The Time-Periodic Solution to a 2D Generalized Ginzburg–Landau Equation

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In this paper, we study a 2D generalized Ginzburg–Landau equation with a periodic boundary condition. The existence and uniqueness of a time-periodic solution to this equation is proved.

1. INTRODUCTION

The 2D generalized Ginzburg–Landau equation

\[ u_t = \rho u + (1 + iv)\Delta u - (1 + i\mu)|u|^{2\sigma} u \\
+ \alpha\lambda_1 \cdot \nabla(|u|^2 u) + \beta(\lambda_2 \cdot \nabla u)|u|^2, \tag{1} \]

in \( \Omega \times \mathbb{R}^+ \), has been studied by many authors. Bartuccelli et al. studied this equation when \( \alpha = \beta = 0 \) [1]. Guo and Wang discussed the global existence and finite-dimensional behavior for Eq. (1) under a periodic boundary condition [2], [3]. In the case where \( \alpha = \beta = 0 \), Doering et al. studied the weak and strong solutions to the initial problem of Eq. (1) in any spatial dimension [4].

In the present paper, we study Eq. (1) with a non-homogeneous term in two spatial dimensions as

\[ u_t = \rho u + (1 + iv)\Delta u - (1 + i\mu)|u|^{2\sigma} u \\
+ \alpha\lambda_1 \cdot \nabla(|u|^2 u) + \beta(\lambda_2 \cdot \nabla u)|u|^2 + f \tag{2} \]

in \( \Omega \times \mathbb{R}^+ \), where \( \Omega = [0, L] \times [0, L] \), \( \rho, \sigma, L \) are positive numbers, \( \alpha, \beta, v, \mu \) are real numbers, and \( \lambda_1, \lambda_2 \) are real vectors. We assume that
the function \( f \) is \( \omega \)-periodic in time \( t \) and

the function \( u(x, t) \) is \( \Omega \)-periodic in the spatial variable \( x \).

We shall prove that Eq. (2) with a periodic boundary condition (3) has a time-periodic solution.

We study the problem by the Galerkin method and Leray–Schauder fixed point theorem. The outline of this paper is as follows: In Section 2, we give the work space and write the problem which we want to discuss into an abstract problem (B) (see later) in the work space. We prove the existence of an approximate solution to problem (B) and give some uniform a priori estimates needed in the proof in Sections 3 and 4, respectively. Other uniform a priori estimates needed when we prove the convergence of sequence of the approximate solutions are given in Section 5. In Section 6, we prove the main result obtained in the paper.

Throughout this paper, we denote the norm of \( L^2(\Omega) \) with the usual inner product \( \langle \cdot, \cdot \rangle \) by \( \| \cdot \| \) and the norm of any Banach space \( X \) by \( \| \cdot \|_X \).

2. WORK SPACE AND ABSTRACT PROBLEM

We note that

\[ L^2_{per} = \{ u \in L^2(\Omega), \ u \text{ is a } \Omega\text{-periodic function} \} \]

and the norm in \( L^2_{per} \) is the same as that in \( L^2(\Omega) \). And

\[ H^k_{per} = \{ u \in H^k(\Omega), \ u \text{ is a } \Omega\text{-periodic function} \} \]

with the norm in \( H^k_{per} \) being the same as that in \( H^k(\Omega) \).

Let \( X \) be a Banach space; we note that

\[ C^k(\omega, X) = \{ f: [0, \infty) \to X, f^{(j)} \text{ is continuous}, j = 0, 1, \ldots, k, \ f \text{ is an } \omega\text{-periodic function} \}. \]

When \( k = 0 \), we replace \( C^0(\omega, X) \) with \( C(\omega, X) \). Let

\[ A: = -[(1 + i\nu)\Delta + d], \]

where \( d \) is a real number and the domain of operator \( A \) is \( D(A) = H^2_{per} \).

And let

\[ N(u) = (\rho - d)u - (1 + i\mu)|u|^{2\sigma}u + \alpha \lambda_1 \cdot \nabla(|u|^2u) + \beta(\lambda_2 \cdot \nabla u)|u|^2. \]

\( N \) is a nonlinear operator from \( H^1_{per} \) to \( L^2_{per} \).

We have the results as follows.
The set of all linearly independent eigenvectors of operator $A$ is an orthogonal basis of the space $L^2_{\text{per}}$, and we can choose a real number $d < 0$ such that $0 \in \rho(A)$, where $\rho(A)$ denotes the resolve set of operators $A$ (see [5]).

The problem which we want to discuss can be written into an abstract problem in the space $C(\omega, L^2_{\text{per}})$ as

$$
\begin{align*}
    u_t + Au &= N(u) + f, \\
    u(t, \cdot) &= u(t + \omega, \cdot),
\end{align*}
$$

(B)

where $f \in C^1(\omega, H^1_{\text{per}})$.

3. APPROXIMATE SOLUTIONS

We study problem (B) by the Galerkin method.

Let $\{\phi_j\}_{j=1}^{\infty}$ be a normal orthogonal basis of the space $H^2_{\text{per}}$, and for each $j$, $\phi_j$ is an eigenvector of the operator $A$. Let $\mu_j$ be the eigenvalue of the operator $A$ corresponding to $\phi_j$, $j = 1, 2, \ldots$. For any number $n$, we note that $H_n = \text{span}\{\phi_1, \phi_2, \ldots, \phi_n\}$.

DEFINITION 3.1 (approximate solution to problem (B)). Let $f \in C^1(\omega, H^1_{\text{per}})$. For any number $n$ and a group of functions $(d_{1n}(t), d_{2n}(t), \ldots, d_{nn}(t))$, where $d_{kn}(t) (k = 1, 2, \ldots, n)$ belongs to $C^1(\omega, \mathbb{C})$ and $\mathbb{C}$ is the set of all complex numbers, the function $u_n(t) = \sum_{k=1}^{n} d_{kn}(t) \phi_k \in C^1(\omega, H_n)$ is called an approximate solution to (B) if it satisfies the equation system as follows

$$
(u_n + Au_n, \phi_j) = (N(u_n) + f, \phi_j), \quad j = 1, 2, \ldots, n.
$$

(4)

In order to prove that (B) has an approximate solution, we need to define an image $F$ in $C^1(\omega, H_n)$. Set $v_n(t) = \sum_{k=1}^{n} c_{kn}(t) \phi_k \in C^1(\omega, H_n)$; we consider the linear equation system as follows

$$
(u_n + Au_n, \phi_j) = (N(v_n) + f, \phi_j), \quad j = 1, 2, \ldots, n,
$$

(5)

where $u_n(t) = \sum_{k=1}^{n} d_{kn}(t) \phi_k$, and $d_{kn}(t) (k = 1, 2, \ldots, n)$ is an undetermined function. Therefore, (5) is an ordinary differential equation system in $(d_{1n}(t), d_{2n}(t), \ldots, d_{nn}(t))$. By the theory of ordinary differential equations, (5) has a unique solution $(d_{1n}(t), d_{2n}(t), \ldots, d_{nn}(t))$, and the image $F$: $v_n \to u_n$ is a continuous and compact image from the space $C^1(\omega, H_n)$ to itself. For the existence of an approximate solution to the problem (B), it is sufficient to prove that the image $F$ has a fixed point in $C^1(\omega, H_n)$. 

We prove the existence of a fixed point of the image \( F \) by the Leray–Schauder fixed point theorem. For this purpose, we induct a group of operators \( F_\lambda \) with the parameter \( \lambda \), where \( 0 \leq \lambda \leq 1 \). Similar to the image \( F \), for each \( \lambda \in [0, 1] \), \( F_\lambda: v_n \to u_n \) is defined by the equation system as follows

\[
(u_{nt} + Au_n, \phi_j) = (\lambda N(v_n) + f, \phi_j), \quad j = 1, 2, \ldots, n. \tag{6}
\]

\( F_\lambda \) also is a continuous and compact image from \( C^1(\omega, H_n) \) to \( C^1(\omega, H_n) \). It is obvious that \( F_1 = F \). As \( \lambda = 0 \), the equation system (6) has a unique solution \( \tilde{u}_n(t) = (\tilde{d}_{1n}(t), \tilde{d}_{2n}(t), \ldots, \tilde{d}_{mn}(t)) \in C^1(\omega, H_n) \). Hence, \( F_0 \) has a unique fixed point \( \tilde{u}_n(t) \) in \( C^1(\omega, H_n) \). For the existence of a fixed point of the image \( F_1 \), using the Leray–Schauder fixed point theorem, we need only to prove that if the equation \( F_\lambda u_n = u_n \) \((0 \leq \lambda \leq 1)\) has a solution \( u_n(t) \), it must satisfy the inequality as follows

\[
\sup_{0 \leq t \leq \omega} \|u_n(t)\| \leq K_1, \tag{7}
\]

where \( K_1 \) is a positive constant which is independent of \( \lambda \) and depends only on \( \alpha, \beta, \rho, \mu, \nu, \sigma, \lambda_1, \lambda_2, \omega, L \), and \( f \).

Therefore, for proving that (B) has an approximate solution, it is sufficient to prove that the inequality (7) holds.

4. PROOF OF INEQUALITY (7)

**Lemma 4.1.** Let \( f \in C^1(\omega, H_{\text{per}}^1) \), if \( F_\lambda u_n = u_n \), \( 0 \leq \lambda \leq 1 \), then there exists a positive constant \( C_1 \) such that

\[
\frac{1}{2} \frac{d}{dt} \|u_n\|^2 + \|\nabla u_n\|^2 - \frac{1}{2} d\|u_n\|^2 \leq C_1,
\]

where \( C_1 \) only depends on \( \alpha, \beta, \rho, \mu, \nu, \sigma, \lambda_1, \lambda_2, \omega, L \), and \( f \).

**Proof.** By \( F_\lambda u_n = u_n \), i.e.,

\[
(u_{nt} + Au_n, \phi_j) = (\lambda (N(u_n) + f), \phi_j), \quad j = 1, 2, \ldots, n. \tag{4\lambda}
\]

multiply each equation system (4\lambda) by \( \frac{d}{dt} \) and sum over \( j \) from 1 to \( n \). This gives us

\[
(u_{nt} + Au_n, u_n) = (\lambda (N(u_n) + f), u_n).
\]

Taking the real part in two sides of the equation yields that

\[
\frac{1}{2} \frac{d}{dt} \|u_n\|^2 + \|\nabla u_n\|^2 - d\|u_n\|^2 + \lambda \int_{\Omega} |u_n|^{2\rho+2} dx = \lambda (\rho - d)\|u_n\|^2 + \lambda \alpha Re \int_{\Omega} (\lambda_1 \cdot \nabla |u_n|^2 u_n) \overline{u_n} dx
\]

\[
+ \lambda \beta Re \int_{\Omega} (\lambda_2 \cdot \nabla u_n) |u_n|^2 \overline{u_n} dx + Re(f, u_n). \tag{8}
\]
Noting that $u_n$ is $\Omega$-periodic, the following equalities hold [2]

$$\text{Re} \int_{\Omega} (\lambda_1 \cdot \nabla (|u_n|^2 u_n)) \bar{u}_n \, dx = 0$$

and

$$\text{Re} \int_{\Omega} (\lambda_2 \cdot \nabla u_n) |u_n|^2 \bar{u}_n \, dx = 0.$$ 

The mark $D$ denotes a constant which depends on $\alpha, \beta, \rho, \mu, \nu, \sigma, \lambda_1, \lambda_2, \omega, L,$ and $f,$ but is independent of $n$ and $\lambda$ in each inequality below, respectively. Since

$$\text{Re}(f, u_n) \leq \epsilon \|u_n\|_2^2 + D(\epsilon),$$

where $\epsilon \in (0, -d)$ is an undetermined constant, and

$$\|u_n\|_2^2 \leq \frac{1}{\rho - d} \int_{\Omega} |u_n|^{2\rho + 2} \, dx + D.$$ 

Using equality (8) and inequality (9) and choosing $\epsilon$ appropriately, the inequality below is true.

$$\frac{1}{2} \frac{d}{dt} \|u_n\|_2^2 + \|\nabla u_n\|_2^2 - \frac{1}{2} \frac{d}{dt} \|u_n\|_2^2 + \lambda \int_{\Omega} |u_n|^{2\rho + 2} \, dx$$

$$\leq \lambda (\rho - d) \|u_n\|_2^2 + D. \quad (11)$$

By (11) + $\lambda \times (10),$ there exists a positive constant $C_1 = C_1(\alpha, \beta, \rho, \mu, \nu, \sigma, \lambda_1, \lambda_2, \omega, L, f)$ such that

$$\frac{1}{2} \frac{d}{dt} \|u_n\|_2^2 + \|\nabla u_n\|_2^2 - \frac{1}{2} \frac{d}{dt} \|u_n\|_2^2 \leq C_1.$$ 

This completes the proof of Lemma 4.1.

Using Lemma 4.1, we can prove the inequality (7). In fact, if $F_1 u_n = u_n,$ then

$$\frac{1}{2} \frac{d}{dt} \|u_n\|_2^2 - \frac{1}{2} \frac{d}{dt} \|u_n\|_2^2 \leq C_1.$$ 

Noting that $u_n$ is a $\omega$-periodic function in $t,$ we have

$$-\frac{1}{2} \frac{d}{dt} \int_0^{\omega} \|u_n\|^2 \, dt \leq C_1 \omega.$$ 

Therefore, there exists a $t^* \in [0, \omega)$ such that $\|u_n(t^*)\|^2 \leq -2C_1/d.$ By Lemma 4.1, $\frac{1}{2} \frac{d}{dt} \|u_n\|^2 \leq C_1.$ Integrating the inequality about $t$ from $t^*$ to $t(\in [t^*, t^* + \omega))$ reduces that

$$\|u_n\|^2 \leq 2C_1 \omega + \|u_n(t^*)\|^2 \leq 2C_1 \omega - 2C_1/d;$$

i.e., there exists a positive constant $K_1 = K_1(\alpha, \beta, \rho, \mu, \nu, \sigma, \lambda_1, \lambda_2, \omega, L, f),$ which is independent of $\lambda$ and $n,$ such that inequality (7) holds. By the Leray–Schauder fixed point theorem, $F_1$ has a fixed point. Hence we have the theorem below.
5. A PRIORI ESTIMATES OF THE DERIVATIVE

We have proved that (B) has a sequence of approximate solutions \( \{u_n\}_{n=1}^{\infty} \). We want to prove that the sequence converges and the limit is a solution to (B). For this purpose, we need to give some a priori estimates about \( u_n(t) \).

**Lemma 5.1.** Let \( f \in C^1(\omega, H^1_{per}) \). If \( F_{\lambda} u_n = u_n \), and there exists a positive number \( \delta \) such that the inequality below holds

\[
\frac{7}{3} \leq \sigma \leq \frac{1}{\sqrt{1 + \left(\frac{\mu - \nu \delta}{1 + \delta^2}\right)^2}} - 1
\]

then, for any \( \varepsilon > 0 \), there exist positive constants \( K_2 \) and \( K_3 \) which depend only on \( \alpha, \beta, \rho, \mu, \nu, \sigma, \lambda_1, \lambda_2, \omega, L \), and \( f \), and \( K_2 \) depends on \( \varepsilon \) also, such that the inequality holds as follows

\[
\frac{1}{2(1 + \sigma)} \frac{d}{dt} \int_{\Omega} |u_n|^{2\sigma + 2} dx
\]

\[
\leq -\frac{1}{2} \int_{\Omega} |u_n|^{4\sigma + 2} dx + \varepsilon \|\Delta u_n\|^2 + \varepsilon \|\nabla u_n\|^4
\]

\[
- \frac{1}{4} \int_{\Omega} |u_n|^{2\sigma + 2} \left( (1 + 2\sigma) |\nabla|u_n|^2|^2 - 2\nu \sigma |\nabla|u_n|^2| \cdot i(u_n \nabla \bar{u}_n - \bar{u}_n \nabla u_n) + |u_n \nabla \bar{u}_n - \bar{u}_n \nabla u_n|^2 \right) dx
\]

\[
+ K_2 + K_3 + Re\left( f \cdot |u_n|^{2\sigma} u_n \right).
\]

**Proof.** Multiply each equation in (4) by \( (\sum_{j=1}^{n} |u_n(t)|^{2\sigma})^{\sigma} \frac{d}{dt} d_{\rho}(t) \) and sum over \( j \) from \( j = 1 \) to \( n \); we have that

\[
\int_{\Omega} \frac{\partial u_n}{\partial t} |u_n|^{2\sigma} \bar{u}_n dx
\]

\[
= \rho \int_{\Omega} |u_n|^{2\sigma + 2} dx + (1 + iv) \int_{\Omega} \Delta u_n \cdot |u_n|^{2\sigma} \bar{u}_n dx
\]

\[- (1 + i\mu) \int_{\Omega} |u_n|^{4\sigma + 2} dx + \alpha \int_{\Omega} (\lambda_1 \cdot \nabla(|u_n|^2 u_n)) |u_n|^{2\sigma} \bar{u}_n dx
\]

\[+ \beta \int_{\Omega} (\lambda_2 \cdot \nabla u_n) |u_n|^{2\sigma + 2} \bar{u}_n dx + \int_{\Omega} f \cdot |u_n|^{2\sigma} \bar{u}_n dx.
\]
Taking the real part of the equality yields that
\[
\frac{1}{2(1 + \sigma)} \frac{d}{dt} \int_\Omega |u_n|^{2\sigma+2} \, dx
\]
\[
= \rho \int_\Omega |u_n|^{2\sigma+2} \, dx - \int_\Omega |\nabla u_n|^2 |u_n|^{2\sigma} - \frac{1}{2} \sigma \int_\Omega |u_n|^{2\sigma-2} |\nabla u_n|^2 \\
+ \frac{\nu}{2} \int_\Omega |u_n|^{2\sigma-2} |\nabla u_n|^2 \cdot i(u_n \nabla \bar{u}_n - \bar{u}_n \nabla u_n) - \int_\Omega |u_n|^{4\sigma+2} \\
+ \text{Re} \alpha \int_\Omega (\lambda_1 \cdot \nabla (|u_n|^2 u_n)) |u_n|^{2\sigma} \bar{u}_n \, dx \\
+ \text{Re} \beta \int_\Omega (\lambda_2 \cdot \nabla u_n) |u_n|^{2\sigma+2} \bar{u}_n \, dx + \text{Re} \int_\Omega f \cdot |u_n|^{2\sigma} \bar{u}_n \, dx. 
\]

The following proof is similar to that of Lemma 2.2 in [2] and we abridge it here. This completes the proof of Lemma 5.1.

**Lemma 5.2.** Assume that the conditions of Lemma 5.1 hold. Then there exists a positive constant \( K_4 \) which only depends on \( \alpha, \beta, \rho, \nu, \sigma, \lambda_1, \lambda_2, \omega, L, \) and \( f \), such that
\[
\sup_{0 \leq t \leq \omega} \| \nabla u_n \| \leq K_4. 
\]

**Proof.** By the equation \( Fu_n = u_n \), (4) holds. Multiply each equation in (4) by \(- (\frac{\mu + \sigma}{1 + \sigma}) d\mu(t)\) and sum over \( j \) from \( j = 1 \) to \( n \), we have that
\[
(u_n + Au_n, \Delta u_n) = (N(u_n) + f, \Delta u_n). 
\]
Taking the real part in two sides of the identity, noting that \( u_n \) is a \( \Omega \)-periodic function, we have
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u_n \|^2 + \| \Delta u_n \|^2
\]
\[
= \text{Re} (1 + i\mu) \int_\Omega |u_n|^{2\sigma} u_n \Delta \bar{u}_n \, dx - \alpha \text{Re} \int_\Omega (\lambda_1 \cdot \nabla (|u_n|^2 u_n)) \Delta \bar{u}_n \, dx \\
- \beta \text{Re} \int_\Omega (\lambda_2 \cdot \nabla u_n) |u_n|^2 \Delta \bar{u}_n \, dx + \text{Re}(f, \Delta u_n) + \rho \| \nabla u_n \|^2. 
\]
By
\[
|u_n|^2 |\nabla u_n|^2 = \frac{1}{4} |\nabla |u_n|^2|^2 + \frac{1}{4} |u_n \nabla \bar{u}_n - \bar{u}_n \nabla u_n|^2 
\]
we have
\[
\text{Re} (1 + i\mu) \int_\Omega |u_n|^{2\sigma} u_n \Delta \bar{u}_n \, dx \\
= - \frac{1}{4} \int_\Omega |u_n|^{2\sigma-2} ((1 + 2\sigma)|\nabla |u_n|^2|^2 - 2\mu \sigma |\nabla u_n|^2 \\
\cdot i(u_n \nabla \bar{u}_n - u_n \nabla \bar{u}_n) + |u_n \nabla \bar{u}_n - \bar{u}_n \nabla u_n|^2) \, dx. 
\] (12)
We shall use these inequalities as

\[ \|u\|^2 \leq k \|u\|_{H^s}^2 \|u\|^{1/2}, \text{ for any } u \in H^1(\Omega) \]  
(13)

\[ \|u\|_8 \leq k \|u\|_{H^s}^6 \|u\|^{1-\theta}, \text{ for any } u \in H^2(\Omega), \]  
(14)

where \( \theta = \frac{8-q}{4q} \) as \( 1 < q < 8 \) and \( \theta = 0 \) as \( q \geq 8 \).

Let \( \bar{D}(s) \) describe a positive constant which depends on real number \( s \) \((0 < s \leq 1)\) and may depend on \( \alpha, \beta, \rho, \nu, \sigma, \lambda_1, \lambda_2, \omega, L, \) and \( f \). Let \( D \) describe a positive constant depending on \( \alpha, \beta, \rho, \nu, \sigma, \lambda_1, \lambda_2, \omega, L, \) or \( f \). Using

\[ -\beta Re \int_\Omega (\lambda_2 \cdot \nabla u_n) |u_n|^2 \Delta u_n \, dx \]

\[ \leq |\beta\lambda_2| \int_\Omega |\nabla u_n| |u_n|^2 |\Delta u_n| \, dx \leq (\text{using Hölder's inequality}) \]

\[ \leq D \|\Delta u_n\| \|\nabla u_n\|_4 |u_n|_8^2 \]

\[ \leq D \|\Delta u_n\|_H^{1/2} \|\nabla u_n\|_H^{1/2} |u_n|_q \leq (\text{using (14)}) \]

\[ \leq D \|u_n\|_H^{2\theta+3/2} \|\nabla u_n\|_H^{2\theta+3/2} \|u_n\|_q^{2(1-\theta)} \leq (\text{using Young's inequality}) \]

\[ \leq \gamma \|u_n\|_H^{2\theta+3/2} \tilde{D}(\gamma) \|\nabla u_n\|_H^{(2/1-4\theta)\|u_n\|_q^{(8(1-\theta)/1-4\theta)}} \]

\[ \leq (\text{using Young's inequality}) (q > 3 \text{ and } 0 < \gamma \leq 1) \]

\[ \leq \gamma \|u_n\|_H^{2\theta+3/2} + \gamma \|\nabla u_n\|^4 + \tilde{D}(\gamma) \|u_n\|_q^{(16(1-\theta)/1-8\theta)} \]

\[ \leq (\text{using Young's inequality}) (q > \frac{14}{3} \text{ and } 0 < \gamma \leq 1) \]

\[ \leq \gamma C \|\Delta u_n\|^2 + \gamma \|\nabla u_n\|^4 + \gamma |u_n|_q^q + \tilde{D}(\gamma) \quad (q > \frac{34}{3} \text{ and } 0 < \gamma \leq 1) \]

\[ \leq \gamma C \|\Delta u_n\|^2 + \gamma \|\nabla u_n\|^4 + \gamma |u_n|_q^q + \tilde{D}(\gamma) \leq (\text{let } q = 4\sigma + 2 > \frac{34}{3}) \]

\[ = \gamma C \|\Delta u_n\|^2 + \gamma \|\nabla u_n\|^4 + \gamma |u_n|_{4\sigma+2}^4 + \tilde{D}(\gamma) \]

\[ \leq (\text{let } \gamma \text{ be small enough}) (\sigma > \frac{3}{2} \text{ as } q = 4\sigma + 2 > \frac{34}{3}) \]

\[ \leq \frac{e}{2} (\|\Delta u_n\|^2 + \|\nabla u_n\|^4 + |u_n|_{4\sigma+2}^4) + \tilde{D}(e) \quad (0 < e \leq 1). \]

Similarly,

\[ -\alpha Re \int_\Omega (\lambda_1 \cdot \nabla (|u_n|^2 u_n)) \Delta u_n \, dx \]

\[ \leq \frac{e}{2} (\|\Delta u_n\|^2 + \|\nabla u_n\|^4 + |u_n|_{4\sigma+2}^4) + \tilde{D}(e) \quad (0 < e \leq 1). \]

Therefore, the inequality below holds.
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u_n \|^2 + \| \Delta u_n \|^2 \\
\leq \varepsilon \| \Delta u_n \|^2 + 2\varepsilon \| \nabla u_n \|^4 + \varepsilon \| u_n \|^{4\sigma+2} - \frac{1}{4} \int_{\Omega} |u_n|^{2\sigma-2} \\
\cdot \left( (1 + 2\sigma) |\nabla u_n|^2 - 2\mu\sigma |u_n|^2 \right) \\
\cdot \left( i(\overline{u}_n \nabla u_n - u_n \overline{\nabla u_n}) + \left| \overline{u}_n \nabla u_n - u_n \overline{\nabla u_n} \right|^2 \right) + \tilde{D}(\varepsilon). \quad (15)
\]

Multiplying the inequality in Lemma 5.1 by \(\delta^2\), then the resulting addition to (15), we obtain that
\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla u_n \|^2 + \frac{\delta^2}{1 + \sigma} \int_{\Omega} |u_n|^{2\sigma+2} \right) + \| \Delta u_n \|^2 + \frac{1}{2} \delta^2 \int_{\Omega} |u_n|^{4\sigma+2} \\
\leq \varepsilon(1 + \delta^2) \| \Delta u_n \|^2 + \varepsilon(2 + \delta^2) \| \nabla u_n \|^4 + \varepsilon \int_{\Omega} |u_n|^{4\sigma+2} \\
+ \tilde{D}(\varepsilon) + Re(f \cdot |u_n|^{2\sigma} u_n) + Re(f \cdot \Delta u_n) \\
- \frac{1}{4} \int_{\Omega} |u_n|^{2\sigma-2} \left[ (1 + 2\sigma)(1 + \delta^2) |\nabla u_n|^2 \right] \\
+ 2\sigma(\nu\delta^2 - \mu) |\nabla u_n|^2 \cdot i \left( \overline{u}_n \nabla u_n - u_n \overline{\nabla u_n} \right) \\
+ (1 + \delta^2) \left| \overline{u}_n \nabla u_n - u_n \overline{\nabla u_n} \right|^2. \quad (16)
\]

By the following inequalities
\[
Re(f \cdot |u_n|^{2\sigma} u_n) \leq \varepsilon \int_{\Omega} |u_n|^{4\sigma+2} + \tilde{D}(\varepsilon)
\]
and
\[
|Re(f \cdot \Delta u_n)| \leq \varepsilon \| \Delta u_n \|^2 + \tilde{D}(\varepsilon)
\]
we can choose \(\varepsilon\) small enough such that \(\varepsilon(1 + \delta^2) + \varepsilon(2 + \delta^2) + \varepsilon < \frac{1}{2}\) and \(2\varepsilon < \frac{1}{2}\delta^2\). On the other hand, we denote the last term of the right-hand side of (16) by \(I\). Then (16) reduces to
\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla u_n \|^2 + \frac{\delta^2}{1 + \sigma} \int_{\Omega} |u_n|^{2\sigma+2} \right) + \frac{1}{2} \| \Delta u_n \|^2 + \frac{1}{4} \delta^2 \int_{\Omega} |u_n|^{4\sigma+2} \leq D + I.
\]

We can take \(\delta\) appropriately such that \(I \leq 0\). In fact, it is needed only that the matrix
\[
E = \begin{pmatrix}
(1 + 2\sigma)(1 + \delta^2) & \sigma(\nu\delta^2 - \mu) \\
\sigma(\nu\delta^2 - \mu) & 1 + \delta^2
\end{pmatrix}
\]
is non-negative definite. Further, if \(|E| \geq 0\), then \(I \leq 0\). Therefore, we need only to take \(\delta\) such that \(|E| \geq 0\). This is available.
Hence, we have that
\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla u_n \|^2 + \frac{\delta^2}{1 + \sigma} \int_\Omega |u_n^{2\sigma+2}| \, dx \right) + \frac{1}{2} \| \Delta u_n \|^2 + \frac{1}{2} \int_\Omega |u_n^{4\sigma+2}| \, dx \leq D.
\]

Noting (7), the inequality below holds.
\[
\frac{\delta^2}{2(1 + \sigma)} \int_\Omega |u_n^{2\sigma+2}| \, dx \leq \frac{1}{4} \delta^2 \int_\Omega |u_n^{4\sigma+2}| \, dx + D.
\]
Therefore,
\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla u_n \|^2 + \frac{\delta^2}{1 + \sigma} \int_\Omega |u_n^{2\sigma+2}| \, dx \right) + \frac{1}{2} \left( \| \nabla u_n \|^2 + \frac{\delta^2}{1 + \sigma} \int_\Omega |u_n^{2\sigma+2}| \, dx \right) \leq D.
\] (17)

Integrating the inequality about \( t \) from 0 to \( \omega \), we obtain
\[
\int_0^\omega \left( \| \nabla u_n \|^2 + \frac{\delta^2}{1 + \sigma} \int_\Omega |u_n^{2\sigma+2}| \, dx \right) \, dt \leq D.
\]
Hence there exists \( t^* \in [0, \omega] \) such that
\[
\| \nabla u_n(t^*) \|^2 + \frac{\delta^2}{1 + \sigma} \int_\Omega |u_n(t^*)^{2\sigma+2}| \, dx \leq D.
\]
Integrating (17) about \( t \) from \( t^* \) to \( t \) \((\in [t^*, t^* + \omega])\), we have
\[
\| \nabla u_n \|^2 + \frac{\delta^2}{1 + \sigma} \int_\Omega |u_n^{2\sigma+2}| \, dx \leq D.
\]
Therefore, there exists a constant \( K_4 \) which only depends on \( \alpha, \beta, \rho, \mu, \nu, \sigma, \lambda_1, \lambda_2, \omega, L, \) and \( f \) such that
\[
\sup_{0 \leq t \leq \omega} \| \nabla u_n \| \leq K_4.
\]
This completes the proof of Lemma 5.2.

**Lemma 5.3.** Assume that the conditions of Lemma 5.1 hold. Then there exists a positive constant \( K_5 \) which only depends on \( \alpha, \beta, \rho, \mu, \nu, \sigma, \lambda_1, \lambda_2, \omega, L, \) and \( f \) such that
\[
\sup_{0 \leq t \leq \omega} \| \Delta u_n \| \leq K_5.
\]
Proof. By the equation \( Fu_n = u_n \), (4) holds. Multiply each equation system in (4) by \( (\frac{\mu_j + d}{1 + iv})^2 d^j_m(t) \) and sum over \( j \) from 1 to \( n \), we have that
\[
(u_{nt} + Au_n, \Delta^2 u_n) = (N(u_n) + f, \Delta^2 u_n).
\]
Taking the real part in two sides of the equation, we have that
\[
\frac{1}{2} \frac{d}{dt} \| \Delta u_n \|^2 = \rho \| \Delta u_n \|^2 - \| \nabla \Delta u_n \|^2 - \text{Re}(1 + i\mu) \int_\Omega |u_n|^2 u_n \Delta^2 \overline{u_n} dx
\]
\[
+ \alpha \text{Re} \int_\Omega (\lambda_1 \cdot \nabla (|u_n|^2 u_n)) \Delta^2 \overline{u_n} dx
\]
\[
+ \beta \text{Re} \int_\Omega (\lambda_2 \cdot \nabla u_n)|u_n|^2 \Delta^2 \overline{u_n} dx + \text{Re}(f, \Delta^2 u_n).
\]
Using the inequality (7) and Lemma 5.2, we have the result as follows: There exists a constant \( \hat{C} = \hat{C}(\alpha, \beta, \rho, \mu, \nu, \sigma, L, \omega, \lambda_1, \lambda_2, \omega, L, f) \) such that
\[
\frac{d}{dt} \| \Delta u_n \|^2 + \| \nabla \Delta u_n \|^2 + \| \Delta u_n \|^2 \leq \hat{C} + \| f \|^2_{H^1}. \tag{18}
\]
Hence,
\[
\int_0^\omega \| \Delta u_n \|^2 dx \leq (\hat{C} + \| f \|^2_{H^1}) \omega.
\]
Applying the mean value theorem for the integral, there is a number \( t^{**} \in [0, \omega) \), which satisfies that
\[
\| \Delta u_n(t^{**}) \|^2 \leq \hat{C} + \| f \|^2_{H^1}. \tag{19}
\]
By the inequality (18), we have that
\[
\frac{d}{dt} \| \Delta u_n \|^2 \leq \hat{C} + \| f \|^2_{H^1}. \tag{20}
\]
Integrating the inequality (20) about \( t \) from \( t^{**} \) to \( t \in [t^{**}, t^{**} + \omega) \) obtains that
\[
\| \Delta u_n(t) \|^2 \leq \| \Delta u_n(t^{**}) \|^2 + (\hat{C} + \| f \|^2_{H^1}) \omega.
\]
Let \( K_5 = (\hat{C} + \| f \|^2_{H^1})(\omega + 1) \); we have that
\[
\sup_{0 \leq t \leq \omega} \| \Delta u_n \| \leq K_5.
\]
This completes the proof of Lemma 5.3.
Remark 1. By inequality (7) and Lemma 5.3, we obtain that there is a positive constant $K_6$ which only depends on $\alpha, \beta, \rho, \mu, \nu, \sigma, \lambda_1, \lambda_2, \omega, L,$ and $f$ such that

$$\sup_{0 \leq t \leq \omega} \|u_n\|_\infty \leq K_6.$$ 

Lemma 5.4. Assume that the conditions of Lemma 5.1 hold. Then there exists a positive constant $K_7$ which only depends on $\alpha, \beta, \rho, \mu, \nu, \sigma, \lambda_1, \lambda_2, \omega, L,$ and $f$ such that

$$\sup_{0 \leq t \leq \omega} \|u_{nt}\| \leq K_7.$$ 

Proof. By the equation $F_{u_n} = u_n$, (4) holds. Multiply each equation system in (4) by $d_j(t)$ and sum over $j$ from $j = 1$ to $n$, we have that

$$(u_{nt} + Au_n, u_{nt}) = (N(u_n) + f, u_{nt}).$$

Taking the real part in two sides of the equation, we have

$$\|u_{nt}\|^2 \leq \sqrt{1 + \nu^2} \|\Delta u_n\| \cdot \|u_{nt}\| + \|N(u_n) + f + du_n\| \cdot \|u_{nt}\|.$$ (21)

Using the inequality $\|u_n\|_\infty \leq K_6$ and the definition of $N(u_n)$, we have

$$\|N(u_n) + f + du_n\| \leq \rho K_6 + K_6^{2\sigma + 1} + |\alpha\lambda_1| \cdot 3K_6^2 \|
\cdot \|u_n\| + \|f\|$$

$$\leq D.$$ 

Therefore, by inequality (21) and Lemma 5.3, there exists a positive constant $K_7$ which only depends on $\alpha, \beta, \rho, \mu, \nu, \sigma, \lambda_1, \lambda_2, \omega, L,$ and $f$ such that

$$\sup_{0 \leq t \leq \omega} \|u_{nt}\| \leq K_7.$$ 

This completes the proof of Lemma 5.4.

6. MAIN THEOREM

Using the existence of an approximate solution and the a priori estimates above, we can prove the main theorem in this paper:

Theorem 6.1. Assume that $f \in C^1(\omega, H^1_{per})$. If there exists a positive number $\delta$ such that

$$\frac{7}{3} \leq \sigma \leq \frac{1}{\sqrt{1 + \left(\frac{\mu + \delta}{1 + \delta}\right)^2} - 1},$$

then Eq. (3) has a solution $u$ in $C^1(\omega, H^1_{per})$. 

Proof. For any number \( n \), we proved that Eq. (3) has an approximate solution \( u_n(t) \)—i.e., the system (4) holds—and we have some estimates about the norm of \( u_n(t) \). For each fixed \( t \), the uniform boundedness of the norms \( \| u_n \|_{H^1_{\text{per}}} \) in the space \( H^1_{\text{per}} \), makes it possible to choose a subsequence \( \{ u_{n_k}(t) \} \) which converges weakly to some element \( u(t) \in H^1_{\text{per}} \). We shall prove that \( u \) is a solution to problem (3).

In fact, by \( \{ u_{n_k}(t) \} \) converging weakly to \( u(t) \) in \( H^1_{\text{per}} \), the results below are true: for any \( t \in [0, \omega) \),

\[
\begin{align*}
    u_{n_k}(t) & \to u(t), \ (k \to \infty) \quad \text{weakly in } L^2_{\text{per}} \\
    u_{n_k,x}(t) & \to u_x(t), \ (k \to \infty) \quad \text{weakly in } L^2_{\text{per}} \\
    u_{n_k,y}(t) & \to u_y(t), \ (k \to \infty) \quad \text{weakly in } L^2_{\text{per}},
\end{align*}
\]

(22)

Since \( H^1_{\text{per}} \) is embedded compactly in \( L^2(\Omega) \), we can choose a subsequence of \( \{ u_{n_k}(t) \} \), and still denote it as \( \{ u_{n_k}(t) \} \) for convenience, such that for any \( t \in [0, \omega) \),

\[
    u_{n_k}(t) \to u(t), \ (k \to \infty) \quad \text{strongly in } L^2_{\text{per}}
\]

(23)

and

\[
    u_{n_k}(t) \to u(t), \ (k \to \infty) \quad \text{a.e. } \Omega.
\]

(24)

By inequality (7), Lemma 5.2, and Lemma 5.3, for any \( t \in [0, \omega) \), \( \{ u_{n_k}(t) \} \) is the uniform bound in \( H^2_{\text{per}}(\Omega) \). Therefore, we can choose a subsequence of \( \{ u_{n_k}(t) \} \), and still denote it as \( \{ u_{n_k}(t) \} \) for convenience, such that \( \{ u_{n_k}(t) \} \) converges weakly in \( H^2_{\text{per}}(\Omega) \), especially when, for any \( t \in [0, \omega) \),

\[
    \Delta u_{n_k}(t) \to \Delta u(t), \ (k \to \infty) \quad \text{weakly in } L^2_{\text{per}}.
\]

(25)

On the other hand, \( \{ \partial_x u_{n_k}(t) \} \) and \( \{ \partial_y u_{n_k}(t) \} \) are uniform bounded in \( H^1_{\text{per}} \) by the uniform boundedness of \( \{ u_{n_k}(t) \} \) in \( H^2_{\text{per}}(\Omega) \). From this, we can choose a subsequence of \( \{ u_{n_k}(t) \} \), and still denote it as \( \{ u_{n_k}(t) \} \) for convenience, such that for any \( t \in [0, \omega) \),

\[
    \partial_x u_{n_k}(t) \to w_1(t), \ (k \to \infty) \quad \text{strongly in } L^2_{\text{per}}
\]

\[
    \partial_y u_{n_k}(t) \to w_2(t), \ (k \to \infty) \quad \text{strongly in } L^2_{\text{per}},
\]

where \( w_1(t) \) and \( w_2(t) \) are some two elements in \( H^1_{\text{per}} \). And

\[
    \nabla u_{n_k}(t) \to \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \ (k \to \infty) \quad \text{a.e. in } \Omega.
\]

(26)

Using (22) and (26), we have

\[
    \nabla u_{n_k}(t) \to \nabla u(t), \ (k \to \infty) \quad \text{a.e. in } \Omega.
\]

(27)
Noting inequality (7) and Lemmas 5.2 and 5.3,
\[ \|N(u_{n_k}(t))\| \leq \tilde{C}, \]
where \( \tilde{C} \) is a constant to depend only on \( \alpha, \beta, \rho, \mu, v, \sigma, \lambda_1, \lambda_2, \omega, L, \) and \( f \). By (24) and (27), we have
\[ N(u_{n_k}(t)) \to N(u(t)), \quad (k \to \infty) \quad \text{a.e. in } \Omega. \]

Using Lemma 1.3 in [6],
\[ N(u_{n_k}(t)) \rightharpoonup N(u(t)), \quad (k \to \infty) \quad \text{weakly in } L^2_{\text{per}}. \] (28)

From that \( u_n \in C^1(\omega, H^1_{\text{per}}) \), Lemma 5.4, and (22), we have that \( u(t) \in C^1(\omega, H^2_{\text{per}}) \), and for any \( t \in [0, \omega) \),
\[ u_{n_k}(t) \to u(t), \quad (k \to \infty) \quad \text{weakly in } L^2_{\text{per}}. \] (29)

Let us multiply each equation in (4) by any \( C_j \in C^1(\omega, \mathbb{C}) \) and sum over \( j \) from 1 to \( n \). This gives us
\[ (u_{n_k} + Au_n, \eta) = (N(u_n) + f, \eta) \quad \text{for any } \eta \in C^1(\omega, H_n). \]

For any fixed \( k_0 \), by \( H_{n_0} \subset H_{n_{k_0+1}} \subset \cdots \), we have that, as \( k \geq k_0 \),
\[ (u_{n_k} + Au_n, \eta) = (N(u_{n_k}) + f, \eta) \quad \text{for any } \eta \in C^1(\omega, H_{n_k}). \] (30)

Using (25), (28), and (29), taking the limit \( k \to \infty \) in (30), we obtain that for any \( \eta \in C^1(\omega, H_{n_0}) \),
\[ (u + Au, \eta) = (N(u) + f, \eta). \] (31)

The number \( k_0 \) here is arbitrary, such that (31) holds for all \( \eta \in C^1(\omega, \bigcup_{n=1}^{\infty} H_n) \). Since \( \bigcup_{n=1}^{\infty} H_n \) is dense in \( L^2_{\text{per}} \), the function \( u(t) \) satisfies (31) for all \( \eta \in C^1(\omega, L^2_{\text{per}}) \); i.e., \( u(t) \) is a solution to problem (3). This completes the proof of Theorem 6.1.

REFERENCES