(u, x, y) \) with \( \det(\tilde{R}(u, x, y)) \neq 0 \) be an \( n^2 \times n^2 \) complex meromorphic function matrix with three independence complex variables \( u, x, y \). Here, and

© Project supported by the National Natural Science Foundations of China (Grant Nos. 10171027 and 10271097).
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doi:10.1016/j.jmaa.2006.02.017
throughout this paper, we use $f(u, x, y) \neq 0$ to signify that $f(u, x, y)$ is not the zero function $0$, but it may have zero points in $\mathbb{C}^3$. We define

$$
\check{R}^{12}(u, x, y) = \check{R}(u, x, y) \otimes E, \quad \check{R}^{23}(u, x, y) = E \otimes \check{R}(u, x, y),
$$

where $\otimes$ means the tensor product of two matrices. By the quantum coloured Yang–Baxter equation (QCYBE) we mean the matrix equation

$$
\check{R}^{12}(u, x, y) \check{R}^{23}(u + v, x, z) \check{R}^{12}(v, y, z) = \check{R}^{23}(v, y, z) \check{R}^{12}(u + v, x, z) \check{R}^{23}(u, x, y).
$$

(1.2)

The QCYBE (1.2) depends on both the spectral parameters $u, v$ and the colour parameters $x, y, z$, and when it is independent of the colour parameters $x, y, z$, it reduces to the usual YBE (1.1). This type of YBE is a very important equation in mathematical physics. It is relevant to statistics physics, quantum groups, quantum field theory, lower dimension topology, knot theory and etc. (see, e.g., [1, 2, 4, 6–14, 19]). For the QCYBE, one of the most important basic problems is to solve it. For instance, it is known that to give an exact solution of a statistical model from physics, one must solve the corresponding YBE. So, up to now, a lot of research interest have been paid to find exact solutions for this type of YBE (see, e.g., [5, 9, 12, 15]). And for the most interesting example in physics with $\check{R}(u, x, y)$ to be the so-called eight-vertex type of the form

$$
\check{R}(u, x, y) = \begin{pmatrix}
* & 0 & 0 & * \\
0 & * & * & 0 \\
0 & * & * & 0 \\
* & 0 & 0 & *
\end{pmatrix},
$$

many useful results have been achieved (see, e.g., [16]).

Using the bi-indices symbols $(1, 1), \ldots, (1, n), \ldots, (n, 1), \ldots, (n, n)$ for the rows and columns in the matrix $\check{R}(u, x, y)$, we can write

$$
\check{R}(u, x, y) = \begin{pmatrix}
\check{R}_{11}(u, x, y) & \cdots & \check{R}_{1n}(u, x, y) \\
\vdots & \ddots & \vdots \\
\check{R}_{n1}(u, x, y) & \cdots & \check{R}_{nn}(u, x, y)
\end{pmatrix} = \sum_{i,j=1}^{n} E_{ij} \otimes \check{R}_{ij}(u, x, y),
$$

where

$$
E_{ij} = \begin{pmatrix}
\delta_{i1}\delta_{1j} & \cdots & \delta_{i1}\delta_{nj} \\
\vdots & \ddots & \vdots \\
\delta_{ni}\delta_{1j} & \cdots & \delta_{ni}\delta_{nj}
\end{pmatrix} = e_i e_j, \quad 1 \leq i, j \leq n,
$$

with $e_i$ being the $i$th unit row vector, and

$$
\check{R}_{ij}(u, x, y) = \begin{pmatrix}
\check{r}_{i1j}(u, x, y) & \cdots & \check{r}_{i1n}(u, x, y) \\
\vdots & \ddots & \vdots \\
\check{r}_{inj}(u, x, y) & \cdots & \check{r}_{inn}(u, x, y)
\end{pmatrix} = \sum_{k,\ell=1}^{n} \check{r}_{ikj}(u, x, y) E_{k\ell}.
$$

Thus

$$
\check{R}(u, x, y) = \sum_{i,j,k,\ell=1}^{n} \check{r}_{ij\ell}(u, x, y) E_{ij} \otimes E_{k\ell}.
$$
It is well known that
\[ E_{ij} E_{k\ell} = \delta_{jk} E_{i\ell}, \quad 1 \leq i, j, k, \ell \leq n. \]

By this and using the matrix entries \( \tilde{r}^{jk}_{il}(u, x, y), 1 \leq i, j, k, \ell \leq n \), the matrix equation (1.2) can be transformed to the following function equation system
\[
\sum_{p, q, r = 1}^{n} \tilde{y}^{pq}_{be}(u, x, y) \tilde{y}^{ir}_{ac}(u + v, x, z) \tilde{y}^{jk}_{rp}(v, y, z) = \sum_{p, q, r = 1}^{n} \tilde{y}^{qr}_{bc}(u, x, y) \tilde{y}^{ir}_{aq}(u + v, x, z) \tilde{y}^{jk}_{rp}(v, y, z), \quad (1.3)
\]
where \( i, j, k, a, b, c = 1, \ldots, n \).

Clearly, (1.3) consists of \( n^6 \) equations with \( n^4 \) meromorphic functions as unknowns, and our problem now becomes to solve the nonlinear function equation system (1.3). Obviously, it is difficult to give efficient systematic methods to obtain the general solution of (1.3).

The purpose of this series of papers is to display a new method for solving this type of QCYBE, and to give an exact general solution of Eq. (1.2) when \( \tilde{R}(u, x, y) \) is an \( 4 \times 4 \) upper triangle matrix
\[
\tilde{R}(u, x, y) = \begin{pmatrix}
\tilde{r}^{11}_{11} & \tilde{r}^{11}_{12} & \tilde{r}^{11}_{13} & \tilde{r}^{11}_{14} \\
\tilde{r}^{12}_{11} & \tilde{r}^{12}_{12} & \tilde{r}^{12}_{13} & \tilde{r}^{12}_{14} \\
\tilde{r}^{13}_{11} & \tilde{r}^{13}_{12} & \tilde{r}^{13}_{13} & \tilde{r}^{13}_{14} \\
\tilde{r}^{14}_{11} & \tilde{r}^{14}_{12} & \tilde{r}^{14}_{13} & \tilde{r}^{14}_{14}
\end{pmatrix},
\]
which is abbreviated as, throughout this paper,
\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix},
\quad (1.4)
\]
where
\[
a_{ij}(u, x, y) = 0 \quad \text{for} \quad 1 \leq j < i \leq 4, \quad \text{and} \quad a_{ii}(u, x, y) \neq 0, \quad \text{for} \quad 1 \leq i \leq 4. \quad (1.5)
\]

We note here that our method also applies to the lower triangle case. So our results will contain some cases of the eight-vertex type solutions as a special case. Notice that, by (1.3), if \( f(u, x, y) \neq 0 \) is a meromorphic function and if \( \tilde{R}(u, x, y) \) satisfies (1.2), then \( f(u, x, y) \tilde{R}(u, x, y) \) also satisfies (1.2). Thus, for \( \tilde{r}^{11}_{11}(u, x, y) \neq 0 \), to find a general solution of the QCYBE (1.2), we can assume without loss of generality that
\[
a_{11}(u, x, y) = \tilde{r}^{11}_{11}(u, x, y) = 1.
\]
Throughout this paper, we will always use the symbol
\[
p(u, x, y) = a_{44}(u, x, y) - a_{22}(u, x, y)a_{33}(u, x, y). \quad (1.6)
\]

We will divide our discussion into four parts according as the functions \( p(u, x, y) \) and \( a_{23}(u, x, y) \) are zero or not. The discussion for the case of \( a_{23}(u, x, y) \neq 0 \) forms the theme of the present paper. Our main result in this paper is the following
Theorem. Assume that
\[ a_{23}(u, x, y) \neq 0, \quad p(u, x, y) = a_{44}(u, x, y) - a_{22}(u, x, y)a_{33}(u, x, y) \neq 0. \]

Then, when all the \( a_{ij}(u, x, y) \) are transformed by the multiplication by the function so that \( a_{11}(u, x, y) = 1 \), the general solution of the quantum coloured Yang–Baxter equation (1.2) can be expressed as
\[
\begin{align*}
a_{22}(u, x, y) &= \exp(\alpha_2 u)M_2(x)/M_2(y), \\
a_{33}(u, x, y) &= \exp(\alpha_3 u)M_3(x)/M_3(y), \\
a_{44}(u, x, y) &= M_4(x)/M_4(y), \\
a_{12}(u, x, y) &= a_{22}(u, x, y)L(y) - L(x), \quad (E1) \\
a_{23}(u, x, y) &= cM_4(y)p(u, x, y), \quad (E2) \\
a_{34}(u, x, y) &= a_{44}(u, x, y)L(y) - a_{33}(u, x, y)L(x), \quad (E3) \\
a_{13}(u, x, y) &= a_{33}(u, x, y)L(x) - L(y) + L(x)L(y) + a_{44}(u, x, y)L(x)L(y) \\
&\quad - a_{23}(u, x, y)L(x)L(y) - a_{22}(u, x, y)L(y)L(y), \quad (E4)
\end{align*}
\]

where \( \alpha_2, \alpha_3 \) and \( c \neq 0 \) are complex constants, \( M_2(x) \neq 0, M_3(x) \neq 0, M_4(x) \neq 0 \) and \( L(x) \) are arbitrary meromorphic functions of complex variable \( x \in \mathbb{C} \) such that \( \frac{M_4(x)}{M_2(x)M_3(x)} \) is not a constant when \( \alpha_2 + \alpha_3 = 0 \).

2. Classification of (1.3)

To solve equation system (1.3), we first group the equations in it, with the \( a' \)s in (1.4) instead of the \( \hat{r} \)'s, into the following five categories (B1)–(B5). This can be done as follows. We first display all the \( 2^8 = 64 \) equations from (1.3), each of which has eight terms on both sides. Secondly, we use 0 instead of \( a_{ij}(u, x, y) \) for \( 1 \leq j < i \leq 4 \) getting “shorter” equations in which the number of terms on both sides less than \( 2 \times 8 = 16 \) in total. Then we omit the trivial equations, and get 24 nontrivial remaining ones. Finally, we group the 24 remaining equations according to the statement at the beginning of each category and get the equation systems (B1)–(B5) as follows.

(B1) The function equations only with diagonal unknowns:
\[
\begin{align*}
a_{22}(u + v, x, z) &= a_{22}(u, x, y)a_{22}(v, y, z), \quad (E1) \\
a_{33}(u + v, x, z) &= a_{33}(u, x, y)a_{33}(v, y, z), \quad (E2) \\
a_{44}(u + v, x, z) &= a_{44}(u, x, y)a_{44}(v, y, z). \quad (E3)
\end{align*}
\]

(B2) The function equations only with unknowns on the diagonal and on the line \( a_{12}, a_{23} \) and \( a_{34} \) in the matrix (1.4):
\[
\begin{align*}
a_{12}(u + v, x, z) &= a_{12}(u, x, y) + a_{22}(u, x, y)a_{12}(v, y, z), \quad (E4) \\
a_{23}(u + v, x, z) &= a_{23}(u, x, y) + a_{22}(u, x, y)a_{33}(u, x, y)a_{23}(v, y, z), \quad (E5) \\
a_{34}(u + v, x, z) &= a_{33}(v, y, z)a_{34}(u, x, y) + a_{44}(u, x, y)a_{34}(v, y, z), \quad (E6)
\end{align*}
\]
\[ a_{44}(v, y, z)a_{23}(u + v, x, z) = a_{22}(v, y, z)a_{33}(v, y, z)a_{23}(u, x, y) \]
\[ + a_{44}(u + v, x, z)a_{23}(v, y, z), \quad (E7) \]
\[ a_{33}(v, y, z)a_{23}(u + v, x, z)a_{12}(u, x, y) + a_{22}(u, x, y)a_{23}(u + v, x, z)a_{34}(v, y, z) \]
\[ = a_{33}(v, y, z)a_{23}(u, x, y)a_{12}(u + v, x, z) \]
\[ + a_{22}(u, x, y)a_{23}(v, y, z)a_{34}(u + v, x, z). \quad (E8) \]

\[(B3)\] The function equations with unknown \(a_{13}\) and the unknowns in (B2):
\[ a_{13}(u + v, x, z) - a_{23}(v, y, z)a_{13}(u + v, x, z) \]
\[ = a_{33}(v, y, z)a_{13}(u, x, y) + a_{13}(v, y, z) - a_{23}(u + v, x, z)a_{13}(v, y, z) \]
\[ + a_{33}(u + v, x, z)a_{23}(u, x, y), \quad (E9) \]
\[ a_{22}(v, y, z)a_{13}(u + v, x, z) \]
\[ = a_{13}(u, x, y) + a_{22}(u + v, x, z)a_{33}(u, x, y)a_{13}(v, y, z) + a_{33}(u, x, y)a_{12}(u + v, x, z) \]
\[ - a_{22}(v, y, z)a_{33}(u + v, x, z)a_{12}(u, x, y) - a_{12}(v, y, z) \]
\[ + a_{23}(u, x, y)a_{12}(v, y, z), \quad (E10) \]
\[ a_{44}(v, y, z)a_{13}(u + v, x, z) \]
\[ = a_{33}(v, y, z)a_{13}(u, x, y) + a_{44}(u + v, x, z)a_{13}(v, y, z) \]
\[ + a_{33}(v, y, z)a_{23}(u, x, y)a_{12}(v, y, z) - a_{33}(v, y, z)a_{44}(v, y, z)a_{34}(u, x, y) \]
\[ - a_{34}(v, y, z) + a_{34}(u + v, x, z). \quad (E11) \]

\[(B4)\] The function equations with unknowns \(a_{13}, a_{24}\) and the unknowns in (B2):
\[ a_{24}(u + v, x, z) \]
\[ = a_{24}(u, x, y) + a_{22}(u, x, y)a_{44}(u, x, y)a_{24}(v, y, z) \]
\[ + a_{22}(u, x, y)a_{34}(u, x, y)a_{23}(v, y, z) + a_{44}(u, x, y)a_{12}(u + v, x, z) \]
\[ - a_{44}(u + v, x, z)a_{12}(u, x, y) - a_{22}(u, x, y)a_{12}(v, y, z), \quad (E12) \]
\[ a_{33}(u, x, y)a_{44}(v, y, z)a_{24}(u + v, x, z) \]
\[ = a_{22}(v, y, z)a_{33}(u + v, x, z)a_{24}(u, x, y) + a_{44}(u, x, y)a_{44}(u + v, x, z)a_{24}(v, y, z) \]
\[ + a_{22}(v, y, z)a_{44}(u, x, y)a_{34}(u + v, x, z) + a_{44}(u + v, x, z)a_{23}(v, y, z)a_{34}(u, x, y) \]
\[ - a_{44}(v, y, z)a_{44}(u + v, x, z)a_{34}(u, x, y) - a_{22}(u + v, x, z)a_{33}(u, x, y)a_{34}(v, y, z), \quad (E13) \]
\[ a_{44}(v, y, z)a_{23}(u, x, y)a_{24}(u + v, x, z) - a_{44}(u + v, x, z)a_{24}(u + v, x, z) \]
\[ = a_{44}(v, y, z)a_{23}(u + v, x, z)a_{24}(u, x, y) - a_{44}(v, y, z)a_{44}(u + v, x, z)a_{24}(u, x, y) \]
\[ - a_{22}(u, x, y)a_{44}(u + v, x, z)a_{24}(v, y, z) - a_{22}(u + v, x, z)a_{23}(u, x, y)a_{34}(v, y, z), \quad (E14) \]
\[ a_{22}(u + v, x, z)a_{13}(u + v, x, z) \]
\[ = a_{22}(u + v, x, z)a_{33}(v, y, z)a_{13}(u, x, y) - a_{22}(u + v, x, z)a_{13}(v, y, z) \]
\[ = a_{23}(v, y, z)a_{24}(u + v, x, z) - a_{22}(u, x, y)a_{23}(u + v, x, z)a_{24}(v, y, z) \]
\[ a_{23}(u, x, y) = a_{22}(u, x, y) + a_{33}(u + v, x, z) - a_{33}(u, x, y) + a_{23}(u + v, x, z) a_{12}(u, x, y) + a_{44}(u + v, x, z) a_{23}(v, y, z) a_{12}(u, x, y), \]
\[ (E15) \]
\[ a_{33}(u, x, y) - a_{33}(v, y, z) a_{24}(u + v, x, z) - a_{33}(u + v, x, z) a_{44}(v, y, z) a_{24}(u, x, y) \]
\[ a_{22}(u, x, y) a_{33}(u + v, x, z) - a_{33}(u, x, y) a_{44}(v, y, z) a_{24}(u, x, y) \]
\[ = a_{44}(v, y, z) a_{23}(u, x, y) a_{13}(u + v, x, z) - a_{33}(v, y, z) a_{23}(u + v, x, z) a_{13}(u, x, y) \]
\[ + a_{23}(u, x, y) a_{34}(v, y, z) - a_{23}(u, x, y) a_{34}(v, y, z) a_{23}(u + v, x, z), \]
\[ (E16) \]
\[ a_{33}(u, x, y) a_{23}(v, y, z) a_{24}(u + v, x, z) - a_{23}(u + v, x, z) a_{24}(v, y, z) \]
\[ + a_{23}(u, x, y) a_{23}(u + v, x, z) a_{24}(v, y, z) \]
\[ = -a_{22}(u, x, y) a_{23}(u, x, y) a_{13}(u + v, x, z) + a_{44}(v, y, z) a_{23}(u + v, x, z) a_{13}(u, x, y) \]
\[ - a_{23}(v, y, z) a_{23}(u + v, x, z) a_{13}(u, x, y). \]
\[ (E17) \]

Note that (E12)–(E14) contain \( a_{24} \), but do not contain \( a_{13} \). However, (E15)–(E17) contain both \( a_{13} \) and \( a_{24} \).

(B5) The remaining ones from (1.3), which all contain the unknown \( a_{14} \):
\[ a_{23}(v, y, z) a_{14}(u + v, x, z) \]
\[ = a_{22}(u, x, y) a_{23}(u + v, x, z) a_{14}(v, y, z) + a_{12}(u, x, y) a_{13}(u + v, x, z) \]
\[ + a_{22}(u, x, y) a_{12}(v, y, z) a_{13}(u + v, x, z) - a_{33}(v, y, z) a_{12}(u + v, x, z) a_{13}(u, x, y) \]
\[ + a_{23}(u + v, x, z) a_{12}(u, x, y) a_{13}(v, y, z) - a_{12}(u + v, x, z) a_{13}(v, y, z) - a_{23}(v, y, z) a_{12}(u + v, x, z) a_{13}(u, x, y) \]
\[ - a_{23}(v, y, z) a_{12}(u, x, y) a_{34}(u + v, x, z), \]
\[ (E18) \]
\[ a_{22}(v, y, z) a_{14}(u + v, x, z) \]
\[ = a_{14}(u, x, y) + a_{22}(u + v, x, z) a_{44}(u, x, y), \]
\[ (E19) \]
\[ a_{44}(v, y, z) a_{23}(u, x, y) a_{14}(u + v, x, z) \]
\[ = a_{33}(v, y, z) a_{23}(u + v, x, z) a_{14}(u, x, y) + a_{33}(v, y, z) a_{34}(u, x, y) a_{24}(u + v, x, z) \]
\[ + a_{44}(v, y, z) a_{34}(v, y, z) a_{24}(u + v, x, z) + a_{23}(u + v, x, z) a_{34}(v, y, z) a_{24}(u, x, y) \]
\[ - a_{44}(v, y, z) a_{34}(u + v, x, z) a_{24}(u, x, y) - a_{22}(u, x, y) a_{34}(u + v, x, z) a_{24}(v, y, z) \]
\[ - a_{23}(u, x, y) a_{34}(v, y, z) a_{12}(u + v, x, z), \]
\[ (E20) \]
\[ a_{33}(u, x, y) a_{44}(v, y, z) a_{14}(u + v, x, z) \]
\[ = a_{33}(u + v, x, z) a_{14}(u, x, y) + a_{44}(u, x, y), \]
\[ (E21) \]
Lemma 2.1. The general solution of the function equation system (B1) is given by

\begin{align*}
& a_{22}(u, x, y) = \exp(\alpha_2 u) M_2(x)/M_2(y), \\
& a_{33}(u, x, y) = \exp(\alpha_3 u) M_3(x)/M_3(y), \\
& a_{44}(u, x, y) = \exp(\alpha_4 u) M_4(x)/M_4(y),
\end{align*}

where \( M_i(x) \neq 0 \) are meromorphic functions, and \( \alpha_i \) are arbitrary complex constants, \( i = 2, 3, 4 \).

Proof. In view of the similarity of (E1)–(E3), it is sufficient to find the general solution of the function equation

\[ f(u + v, x, z) = f(u, x, y) f(v, y, z), \]

where \( f(u, x, y) \neq 0 \) is an unknown meromorphic function. Let \( g(u, x, y) = \log f(u, x, y) \), then

\[ g(u + v, x, z) = g(u, x, y) + g(v, y, z). \]
Here, the log function is taken to satisfy \( \log 1 = 0 \), and the domain of \( g(u, x, y) \) is that of \( f(u, x, y) \) excluding the zero points in \( \mathbb{C}^3 \). We may assume that \( (0, 0, 0) \) is in the domain of \( g(u, x, y) \) without loss of generality. By setting \( x = y = z = 0 \) in (2.2) we get
\[
g(u + v, 0, 0) = g(u, 0, 0) + g(v, 0, 0),
\]
and so prove that \( g(u, 0, 0) = \alpha u \), where \( \alpha \) is a complex constant. Let \( f(0, x, 0) = M(x) \), \( f(0, 0, x) = M_0(x) \). Note that \( M(0) = f(0, 0, 0) = 1 \). By (2.2) with \( v = 0 \) and \( y = 0 \) we have \( g(u, x, z) = g(u, x, 0) + \log M_0(z) \); and (2.2) with \( u = y = z = 0 \) yields \( g(v, x, 0) = \log M(x) + g(v, 0, 0) \). Inserting this with \( v \) replaced by \( u \) into the former, and replacing \( z \) by \( y \), we get
\[
g(u, x, y) = g(u, 0, 0) + \log M(x) + \log M_0(y).
\]
Using (2.4) to rewrite (2.2) we get
\[
g(u + v, 0, 0) = g(u, 0, 0) + g(v, 0, 0) + \log M(y) + \log M_0(y).
\]
This together with (2.3) proves \( M(y)M_0(y) = 1 \). Therefore (2.4) gives \( g(u, x, y) = \alpha u + \log M(x) - \log M(y) \), and so the general solution for \( f(u, x, y) \) is given by
\[
f(u, x, y) = \exp(\alpha u)M(x)/M(y),
\]
where \( M(x) \) is an arbitrary meromorphic function satisfying \( M(0) = 1 \), so \( M(x) \neq 0 \), as desired. \( \square \)

**Note.** The novelty of our Lemma 2.1 compared with (1.8) of [15] is that we do not need the differentiability of \( f(u, x, y) \) in our proof.

### 3. The solutions of the function equation system (B2)

In the sequel, we need to assume further that
\[
p(u, x, y)a_{23}(u, x, y) \neq 0.
\]
In this section, we consider the function equation system (B2). By (1.6), (E3) and (E5), we can write (E7) as \( p(v, y, z)a_{23}(u, x, y) = p(u, x, y)a_{44}(v, y, z)a_{23}(v, y, z) \); using (3.1) this can be rewritten as
\[
\frac{a_{23}(u, x, y)}{p(u, x, y)} = \frac{a_{44}(v, y, z) a_{23}(v, y, z)}{p(v, y, z)}.
\]
Note that the right-hand side of (3.2) is irrelevant to the variables \( u \) and \( x \). So the left-hand side of it, i.e., \( \frac{a_{23}(u, x, y)}{p(u, x, y)} \), is also independent of \( u \) and \( x \) as the right. Thus we have
\[
\frac{a_{23}(u, x, y)}{p(u, x, y)} = L_1(y),
\]
where \( L_1(y) \neq 0 \) is a meromorphic function. This in combination with (3.2) yields
\[
L_1(y) = a_{44}(v, y, z)L_1(z).
\]
Again, using the third equality in (2.1) to replace the \( a_{44}(v, y, z) \) in (3.4), we see that the \( \alpha_4 \) in Lemma 2.1 is, by (3.1),
\[
\alpha_4 = 0,
\]
and so (3.4) becomes \( L_1(y) = \frac{M_4(y)}{M_4(z)} L_1(z) \), or \( \frac{L_1(y)}{M_4(y)} = \frac{L_1(z)}{M_4(z)} \). By this, we get

\[ L_1(y) = c M_4(y), \]

where \( c \) is a nonzero complex constant. Therefore by (3.3),

\[ a_{23}(u, x, y) = c M_4(y) p(u, x, y), \tag{3.6} \]

Next, by (E4)–(E6) and (1.6), we see that (E8) is

\[ a_{23}(u, x, y) [a_{34}(v, y, z) - a_{33}(v, y, z)a_{12}(v, y, z)] - p(u, x, y)a_{23}(v, y, z)a_{34}(v, y, z) \]

\[ = [a_{34}(u, x, y) - a_{33}(u, x, y)a_{12}(u, x, y)]a_{33}(v, y, z)a_{23}(v, y, z). \]

Using (3.6), this can be rewritten as

\[ a_{23}(u, x, y) [a_{34}(v, y, z) - a_{33}(v, y, z)a_{12}(v, y, z) - c^{-1} M_4(y)^{-1}a_{23}(v, y, z)a_{34}(v, y, z)] \]

\[ = [a_{34}(u, x, y) - a_{33}(u, x, y)a_{12}(u, x, y)]a_{33}(v, y, z)a_{23}(v, y, z). \]

In view of \( a_{23}(u, x, y) \neq 0 \), similarly to the above, this implies that there exists a meromorphic function \( L_2(y) \) such that

\[ a_{34}(u, x, y) - a_{33}(u, x, y)a_{12}(u, x, y) = a_{23}(u, x, y)L_2(y), \tag{3.7} \]

and

\[ a_{34}(v, y, z) - a_{33}(v, y, z)a_{12}(v, y, z) - c^{-1} M_4(y)^{-1}a_{23}(v, y, z)a_{34}(v, y, z) \]

\[ = a_{33}(v, y, z)a_{23}(v, y, z)L_2(y). \tag{3.8} \]

By (3.7) we have \( a_{34}(v, y, z) = a_{33}(v, y, z)a_{12}(v, y, z) + a_{23}(v, y, z)L_2(z) \). Using the right-hand side of this to replace the \( a_{34}(v, y, z) \) in (3.8), and then using (3.6), (1.6) and the third equality in (2.1) with \( \alpha_4 = 0 \), (3.8) can be simplified, and leads to

\[ a_{12}(u, x, y) = c M_4(y)a_{22}(u, x, y)L_2(y) - c M_4(x)L_2(x). \]

This together with (3.7), (3.6) and (1.6) gives \( a_{34}(u, x, y) = c M_4(y)a_{44}(u, x, y)L_2(y) - c M_4(x)a_{33}(u, x, y)L_2(x) \). Let \( L(x) = c M_4(x)L_2(x) \). We conclude that, under (3.1),

\[ a_{34}(u, x, y) = a_{44}(u, x, y)L(y) - a_{33}(u, x, y)L(x), \tag{3.9} \]

and

\[ a_{12}(u, x, y) = a_{22}(u, x, y)L(y) - L(x). \tag{3.10} \]

Finally, we can summarize the above discussions to give the following

**Lemma 3.1.** Let \( a_{22}(u, x, y) \), \( a_{33}(u, x, y) \), \( a_{44}(u, x, y) \) and \( M_4(y) \) be as in Lemma 2.1 and \( p(u, x, y) \) be given by (1.6). Assume (3.1). Then the general solution of the function equation system (B2) can be expressed as

\[ a_{12}(u, x, y) = a_{22}(u, x, y)L(y) - L(x), \]

\[ a_{23}(u, x, y) = c M_4(y) p(u, x, y), \]

\[ a_{34}(u, x, y) = a_{44}(u, x, y)L(y) - a_{33}(u, x, y)L(x), \]

where \( L(x) \) is a meromorphic function, \( c \) is a nonzero complex constant.
4. The function equation system (B3)

In this section, we consider (B3) to give the general solution for $a_{13}(u, x, y)$. Consider the difference of $(E11) \times a_{22}(v, y, z)$ and $(E10) \times a_{44}(v, y, z)$, use $(E1)–(E3), (E4)$ and $(E6)$ for the substitutions of $a_{ii}(u + v, x, z), a_{12}(u + v, x, z)$ and $a_{34}(u + v, x, z)$ respectively, then simplify by $(1.6)$, Lemma 3.1 and the equality $a_{34}(u, x, y) = a_{33}(u, x, y)a_{12}(u, x, y) + p(u, x, y)L(y)$, which follows from $(3.6)$ and $(3.7)$. We finally get

$$
p(u, x, y)[a_{22}(v, y, z)a_{44}(v, y, z)a_{12}(v, y, z) + a_{22}(v, y, z)a_{44}(v, y, z)L(z)]
-a_{22}(v, y, z)a_{33}(v, y, z)a_{44}(v, y, z)L(y) - p(v, y, z)cM_4(y)a_{12}(v, y, z)] = p(v, y, z)[L(y) + a_{13}(u, x, y) - a_{33}(u, x, y)L(x)],
$$

where $L(x)$ and $c$ are as in Lemma 3.1. In view of $p(u, x, y) \neq 0$, this implies that there exists a meromorphic function $L_3(y)$ such that

$$a_{13}(u, x, y) = a_{33}(u, x, y)L(x) - L(y) + p(u, x, y)L_3(y),$$

and

$$cM_4(y)a_{12}(v, y, z) = a_{22}(v, y, z)a_{44}(v, y, z)L_3(z) = L_3(y).$$

Since $a_{12}(v, y, z) = a_{22}(v, y, z)L(z) - L(y)$ by $(3.10)$, we get from the later, using $a_{44}(v, y, z) = \frac{M_3(y)}{M_4(y)}$,

$$L_3(y) - cM_4(y)L_3(y) = a_{22}(v, y, z)a_{44}(v, y, z)[L_3(z) - cM_4(z)L(z)].$$

By $(2.1)$ with $\alpha_4 = 0$, this can be written further as

$$\frac{L_3(y) - cM_4(y)L_3(z)}{M_2(y)M_4(y)} = \exp(\alpha_2 v) \frac{L_3(z) - cM_4(z)L(z)}{M_2(z)M_4(z)}.$$

This proves that $L_3(y) = cM_4(y)L_3(y) + c'M_2(y)M_4(y)$, where $c'$ is a complex constant such that $c'a_2 = 0$. Using $c'$ to replace $c'$, where $c$ is as in Lemma 3.1, we get, by $(3.6)$,

$$a_{13}(u, x, y) = a_{33}(u, x, y)L(x) - L(y) + a_{23}(u, x, y)L(y) + c'M_2(y)a_{23}(u, x, y), \quad (4.1)$$

where $c'$ is also a complex constant such that

$$c'a_2 = 0.$$

Obviously, $(E10)$ and $(E11)$ hold for the solution of $a_{13}$ given by $(4.1)$. Next, for the validity of $(E9)$ to the solution of $a_{13}$ given by $(4.1)$, we substitute the unknown $a_{13}$ by $(4.1)$, then cancel and simplify by Lemma 3.1, $(E2), (E3), (E5)$ and the first equality in $(2.1)$, getting

$$c'a_{23}(u, x, y)[1 - \exp(-\alpha_2 v)a_{22}(v, y, z)a_{33}(v, y, z)]
= c'a_{23}(v, y, z)[1 - a_{22}(u, x, y)a_{33}(u, x, y)]. \quad (4.2)$$

If $c' = 0$, then $(4.2)$ (so $(E9)$) always holds. If $c' \neq 0$ (so $a_2 = 0$), then using $a_{23}(u, x, y) \neq 0$ we see that, by $(4.2)$, there is a complex constant $c''$ such that $1 - a_{22}(u, x, y)a_{33}(u, x, y) = c''a_{23}(u, x, y)$. We conclude the above arguments to give the following lemma.

**Lemma 4.1.** Let $a_{22}(u, x, y), a_{33}(u, x, y), M_2(y)$ be as in Lemma 2.1, and $a_{23}(u, x, y), L(x)$ as in Lemma 3.1. Assume $(3.1)$. Then the general solution of $a_{13}(u, x, y)$ to the function equation system $(B3)$ can be expressed as

$$a_{13}(u, x, y) = a_{33}(u, x, y)L(x) - L(y) + a_{23}(u, x, y)L(y) + c'M_2(y)a_{23}(u, x, y). \quad (4.3)$$
Here $c'$ is a complex constant such that $c'\alpha = 0$ with $\alpha$ as in (2.1), and that, if $c' \neq 0$ (so $\alpha = 0$), there exists a complex constant $c''$ satisfying $1 - a_{22}(u, x, y)a_{33}(u, x, y) = c''a_{23}(u, x, y)$.

5. The function equation system (B4)

In this section, we consider the general solution of (B4). By (E12), and using (E1)–(E4), (E6), (1.6), (3.2) and (3.9), we can write (E13) as

$$p(u, x, y)a_{44}(u, x, y)[a_{44}(v, y, z)L(y) - a_{44}(v, y, z)a_{23}(v, y, z)L(y) - a_{22}(v, y, z)a_{44}(v, y, z)L(z)] = p(v, y, z)[a_{33}(u, x, y)a_{44}(u, x, y)L(x) - a_{33}(u, x, y)a_{24}(u, x, y) - a_{22}(u, x, y)a_{33}(u, x, y)L(y) - a_{33}(u, x, y)a_{23}(u, x, y)L(x)].$$

Thus there exists a meromorphic function $L_4(y)$ such that

$$a_{24}(u, x, y) = a_{44}(u, x, y)L(x) - a_{22}(u, x, y)L(y) - a_{33}(-u, y, x)p(u, x, y)a_{44}(u, x, y)L_4(y)$$

and $a_{33}(-u, y, x)a_{44}(u, x, y) = c'p(u, x, y)M_4(x)M_4(y)/M_3(x)$. The later together with (2.1) with $\alpha_3 = 0$ implies that $L_4(x) = c_0\frac{M_4(x)^2}{M_3(x)}$, thus

$$a_{24}(u, x, y) = a_{44}(u, x, y)L(x) - a_{23}(u, x, y)L(x) - a_{22}(u, x, y)L(y) - c_0p(u, x, y)\frac{M_4(x)M_4(y)}{M_3(x)}$$

(5.1)

where $c_0$ is a complex constant such that $c_0\alpha_3 = 0$. Now, (E12) clearly holds to the solution of $a_{24}(u, x, y)$ given by (5.1). Again substituting (5.1) into (E14), then cancelling and simplifying by (E5), (3.2) and (3.9), we get

$$-c_0p(u + v, x, z)a_{44}(v, y, z)a_{23}(u, x, y)M_4(x)M_4(z)/M_3(x)$$

$$+ c_0p(u + v, x, z)a_{44}(u + v, x, z)M_4(x)M_4(z)/M_3(x)$$

$$= -c_0p(u, x, y)a_{44}(v, y, z)a_{23}(u + v, x, z)M_4(x)M_4(y)/M_3(x)$$

$$+ c_0p(u, x, y)a_{44}(v, y, z)a_{44}(u + v, x, z)M_4(x)M_4(y)/M_3(x)$$

$$+ c_0p(v, y, z)a_{22}(u, x, y)a_{44}(u + v, x, z)M_4(y)M_4(z)/M_3(y).$$

(5.2)

Note that, by (1.6) and (E1)–(E3), we have

$$p(u + v, x, z) = p(u, x, y)a_{44}(v, y, z) + a_{22}(u, x, y)a_{33}(u, x, y)p(v, y, z),$$

(5.3)

and, by (E5) and (3.2), we have

$$p(u, x, y)a_{44}(v, y, z)a_{23}(u + v, x, z) = p(u, x, y)a_{44}(v, y, z)a_{23}(u, x, y) + p(v, y, z)a_{22}(u, x, y)a_{33}(u, x, y)a_{23}(u, x, y).$$

These together with $a_{44}(v, y, z) = M_4(y)/M_4(z)$ ensure a cancellation of the first term on the right-hand side of (5.2) and the first term on the left, so (5.2) becomes, by (5.3),
respectively, the above can be rewritten as, by (2.1),

\[ c_0\left(p(u, x, y)a_{44}(v, y, z) + a_{22}(u, x, y)a_{33}(u, x, y)p(v, y, z)\right) \]
\[ \times a_{44}(u + v, x, z)M_4(x)M_4(z)/M_3(x) \]
\[ = c_0p(u, x, y)a_{44}(v, y, z)a_{44}(u + v, x, z)M_4(x)M_4(y)/M_3(x) \]
\[ + c_0p(v, y, z)a_{22}(u, x, y)a_{44}(u + v, x, z)M_4(y)M_4(z)/M_3(y). \]

This can be rewritten as, by (E3) and (2.1),

\[ c_0\left[a_{44}(u, x, y) - \exp(-\alpha_3u)\right]a_{22}(u, x, y)a_{33}(u, x, y)p(v, y, z) \]
\[ = c_0\left[a_{44}(v, y, z) - 1\right]a_{44}(u, x, y)p(v, y, z). \] \hspace{1cm} (5.4)

To ensure the validity of (E15), we use Lemma 4.1 and (5.1) for the substitution of \( a_{13} \) and \( a_{24} \) respectively, then use (E1), (E2), (E5), (1.6), (3.2) and (3.10) to cancel and simplify, getting

\[ c'a_{22}(u + v, x, z)a_{23}(u + v, x, z)M_2(z) \]
\[ - c'a_{22}(u + v, x, z)a_{33}(v, y, z)a_{23}(u, x, y)M_2(y) \]
\[ - c'a_{22}(u + v, x, z)a_{23}(v, y, z)M_2(z) \]
\[ = -c_0p(u + v, x, z)a_{23}(v, y, z)M_4(x)M_4(z)/M_3(x) \]
\[ + c_0p(v, y, z)a_{22}(u, x, y)a_{23}(u + v, x, z)M_4(y)M_4(z)/M_3(y). \]

Again, using (E5) and (3.6) to replace the \( a_{23}(u + v, x, z) \) on the left- and the right-hand side respectively, the above can be rewritten as, by (2.1),

\[ c'M_2(x)\left[\exp(\alpha_2u)a_{22}(v, y, z)a_{33}(v, y, z) - \exp(\alpha_2(u + v))\right]a_{23}(u, x, y) \]
\[ - c'M_2(x)\left[\exp(\alpha_2(u + v))a_{22}(u, x, y)a_{33}(u, x, y) - \exp(\alpha_2(u + v))\right]a_{23}(v, y, z) \]
\[ = cc_0\frac{M_4(y)M_4(z)^2}{M_3(x)}p(u + v, x, z)p(v, y, z) \]
\[ \times (a_{44}(u, x, y) - \exp(-\alpha_3u)a_{22}(u, x, y)a_{33}(u, x, y)). \] \hspace{1cm} (5.5)

Next, using Lemma 4.1 and (5.1) for the substitution of \( a_{13} \) and \( a_{24} \), then cancelling and simplifying by (E1)–(E3), (E5) and (3.9), we can write (E16) as

\[ -c_0p(u + v, x, z)a_{33}(u, x, y)a_{33}(v, y, z)M_4(x)M_4(z)/M_3(x) \]
\[ + c_0p(u, x, y)a_{33}(u, x, y)a_{33}(v, y, z)a_{44}(v, y, z)M_4(x)M_4(y)/M_3(x) \]
\[ + c_0p(v, y, z)a_{22}(u, x, y)a_{33}(u, x, y)a_{33}(v, y, z)M_4(y)M_4(z)/M_3(y) \]
\[ = c'M_2(z)a_{44}(v, y, z)a_{23}(u, x, y)a_{23}(u + v, x, z) \]
\[ + c'M_2(y)a_{33}(v, y, z)a_{23}(u, x, y)a_{23}(u + v, x, z). \] \hspace{1cm} (5.6)

By (5.3) and (2.1), this can be written further as

\[ c_0p(u, x, y)a_{44}(u + v, x, z)\exp(\alpha_3(u + v))(a_{44}(v, y, z) - 1)M_4(y)M_4(z)/M_3(z) \]
\[ + c_0p(v, y, z)a_{22}(u, x, y)a_{33}(u, x, y)(\exp(\alpha_3 v) - \exp(\alpha_3(u + v))a_{44}(u, x, y)) \]
\[ \times M_4(y)M_4(z)/M_3(z) \]
\[ = c'M_2(z)a_{23}(u, x, y)a_{23}(u + v, x, z) \]
\[ \times (a_{44}(v, y, z) - \exp(-\alpha_2 v)a_{22}(v, y, z)a_{33}(v, y, z)). \] \hspace{1cm} (5.7)
To ensure the validity of (E17), we also use Lemma 4.1 and (5.1) to substitute $a_{13}$ and $a_{24}$, then cancel and simplify by (E1)–(E3), (E5), (1.6) and (3.2), getting

$$
-c_0 p(u + v, x, z)a_{23}(u, x, y)a_{23}(v, y, z)M_4(x)M_4(z)/M_3(x)
+ c_0 p(v, y, z)a_{23}(u + v, x, z)M_4(y)M_4(z)/M_3(y)
- c_0 p(v, y, z)a_{23}(u, x, y)a_{23}(u + v, x, z)M_4(y)M_4(z)/M_3(y)
= -c'M_2(z)a_{22}(v, y, z)a_{23}(u, x, y)a_{23}(u + v, x, z)
+ c'M_2(y)a_{44}(v, y, z)a_{23}(u, x, y)a_{23}(u + v, x, z)
- c'M_2(y)a_{23}(v, y, z)a_{23}(u, x, y)a_{23}(u + v, x, z).
$$

(5.8)

By (E5), (5.3) and (3.2), we have

$$
p(v, y, z)a_{23}(u + v, x, z) = p(u + v, x, z)a_{23}(v, y, z).
$$

(5.9)

So by (2.1), the left-hand side of (5.8) is

$$
= c_0 p(u + v, x, z)a_{23}(v, y, z)(1 - \exp(\alpha_3 u)a_{44}(u, x, y) - a_{23}(u, x, y))
\times M_4(y)M_4(z)/M_3(y),
$$

and the right-hand side is

$$
= c'M_2(y)a_{23}(u, x, y)a_{23}(u + v, x, z)(a_{44}(v, y, z) - \exp(\alpha_2 v) - a_{23}(v, y, z)).
$$

Thus, (5.8), or (E17), now becomes

$$
c_0 p(u + v, x, z)a_{23}(v, y, z)(1 - \exp(\alpha_3 u)a_{44}(u, x, y) - a_{23}(u, x, y))M_4(y)M_4(z)
= c'M_2(y)a_{23}(u, x, y)a_{23}(u + v, x, z)(a_{44}(v, y, z) - \exp(\alpha_2 v) - a_{23}(v, y, z)).
$$

(5.10)

Now we divide our discussion into the following 4 cases to prove that (4.3) and (5.1) can be the solutions of $a_{13}$ and $a_{24}$ respectively if and only if both $c'$ and $c_0$ in them are 0.

Case (i). Assume that $c_0 = c' = 0$. Then, for the solutions of $a_{13}$ and $a_{24}$ given by (4.3) and (5.1) respectively, (5.4), (5.5), (5.7) and (5.10) clearly hold, so assure the validity of (E14)–(E17).

Case (ii). Assume that $c_0 \neq 0, c' = 0$. Then by $c_0 \alpha_3 = 0$ there must hold $\alpha_3 = 0$, so the right-hand side of (5.5) does not equal to 0 by assumption (3.1). However, the left-hand side of (5.5) is clearly $= 0$ since $c' = 0$ in this case. This assures that there is no this kind of solutions with $c_0 \neq 0, c' = 0$.

Case (iii). Assume that $c_0 = 0, c' \neq 0$. Then we must have $\alpha_2 = 0$ by Lemma 4.1, so the right-hand side of (5.7) is not equal to 0 by (3.1), but the left is by $c_0 = 0$. This also shows that there certainly does not exist this kind of solutions.

Case (iv). Assume that $c_0 c' \neq 0$. Then we have $\alpha_2 = \alpha_3 = 0$, and so the right-hand side of (5.7) is $\neq 0$ by (3.1). However, the left-hand side of it is

$$
h(u, v, x, y, z)c_0 M_4(y)M_4(z)/M_3(z),
$$

(5.11)

where

$$
h(u, v, x, y, z) = p(u, x, y)a_{44}(u + v, x, z)(a_{44}(v, y, z) - 1)
+ p(v, y, z)a_{22}(u, x, y)a_{33}(u, x, y)(1 - a_{44}(u, x, y)).
$$

(5.11)
Again, in this case, (5.4) becomes
\[
\begin{align*}
[a_{44}(u, x, y) - 1]a_{22}(u, x, y)a_{33}(u, x, y)p(v, y, z) \\
= [a_{44}(v, y, z) - 1]a_{44}(v, y, z)a_{44}(u, x, y)p(u, x, y).
\end{align*}
\]
This implies that there exists a meromorphic function \( L(y) \) satisfying
\[
[a_{44}(u, x, y) - 1]a_{44}(u, x, y) = p(u, x, y)L_5(x) \quad (5.12)
\]
and \( a_{22}(u, x, y)a_{33}(u, x, y)L_5(x) = a_{44}(u, x, y)^2L_5(y) \). The later together with (2.1) ensures further that
\[
L_5(x) = c_0' \frac{M_4(x)^2}{M_2(x)M_3(x)},
\]
where \( c_0' \) is a complex constant. This in combination with (5.12) and (3.6) gives
\[
a_{44}(u, x, y) - 1 = c^{-1}c_0'a_{23}(u, x, y)\frac{M_4(x)}{M_2(x)M_3(x)}.
\]
Using this, together with (E3), we get
\[
\begin{align*}
h(u, v, x, y, z) &= p(u, x, y)a_{44}(u, x, y)a_{44}(v, y, z)a_{23}(v, y, z)c^{-1}c_0'\frac{M_4(y)}{M_2(y)M_3(y)} \\
- p(v, y, z)a_{22}(u, x, y)a_{33}(u, x, y)a_{23}(u, x, y)c^{-1}c_0'\frac{M_4(x)}{M_2(x)M_3(x)} \\
&= 0.
\end{align*}
\]
Thus, (5.11), and so the left-hand side of (5.7) is \( = 0 \). This again shows that there is no solution for \( a_{13} \) and \( a_{24} \) of the forms given by (4.3) and (5.1) respectively when \( c'c_0 \neq 0 \).

In conclusion, we proved the following

**Lemma 5.1.** Let \( a_{22}(u, x, y), a_{33}(u, x, y), a_{44}(u, x, y) \) be as in Lemma 2.1, and \( L(x) \), \( a_{23}(u, x, y) \) be as in Lemma 3.1. Assume (3.1). Then the general solution, for the unknowns \( a_{13} \) and \( a_{24} \), of the function equation systems (B3) and (B4) can be given by
\[
a_{13}(u, x, y) = a_{33}(u, x, y)L(x) - L(y) + a_{23}(u, x, y)L(y),
\]
and
\[
a_{24}(u, x, y) = a_{44}(u, x, y)L(x) - a_{23}(u, x, y)L(x) - a_{22}(u, x, y)L(y).
\]

6. The function equation system (B5)

In this last section, we consider the general solution for \( a_{14}(u, x, y) \) in (B5), and then give a general solution for the coloured quantum Yang–Baxter equation (1.2). Using (E19) to replace the \( a_{14}(u + v, x, z) \) in (E21), then replacing all the \( a_{12}, a_{34} \) and \( a_{13}, a_{24} \) on both sides by Lemmas 3.1 and 5.1, and then cancelling and simplifying by (E1)–(E3), (1.6) and (3.2), we arrive at
\[
\begin{align*}
p(v, y, z)a_{33}(u, x, y)[a_{14}(u, x, y) + a_{33}(u, x, y)L(x)L(x) - L(x)L(y) \\
- a_{44}(u, x, y)L(x)L(y) + a_{23}(u, x, y)L(x)L(y) + a_{22}(u, x, y)L(y)L(y)] \\
= p(u, x, y)a_{44}(u, x, y)a_{22}(v, y, z)a_{44}(v, y, z)
\end{align*}
\]
\[ \times [a_{14}(v, y, z) + a_{33}(v, y, z)L(y)L(y) - L(y)L(z) \\
- a_{44}(v, y, z)L(y)L(z) + a_{23}(v, y, z)L(y)L(z) + a_{22}(v, y, z)L(z)L(z)]. \]

This together with (3.1) implies that there exists a meromorphic function \( L_6(y) \) such that

\[
\begin{align*}
 a_{33}(u, x, y) & [a_{14}(u, x, y) + a_{33}(u, x, y)L(x)L(x) - L(x)L(y) \\
- a_{44}(u, x, y)L(x)L(y) + a_{23}(u, x, y)L(x)L(y) + a_{22}(u, x, y)L(y)L(y)] \\
& = p(u, x, y)a_{44}(u, x, y)L_6(y),
\end{align*}
\]

(6.1)

and \( a_{33}(v, y, z)L_6(y) = a_{22}(v, y, z)a_{44}(v, y, z)^2L_6(z) \). By (2.1) the later can be written further as

\[
\exp((\alpha_3 - \alpha_2)v) \frac{L_6(y)}{M_2(y)M_4(y)^2} = \frac{L_6(z)}{M_2(z)M_4(z)^2}.
\]

This shows that

\[
L_6(y) = c_1 - \frac{M_2(y)M_4(y)^2}{M_3(y)},
\]

where \( c_1 \) is a complex constant such that \( c_1\alpha_2 = c_1\alpha_3 \). Hence by (6.1) and (2.1),

\[
\begin{align*}
a_{14}(u, x, y) & = -a_{33}(u, x, y)L(x)L(x) + L(x)L(y) + a_{44}(u, x, y)L(x)L(y) \\
& - a_{23}(u, x, y)L(x)L(y) - a_{22}(u, x, y)L(y)L(y) \\
& + c_1 p(u, x, y)\frac{M_2(y)M_4(y)M_4(y)}{M_3(x)} \exp(-\alpha_3 u).
\end{align*}
\]

(6.2)

Next we consider (E18). Substituting the unknowns \( a_{14}, a_{13}, a_{24}, a_{12} \) and \( a_{34} \) on both sides of (E18), by (6.2), Lemmas 5.1 and 3.1 respectively, then cancelling and simplifying by (E1)–(E3), (E5) and (3.2), we get

\[
\begin{align*}
c_1a_{23}(v, y, z)p(u + v, x, z) \exp(-\alpha_3(u + v)) & M_2(z)M_4(x)M_4(z)/M_3(x) \\
& = c_1a_{22}(u, x, y)a_{23}(u + v, x, z)p(v, y, z) \exp(-\alpha_3v)M_2(z)M_4(y)M_4(z)/M_3(y).
\end{align*}
\]

Using (2.1), (5.9) and (3.1), this can be written further as

\[
c_1a_{44}(u, x, y) = c_1a_{22}(u, x, y)a_{33}(u, x, y).
\]

Therefore we derive \( c_1 = 0 \) by (3.1), and by (6.2) we get

\[
\begin{align*}
a_{14}(u, x, y) & = -a_{33}(u, x, y)L(x)L(x) + L(x)L(y) + a_{44}(u, x, y)L(x)L(y) \\
& - a_{23}(u, x, y)L(x)L(y) - a_{22}(u, x, y)L(y)L(y).
\end{align*}
\]

(6.3)

Now (E18), (E19) and (E21) clearly hold by the procedure producing (6.3). Further, the validity of (E20) and (E22)–(E24) to the solutions given by (6.3), Lemmas 3.1 and 5.1 can be checked straightforward: indeed, we only need to substitute the unknowns \( a_{14}, a_{12}, a_{34}, a_{13} \) and \( a_{24} \) by (6.3), Lemmas 3.1 and 5.1, then cancel by (E1)–(E3), (E5) and (3.2), as we did for (E18) above. So we omit the details of the computations.

Finally, collecting the results of (6.3) and Lemmas 2.1, 3.1 and 5.1, and noting \( p(u, x, y) \neq 0 \), we prove the theorem of this paper given at the end of Section 1.
References