Journal of Computational and Applied Mathematics 39 (1992) 95–101
North-Holland

Letter Section

A note on inverses of power series

M. Ptak, A. Rutkowska and J. Szczurek

Institute of Mathematics and Institute of Higher Geodesy, University of Agriculture, Ul. 18 Stycznia 6, 30-045 Kraków, Poland

Received 1 August 1990
Revised 25 July 1991

Abstract

Ptak, M., A. Rutkowska and J. Szczurek, A note on inverses of power series, Journal of Computational and Applied Mathematics 39 (1992) 95–101.

We show the recurrence formula for coefficients of an inverse of a power series of two variables. This problem arises from geodesy.

Keywords: Homogeneous polynomial, power series.

In the following we will consider an inverse of a power series of two variables. The problem arises from geodesy, where the Gauss–Krüger mapping, a function from the surface of the earth's ellipsoid to a subset of $\mathbb{R}^2$, is considered. It is usually given by a power series of two variables, the longitude and the latitude. The coefficients are calculated on the basis of the constants of the ellipsoid. The inverse problem of finding the inverse function to the Gauss–Krüger mapping is also studied. It can also be given by a power series. It is interesting from the geodesy point of view to compute the coefficients of the inverse series. In the era of computers, it means to find an explicit formula for them. Our purpose is to prove the recurrence formula (1). Let us observe that one can make an algorithm for it, using, for example, [4].

Choose the norm in $\mathbb{R}^2$ which gives rectangles as balls, i.e., $\|(x, y)\| = \max\{|x|, \lambda|y|\}$, where $\lambda > 0$ is fixed. Let us recall that a power series $\sum_{p,q=0}^{\infty} a_{pq} f^p l^q$, $a_{pq} \in \mathbb{R}^2$, converges on an open set G to a function $\psi : G \to \mathbb{R}^2$ if

$$\sum_{p,q=0}^{p=m,q=n} a_{pq} f^p l^q \rightarrow \psi(f, l),$$

if $m, n \to \infty$ for all $(f, l) \in G$. In what follows, I_k denotes a set $\{1, 2, \dots, k\}$.

Now we can formulate our main result.

Theorem 1. *Suppose that a power series* $\sum_{p,q=0}^{\infty} a_{pq} f^p l^q$, $a_{pq} \in \mathbb{R}^2$, *converges on* $K(0, R)$ *to a function* $\psi : K(0, R) \to V \stackrel{\text{def}}{=} \psi(K(0, R)) \subset \mathbb{R}^2$. *Assume also that* $a_{00} = 0$ *and*

$$\det\begin{bmatrix} a_{10} \\ a_{01} \end{bmatrix} \neq 0.$$

Then ψ *is invertible on* $K(0, R)$ *and the inverse* $\phi : V \to K(0, R)$ *is given as a power series* $\sum_{s,t=0}^{\infty} \alpha_{st} x^s y^t$, $\alpha_{st} \in \mathbb{R}^2$, *converging on* V. *Moreover*, $\alpha_{00} = 0$,

$$\phi_1 \stackrel{\text{def}}{=} \begin{bmatrix} \alpha_{10} \\ \alpha_{01} \end{bmatrix} = \begin{bmatrix} a_{10} \\ a_{01} \end{bmatrix}^{-1},$$

and for $s + t > 1$,

$$\alpha_{st} = -\phi_1\left(\sum_{\substack{k=2,\dots,s+t \\ p+q=k}} \frac{p!q!}{k!}\left(\sum_{\substack{\chi : I_k \to \{1,2\} \\ \operatorname{card}\chi^{-1}(1)=p}} \sum_{\substack{s_1+\cdots+s_k=s \\ t_1+\cdots+t_k=t \\ s_i+t_i \geq 1}} (\alpha_{s_1t_1})_{\chi(1)} \cdots (\alpha_{s_kt_k})_{\chi(k)}\right) a_{pq}\right), \tag{1}$$

where χ *sets the number of a coordinate* $\alpha_{s_it_i}$ (*i.e.*, $(\alpha_{s_it_i})_{\chi(i)}$ *is the* $\chi(i)$ *coordinate of* $\alpha_{s_it_i}$).

The recurrence formula (1) gives us the possibility to calculate the coefficients α_{st} of the inverse series knowing the coefficients a_{pq} of the given series. Precisely, α_{st} depends, for fixed s, t, on a_{pq} and on $\alpha_{s't'}$ for s', t' such that $s' + t' < s + t$.

Let X, Y be normed finite-dimensional vector spaces. We state a result that will be of use later.

Proposition 2. *Suppose that a series* $\sum_{k=0}^{\infty} \psi_k$ *of homogeneous polynomials* $\psi_k : X \to Y$ *converges to a function* $\psi : K(0, \rho) \to V \stackrel{\text{def}}{=} \psi(K(0, \rho)) \subset Y$ *on* $K(0, \rho)$, *where* $0 < \rho \leq (\limsup_{k\to\infty} \sqrt[k]{\|\psi_k\|})^{-1}$. *If* $\psi(0) = 0$ *and* ψ_1 *is invertible, then* ψ *is invertible and the inverse* $\phi : V \to K(0, \rho)$ *is given as a series* $\sum_{n=0}^{\infty} \phi_n$ *of homogeneous polynomials* $\phi_n : Y \to X$ *converging on* V.

The above proposition strenghtens [3, Theorem 107], where the inverse of a power series on $\mathbb{R}$ is studied. The main idea of the proof of [3, Theorem 107] works also here. Thus we omit the proof.

Proof of Theorem 1. We start with defining the homogeneous polynomials $\psi_k(f, l) = \sum_{p+q=k} a_{pq} f^p l^q$ for $k = 0, 1, 2, \dots$. Reference [2, Chapter XI, §5, Theorem 7] shows that $\sum_{k=0}^{\infty} \psi_k(f, l)$ converges to ψ on $K(0, R)$. It is easy to see that $R \leq (\limsup_{k\to\infty} \sqrt[k]{\|\psi_k\|})^{-1}$. Applying Proposition 2, there is $\phi : V \to K(0, R)$, inverse of ψ, and ϕ is given as a series $\sum_{n=0}^{\infty} \phi_n$ of homogeneous polynomials $\phi_n : Y \to X$ converging on V. We may write

$$\phi_n(x, y) = \sum_{s+t=n} \alpha_{st} x^s y^t, \tag{2}$$

for $(x, y) \in V$. Reference [2, Chapter XI, §5, Theorem 7] shows also that $\sum_{s,t=0}^{\infty} \alpha_{st} x^s y^t$ converges to ϕ on V. Hence, it remains to prove the formula for α_{st}.

Let $h: X \to Y$ be any function. For $u_1 \in X$ we define the function $\Delta_{u_1} h: X \to Y$ as $(\Delta_{u_1} h)(u) = h(u + u_1) - h(u)$. Moreover, for $u_1, \ldots, u_k \in X$, we put $\Delta_{u_1 \ldots u_k} h = \Delta_{u_1}(\Delta_{u_2 \ldots u_k} h)$. By induction, it is possible to prove the next lemma.

Lemma 3. *If $h: X \to Y$ is any function, then*

$$\Delta_{u_1 \ldots u_k} h(u) = (-1)^k h(u) + \sum_{r=1}^{k} (-1)^{k-r} \sum_{1 \leqslant \nu_1 < \cdots < \nu_r \leqslant k} h\left(\sum_{i=1}^{r} u_{\nu_i} + u \right).$$

Applying [1, formula 7.4.6] for a composition of two polynomials, we can see that the nth homogeneous polynomial of the composition of ψ and ϕ is the following:

$$\sum_{i_1 + \cdots + i_k = n} \tilde{\psi}_k\left(\phi_{i_1}(\cdot), \ldots, \phi_{i_k}(\cdot)\right),$$

where $\tilde{\psi}_k : (\mathbb{R}^2)^k \to \mathbb{R}^2$ is the unique k-linear symmetric function corresponding to the homogeneous polynomial ψ_k, i.e.,

$$\psi_k(u) = \tilde{\psi}_k(\underbrace{u, \ldots, u}_{k \text{ times}}).$$

Since $\psi \circ \phi = \text{id}$, we have

$$\phi_0 = 0, \qquad \phi_1 = \psi_1^{-1}, \qquad \phi_n = -\phi_1 \circ \sum_{\substack{k=2,\ldots,n \\ i_1 + \cdots + i_k = n \\ i_\iota \geqslant 1}} \tilde{\psi}_k \circ (\phi_{i_1}, \ldots, \phi_{i_k}). \tag{3}$$

In [1, Theorem 6.3.1] it was shown that $\Delta_{u_1 \ldots u_k} \psi_k$ is a constant function for any $u_1, \ldots, u_k \in \mathbb{R}^2$ and

$$\tilde{\psi}_k(u_1, \ldots, u_k) = \frac{1}{k!} \Delta_{u_1 \ldots u_k} \psi_k.$$

Hence, by Lemma 3 we obtain

$$\begin{aligned} \tilde{\psi}_k(u_1, \ldots, u_k) &= \frac{1}{k!} \Delta_{u_1, \ldots, u_k} \psi_k(0) \\ &= \frac{1}{k!} \sum_{r=1}^{k} (-1)^{k-r} \sum_{1 \leqslant \nu_1 < \cdots < \nu_r \leqslant k} \psi_k\left(\sum_{i=1}^{r} u_{\nu_i} \right). \end{aligned}$$

Let $u_i = (f_i, l_i) \in K(0, R)$, thus

$$\begin{aligned} &\tilde{\psi}_k((f_1, l_1), \ldots, (f_k, l_k)) \\ &\quad = \frac{1}{k!} \sum_{r=1}^{k} (-1)^{k-r} \sum_{1 \leqslant \nu_1 < \cdots < \nu_r \leqslant k} \psi_k\left(\sum_{i=1}^{r} f_{\nu_i}, \sum_{i=1}^{r} l_{\nu_i} \right) \end{aligned}$$

$$= \frac{1}{k!}\sum_{r=1}^{k}(-1)^{k-r}\sum_{1\leqslant\nu_1<\cdots<\nu_r\leqslant k}\sum_{\substack{p+q=k\\p,q\geqslant 0}}\left(\left(\sum_{i=1}^{r} f_{\nu_i}\right)^{p}\left(\sum_{i=1}^{r} l_{\nu_i}\right)^{q}\right)a_{pq}$$

$$= \sum_{\substack{p+q=k\\p,q\geqslant 0}}\frac{1}{k!}\left(\sum_{r=1}^{k}(-1)^{k-r}\sum_{1\leqslant\nu_1<\cdots<\nu_r\leqslant k}\left(\sum_{i=1}^{r} f_{\nu_i}\right)^{p}\left(\sum_{i=1}^{r} l_{\nu_i}\right)^{q}\right)a_{pq}.$$

Let

$$S \stackrel{\text{def}}{=} \sum_{r=1}^{k}(-1)^{k-r}\sum_{1\leqslant\nu_1<\cdots<\nu_r\leqslant k}\left(\sum_{i=1}^{r} f_{\nu_i}\right)^{p}\left(\sum_{i=1}^{r} l_{\nu_i}\right)^{q}.$$

It is known that

$$(a_1+\cdots+a_m)^n = \sum_{k_1+\cdots+k_m=n}\frac{n!}{k_1!\cdots k_m!}a_1^{k_1}\cdots a_m^{k_m},$$

thus

$$S = \sum_{r=1}^{k}(-1)^{k-r}\sum_{1\leqslant\nu_1<\cdots<\nu_r\leqslant k}\left(\sum_{\substack{p_1+\cdots+p_r=p\\p_i\geqslant 0}}\frac{p!}{\bar{p}!}f_{\nu_1}^{p_1}\cdots f_{\nu_r}^{p_r}\right)$$

$$\times\left(\sum_{\substack{q_1+\cdots+q_r=q\\q_i\geqslant 0}}\frac{q!}{\bar{q}!}l_{\nu_1}^{q_1}\cdots l_{\nu_r}^{q_r}\right)$$

$$= \sum_{r=1}^{k}(-1)^{k-r}\sum_{1\leqslant\nu_1<\cdots<\nu_r\leqslant k}\sum_{\substack{p_1+\cdots+p_r=p\\q_1+\cdots+q_r=q\\p_i,q_i\geqslant 0}}\frac{p!q!}{\bar{p}!\bar{q}!}f_{\nu_1}^{p_1}\cdots f_{\nu_r}^{p_r}l_{\nu_1}^{q_1}\cdots l_{\nu_r}^{q_r},$$

where $\bar{p}! = p_1!\cdots p_r!$, $\bar{q}! = q_1!\cdots q_r!$. Fix r and let

$$S_r \stackrel{\text{def}}{=} \sum_{1\leqslant\nu_1<\cdots<\nu_r\leqslant k}\sum_{\substack{p_1+\cdots+p_r=p\\q_1+\cdots+q_r=q\\p_i,q_i\geqslant 0}}\frac{p!q!}{\bar{p}!\bar{q}!}f_{\nu_1}^{p_1}\cdots f_{\nu_r}^{p_r}l_{\nu_1}^{q_1}\cdots l_{\nu_r}^{q_r}.$$

Now put $\tilde{q} = (\tilde{q}_1,\ldots,\tilde{q}_k)$, $\tilde{p} = (\tilde{p}_1,\ldots,\tilde{p}_k)$, where $\tilde{p}_i = p_i$ and $\tilde{q}_i = q_i$ for $i \in \{\nu_1,\ldots,\nu_r\}$ and $\tilde{p}_i = \tilde{q}_i = 0$, otherwise. Hence

$$S_r = \sum_{\substack{\nu: I_r\to I_k\\ \text{increasing}}}\sum{}'\frac{p!q!}{\tilde{p}!\tilde{q}!}f_1^{\tilde{p}_1}\cdots f_k^{\tilde{p}_k}l_1^{\tilde{q}_1}\cdots l_k^{\tilde{q}_k},$$

where $\sum'$ denotes the sum over all $\tilde{p}, \tilde{q} : I_k \to I_k \cup \{0\}$ such that $\tilde{p}_1+\cdots+\tilde{p}_k = p$, $\tilde{q}_1+\cdots+\tilde{q}_k = q$. Put also $f^{\tilde{p}} = f_1^{\tilde{p}_1}\cdots f_k^{\tilde{p}_k}$, $l^{\tilde{q}} = l_1^{\tilde{q}_1}\cdots l_k^{\tilde{q}_k}$, $|\tilde{p}| = \tilde{p}_1+\cdots+\tilde{p}_k$, $|\tilde{q}| =$

$\tilde{q}_1 + \cdots + \tilde{q}_k$. But one can see that

$$S_r = \sum_{\delta=k-r}^{k-1} \sum_{\substack{\text{all } \tilde{p},\tilde{q} \text{ having} \\ \delta \text{ noughts}}} \frac{p!q!}{\tilde{p}!\tilde{q}!} f^{\tilde{p}} l^{\tilde{q}} \sum_{\substack{\nu: I_r \to I_k \\ I_k - \nu(I_r) \subset \{i: \tilde{p}_i = \tilde{q}_i = 0\}}} 1$$

$$= \sum_{\delta=k-r}^{k-1} \sum_{\substack{\text{all } \tilde{p},\tilde{q} \text{ having} \\ \delta \text{ noughts}}} \binom{\delta}{k-r} \frac{p!q!}{\tilde{p}!\tilde{q}!} f^{\tilde{p}} l^{\tilde{q}}.$$

Thus

$$S = \sum_{r=1}^{k} (-1)^{k-r} S_r$$

$$= \sum_{r=1}^{k} \sum_{\delta=k-r}^{k-1} \sum_{\substack{\text{all } \tilde{p},\tilde{q} \text{ having} \\ \delta \text{ noughts}}} (-1)^{k-r} \binom{\delta}{k-r} \frac{p!q!}{\tilde{p}!\tilde{q}!} f^{\tilde{p}} l^{\tilde{q}}.$$

Now we will change the order of the sums. Let $\delta_{\tilde{p}\tilde{q}} = \text{card}\{i: \tilde{p}_i = \tilde{q}_i = 0\}$. If $\tilde{p}$, $\tilde{q}$ are fixed, then r goes from $k - \delta_{\tilde{p}\tilde{q}}$ to k.

Hence,

$$S = \sum_{\substack{\tilde{p},\tilde{q}: I_k \to I_k \cup \{0\} \\ |\tilde{p}|=p, |\tilde{q}|=q}} \sum_{r=k-\delta_{\tilde{p}\tilde{q}}}^{k} (-1)^{k-r} \binom{\delta_{\tilde{p}\tilde{q}}}{k-r} \frac{p!q!}{\tilde{p}!\tilde{q}!} f^{\tilde{p}} l^{\tilde{q}}$$

$$= \sum_{\substack{\tilde{p},\tilde{q}: I_k \to I_k \cup \{0\} \\ |\tilde{p}|=p, |\tilde{q}|=q}} \frac{p!q!}{\tilde{p}!\tilde{q}!} f^{\tilde{p}} l^{\tilde{q}} \sum_{r=k-\delta_{\tilde{p}\tilde{q}}}^{k} (-1)^{k-r} \binom{\delta_{\tilde{p}\tilde{q}}}{k-r}.$$

Let us denote the last sum by S_0. Then

$$S_0 = \sum_{s=0}^{\delta_{\tilde{p}\tilde{q}}} \binom{\delta_{\tilde{p}\tilde{q}}}{s} (-1)^s = \sum_{s=0}^{\delta_{\tilde{p}\tilde{q}}} \binom{\delta_{\tilde{p}\tilde{q}}}{s} (-1)^s 1^{\delta_{\tilde{p}\tilde{q}}-s} = (1-1)^{\delta_{\tilde{p}\tilde{q}}}.$$

If $\delta_{\tilde{p}\tilde{q}} = 0$, then $S_0 = 1$ and $S_0 = 0$, otherwise. But $\delta_{\tilde{p}\tilde{q}} = 0$ means that $\tilde{p}_i \neq 0$ or $\tilde{q}_i \neq 0$ for all i, so $\tilde{p}_i + \tilde{q}_i \geqslant 1$ for all i. However, if $p_{i_0} + q_{i_0} \geqslant 2$ for any i_0, then $p + q \geqslant k + 1$, but $p + q = k$. Hence $\tilde{p}_i = 1$ or $\tilde{q}_i = 1$ and $\tilde{p}^{-1}(1) \cap \tilde{q}^{-1}(1) = \emptyset$. Since $\tilde{p}! = \tilde{q}! = 1$, if Σ'' is the sum over all $\tilde{p}$, $\tilde{q}: I_k \to \{0, 1\}$ such that $\tilde{p}^{-1}(1) \cup \tilde{q}^{-1}(1) = I_k$, $\tilde{p}^{-1}(1) \cap \tilde{q}^{-1}(1) = \emptyset$, $|\tilde{p}| = p$, $|\tilde{q}| = q$, we have

$$S = \sum\nolimits'' p!q! f^{\tilde{p}} l^{\tilde{q}} = \sum_{\substack{1 \leqslant \nu_1 < \cdots < \nu_p \leqslant k \\ 1 \leqslant \mu_1 < \cdots < \mu_q \leqslant k \\ \nu_i \neq \mu_j \text{ for all } (i,j)}} p!q! f_{\nu_1} \cdots f_{\nu_p} l_{\mu_1} \cdots l_{\mu_q}.$$

Finally,

$$\tilde{\psi}_k((f_1, l_1), \ldots, (f_k, l_k)) = \sum_{\substack{p+q=k \\ p,q \geqslant 0}} \left(\frac{p!q!}{k!} \sum_{\substack{1 \leqslant \nu_1 < \cdots < \nu_p \leqslant k \\ 1 \leqslant \mu_1 < \cdots < \mu_q \leqslant k \\ \nu_i \neq \mu_j \text{ for all } (i,j)}} f_{\nu_1} \cdots f_{\nu_p} l_{\mu_1} \cdots l_{\mu_q} \right) a_{pq}.$$

Hence, using (3) we have

$$\phi_n(x,y) = -\phi_1\left(\sum_{\substack{k=2,\dots,n\\ i_1+\cdots+i_k=n}} \tilde{\psi}_k\left(\sum_{\substack{s_1+t_1=i_1\\ s_1,t_1\geq 0}} \alpha_{s_1t_1}x^{s_1}y^{t_1},\dots,\sum_{\substack{s_k+t_k=i_k\\ s_k,t_k\geq 0}} \alpha_{s_kt_k}x^{s_k}y^{t_k}\right)\right)$$

$$= -\phi_1\left(\sum_{\substack{k=2,\dots,n\\ i_1+\cdots+i_k=n}} \sum_{\substack{p+q=k\\ p,q\geq 0}} \frac{p!q!}{k!}\left(\sum_{\substack{\chi:I_k\to\{1,2\}\\ \operatorname{card}\chi^{-1}(1)=p}}\left(\sum_{\substack{s_1+t_1=i_1\\ s_1,t_1\geq 0}}(\alpha_{s_1t_1})_{\chi(1)}x^{s_1}y^{t_1}\right)\right.\right.$$

$$\left.\left.\times\cdots\times\left(\sum_{\substack{s_k+t_k=i_k\\ s_k,t_k\geq 0}}(\alpha_{s_kt_k})_{\chi(k)}x^{s_k}y^{t_k}\right)\right)a_{pq}\right),$$

where $(\alpha_{s_it_i})_{\chi(i)}$ is the $\chi(i)$ coordinate of $\alpha_{s_it_i}$. Thus

$$\phi_n(x,y) = -\phi_1\left(\sum_{\substack{k=2,\dots,n\\ i_1+\cdots+i_k=n}} \sum_{\substack{p+q=k\\ p,q\geq 0}} \frac{p!q!}{k!}\right.$$

$$\left.\times\left(\sum_{\substack{\chi:I_k\to\{1,2\}\\ \operatorname{card}\chi^{-1}(1)=p}}\sum_{\substack{j=1,\dots,k\\ s_j+t_j=i_j}}(\alpha_{s_1t_1})_{\chi(1)}\cdots(\alpha_{s_kt_k})_{\chi(k)}x^{s_1+\cdots+s_k}y^{t_1+\cdots+t_k}\right)a_{pq}\right)$$

$$= -\phi_1\left(\sum_{\substack{k=2,\dots,n\\ p+q=k}}\frac{p!q!}{k!}\right.$$

$$\left.\times\left(\sum_{\substack{\chi:I_k\to\{1,2\}\\ \operatorname{card}\chi^{-1}(1)=p}}\sum_{s+t=n}\sum_{\substack{s_1+\cdots+s_k=s\\ t_1+\cdots+t_k=t\\ s_i+t_i\geq 1}}(\alpha_{s_1t_1})_{\chi(1)}\cdots(\alpha_{s_kt_k})_{\chi(k)}x^sy^t\right)a_{pq}\right)$$

$$= \sum_{s+t=n} -\phi_1\left(\sum_{\substack{k=2,\dots,n\\ p+q=k}}\frac{p!q!}{k!}\right.$$

$$\left.\times\left(\sum_{\substack{\chi:I_k\to\{1,2\}\\ \operatorname{card}\chi^{-1}(1)=p}}\sum_{\substack{s_1+\cdots+s_k=s\\ t_1+\cdots+t_k=t\\ s_i+t_i\geq 1}}(\alpha_{s_1t_1})_{\chi(1)}\cdots(\alpha_{s_kt_k})_{\chi(k)}\right)a_{pq}\right)x^sy^t.$$

Comparing the above with (2) we obtain (1) and the proof has been finished. □

Remark 4. For the sake of simplicity of notation, Theorem 1 concerned a power series of two variables. However, it can be easily generalized to a power series of n variables.

References

[1] H. Cartan, *Calcul Différentiel. Form Différentielles* (Hermann, Paris, 1967).
[2] G.M. Fichtengolc, *Lectures on Differential and Integral Calculus* (Nauka, Moscow, 1969, in Russian).
[3] H. Knopp, *Theorie und Anwendung der Unendlichen Reihen* (Springer, Berlin, 1922).
[4] E.M. Reingold, J. Nievergelt and N. Deo, *Combinatorial Algorithms. Theory and Practice* (Prentice-Hall, Englewood Cliffs, NJ, 1977).