Global existence and uniqueness of Schrödinger maps in dimensions \( d \geq 4 \)✩

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Abstract

In dimensions \( d \geq 4 \), we prove that the Schrödinger map initial-value problem

\[
\begin{aligned}
\partial_t s &= s \times \Delta s \quad \text{on } \mathbb{R}^d \times \mathbb{R}; \\
s(0) &= s_0
\end{aligned}
\]

admits a unique solution \( s : \mathbb{R}^d \times \mathbb{R} \to \mathbb{S}^2 \leftrightarrow \mathbb{R}^3 \), \( s \in C(\mathbb{R} : H^\infty_Q) \), provided that \( s_0 \in H^\infty_Q \) and \( \| s_0 - Q \|_{H^{d/2}} \ll 1 \), where \( Q \in \mathbb{S}^2 \).

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1. Introduction

In this paper we consider the Schrödinger map initial-value problem

\[
\begin{aligned}
\partial_t s &= s \times \Delta s & \text{on } \mathbb{R}^d \times \mathbb{R}; \\
s(0) &= s_0,
\end{aligned}
\]

where \(d \geq 4\) and \(s : \mathbb{R}^d \times \mathbb{R} \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3\) is a continuous function. The Schrödinger map equation has a rich geometric structure and arises naturally in a number of different ways; we refer the reader to [19] for details.

For \(\sigma \geq 0\) and \(n \in \{1, 2, \ldots\}\) let \(H^\sigma = H^\sigma(\mathbb{R}^d; \mathbb{C}^n)\) denote the Banach spaces of \(\mathbb{C}^n\)-valued Sobolev functions on \(\mathbb{R}^d\), i.e.

\[
H^\sigma = \left\{ f : \mathbb{R}^d \to \mathbb{C}^n ; \| f \|_{H^\sigma} = \left[ \sum_{l=1}^n \| \mathcal{F}(d)(f_l) \cdot (|\xi|^2 + 1)^{\sigma/2} \|_{L^2}^2 \right]^{1/2} < \infty \right\},
\]

where \(\mathcal{F}(d)\) denotes the Fourier transform on \(L^2(\mathbb{R}^d)\). For \(\sigma \geq 0\), \(n \in \{1, 2, \ldots\}\), and \(f \in H^\sigma(\mathbb{R}^d; \mathbb{C}^n)\), we define

\[
\| f \|_{H^\sigma} = \left[ \sum_{l=1}^n \| \mathcal{F}(d)(f_l)(\xi) \cdot |\xi|^{\sigma} \|_{L^2}^2 \right]^{1/2}.
\]

For \(\sigma \geq 0\) and \(Q = (Q_1, Q_2, Q_3) \in \mathbb{S}^2\) we define the complete metric space

\[
H^\sigma_Q = H^\sigma_Q(\mathbb{R}^d; \mathbb{S}^2 \hookrightarrow \mathbb{R}^3) = \left\{ f : \mathbb{R}^d \to \mathbb{R}^3 ; \| f(x) \| = 1 \text{ and } f - Q \in H^\sigma \right\},
\]

with the induced distance

\[
d^\sigma_Q(f, g) = \| f - g \|_{H^\sigma}.
\]
For simplicity of notation, we let \( \| f \|_{H^\sigma_Q} = d^\sigma_Q(f, Q) \) for \( f \in H^\sigma_Q \). Let \( \mathbb{Z}_+ = \{0, 1, \ldots\} \). For \( n \in \{1, 2, \ldots\} \) and \( Q \in \mathbb{S}^2 \) we define the complete metric spaces

\[
H^\infty = H^\infty(\mathbb{R}^d; \mathbb{C}^n) = \bigcap_{\sigma \in \mathbb{Z}_+} H^\sigma \quad \text{and} \quad H^\infty_Q = \bigcap_{\sigma \in \mathbb{Z}_+} H^\sigma_Q,
\]

with the induced distances. Our main theorem concerns global existence and uniqueness of solutions of the initial-value problem (1.1) for data \( s_0 \in H^\infty_Q \), with \( \|s_0 - Q\|_{\dot{H}^{d/2}} \ll 1 \).

**Theorem 1.1.** Assume \( d \geq 4 \) and \( Q \in \mathbb{S}^2 \). Then there is \( \varepsilon_0 = \varepsilon_0(d) > 0 \) such that for any \( s_0 \in H^\infty_Q \) with \( \|s_0 - Q\|_{\dot{H}^{d/2}} \leq \varepsilon_0 \) there is a unique solution

\[
s = S_Q(s_0) \in C(\mathbb{R} : H^\infty_Q)
\]

of the initial-value problem (1.1). Moreover

\[
\sup_{t \in \mathbb{R}} \|s(t) - Q\|_{\dot{H}^{d/2}} \leq C \|s_0 - Q\|_{\dot{H}^{d/2}}, \tag{1.5}
\]

and

\[
\sup_{t \in [-T,T]} \|s(t)\|_{H^\sigma_Q} \leq C(\sigma, T, \|s_0\|_{H^\sigma_Q}) \tag{1.6}
\]

for any \( T \in [0, \infty) \) and \( \sigma \in \mathbb{Z}_+ \).

**Remark.** We prove in fact a slightly stronger statement: there is \( \sigma_0 \in [d/2, \infty) \cap \mathbb{Z} \) sufficiently large such that for any \( s_0 \in H^\sigma_Q \) with \( \|s_0 - Q\|_{\dot{H}^{d/2}} \leq \varepsilon_0 \) there is a unique solution

\[
s = S_Q(s_0) \in C(\mathbb{R} : H^{\sigma_0-1}_Q) \cap L^\infty(\mathbb{R} : H^{\sigma_0}_Q)
\]

of the initial-value problem (1.1). Moreover, the bounds (1.5) and (1.6) (assuming \( s_0 \in H^\sigma_Q \), \( \sigma \in \mathbb{Z}_+ \)) still hold.

The main point of Theorem 1.1 is the global (in time) existence of solutions. Its direct analogue in the setting of wave maps is the work of Tao [23] (see also [13–15,18,21,24–27] for other local and global existence (or well-posedness) theorems for wave maps). However, our proof of Theorem 1.1 is closer to that of [18,21].

The initial-value problem (1.1) has been studied extensively (also in the case in which the sphere \( \mathbb{S}^2 \) is replaced by more general targets). It is known that sufficiently smooth solutions exist locally in time, even for large data (see, for example, [3,5,16,22] and the references therein). Such theorems for (local in time) smooth solutions are proved using delicate geometric variants of the energy method. For low-regularity data, the initial-value problem (1.1) has been studied indirectly using the “modified Schrödinger map equations” (see, for example, [3,9–11,17,19,20]) and certain enhanced energy methods.

In [8], Ionescu–Kenig realized that the initial-value problem (1.1) can be analyzed perturbatively using the stereographic model, in the case of “small data” (i.e. data that takes values in a small neighborhood of a point on the sphere), and proved local well posedness for small data.
in $H^\sigma_Q$, $\sigma > (d + 1)/2$, $d \geq 2$. The resolution spaces constructed in [8] (see also [6] for the 1-dimensional version of these spaces) are based on directional $L^{p,q}_e$ physical spaces, which are related to local smoothing; in particular, the nonlinear analysis is based on local smoothing and the simple inclusion

$$L_e^{\infty,2} \cdot L_e^{2,\infty} \cdot L_e^{2,\infty} \subseteq L_e^{1,2}.$$ 

We use the same resolution spaces and this simple inclusion in the perturbative analysis in Section 3 in this paper.

Slightly later and independently, Bejenaru [1] also realized that the stereographic model can be used for perturbative analysis, and proved local well posedness for small data in $H^\sigma$, in the full subcritical range $\sigma > d/2$, $d \geq 2$. In the stereographic model Bejenaru observed, apparently for the first time in the setting of Schrödinger maps, that the gradient part of the nonlinearity has a certain null structure (similar to the null structure of wave maps, observed by S. Klainerman).\(^1\)

The resolution spaces used in [1] for the perturbative argument are different from those of [8]; these resolution spaces are based on the construction of suitably normalized wave packets, and had been previously used by Bejenaru in other subcritical problems (see [2] and the references therein).

In [7] Ionescu–Kenig proved the first global (in time) well-posedness theorem for small data in the critical Besov spaces $\dot{B}^{d/2}_Q$, in dimensions $d \geq 3$, using certain technical modifications of the resolution spaces of [8] and the null structure observed in [1]. As explained in [7], the main difficulty in proving this result in dimension $d = 2$ is the logarithmic failure of the scale-invariant $L^{2,\infty}_e$ estimate.

Unlike its Besov analogue, the condition $\|s_0 - Q\|_{\dot{H}^{d/2}_Q} \ll 1$ in Theorem 1.1 does not guarantee that the data $s_0$ takes values in a small neighborhood of $Q$. Because of this, the stereographic model used in [1,7,8] is not relevant, and it does not appear possible to prove Theorem 1.1 using a direct perturbative construction. We construct the solution $s$ indirectly, using a priori estimates: we start with a solution $s \in C([-T, T] : H^\infty_Q)$ of (1.1), where $T = T(\|s_0\|_{H^\sigma_Q}) > 0$, $\sigma_0$ sufficiently large, and transfer the quantitative bounds on the function $s$ at time 0 to suitable quantitative bounds on the functions $\psi_m$ at time 0 (the functions $\psi_m$ are solutions of the modified Schrödinger map equations, see Section 2). Then we study the modified Schrödinger map equations perturbatively, and prove uniform quantitative bounds on the functions $\psi_m$ at all times $t \in [-T, T]$. Finally, we transfer these bounds back to the solution $s$; this gives uniform quantitative bounds on $s$ at all times $t \in [-T, T]$, which allow us to extend the solution $s$ up to time $T = 1$. By scaling, we can construct a global solution.

The rest of the paper is organized as follows: in Section 2 we explain how to derive the modified Schrödinger map equations (MSM),\(^2\) and prove quantitative bounds on the solutions $\psi_m$ of the MSM at time $t = 0$. In Section 3 we use a perturbative argument and the resolution spaces defined in [8] (and some of their properties) to prove bounds on the solutions $\psi_m$ of the MSM on the time interval $[-T, T]$. The proofs of some of the technical nonlinear bounds are deferred to Section 5. In Section 4 we transfer the bounds on $\psi_m$ to a priori bounds on solution $s$ of (1.1), and use a local existence theorem to close the argument.

---

\(^1\) This null structure was not observed in the earlier paper of Ionescu–Kenig [8]; without this null structure the restriction $\sigma > (d + 1)/2$ in [8] is necessary for the perturbative argument.

\(^2\) The MSM were first derived in [3], using orthonormal frames, and [19], using the stereographic projection.
We will always assume in the rest of the paper that \( d \geq 3 \) (we have not constructed yet suitable resolution spaces in dimension \( d = 2 \)). In Section 3.3 and Sections 4 and 5 we assume the stronger restriction \( d \geq 4 \); the reason for this restriction is mostly technical, as it leads to simple proofs of the nonlinear estimates in Lemma 3.5. In many estimates, we will use the letter \( C \) to denote constants that may depend only on the dimension \( d \).

2. The modified Schrödinger map

In this section we give a self-contained derivation of the modified Schrödinger map equations, using orthonormal frames.\(^3\) In the context of wave maps, orthonormal frames have been used in [4,14,18,21] etc. In the context of Schrödinger maps, orthonormal frames (on the pullback of \( T^*M \) under the solution \( s \)) have been used for the first time in [3] to construct the modified Schrödinger map equations. See also [16]. Complete expositions of this construction have been presented by J. Shatah on several occasions.

In this section we assume \( d \geq 3 \) (some technical changes are needed in dimension \( d = 2 \), but we will not discuss them here).

2.1. A topological construction

Assume \( n \in [1, \infty) \cap \mathbb{Z}, a_1, \ldots, a_n \in [0, \infty) \), and let

\[
D^n = [-a_1, a_1] \times \cdots \times [-a_n, a_n].
\]

For \( n = 0 \) let \( D^0 = \{0\} \).

Lemma 2.1. Assume \( n \geq 0 \) and \( s : D^n \rightarrow S^2 \) is a continuous function. Then there is a continuous function \( v : D^n \rightarrow S^2 \) with the property that

\[
s(x) \cdot v(x) = 0 \quad \text{for any } x \in D^n.
\]

Proof. We argue by induction over \( n \) (the case \( n = 0 \) is trivial). Since \( s \) is continuous, there is \( \epsilon > 0 \) with the property that

\[
|s(x) - s(y)| \leq 2^{-10} \quad \text{for any } x, y \in D^n \text{ with } |x - y| \leq \epsilon.
\]

For \( x \in D^n \) we write \( x = (x', x_n) \in D^{n-1} \times [-a_n, a_n] \). For any \( b \in [-a_n, a_n] \) let \( D^n_b = D^{n-1} \times [-a_n, b] = \{x = (x', x_n) \in D^n : x_n \in [-a_n, b]\} \). By the induction hypothesis, we can define \( v : D^n_{-a_n} \rightarrow S^2 \) continuous such that

\[
s(x) \cdot v(x) = 0 \quad \text{for any } x \in D^n_{-a_n}.
\]

We extend now the function \( v \) to \( D^n \). With \( \epsilon \) as in (2.1), it suffices to prove that if \( b, b' \in [-a_n, a_n], 0 \leq b' - b \leq \epsilon, v : D^n_b \rightarrow S^2 \) is continuous, and \( s(x) \cdot v(x) = 0 \) for any \( x \in D^n_b \), then \( v \) can be extended to a continuous function \( \tilde{v} : D^n_{b'} \rightarrow S^2 \) such that \( s(x) \cdot \tilde{v}(x) = 0 \) for any \( x \in D^n_{b'} \).

\(^3\) This elementary construction was suggested to us by T. Tao.
Let
\[ \mathcal{R} = \{(u_1, u_2) \in \mathbb{R}^3 \times \mathbb{R}^3 : |u_1|, |u_2| \in (1/2, 2) \text{ and } |u_1 \cdot u_2| < 2^{-5}\}, \] (2.2)
and let \( N : \mathcal{R} \to S^2 \) denote the smooth function
\[ N[u_1, u_2] = \frac{u_1 - ((u_1 \cdot u_2)/|u_2|^2) u_2}{|u_1 - ((u_1 \cdot u_2)/|u_2|^2) u_2|}. \] (2.3)
So \( N[u_1, u_2] \) is a unit vector orthogonal to \( u_2 \) in the plane generated by the vectors \( u_1 \) and \( u_2 \).

We construct now the extension \( \tilde{v} : D_{i,n}^2 \to S^2 \). For \( x' \in D_{i,n}^{n-1} \) and \( x_n \in [-a_n, b'] \) let
\[ \tilde{v}(x', x_n) = \begin{cases} 
N[v(x', b), s(x', x_n)] & \text{if } x_n \in [b, b'] ; \\
v(x', x_n) & \text{if } x_n \in [-a_n, b]. 
\end{cases} \]
In view of (2.1), the function \( \tilde{v} : D_{i,n}^2 \to S^2 \) is well defined, continuous, and \( s(x) \cdot \tilde{v}(x) = 0 \) for any \( x \in D_{i,n}^2 \). This completes the proof of Lemma 2.1.

**Lemma 2.2.** Assume \( T \in [0, 2] \), \( Q, Q' \in S^2 \), \( Q \cdot Q' = 0 \), and \( s : \mathbb{R}^d \times [-T, T] \to S^2 \) is a continuous function with the property that
\[ \lim_{x \to \infty} s(x, t) = Q \quad \text{uniformly in } t \in [-T, T]. \]
Then there is a continuous function \( v : \mathbb{R}^d \times [-T, T] \to S^2 \) with the property that
\[ \left\{ \begin{array}{l}
\lim_{x \to \infty} s(x, t) = 0 \\
\lim_{x \to \infty} v(x, t) = Q'
\end{array} \right. \quad \text{for any } (x, t) \in \mathbb{R}^d \times [-T, T]. \]

**Proof.** We fix \( R > 0 \) such that
\[ |s(x, t) - Q| \leq 2^{-10} \quad \text{if } |x| \geq R \text{ and } t \in [-T, T]. \]
Using Lemma 2.1, we can define a continuous function \( v_0 : B_R \times [-T, T] \to S^2 \) such that \( s(x, t) \cdot v_0(x, t) = 0 \) for \( (x, t) \in B_R \times [-T, T] \), where \( B_R = \{ x \in \mathbb{R}^d : |x| \leq R \} \). Let \( S_R = \{ x \in \mathbb{R}^d : |y| = R \} \) and \( S^1_Q = \{ x \in S^2 : x \cdot Q = 0 \} \). We define the continuous function
\[ w : S_R \times [-T, T] \to S^1_Q, \qquad w(y, t) = \frac{(s(y, t) \cdot Q)v_0(y, t) - (v_0(y, t) \cdot Q)s(y, t)}{|(s(y, t) \cdot Q)v_0(y, t) - (v_0(y, t) \cdot Q)s(y, t)|}, \]
so \( w(y, t) \) is a vector in \( S^1_Q \) and in the plane generated by \( s(y, t) \) and \( v_0(y, t) \). Since \( d \geq 3 \), the space \( S_R \times [-T, T] \) is simply connected (and compact), thus the function \( w \) is homotopic to a constant function. Thus there is a continuous function
\[ \tilde{w} : S_R \times [-T, T] \times [1, 2] \to S^1_Q \quad \text{such that } \tilde{w}(y, t, 1) = w(y, t) \text{ and } \tilde{w}(y, t, 2) = Q'. \]
With \( N \) as in (2.3), we define
\[ v_1(x, t) = N[\tilde{w}(Rx/|x|, t, |x|/R), s(x, t)] \]
for $|x| \in [R, 2R]$, and

$$v_2(x, t) = N[Q', s(x, t)]$$

for $|x| \geq 2R$. The function $v$ in Lemma 2.2 is obtained by gluing the functions $v_0$, $v_1$, and $v_2$. \qed

2.2. Derivation of the modified Schrödinger map equations

Assume now that $T \in [0, 1]$, $Q, Q' \in S^2$, and $Q \cdot Q' = 0$. Assume that

$$\left\{ \begin{array}{l}
s \in C([-T, T] : H^\infty_Q) ; \\
\partial t s \in C([-T, T] : H^\infty) .
\end{array} \right. \tag{2.4}$$

We extend the function $s$ to a function $\tilde{s} \in C([-T - 1, T + 1] : H^\infty_Q)$ by setting $\tilde{s}(., t) = s(., T)$ if $t \in [T, T + 1]$ and $\tilde{s}(., t) = s(., -T)$ if $t \in [-T - 1, -T]$. Clearly, the function $\tilde{s} : \mathbb{R}^d \times [-T - 1, T + 1] \to S^2$ is continuous and $\lim_{x \to \infty} \tilde{s}(x, t) = Q$ uniformly in $t$. We apply Lemma 2.2 to construct a continuous function $\tilde{v} : \mathbb{R}^d \times [-T - 1/2, T + 1/2] \to S^2$ such that $\tilde{s} \cdot \tilde{v} \equiv 0$ and $\lim_{x \to \infty} \tilde{v}(x, t) = Q'$ uniformly in $t$.

We regularize now the function $\tilde{v}$. Let $\varphi : \mathbb{R}^d \times \mathbb{R} \to [0, \infty)$ denote a smooth function supported in the ball $\{(x, t) : |x|^2 + t^2 \leq 1\}$ with $\int_{\mathbb{R}^d \times \mathbb{R}} \varphi \, dx \, dt = 1$. Since $\tilde{v}$ is a uniformly continuous function, there is $\epsilon = \epsilon(\tilde{v})$ with the property that

$$|\tilde{v}(x, t) - (\tilde{v} \ast \varphi_\epsilon)(x, t)| \leq 2^{-20}$$

for any $(x, t) \in \mathbb{R}^d \times [-T - 1/2, T + 1/2]$, where $\varphi_\epsilon(x, t) = \epsilon^{-d-1} \varphi(x/\epsilon, t/\epsilon)$. Using a partition of 1, we replace smoothly $(\tilde{v} \ast \varphi_\epsilon)(x, t)$ with $Q'$ for $|x|$ large enough. Thus we have constructed a smooth function $v' : \mathbb{R}^d \times (-T - 1/2, T + 1/2) \to \mathbb{R}^3$ with the properties

$$\left\{ \begin{array}{l}
|v'(x, t)| \in [1 - 2^{-10}, 1 + 2^{-10}] \quad \text{for any } (x, t) \in \mathbb{R}^d \times [-T, T] ; \\
|v'(x, t) \cdot s(x, t)| \leq 2^{-10} \quad \text{for any } (x, t) \in \mathbb{R}^d \times [-T, T] ; \\
v'(x, t) = Q' \quad \text{for } |x| \text{ large enough and } t \in [-T, T].
\end{array} \right. \tag{2.5}$$

With $N$ as in (2.3), we define

$$v(x, t) = N[v'(x, t), s(x, t)].$$

In view of (2.5), the continuous function $v : \mathbb{R}^d \times [-T, T] \to S^2$ is well defined, $s(x, t) \cdot v(x, t) \equiv 0$, and

$$\left\{ \begin{array}{l}
\partial_m v \in C([-T, T] : H^\infty) \quad \text{for } m = 1, \ldots, d; \\
\partial t v \in C([-T, T] : H^\infty).
\end{array} \right. \tag{2.6}$$

Given $s$ as in (2.4) and $v$ as in (2.6), we define

$$w(x, t) = s(x, t) \times v(x, t).$$
Since $H^\sigma$ is an algebra for $\sigma > d/2$, we have
\[
\begin{cases}
\partial_m w \in C([-T, T] : H^\infty) & \text{for } m = 1, \ldots, d; \\
\partial_t w \in C([-T, T] : H^\infty).
\end{cases}
\]
(2.7)

To summarize, given a function $s$ as in (2.4) we have constructed continuous functions $v, w : \mathbb{R}^d \times [-T, T] \to \mathbb{S}^2$ such that $s \cdot v = s \cdot w = v \cdot w \equiv 0$, and (2.6) and (2.7) hold.

We use now the functions $v$ and $w$ to construct a suitable Coulomb gauge. Let
\[
A_m = (\partial_m v) \cdot w = -(\partial_m w) \cdot v \quad \text{for } m = 1, \ldots, d.
\]
Clearly, the functions $A_m$ are real-valued,
\[
A_m \in C([-T, T] : H^\infty) \quad \text{and} \quad \partial_t A_m \in C([-T, T] : H^\infty).
\]
(2.8)

We would like to modify the functions $v$ and $w$ such that $\sum_{m=1}^d \partial_m A_m \equiv 0$. Let
\[
\begin{cases}
v' = (\cos \chi) v + (\sin \chi) w; \\
w' = (-\sin \chi) v + (\cos \chi) w,
\end{cases}
\]
for some function $\chi : \mathbb{R}^d \times [-T, T] \to \mathbb{R}$ to be determined. Then, using the orthonormality of $v$ and $w$ (which gives $\partial_m v \cdot v = \partial_m w \cdot w \equiv 0$),
\[
A'_m = (\partial_m v') \cdot w' = A_m + \partial_m \chi.
\]

The condition $\sum_{m=1}^d \partial_m A'_m \equiv 0$ gives
\[
\Delta \chi = -\sum_{m=1}^d \partial_m A_m.
\]

Thus we define $\chi$ by the formula
\[
\chi(x, t) = c \int_{\mathbb{R}^d} e^{i x \cdot \xi} |\xi|^{-2} \sum_{m=1}^d (i \xi_m) \mathcal{F}(d)(A_m)(\xi, t) d\xi.
\]

The integral defining the function $\chi$ converges absolutely since $A_m \in C([-T, T] : H^\infty)$ and $d \geq 3$. Using (2.8), it follows that $\chi : \mathbb{R}^d \times [-T, T] \to \mathbb{R}$ is a bounded, continuous function, $\partial_m \chi \in C([-T, T] : H^\infty)$ and $\partial_t \chi \in C([-T, T] : H^\infty)$. To summarize, we proved the following proposition:

Proposition 2.3. Assume $T \in [0, 1]$, $Q \in \mathbb{S}^2$, and
\[
\begin{cases}
s \in C([-T, T] : H^\infty_Q) ; \\
\partial_t s \in C([-T, T] : H^\infty).
\end{cases}
\]
(2.9)
Then there are continuous functions \( v, w : \mathbb{R}^d \times [-T, T] \to S^2, s \cdot v = 0, w = s \times v \), such that
\[
\partial_m v, \partial_m w \in C([-T, T] : H^\infty) \quad \text{for} \quad m = 0, 1, \ldots, d,
\] (2.10)
where \( \partial_0 = \partial_t \). In addition,
\[
\text{if} \quad A_m = (\partial_m v) \cdot w \quad \text{for} \quad m = 1, \ldots, d, \quad \text{then} \quad \sum_{j=1}^d \partial_j A_m \equiv 0.
\] (2.11)

Assume now that \( s, v, w \) are as in Proposition 2.3. In addition to the functions \( A_m \), we define the continuous functions \( \psi_m : \mathbb{R}^d \times [-T, T] \to \mathbb{C} \), \( m = 1, \ldots, d \),
\[
\psi_m = (\partial_m s) \cdot v + i(\partial_m s) \cdot w.
\] (2.12)
Let \( \partial_0 = \partial_t \). We also define the continuous functions \( A_0 : \mathbb{R}^d \times [-T, T] \to \mathbb{R} \) and \( \psi_0 : \mathbb{R}^d \times [-T, T] \to \mathbb{C} \),
\[
\begin{cases}
\psi_0 = (\partial_0 s) \cdot v + i(\partial_0 s) \cdot w; \\
A_0 = (\partial_0 v) \cdot w = -(\partial_0 w) \cdot v.
\end{cases}
\] (2.13)
Clearly, \( \psi_m, A_m \in C([-T, T] : H^\infty) \) for \( m = 0, 1, \ldots, d \), and \( \partial_t \psi_m, \partial_t A_m \in C([-T, T] : H^\infty) \) for \( m = 1, \ldots, d \). In view of the orthonormality of \( s, v, w \), for \( m = 0, 1, \ldots, d \)
\[
\begin{cases}
\partial_m s = \Re(\psi_m) v + \Im(\psi_m) w; \\
\partial_m v = -\Re(\psi_m) s + A_m w; \\
\partial_m w = -\Im(\psi_m) s - A_m v.
\end{cases}
\] (2.14)
A direct computation using the orthonormality of \( s, v, w \) gives
\[
(\partial_l + i A_l) \psi_m = (\partial_m + i A_m) \psi_l \quad \text{for any} \quad m, l = 0, 1, \ldots, d.
\] (2.15)
A direct computation also shows that
\[
\partial_l A_m - \partial_m A_l = \Im(\psi_l \overline{\psi}_m) \quad \text{for any} \quad m, l = 0, 1, \ldots, d.
\] (2.16)
We combine these identities with the Coulomb gauge condition \( \sum_{m=1}^d \partial_m A_m \equiv 0 \) and solve the div-curl system for each \( t \) fixed. The result is
\[
\Delta A_m = -\sum_{l=1}^d \partial_l \Im(\psi_m \overline{\psi}_l) \quad \text{for} \quad m = 1, \ldots, d.
\] (2.17)
Thus, using (2.17), for \( m = 1, \ldots, d \),
\[
A_m = \nabla^{-1} \left[ \sum_{l=1}^d R_l \Im(\psi_m \overline{\psi}_l) \right],
\] (2.18)
where \( R_l \) denotes the Riesz transform defined by the Fourier multiplier \( \xi \to i\xi_l/|\xi| \) and \( \nabla^{-1} \) is the operator defined by the Fourier multiplier \( \xi \to |\xi|^{-1} \).

Assume now that the function \( s \) satisfies the identity
\[
\partial_t s = s \times \Delta s \quad \text{on } \mathbb{R}^d \times [-T, T],
\]
(2.19)
in addition to (2.9). For \( m = 0, 1, \ldots, d \) we define the covariant derivatives \( D_m = \partial_m + iA_m \).

Using the definition,
\[
\psi_0 = (s \times \Delta s) \cdot v + i(s \times \Delta s) \cdot w.
\]
In addition, using (2.14),
\[
\partial_t^2 s = \left( \partial_m \Re(\psi_m) - A_m \cdot \Im(\psi_m) \right) v + \left( \partial_m \Im(\psi_m) + A_m \cdot \Re(\psi_m) \right) w - |\psi_m|^2 s.
\]
Thus, using \( s \times v = w, s \times w = -v \),
\[
\psi_0 = -\sum_{m=1}^d \left( \partial_m \Im(\psi_m) + A_m \cdot \Re(\psi_m) \right) + i \sum_{m=1}^d \left( \partial_m \Re(\psi_m) - A_m \cdot \Im(\psi_m) \right)
\]
\[
= i \sum_{m=1}^d D_m \psi_m.
\]
(2.20)

We use now (2.15) and (2.16) to convert (2.20) into a nonlinear Schrödinger equation. We rewrite the identities (2.15) and (2.16) in the form
\[
\begin{align*}
D_l \psi_m &= D_m \psi_l \quad \text{for any } m, l = 0, 1, \ldots, d; \\
D_l D_m f - D_m D_l f &= i \Im(\psi_l \overline{\psi_m}) f \quad \text{for any } m, l = 0, 1, \ldots, d.
\end{align*}
\]
Thus, using (2.20), for \( m = 1, \ldots, d \),
\[
D_0 \psi_m = D_m \psi_0 = i \sum_{l=1}^d D_m D_l \psi_l = i \sum_{l=1}^d D_l D_m \psi_l - \sum_{l=1}^d \Im(\psi_m \overline{\psi_l}) \psi_l
\]
\[
= i \sum_{l=1}^d D_l D_l \psi_m - \sum_{l=1}^d \Im(\psi_m \overline{\psi_l}) \psi_l.
\]
Thus, using again (2.11), for \( m = 1, \ldots, d \),
\[
(i \partial_t + \Delta_x) \psi_m = -2i \sum_{l=1}^d A_l \cdot \partial_l \psi_m + \left( A_0 + \sum_{l=1}^d A_l^2 \right) \psi_m - i \sum_{l=1}^d \Im(\psi_m \overline{\psi_l}) \psi_l.
\]
(2.21)

We find now the coefficient \( A_0 \). Using (2.16) and (2.11),
\[
\Delta A_0 = \sum_{l=1}^d \partial_l (\partial_0 A_l + \Im(\psi_l \overline{\psi_0})) = \sum_{l=1}^d \partial_l \Im(\psi_l \overline{\psi_0}).
\]
(2.22)
Using (2.20), (2.15) and the identity \( \overline{\psi} \cdot Dm \psi_m = \partial_m (\overline{\psi} \psi_m) - \psi_m \cdot \overline{Dm} \psi_1 \),

\[
\Im(\psi_l \bar{\psi}_0) = -d \sum_{m=1}^d \Re(\psi_l \cdot Dm \psi_m) = -d \sum_{m=1}^d \partial_m \Re(\overline{\psi} \psi_m) + d \sum_{m=1}^d \Re(\psi_m \cdot Dm \overline{\psi}_1)
\]

\[
= -d \sum_{m=1}^d \partial_m \Re(\overline{\psi} \psi_m) + \frac{1}{2} \partial_l \left( \sum_{m=1}^d \psi_m \overline{\psi}_m \right).
\]

It follows from (2.22) that

\[
\Delta A_0 = -d \sum_{m,l=1}^d \partial_l \partial_m \Re(\overline{\psi} \psi_m) + \frac{1}{2} \Delta \left( \sum_{m=1}^d \psi_m \overline{\psi}_m \right).
\]

Thus

\[
A_0 = \sum_{m,l=1}^d R_l R_m \left( \Re(\overline{\psi} \psi_m) \right) + \frac{1}{2} \sum_{m=1}^d \psi_m \overline{\psi}_m.
\]

(2.23)

**Proposition 2.4.** Assume \( s, v, w, \) and \( A_m, m = 1, \ldots, d, \) are as in Proposition 2.3. Assume in addition that the function \( s \) satisfies the identity

\[
\partial_t s = s \times \Delta s \quad \text{on} \quad \mathbb{R}^d \times [-T, T].
\]

For \( m = 1, \ldots, d \) let

\[
\psi_m = (\partial_m s) \cdot v + i (\partial_m s) \cdot w \quad \text{on} \quad \mathbb{R}^d \times [-T, T].
\]

(2.24)

Then \( \psi_m, A_m, \partial_t \psi_m, \partial_t A_m \in C([-T, T] : H^\infty) \) and

\[
\begin{cases}
(\partial_t + i A_l) \psi_m = (\partial_m + i A_m) \psi_l & \text{for any } m, l = 1, \ldots, d; \\
A_m = \nabla^{-1} [\sum_{l=1}^d R_l (\Im(\psi_l \overline{\psi}_m))] & \text{for any } m = 1, \ldots, d,
\end{cases}
\]

(2.25)

where \( R_l \) denotes the Riesz transform defined by the Fourier multiplier \( \xi \to i \xi_l / |\xi| \) and \( \nabla^{-1} \) is the operator defined by the Fourier multiplier \( \xi \to |\xi|^{-1} \). In addition, the functions \( \psi_m \) satisfy the system of nonlinear Schrödinger equations

\[
(i \partial_t + \Delta_x) \psi_m = -2i \sum_{l=1}^d A_l \cdot \partial_t \psi_m + \left( A_0 + \sum_{l=1}^d A_l^2 \right) \psi_m + i \sum_{l=1}^d \Im(\psi_l \overline{\psi}_m) \psi_l,
\]

(2.26)

for \( m = 1, \ldots, d, \) where

\[
A_0 = \sum_{l,l'} R_l R_{l'} \left( \Re(\overline{\psi} \psi_m) \right) + \frac{1}{2} \sum_{m=1}^d \psi_m \overline{\psi}_m.
\]

(2.27)
2.3. A quantitative estimate

We prove now quantitative estimates for the functions $\psi_m$.

**Lemma 2.5.** With the notation in Propositions 2.3 and 2.4, if the function $s_0(x) = s(x, 0)$ has the additional property $\|s_0 - Q\|_{H^{d/2}} \leq 1$ and $\sigma_0 = d + 10$, then for $m = 1, \ldots, d$,

\[
\begin{align*}
\|\psi_m(\cdot, 0)\|_{H^{(d-2)/2}} & \leq C \cdot \|s_0 - Q\|_{H^{d/2}}; \\
\|\psi_m(\cdot, 0)\|_{H^{\sigma'-1}} & \leq C(\|s_0\|_{H_{Q'}}) \quad \text{for any } \sigma' \in [1, \sigma_0] \cap \mathbb{Z}.
\end{align*}
\]  

(2.28)

**Proof.** The main difficulty is that our construction does not give effective control of the Sobolev norms of $v$ and $w$ in terms of the norms of $s$. We argue indirectly, using a bootstrap argument and the identities (2.14), (2.24), and (2.25). For $\sigma \in [-1, \infty)$ let $\nabla^\sigma$ denote the operator (acting on functions in $H^\infty$) defined by the Fourier multiplier $\xi \mapsto |\xi|^\sigma$. For $\sigma \in [-1/2, d/2]$ let $p_\sigma = d/(\sigma + 1)$. Then, in view of the Sobolev imbedding theorem (recall $d \geq 3$),

\[
\|\nabla^\sigma f\|_{L^{p_\sigma}} \leq C \|\nabla^\sigma' f\|_{L^{p_\sigma'}} \quad \text{if } -1/2 \leq \sigma \leq \sigma' \leq d/2 \text{ and } f \in H^\infty.  
\]  

(2.29)

Let $s_0(x) = s(x, 0)$, $v_0(x) = v(x, 0)$, $w_0(x) = w(x, 0)$, $\psi_{m,0}(x) = \psi_m(x, 0)$, and $A_{m,0}(x) = A_m(x, 0)$, and let $\epsilon_0 = \|s_0 - Q\|_{H^{d/2}} \leq 1$. To start our bootstrap argument, we use (2.24), (2.29) and the fact that $|v_0| = |w_0| = 1$ to obtain

\[
\|\psi_{m,0}\|_{L^{p_0}} \leq C \epsilon_0 \quad \text{for } m = 1, \ldots, d.
\]

Then, using (2.25),

\[
\|\nabla^1 A_{m,0}\|_{L^{p_1}} \leq C \epsilon_0 \quad \text{for } m = 1, \ldots, d.
\]

Thus, using (2.29), $\|A_{m,0}\|_{L^{p_0}} \leq C \epsilon_0$ for $m = 1, \ldots, d$. We use now the identity (2.14) and the fact that for $f \in H^\infty$

\[
\|\nabla^n f\|_{L^p} \approx \sum_{n_1 + \cdots + n_d = n} \|\partial_1^{n_1} \cdots \partial_d^{n_d} f\|_{L^p} \quad \text{if } n \in \mathbb{Z}_+ \text{ and } p \in [p_d/2, p_{-1/2}].
\]  

(2.30)

Thus

\[
\|\nabla^1 v_0\|_{L^{p_0}} + \|\nabla^1 w_0\|_{L^{p_0}} \leq C \epsilon_0.
\]

Therefore

\[
\sum_{m=1}^d \|\psi_{m,0}\|_{L^{p_0}} + \sum_{m=1}^d \|\nabla^1 A_{m,0}\|_{L^{p_1}} + \|\nabla^1 v_0\|_{L^{p_0}} + \|\nabla^1 w_0\|_{L^{p_0}} \leq C \epsilon_0.  
\]  

(2.31)

We prove now that
\[
\sum_{m=1}^{d} \left\| \nabla^{n} \psi_{m,0} \right\|_{L^{p_{m}}} + \sum_{m=1}^{d} \left\| \nabla^{n+1} A_{m,0} \right\|_{L^{p_{n+1}}} + \left\| \nabla^{n+1} v_{0} \right\|_{L^{p_{n}}} + \left\| \nabla^{n+1} w_{0} \right\|_{L^{p_{n}}} \leq C \epsilon_{0},
\]

(2.32)

for any \( n \in \mathbb{Z} \cap [0, (d - 2)/2] \). We argue by induction over \( n \). The case \( n = 0 \) was already proved in (2.31). Assume \( n \geq 1 \) and (2.32) holds for any \( n' \in [0, n - 1] \cap \mathbb{Z} \). Using (2.24), (2.30), and the induction hypothesis

\[
\left\| \nabla^{n} \psi_{m,0} \right\|_{L^{p_{n}}} \leq C \left\| \nabla^{n+1} s_{0} \right\|_{L^{p_{n}}} \cdot \left\| v_{0} \right\|_{L^{\infty}} + C \sum_{n'=0}^{n-1} \left\| \nabla^{n-n'} s_{0} \right\|_{L^{p_{n-n'-1}}} \cdot \left\| \nabla^{n'} v_{0} \right\|_{L^{p_{n'}}},
\]

which suffices to control the first term in the left-hand side of (2.32). For the second term, using (2.25) and (2.30),

\[
\left\| \nabla^{n+1} A_{m,0} \right\|_{L^{p_{n+1}}} \leq C \sum_{l,l'=1}^{d} \sum_{n'=0}^{n} \left\| \nabla^{n'} \psi_{l,0} \right\|_{L^{p_{n'}}} \cdot \left\| \nabla^{n-n'} \psi_{l',0} \right\|_{L^{p_{n-n'}}},
\]

which suffices in view of the induction hypothesis and the bound on the first term proved before. The bound on the last two terms in the left-hand side of (2.32) follows in a similar way, using (2.14), (2.30), and the bound on the first two terms.

If \( d \) is even then (2.32) suffices to prove the first inequality in (2.28), simply by taking \( n = (d - 2)/2 \). If \( d \) is odd, the bounds (2.32) with \( n = (d - 3)/2 \) and (2.29) give

\[
\left\| \nabla^{\sigma+1} v_{0} \right\|_{L^{p_{\sigma}}} + \left\| \nabla^{\sigma+1} w_{0} \right\|_{L^{p_{\sigma}}} \leq C \epsilon_{0} \quad \text{for } \sigma \in \left[-1/2, (d - 3)/2\right].
\]

(2.33)

In view of the hypothesis and (2.29), we also have the bound

\[
\left\| \nabla^{\sigma+1} s_{0} \right\|_{L^{p_{\sigma}}} \leq C \epsilon_{0} \quad \text{for } \sigma \in \left[-1/2, (d - 2)/2\right].
\]

(2.34)

We need the following Leibniz rule (a particular case of [12, Theorem A.8]):

\[
\left\| \nabla^{1/2} (fg) - f \nabla^{1/2} g \right\|_{L^{2}} \leq C \left\| \nabla^{1/2} g \right\|_{L^{q_{1}}} \cdot \left\| f \right\|_{L^{q_{2}}}
\]

(2.35)

if \( 1/q_{1} + 1/q_{2} = 1/2 \) and \( q_{1}, q_{2} \in [p_{d}/2, p_{-1/2}] \). Then, using (2.24) and (2.30)

\[
\left\| \nabla^{(d-2)/2} \psi_{m,0} \right\|_{L^{2}} \leq C \sum_{u_{0} \in \{v_{0}, w_{0}\}} \sum_{n=0}^{(d-3)/2} \left\| \nabla^{1/2} \left( \partial_{m} D^{n} s_{0} \cdot D^{(d-3)/2-n} u_{0} \right) \right\|_{L^{2}},
\]

where \( D^{n} \) denotes any derivative of the form \( \partial^{n_{1}}_{1} \cdots \partial^{n_{d}}_{d} \), with \( n_{1} + \cdots + n_{d} = n \). The first inequality in (2.28) then follows from (2.33)–(2.35) and the fact that \( |u_{0}| \equiv 1 \).

For the second inequality in (2.28), we notice first that \( \left\| \psi_{m,0} \right\|_{H^{0}} \leq C \cdot \left\| s_{0} \right\|_{H^{0}_{q}}, \) since \( |v_{0}| = |w_{0}| \equiv 1 \). In view of the first inequality in (2.28), we may assume \( \sigma' \geq (d + 1)/2 \). We use a similar argument as before: the bootstrap inequality that replaces (2.32) is

\[
\sum_{m=1}^{d} \| \nabla^n \psi_{m,0} \|_{L^2 \cap L^p_{n-\sigma'+d/2}} + \sum_{m=1}^{d} \| \nabla^n A_{m,0} \|_{L^2 \cap L^p_{n-\sigma'+d/2}}
\]

\[
+ \sum_{u_0 \in \{ \nu_0, w_0 \}} \| \nabla^{n+1} u_0 \|_{L^2 \cap L^p_{n-\sigma'+d/2}} \leq C(\| s_0 \|_{H^{\sigma'}}).
\]  

(2.36)

for any \( n \in [0, \sigma' - 1] \cap \mathbb{Z} \), where \( p_{\sigma} = p_{-1/2} = 2d \) if \( \sigma \leq -1/2 \). As before, the bound (2.36) follows by induction over \( n \), using the identities (2.14), (2.24), and (2.25), and the inequalities (2.29), (2.30), and

\[
\sum_{n_1 + \ldots + n_d \leq \sigma'-(d+1)/2} \| \partial_1^{n_1} \ldots \partial_d^{n_d} s_0 \|_{L^\infty} \leq C(\| s_0 \|_{H^{\sigma'}}).
\]

The second inequality in (2.28) follows from the bound (2.36) with \( n = \sigma' - 1 \). \( \square \)

3. Perturbative analysis of the modified Schrödinger map

In this section we analyze the Schrödinger map system derived in Propositions 2.3 and 2.4. In the rest of this section we assume \( d \geq 3 \); this restriction is used implicitly in many estimates.

3.1. The resolution spaces and their properties

In this subsection we define our main normed spaces and summarize some of their basic properties. These resolution spaces have been used in [8] and, with slight modifications, in [7], and we will refer to these papers for most of the proofs.

Let \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote the Fourier transform and the inverse Fourier transform operators on \( L^2(\mathbb{R}^{d+1}) \). For \( l = 1, \ldots, d \) let \( \mathcal{F}_l \) and \( \mathcal{F}_l^{-1} \) denote the Fourier transform and the inverse Fourier transform operators on \( L^2(\mathbb{R}^d) \). We fix \( \eta_0 : \mathbb{R} \to [0, 1] \) a smooth even function supported in the set \( \{ \mu \in \mathbb{R} : |\mu| \leq 8/5 \} \) and equal to 1 in the set \( \{ \mu \in \mathbb{R} : |\mu| \leq 5/4 \} \). Then we define

\[
\eta_j(\mu) = \eta_0(\mu/2^j) - \eta_0(\mu/2^{j-1}),
\]

(3.1)

and \( \eta_k^{(d)} : \mathbb{R}^d \to [0, 1], k \in \mathbb{Z}, \)

\[
\eta_k^{(d)}(\xi) = \eta_0(|\xi|/2^k) - \eta_0(|\xi|/2^{k-1}).
\]

(3.2)

For \( j \in \mathbb{Z}_+ \), we also define \( \eta_{\leq j} = \eta_0 + \cdots + \eta_j \).

For \( k \in \mathbb{Z} \) let \( I_k^{(d)} = \{ \xi \in \mathbb{R}^d : |\xi| \in [2^{k-1}, 2^{k+1}] \} \); for \( j \in \mathbb{Z}_+ \) let \( I_j = \{ \mu \in \mathbb{R} : |\mu| \in [2^{j-1}, 2^{j+1}] \} \) if \( j \geq 1 \) and \( I_j = [-2, 2] \) if \( j = 0 \). For \( k \in \mathbb{Z} \) and \( j \in \mathbb{Z}_+ \) let

\[
D_{k,j} = \{ (\xi, \tau) \in \mathbb{R}^d \times \mathbb{R} : \xi \in I_k^{(d)} \text{ and } |\tau| + |\xi|^2 \in I_j \} \quad \text{and} \quad D_{k,\leq j} = \bigcup_{0 \leq j' \leq j} D_{k,j'}.
\]

For \( k \in \mathbb{Z} \) we define first the normed spaces
\[ X_k = \left\{ f \in L^2(\mathbb{R}^d \times \mathbb{R}) : f \text{ supported in } I_k^{(d)} \times \mathbb{R} \text{ and } \|f\|_{X_k} = \infty \sum_{j=0}^{\infty} 2^{j/2} \|\eta_j(\tau + |\xi|^2) \cdot f\|_{L^2} < \infty \right\}. \quad (3.3) \]

The spaces \( X_k \) are not sufficient for our estimates, due to various logarithmic divergences. For any vector \( e \in S^{d-1} \) let

\[ P_e = \{ \xi \in \mathbb{R}^d : \xi \cdot e = 0 \} \]

with the induced Euclidean measure. For \( p, q \in [1, \infty) \) we define the normed spaces \( L^{p,q}_e = L^{p,q}_e(\mathbb{R}^d \times \mathbb{R}) \),

\[ L^{p,q}_e = \left\{ f \in L^2(\mathbb{R}^d \times \mathbb{R}) : \|f\|_{L^{p,q}_e} = \left[ \int_{\mathbb{R}} \left[ \int_{P_e \times \mathbb{R}} |f(r e + v, t)|^q dv dt \right]^{p/q} dr \right]^{1/p} < \infty \right\}. \quad (3.4) \]

For \( k \in \mathbb{Z} \) and \( j \in \mathbb{Z}_+ \) let

\[ D_{k,j} = \{ (\xi, \tau) \in D_{k,j} : \xi \cdot e \geq 2^{k-20} \} \quad \text{and} \quad D_{k,\leq j} = \bigcup_{0 \leq j' \leq j} D_{k,j}. \]

For \( k \geq 100 \) and \( e \in S^{d-1} \), we define the normed spaces

\[ Y^e_k = \{ f \in L^2(\mathbb{R}^d \times \mathbb{R}) : f \text{ supported in } D_{k,\leq 2k+10} \text{ and } \|f\|_{Y^e_k} = 2^{-k/2} \|\mathcal{F}^{-1}\left[ (\tau + |\xi|^2 + i) \cdot f\right]\|_{L^1_{\xi,2}} < \infty \}. \quad (3.5) \]

For simplicity of notation, we also define \( Y^e_k = \{0\} \) for \( k \leq 99 \).

We fix \( L = L(d) \) large and \( e_1, \ldots, e_L \in S^{d-1} \), \( e_l \neq e_{l'} \) if \( l \neq l' \), such that

\[ \text{for any } e \in S^{d-1} \text{ there is } l \in \{1, \ldots, L\} \text{ such that } |e - e_l| \leq 2^{-100}. \quad (3.6) \]

We assume in addition that if \( e \in \{e_1, \ldots, e_L\} \) then \(-e \in \{e_1, \ldots, e_L\} \). For \( k \in \mathbb{Z} \) we define the normed spaces

\[ Z_k = X_k + Y^{e_{l_1}}_k + \cdots + Y^{e_{l_L}}_k. \quad (3.7) \]

The spaces \( Z_k \) are our main normed spaces.

For \( k \in \mathbb{Z}_+ \) let \( \Xi_k = 2^k \cdot \mathbb{Z}^d \). Let \( \chi^{(1)} : \mathbb{R} \to [0, 1] \) denote an even smooth function supported in the interval \([-2/3, 2/3]\) with the property that

\[ \sum_{n \in \mathbb{Z}} \chi^{(1)}(\xi - n) \equiv 1 \quad \text{on } \mathbb{R}. \]
Let \( \chi: \mathbb{R}^d \rightarrow [0, 1] \), \( \chi(\xi) = \chi^{(1)}(\xi_1) \cdots \chi^{(1)}(\xi_d) \). For \( k \in \mathbb{Z}_+ \) and \( n \in \mathbb{Z}_k \) let
\[
\chi_{k,n}(\xi) = \chi\left((\xi - n)/2^k\right).
\]
Clearly, \( \sum_{n \in \mathbb{Z}_k} \chi_{k,n} = 1 \) on \( \mathbb{R}^d \).

We summarize now some of the main properties of the spaces \( Z_k \).

**Proposition 3.1.**

(a) If \( k \in \mathbb{Z} \), \( m \in L^\infty(\mathbb{R}^d) \), \( \mathcal{F}^{-1}_{(d)}(m) \in L^1(\mathbb{R}^d) \), and \( f \in Z_k \), then \( m(\xi) \cdot f \in Z_k \) and
\[
\| m(\xi) \cdot f \|_{Z_k} \leq C \| \mathcal{F}^{-1}_{(d)}(m) \|_{L^1(\mathbb{R}^d)} \cdot \| f \|_{Z_k}.
\]  
(3.8)

(b) If \( k \in \mathbb{Z} \), \( j \in \mathbb{Z}_+ \) and \( f \in Z_k \) then
\[
\| \eta \leq j(\tau + |\xi|^2) \cdot f \|_{Z_k} \leq C \| f \|_{Z_k}.
\]  
(3.9)

(c) If \( k \in \mathbb{Z} \), \( j \in \mathbb{Z}_+ \), and \( f \in Z_k \) then
\[
\| \eta \leq j(\tau + |\xi|^2) \cdot f \|_{Z_k} \leq C \| f \|_{Z_k}.
\]  
(3.10)

(d) If \( k \in \mathbb{Z} \) and \( f \) is supported in \( D_{k,\leq\infty}^e \) for some \( e \in \{e_1, \ldots, e_L\} \) then
\[
\| f \|_{Z_k} \leq C 2^{-k/2} \| \mathcal{F}^{-1}\left[(\tau + |\xi|^2 + i) \cdot f\right] \|_{L^1_x}. 
\]  
(3.11)

(e) (Energy estimate) If \( k \in \mathbb{Z} \) and \( f \in Z_k \) then
\[
\sup_{t \in \mathbb{R}} \| \mathcal{F}^{-1}(f)(.,t) \|_{L^2_x} \leq C \| f \|_{Z_k}.
\]  
(3.12)

(f) (Localized maximal function estimate) If \( k \in \mathbb{Z} \), \( k' \in (-\infty, k + 10d] \cap \mathbb{Z} \), \( f \in Z_k \), and \( e' \in S^{d-1} \) then
\[
\left[ \sum_{n \in \mathbb{Z}_{k'}} \| \mathcal{F}^{-1}(\chi_{k',n}(\xi) \cdot \tilde{f}) \|_{L^{2,\infty}} \right]^{1/2} \leq C 2^{(d-1)k/2} \cdot 2^{-(d-2)(k-k')/2} \left(1 + |k - k'| \right) \cdot \| f \|_{Z_k},
\]  
(3.13)

where \( \mathcal{F}^{-1}(\tilde{f}) \in \{\mathcal{F}^{-1}(f), \mathcal{F}^{-1}(\bar{f})\} \).

(g) (Local smoothing estimate) If \( k \in \mathbb{Z} \), \( e' \in S^{d-1} \), \( l \in [-1, 40] \cap \mathbb{Z} \), and \( f \in Z_k \) then
\[
\| \mathcal{F}^{-1}\left[\tilde{f} \cdot \eta_1((\xi \cdot e')/2^{k-l})\right] \|_{L^{\infty}_x} \leq C 2^{-k/2} \| f \|_{Z_k},
\]  
(3.14)

where \( \mathcal{F}^{-1}(\tilde{f}) \in \{\mathcal{F}^{-1}(f), \mathcal{F}^{-1}(\bar{f})\} \).
The bound (3.8) follows directly from the definitions. The bound (3.9) is proved in [7, Lemma 2.1]. The bound (3.10) is proved in [7, Lemma 2.3]. The bound (3.11) follows from the estimate (2.15) in [7]. The energy estimate (3.12) is proved in [7, Lemma 2.2]. The localized maximal function estimate (3.13) follows from [7, Lemma 4.1] and (3.9). Finally, the localized smoothing estimate (3.14) is proved in [7, Lemma 4.2].

The estimate in part (f) with \( k' = k \) will often be referred to as the “global (3.13).” For \( k' \leq k - C \) we refer to this estimate as the “localized (3.13).”

### 3.2. Linear estimates

We fix a large constant \( \sigma_0, \) say
\[
\sigma_0 = d + 10. \tag{3.15}
\]
For \( \sigma \in [(d - 2)/2, \sigma_0 - 1] \) we define the normed space
\[
\dot{F}^\sigma = \left\{ u \in C(\mathbb{R} : H^\infty) : \| u \|_{\dot{F}^\sigma} = \left[ \sum_{k \in \mathbb{Z}} (2^{2\sigma k} + 2^{(d-2)k}) \| \eta_k^{(d)}(\xi) \cdot \mathcal{F}(u) \|_{Z_k}^2 \right]^{1/2} < \infty \right\}. \tag{3.16}
\]
For \( \sigma \in [(d - 2)/2, \sigma_0 - 1], T \in [0, 1], u \in C([-T, T] : H^\infty), \) and \( T' \in [0, T] \) we define
\[
E_{T'}(u)(t) = \begin{cases} u(t) & \text{if } |t| \leq T'; \\ 0 & \text{if } |t| > T', \end{cases} \tag{3.17}
\]
and
\[
\| u \|_{\dot{N}^\sigma[-T', T']} = \left[ \sum_{k \in \mathbb{Z}} (2^{2\sigma k} + 2^{(d-2)k}) \| \eta_k^{(d)}(\xi) \cdot (\tau + |\xi|^2 + i)^{-1} \cdot \mathcal{F}(E_{T'} u) \|_{Z_k}^2 \right]^{1/2}. \tag{3.18}
\]
The definition (3.3) shows that if \( k \in \mathbb{Z} \) and \( f \) is supported in \( I_k^{(d)} \times \mathbb{R} \) then
\[
\left\| (\tau + |\xi|^2 + i)^{-1} \cdot f \right\|_{Z_k} \leq C \| f \|_{L^2},
\]
thus, for \( \sigma \in [(d - 2)/2, \sigma_0 - 1] \) and \( T_1, T_2 \in [0, T] \)
\[
\| u \|_{\dot{N}^\sigma[-T_1, T_1]} - \| u \|_{\dot{N}^\sigma[-T_2, T_2]} \leq C |T_1 - T_2|^{1/2} \cdot \sup_{t \in [-T, T]} \| u(., t) \|_{H^\sigma}. \tag{3.19}
\]
For \( \phi \in H^\sigma \) let \( W(t)(\phi) \in C(\mathbb{R} : H^\sigma) \) denote the solution of the free Schrödinger evolution.

**Proposition 3.2.** If \( \sigma \in [(d - 2)/2, \sigma_0 - 1] \) and \( \phi \in H^\infty \) then
\[
\| \eta_0(t) \cdot W(t)(\phi) \|_{\dot{F}^\sigma} \leq C \left( \| \phi \|_{\dot{H}^\sigma} + \| \phi \|_{\dot{H}^{(d-2)/2}} \right).
\]
See [7, Lemma 3.1] for the proof.
Proposition 3.3. If $\sigma \in [(d - 2)/2, \sigma_0 - 1]$, $T \in [0, 1]$, and $u \in C([-T, T] : H^\infty)$ then
\[
\| \eta_0(t) \cdot \int_0^t W(t - s)(E_T(u))(s) \, ds \|_{\dot{F}_{\sigma}} \leq C \| u \|_{\dot{N}^\sigma_{[-T,T]}},
\]
where $E_T(u)$ is defined in (3.17).

See [7, Lemma 3.2] for the proof.

3.3. Nonlinear estimates

In this subsection we assume that $d \geq 4$. Assume that $T \in [0, 1]$ and $\psi_m \in C([-T, T] : H^\infty)$, $m = 1, \ldots, d$. Let $\Psi = (\psi_1, \ldots, \psi_d)$ and define
\[
\begin{align*}
A_0 &= \sum_{l, l'}^d R_l R_{l'}(\Re(\overline{\psi}_l \psi_l')) + \frac{1}{2} \sum_{l = 1}^d \psi_l \overline{\psi}_l; \\
A_m &= \nabla^{-1}(\sum_{l = 1}^d R_l[\Im(\psi_m \overline{\psi}_l)])
\end{align*}
\text{for any } m = 1, \ldots, d,
\tag{3.20}
\]
and
\[
\mathcal{N}_m(\Psi) = -2i \sum_{l = 1}^d A_l : \partial_l \psi_m + \left( A_0 + \sum_{l = 1}^d A_l^2 \right) \psi_m + i \sum_{l = 1}^d \Im(\psi_l \overline{\psi}_m) \psi_l.
\tag{3.21}
\]
Clearly, $A_m, \mathcal{N}_m(\Psi) \in C([-T, T] : H^\infty)$ (recall that $d \geq 3$). We assume also that on $\mathbb{R}^d \times [-T, T]$ we have the integral equation
\[
\psi_m(t) = W(t)(\psi_{m,0}) + \int_0^t W(t - s)(\mathcal{N}_m(\Psi)(s)) \, ds,
\tag{3.22}
\]
where $\psi_{m,0} = \psi_m(0)$. In dimensions $d \geq 4$ we will not need the compatibility conditions
\[
(\partial_l + i A_l) \psi_m = (\partial_m + i A_m) \psi_l \quad \text{for any } m, l = 1, \ldots, d.
\]

We define the extensions $\tilde{E}_T(\psi_m) \in C(\mathbb{R} : H^\infty), m = 1, \ldots, d,$
\[
\tilde{E}_T(\psi_m)(t) = \eta_0(t) \cdot W(t)(\psi_{m,0}) + \eta_0(t) \cdot \int_0^t W(t - s)(E_T(\mathcal{N}_m(\Psi))(s)) \, ds.
\tag{3.23}
\]
Using Propositions 3.2 and 3.3, for $\sigma \in [(d - 2)/2, \sigma_0 - 1]$
\[
\| \tilde{E}_T(\psi_m) \|_{\dot{F}_{\sigma}} \leq C \cdot (\| \psi_{m,0} \|_{\dot{H}_{\sigma} \cap \dot{H}_{(d-2)/2}} + \| \mathcal{N}_m(\Psi) \|_{\dot{N}^\sigma_{[-T,T]}}).
\]
Let $\tilde{E}_T(\Psi) = (\tilde{E}_T(\psi_1), \ldots, \tilde{E}_T(\psi_d))$. For $\sigma \in [(d-2)/2, \sigma_0 - 1]$ let

$$\| \tilde{E}_T(\Psi) \|_{\tilde{F}_\sigma} = \sum_{m=1}^d \| \tilde{E}_T(\psi_m) \|_{\tilde{F}_\sigma}. \quad (3.24)$$

The main result of this subsection is the following proposition.

**Proposition 3.4.** Assume $d \geq 4$. Then, for any $\sigma \in [(d-2)/2, \sigma_0 - 1]$ and $m = 1, \ldots, d$,

$$\|\mathcal{N}_m(\Psi)\|_{\tilde{N}_\sigma[-T,T]} \leq C \| \tilde{E}_T(\Psi) \|_{\tilde{F}_\sigma} \left( \| \tilde{E}_T(\Psi) \|^2_{\tilde{F}((d-2)/2)} + \| \tilde{E}_T(\Psi) \|^4_{\tilde{F}((d-2)/2)} \right). \quad (3.25)$$

The rest of this subsection is concerned with the proof of Proposition 3.4. For $\sigma \in [(d-2)/2, \sigma_0 - 1]$ and $k \in \mathbb{Z}$ let

$$\beta_k(\sigma) = \sum_{m=1}^d \sum_{k' \in \mathbb{Z}} 2^{-|k-k'|/10} \cdot (2^{\sigma k'} + 2^{(d-2)k'/2}) \| \eta_k^{(d)}(\xi) \cdot \mathcal{F}(\tilde{E}_T(\psi_m)) \|_{Z_k}. \quad (3.26)$$

Clearly, $\beta_k(\sigma) \leq C2^{k_1-k_2/10} \beta_{k_2}(\sigma)$ for any $k_1, k_2 \in \mathbb{Z}$, and

$$\left[ \sum_{k \in \mathbb{Z}} \beta_k(\sigma)^2 \right]^{1/2} \leq C \| \tilde{E}_T(\Psi) \|_{\tilde{F}_\sigma} \quad \text{for any } \sigma \in [(d-2)/2, \sigma_0 - 1].$$

For $k \in \mathbb{Z}$ let $P_k$ denote the operator defined by the Fourier multiplier $(\xi, \tau) \rightarrow \eta_k^{(d)}(\xi)$, and let $P_{\leq k} = \sum_{k' \leq k} P_{k'}$. For $k \in \mathbb{Z}$ and $n \in \mathbb{Z}_k$ let $\tilde{P}_{k,n}$ denote the operator defined by the Fourier multiplier $(\xi, \tau) \rightarrow \chi_{k,n}(\xi)$.

**Lemma 3.5.** If $d \geq 4$, $k \in \mathbb{Z}$, $\mathbf{e}' \in \mathbb{S}^{d-1}$, $\sigma \in [(d-2)/2, \sigma_0 - 1]$, and

$$F \in \{ E_T(A_0), E_T(A_m^2), E_T(\tilde{\psi}_m \cdot \tilde{\psi}_l) \colon m, l = 1, \ldots, d, \tilde{\psi} \in \{ \psi, \tilde{\psi} \} \} \quad (3.27)$$

then

$$(2^{\sigma k} + 2^{(d-2)k/2}) \| P_k(F) \|_{L^2} \leq C \beta_k(\sigma) \cdot \left( \| \tilde{E}_T(\Psi) \|_{\tilde{F}((d-2)/2)} + \| \tilde{E}_T(\Psi) \|^2_{\tilde{F}((d-2)/2)} \right). \quad (3.28)$$

and

$$\| P_{\leq k}(F) \|_{L^1_{\mathbf{e}' \infty}} \leq C2^k \left( \| \tilde{E}_T(\Psi) \|^2_{\tilde{F}((d-2)/2)} + \| \tilde{E}_T(\Psi) \|^4_{\tilde{F}((d-2)/2)} \right). \quad (3.29)$$

In addition, for $m = 1, \ldots, d$,

$$(2^{\sigma k} + 2^{(d-2)k/2}) \| P_k(E_T(A_m)) \|_{L^2} \leq C2^{-k} \beta_k(\sigma) \cdot \| \tilde{E}_T(\Psi) \|_{\tilde{F}((d-2)/2)}, \quad (3.30)$$

and

$$\| P_{\leq k}(E_T(A_m)) \|_{L^1_{\mathbf{e}' \infty}} \leq C \| \tilde{E}_T(\Psi) \|^2_{\tilde{F}((d-2)/2)}. \quad (3.31)$$
The main reason we assume \( d \geq 4 \) (rather than \( d \geq 3 \)) is to have a simple proof of (3.31). We defer the proof of Lemma 3.5 to Section 5, and complete now the proof of Proposition 3.4. For (3.25) it suffices to prove that

\[
\frac{(2^n + 2^{(d-2)/2})}{2} \cdot \mathcal{F}\left(P_k \left( E_T(\mathcal{N}_\nu(\psi)) \right)\right) Z_k
\]

\[
\leq C_\beta(k) \cdot \left( \left\| \tilde{E}_T(\psi) \right\|_{F(d-2)/2}^2 + \left\| \tilde{E}_T(\psi) \right\|_{F(d-2)/2}^4 \right)
\]

(3.32)

for any \( k \in \mathbb{Z} \). Since \( E_T(\mathcal{N}_\nu(\psi)) \) is a sum of terms of the form \( F \cdot \tilde{E}_T(\psi_m) \) and \( E_T(A_l) \cdot \partial_l \tilde{E}_T(\psi_m) \), where \( F \) is as in (3.27), it suffices to prove that

\[
\left\| \tau + |\xi| + i \right\|^{\sigma(k)} \cdot F \left( P_k \left( \tilde{E}_T(\psi_m) \right) \right) Z_k
\]

\[
\leq C_\beta(k) \cdot \left( \left\| \tilde{E}_T(\psi) \right\|_{F(d-2)/2}^2 + \left\| \tilde{E}_T(\psi) \right\|_{F(d-2)/2}^4 \right)
\]

(3.33)

is dominated by the right-hand side of (3.32) for any \( m, l = 1, \ldots, d \). We always estimate the expressions in (3.33) using (3.11).

We consider first the term \( F \cdot \tilde{E}_T(\psi_m) \), and write

\[
\sum_{|k_1-k| \leq 2} c_\sigma(k) \cdot \left\| \eta_0 |\xi|/2^{k-50} \right\| \cdot \mathcal{F}\left(P_{k_1} \left( \tilde{E}_T(\psi_m) \right)\right) Z_k
\]

(3.35)

and

\[
\sum_{k_1 \geq k-9} c_\sigma(k) \cdot \left\| \eta_0 |\xi|/2^{k-50} \right\| \cdot \mathcal{F}\left(P_{k_1} \left( \tilde{E}_T(\psi_m) \right)\right) Z_k
\]

(3.36)

are dominated by the right-hand side of (3.32).

To bound the expression in (3.35), we may assume that \( \mathcal{F}(P_{k_1} \left( \tilde{E}_T(\psi_m) \right)) \) is supported in \( I_{k_1}^{(d)} \times \mathbb{R} \cap \{(|\xi|, \tau) : |\xi - w| \leq 2^{k_1-50}\} \) for some \( w \in I_{k_1}^{(d)} \). We use the following simple geometric observation (cf. [7, Section 8]): if \( \hat{v}, \hat{w} \in S^{d-1} \) then there is \( e \in \{e_1, \ldots, e_L\} \) such that

\[
e \cdot \hat{v} \geq 2^{-5} \quad \text{and} \quad |e \cdot \hat{w}| \geq 2^{-5}.
\]

(3.37)

We fix \( e \) as in (3.37) (with \( \hat{v} = v/|v| \) and \( \hat{w} = w/|w| \)). Using (3.11), the expression in (3.35) is dominated by
\[ C \sigma (k) \sum_{|k_1 - k| \leq 2} 2^{-k/2} \| P_{\leq k-10} (F) \cdot P_{k_1} (\tilde{E}_T (\psi_m)) \|_{L^4_\infty} \]
\[ \leq C \sigma (k) \sum_{|k_1 - k| \leq 2} 2^{-k/2} \| P_{\leq k-10} (F) \|_{L^4_\infty} \cdot \| P_{k_1} (\tilde{E}_T (\psi_m)) \|_{L^\infty_\infty}, \]

which suffices, in view of (3.14) and (3.29).

To bound the expression in (3.36), we fix \( e \in \{ e_1, \ldots, e_l \} \) such that \( |e - v/|v|| \leq 2^{-100} \) and use (3.11). The second sum in (3.35) is dominated by

\[ C \sigma (k) \sum_{|k_1 - k| \geq k - 9} 2^{-k/2} \| P_{k_1} (F) \cdot P_{\leq k_1 + 20} (\tilde{E}_T (\psi_m)) \|_{L^4_\infty}, \]

where \( M = \| \tilde{E}_T (\psi) \|_{L^{(d-2)/2}} + \| \tilde{E}_T (\psi) \|_{L^{(d-2)/2}}^3 \), and we used the localized (3.13) and (3.28) in the last estimate. This suffices since \( \beta_k (\sigma) \leq C 2^{k_1 - k/10} \beta_k (\sigma) \) and \( d \geq 4 \).

We consider now \( E_T (A_t) \cdot \partial_t \tilde{E}_T (\psi_m) \). We write \( P_k (E_T (A_t) \cdot \partial_t \tilde{E}_T (\psi_m)) \) as

\[ \sum_{|k_1 - k| \leq 2} P_k [P_{\leq k-10} (E_T (A_t)) \cdot P_{k_1} (\partial_t \tilde{E}_T (\psi_m))], \]
\[ + \sum_{k_1 \geq k - 9} P_k [P_{k_1} (E_T (A_t)) \cdot P_{\leq k_1 + 20} (\partial_t \tilde{E}_T (\psi_m))], \]

and argue as before, using (3.31) and (3.30) instead of (3.29) and (3.28).

4. Proof of Theorem 1.1

In this section we assume \( d \geq 4 \).

4.1. A priori estimates

In this subsection we prove the following:

**Proposition 4.1.** Assume that \( \sigma_0 = d + 10 \) is as in (3.15), \( T \in [0, 1] \) and \( s \in C([-T, T], H^\infty_Q) \) is a solution of the initial-value problem

\[ \begin{cases} \partial_t s = s \times \Delta s & \text{on } \mathbb{R}^d \times [-T, T]; \\ s(0) = s_0. \end{cases} \]
If \( \|s_0 - Q\|_{\dot{H}^{d/2}} \leq \varepsilon_0 \ll 1 \) then

\[
\begin{align*}
\sup_{t \in [-T,T]} \|s(t) - Q\|_{\dot{H}^{d/2}} &\leq C\|s_0 - Q\|_{\dot{H}^{d/2}}; \\
\sup_{t \in [-T,T]} \|s(t)\|_{H^\sigma_Q'} &\leq C(\|s_0\|_{H^\sigma_Q'}) \quad \text{for any } \sigma' \in [0, \sigma_0] \cap \mathbb{Z}.
\end{align*}
\]

(4.2)

**Proof.** We construct \( \psi_m, A_m \in C([-T, T] : H^\infty) \) as in Proposition 2.4. In view of Lemma 2.5,

\[
\|\psi_{m,0}\|_{\dot{H}^{(d-2)/2}} \leq C\|s_0 - Q\|_{\dot{H}^{d/2}} \leq C\varepsilon_0.
\]

(4.3)

For any \( T' \in [0, T] \) we define the functions \( E_{T'}(\hat{N}_m(\Psi)) \) and \( \tilde{E}_{T'}(\psi_m) \) as in (3.17) and (3.23). Using Propositions 3.2 and 3.3, for \( \sigma \in [(d-2)/2, \sigma_0 - 1] \) and \( T' \in [0, T] \),

\[
\|\tilde{E}_{T'}(\Psi)\|_{F_\sigma} \leq C \cdot \left( \sum_{m=1}^{d} \|\psi_{m,0}\|_{\dot{H}^\sigma \cap \dot{H}^{(d-2)/2}} + \sum_{m=1}^{d} \|\hat{N}_m(\Psi)\|_{\dot{N}_\sigma[-T', T']} \right).
\]

(4.4)

In addition, using Lemma 3.4, for \( \sigma \in [(d-2)/2, \sigma_0 - 1] \) and \( T' \in [0, T] \),

\[
\sum_{m=1}^{d} \|\hat{N}_m(\Psi)\|_{\dot{N}_\sigma[-T', T']} \leq C \|\tilde{E}_{T'}(\Psi)\|_{\tilde{F}_\sigma} (\|\tilde{E}_{T'}(\Psi)\|_{F_\sigma}^{2} + \|\tilde{E}_{T'}(\Psi)\|_{\tilde{F}_\sigma}^{2}).
\]

(4.5)

The inequality (3.19) shows that the function \( L(T') = \sum_{m=1}^{d} \|\hat{N}_m(\Psi)\|_{\dot{N}_\sigma[-T', T']} \) is continuous on the interval \([0, T]\). Also, \( L(0) = 0 \). Thus we can combine (4.4) and (4.5) (with \( \sigma = (d-2)/2 \)), together with the smallness of \( \|\psi_{m,0}\|_{\dot{H}^{(d-2)/2}} \), to conclude that

\[
\sum_{m=1}^{d} \|\hat{N}_m(\Psi)\|_{\dot{N}_\sigma[-T', T']} \leq C \sum_{m=1}^{d} \|\psi_{m,0}\|_{\dot{H}^{(d-2)/2}} \quad \text{for any } T' \in [0, T].
\]

Using (4.4) again, it follows that

\[
\|\tilde{E}_{T}(\Psi)\|_{\tilde{F}_{(d-2)/2}} \leq C \sum_{m=1}^{d} \|\psi_{m,0}\|_{\dot{H}^{(d-2)/2}} \ll 1.
\]

(4.6)

We combine (4.4) and (4.5) again; using (4.6), for any \( \sigma \in [(d-2)/2, \sigma_0 - 1] \)

\[
\|\tilde{E}_{T}(\Psi)\|_{\tilde{F}_\sigma} \leq C \sum_{m=1}^{d} \|\psi_{m,0}\|_{\dot{H}^\sigma \cap \dot{H}^{(d-2)/2}}.
\]

(4.7)

Using (3.12), it follows that for any \( \sigma \in [(d-2)/2, \sigma_0 - 1] \)

\[
\sum_{m=1}^{d} \sup_{t \in [-T,T]} \|\psi_{m}(t)\|_{\dot{H}^\sigma \cap \dot{H}^{(d-2)/2}} \leq C \sum_{m=1}^{d} \|\psi_{m,0}\|_{\dot{H}^\sigma \cap \dot{H}^{(d-2)/2}}.
\]

(4.8)
We use (4.8) to get a priori estimates on the solution $s$. Using (4.3) and (4.8),

$$
\sum_{m=1}^{d} \sup_{t \in [-T, T]} \| \psi_m(t) \|_{H^{(d-2)/2}} \leq C \| s_0 - Q \| _{\dot{H}^{d/2}}.
$$

We define the operators $\nabla^{\sigma}$, $\sigma \in [-1/2, d/2]$, as in the proof of Lemma 2.5. Let $p_\sigma = d/(\sigma + 1)$. Then, in view of the Sobolev imbedding theorem (recall $d \geq 3$),

$$
\| \nabla^{\sigma} f \|_{L^{p_\sigma}} \leq C \| \nabla^{\sigma'} f \|_{L^{p_\sigma'}} \quad \text{if } -1/2 \leq \sigma \leq \sigma' \leq d/2 \text{ and } f \in H^{\sigma_0-1},
$$

(4.10)

Let $n_0$ denote the smallest integer $\geq (d-2)/2$. Using (4.10), (2.30), and the definition of the coefficients $A_m$,

$$
\| A_m(t) \|_{H^{(d-2)/2}} \leq \| \nabla^{n_0}(A_m(t)) \|_{L^{p_{n_0}}} \leq C \| s_0 - Q \| _{\dot{H}^{d/2}},
$$

(4.11)

for any $t \in [-T, T]$ and $m = 1, \ldots, d$.

To prove estimates on the solution $s$, recall the identity (2.14),

$$
\begin{cases}
\partial_m s = \Re(\psi_m)v + \Im(\psi_m)w; \\
\partial_m v = -\Re(\psi_m)s + A_m w; \\
\partial_m w = -\Im(\psi_m)s - A_m v.
\end{cases}
$$

(4.12)

Since $|s| = |v| = |w| \equiv 1$, we use (4.9), (4.11), and (4.10) to see that

$$
\sum_{m=1}^{d} \left( \| \partial_m(s(t)) \|_{L^{p_0}} + \| \partial_m(v(t)) \|_{L^{p_0}} + \| \partial_m(w(t)) \|_{L^{p_0}} \right) \leq C \| s_0 - Q \| _{\dot{H}^{d/2}},
$$

for any $t \in [-T, T]$. As in the proof of Lemma 2.5, a simple inductive argument using (4.12), (4.9), (4.11), and (2.30) shows that

$$
\sum_{m=1}^{d} \left( \| \nabla^{n}\partial_m(s(t)) \|_{L^{p_n}} + \| \nabla^{n}\partial_m(v(t)) \|_{L^{p_n}} + \| \nabla^{n}\partial_m(w(t)) \|_{L^{p_n}} \right) \leq C \| s_0 - Q \| _{\dot{H}^{d/2}},
$$

(4.13)

for any $n \in \mathbb{Z} \cap [0, (d-2)/2]$ and $t \in [-T, T]$. If $d$ is even, this gives

$$
\| s(t) - Q \| _{\dot{H}^{d/2}} \leq C \| s_0 - Q \| _{\dot{H}^{d/2}} \quad \text{for any } t \in [-T, T].
$$

(4.14)

If $d$ is odd then, using (4.13) with $n = (d-3)/2$ and (4.10), we have

$$
\sum_{m=1}^{d} \left( \| \nabla^{\sigma}\partial_m(s(t)) \|_{L^{p_\sigma}} + \| \nabla^{\sigma}\partial_m(v(t)) \|_{L^{p_\sigma}} + \| \nabla^{\sigma}\partial_m(w(t)) \|_{L^{p_\sigma}} \right) \leq C \| s_0 - Q \| _{\dot{H}^{d/2}}
$$

for any $\sigma \in [-1/2, (d-3)/2]$. The bound (4.14) follows in this case as well, using the Leibniz rule (2.35).
We show now that for \( \sigma' \in [0, \sigma_0] \cap \mathbb{Z} \)

\[
\sup_{t \in [-T, T]} \| s(t) \|_{H_Q^{\sigma'}} \leq C \left( \| s_0 \|_{H_Q^{\sigma'}} \right).
\]  
(4.15)

For this we observe first that we have the conservation law

\[
\| s(t) \|_{H_Q^0} = \| s_0 \|_{H_Q^0} \quad \text{for any} \ t \in [-T, T],
\]  
(4.16)

which follows by integration by parts from the initial-value problem (4.1). Thus, we need to estimate \( \| s(t) - Q \|_{H_Q^{\sigma'}} \) for \( t \in [-T, T] \). Using the first inequality in (4.2), we may assume \( \sigma' \geq (d + 1)/2 \). In view of (2.28) and (4.8)

\[
\sum_{m=1}^{d} \sup_{t \in [-T, T]} \| \psi_m(t) \|_{H^{\sigma'-1}} \leq C \left( \| s_0 \|_{H_Q^{\sigma'}} \right).
\]

In addition, due to the energy conservation law

\[
\sum_{l=1}^{d} \| \partial_l s(t) \|_{L^2}^2 = \sum_{l=1}^{d} \| \partial_l s(0) \|_{L^2}^2,
\]

and the definition \( \psi_m = (\partial_m s) \cdot v + i(\partial_m s) \cdot w \), we control \( \sup_{t \in [-T, T]} \| \psi_m(t) \|_{L^2} \leq C \left( \| s_0 \|_{H_Q^{\sigma'}} \right) \).

Thus

\[
\sum_{m=1}^{d} \sup_{t \in [-T, T]} \| \psi_m(t) \|_{H^{\sigma'-1}} \leq C \left( \| s_0 \|_{H_Q^{\sigma'}} \right).
\]

Using the definition of the coefficients \( A_m \), it follows easily that

\[
\sum_{m=1}^{d} \sup_{t \in [-T, T]} \| A_m(t) \|_{H^{\sigma'-1}} \leq C \left( \| s_0 \|_{H_Q^{\sigma'}} \right).
\]

We combine the last two inequalities, (4.12), and the fact that \(|s| = |v| = |w|\); a simple inductive argument gives \( \sup_{t \in [-T, T]} \| \partial_m s \|_{H^{\sigma'-1}} \leq C \left( \| s_0 \|_{H_Q^{\sigma'}} \right) \), which completes the proof of (4.15).

4.2. Existence and uniqueness of solutions

The uniqueness statement in part (a) is proved in [8, Section 2]: assume \( s, s' \in C([T_1, T_2] : H_Q^{\sigma_0}) \) solve the equation \( \partial_t s = s \times \Delta x s \) on \( \mathbb{R}^d \times [T_1, T_2] \), and \( s(T_1) = s'(T_1) \). Let \( q = s' - s \), so

\[
\begin{align*}
\partial_t q &= (s + q) \times \Delta x (s + q) - s \times \Delta x s \quad \text{on} \ \mathbb{R}^d \times [T_1, T_2]; \\
q(T_1) &= 0.
\end{align*}
\]

We multiply (4.17) by \( q(t) \) and integrate by parts over \( \mathbb{R}^d \) to obtain
\[ \frac{1}{2} \partial_t \left[ \| q(t) \|_{L^2}^2 \right] = \int_{\mathbb{R}^d} \left[ s(t) \times \Delta x q(t) \right] : q(t) \, dx \]
\[ \leq C_s \left( \| q(t) \|_{L^2}^2 + \sum_{l=1}^{d} \| \partial_l q(t) \|_{L^2}^2 \right). \tag{4.18} \]

Then we apply \( \partial_l \) to (4.17), multiply by \( \partial_l q(t) \), add up over \( l = 1, \ldots, d \), and integrate by parts over \( \mathbb{R}^d \). The result is
\[ \frac{1}{2} \partial_t \left[ \sum_{l=1}^{d} \| \partial_l q(t) \|_{L^2}^2 \right] = -\int_{\mathbb{R}^d} \left[ q(t) \times \Delta x s(t) \right] \cdot \Delta x q(t) \, dx \]
\[ \leq C_s \left( \| q(t) \|_{L^2}^2 + \sum_{l=1}^{d} \| \partial_l q(t) \|_{L^2}^2 \right). \tag{4.19} \]

Using (4.18) and (4.19), \( q \equiv 0 \) on \( \mathbb{R}^d \times [T_1, T_2] \), as desired.

To construct the global solution, we need the following local existence result:

**Proposition 4.2.** Assume \( s_0 \in H^\infty_Q \). Then there is \( T_{\sigma_0} = T(\| s_0 \|_{H^\infty_Q}) > 0 \) and a solution \( s \in C([-T_{\sigma_0}, T_{\sigma_0}] : H^\infty_Q) \) of the initial-value problem
\[
\begin{cases}
\partial_t s = s \times \Delta s & \text{on } \mathbb{R}^d \times [-T_{\sigma_0}, T_{\sigma_0}]; \\
\partial_t s(0) = s_0.
\end{cases}
\]

In addition, the time \( T_{\sigma_0} \) can be chosen such that
\[
\begin{align*}
\sup_{t \in [-T_{\sigma_0}, T_{\sigma_0}]} \| s(t) \|_{H^\infty_Q} & \leq C(\| s_0 \|_{H^\infty_Q}); \\
\sup_{t \in [-T_{\sigma_0}, T_{\sigma_0}]} \| s(t) \|_{H^\sigma_Q} & \leq C(\sigma, \| s_0 \|_{H^\infty_Q}) \quad \text{if } \sigma \in [\sigma_0, \infty) \cap \mathbb{Z}. \tag{4.20}
\end{align*}
\]

The local existence Proposition 4.2 is proved, for example, in [16]. Assuming Proposition 4.2, by scale invariance, it suffices to construct the solution \( s \) in Theorem 1.1 on the time interval \([-1, 1]\). In view of Proposition 4.2, there is \( T_{\sigma_0} > 0 \) and a solution \( s \) on the time interval \([-T_{\sigma_0}, T_{\sigma_0}]\). Assume the solution \( s \in C([-T, T] : H^\infty_Q) \) is constructed on some time interval \([-T, T] \), \( T \leq 1 \). In view of Proposition 4.1,
\[ \sup_{t \in [-T, T]} \| s(t) \|_{H^\infty_Q} \leq C(\| s_0 \|_{H^\infty_Q}), \]
uniformly in \( T \). Using Proposition 4.2, the solution \( s \) can be extended to the time interval \([-T - T', T + T']\) for some \( T' = T'(\| s_0 \|_{H^\infty_Q}) > 0 \) (which does not depend on \( T \)). The theorem follows.
5. Proof of Lemma 3.5

We use the notation in Section 3 and assume in this section that \( d \geq 4 \). For simplicity of notation, we let \( \psi \) denote any of the functions \( \tilde{E}_T (\psi_m) \) or \( \overline{E}_T (\psi_m) \), \( m = 1, \ldots, d \), \( A \) denote any of the functions \( A_m \), \( m = 1, \ldots, d \), and \( R \) denote any operator of the form \( R_l R_{l'} \), \( l, l' = 0, 1, \ldots, d \), \( R_0 = I \). With this convention, we show first that for any \( k \in \mathbb{Z} \) and \( \sigma \in [(d - 2)/2, \sigma_0 - 1] \)

\[
(2^{\sigma k} + 2^{(d-2)k/2}) \| P_k (R (\psi \cdot \psi)) \|_{L^2} \leq C \beta_k (\sigma) \| \tilde{E}_T (\Psi) \| \| (d-2)/2 \cdot \]

The left-hand side of (5.1) is dominated by

\[
C (2^{\sigma k} + 2^{(d-2)k/2}) \sum_{|k_1-k| \leq 2} \sum_{k_2 \leq k-4} \| P_{k_1} (\psi) \cdot P_{k_2} (\psi) \|_{L^2} \\
+ C (2^{\sigma k} + 2^{(d-2)k/2}) \sum_{k_1, k_2 \geq k-4} \sum_{|k_1-k_2| \leq 10} \| P_k (P_{k_1} (\psi) \cdot P_{k_2} (\psi)) \|_{L^2}.

Using (3.14), we estimate \( \| P_{k_1} \psi \| \) in \( L^\infty _{e'} \) (after suitable localization), and, using the global (3.13), we estimate \( \| P_{k_2} \psi \| \) in \( L^2 _{e'} \). The bound (5.1) follows since \( \beta_k (\sigma) \leq C 2^{|k_1-k|/10} \beta_k (\sigma) \). The bounds (3.28) for \( F \in \{ E_T (A_0), E_T (\psi_m \cdot \psi_l) : m, l = 1, \ldots, d, \psi \in \{ \psi, \tilde{\psi} \} \) and (3.30) clearly follow from (5.1). Also, it follows from (5.1) that

\[
(2^{\sigma k} + 2^{(d-2)k/2}) \cdot \| P_k (A) \|_{L^\infty _{e'}} \leq C 2^{-k/2} \cdot \beta_k (\sigma) \cdot \| \tilde{E}_T (\Psi) \| \| (d-2)/2 \cdot \]

for any \( e' \in \mathbb{S}^{d-1} \).

We prove now that for any \( e' \in \mathbb{S}^{d-1} \)

\[
\sum_{k \in \mathbb{Z}} 2^{-k} \| P_k (R (\psi \cdot \psi)) \|_{L^1 _{e'}} \leq C \| \tilde{E}_T (\Psi) \| \| (d-2)/2 \cdot \]

For any \( k \in \mathbb{Z} \)

\[
\| P_k (R (\psi \cdot \psi)) \|_{L^1 _{e'}} \leq C \sum_{|k_1-k| \leq 2} \sum_{k_2 \leq k-4} \| P_{k_1} (\psi) \cdot P_{k_2} (\psi) \|_{L^1 _{e'}} \\
+ C \sum_{k_1, k_2 \geq k-4} \| P_k (P_{k_1} (\psi) \cdot P_{k_2} (\psi)) \|_{L^1 _{e'}}. \]

For the first sum in (5.4), we use the global (3.13):

\[
\sum_{|k_1-k| \leq 2} \sum_{k_2 \leq k-4} \| P_{k_1} (\psi) \cdot P_{k_2} (\psi) \|_{L^1 _{e'}} \\
\leq C \sum_{|k_1-k| \leq 2} \sum_{k_2 \leq k-4} (2^{(d-1)k/2} \| P_{k_1} (\psi) \|_{Z_{k_1}}) \cdot (2^{(d-1)k/2} \| P_{k_2} (\psi) \|_{Z_{k_2}}) \\
\leq C 2^k \beta_k ((d - 2)/2)^2.
\]
For the second sum, we use the localized (3.13) and the assumption $d \geq 4$:

$$\left\| P_k(P_{k_1}\psi \cdot P_{k_2}\psi) \right\|_{L_x^1,\infty} \leq C \sum_{n,n' \in \mathcal{E}_k \text{ and } |n-n'| \leq C_2^k} \left\| P_{k,n} P_{k_1}(\psi) \cdot \tilde{P}_{k,n'} P_{k_2}(\psi) \right\|_{L_x^1,\infty}$$

$$\leq C \left[ \sum_{n \in \mathcal{E}_k} \left\| \tilde{P}_{k,n} P_{k_1}(\psi) \right\|_{L_x^{2,\infty}}^2 \right]^{1/2} \cdot \left[ \sum_{n' \in \mathcal{E}_k} \left\| \tilde{P}_{k,n'} P_{k_2}(\psi) \right\|_{L_x^{2,\infty}}^2 \right]^{1/2}$$

$$\leq C 2^{-3|k-k'|/2} \cdot (2^{(d-1)k_1/2} \left\| P_{k_1}(\psi) \right\|_{Z_k}) \cdot (2^{(d-1)k_2/2} \left\| P_{k_2}(\psi) \right\|_{Z_k})$$

$$\leq C 2^k 2^{-|k-k'|/4} \beta_k ((d - 2)/2)^2.$$

The bound (5.3) follows from (5.4) and the last two estimates. The bounds (3.29) for $F \in \{E_T(A_0), E_T(\tilde{\psi}_m \cdot \tilde{\psi}_l)\}$: $m, l = 1, \ldots, d, \tilde{\psi}_m \in \{\psi, \tilde{\psi}\}$, and (3.31) clearly follow from (5.3).

It remains to prove the bounds (3.28) and (3.29) for $F = E_T(A_{m_0}^2)$. We will need the following technical lemma:

**Lemma 5.1.** If $k \in \mathbb{Z}$, $k' \in (-\infty, k + 10d] \cap \mathbb{Z}$, and $e' \in \mathbb{S}^{d-1}$ then

$$\left[ \sum_{n \in \mathcal{E}_{k'}} \left\| \tilde{P}_{k,n} P_k(A) \right\|_{L_x^{2,\infty}}^2 \right]^{1/2} \leq C 2^{k/2} 2^{-3|k-k'|/4} \left\| \tilde{E}_T(\Psi) \right\|_{\tilde{F}(d-2)/2}^2.$$  \hspace{1cm} (5.5)

Assuming Lemma 5.1, for (3.28) it suffices to prove that

$$(2^{\sigma k} + 2^{(d-2)k/2}) \left\| P_k(A \cdot A) \right\|_{L_x^2} \leq C \beta_k(\sigma) \left\| \tilde{E}_T(\Psi) \right\|_{\tilde{F}(d-2)/2}^3.$$  \hspace{1cm} (5.6)

The proof of (5.6) is similar to the proof of (5.1), using the $L_x^{\infty,2}$ estimate in (5.2) and the global (that is $k' = k$) $L_x^{2,\infty}$ estimate in (5.5). For (3.29) it suffices to prove that

$$\left\| P_k(A \cdot A) \right\|_{L_x^{1,\infty}} \leq C 2^k \left\| \tilde{E}_T(\Psi) \right\|_{\tilde{F}(d-2)/2}^4,$$  \hspace{1cm} (5.7)

for any $k \in \mathbb{Z}$ and $e' \in \mathbb{S}^{d-1}$. The proof of (5.7) is similar to the proof of (5.3), using the localized $L_x^{\infty,\infty}$ estimate in (5.5).

**Proof of Lemma 5.1.** In view of the definitions, we may assume $k' \leq k - 10d$ and it suffices to prove that

$$\left[ \sum_{n \in \mathcal{E}_{k'}} \left\| \tilde{P}_{k',n} P_k(\psi \cdot \psi) \right\|_{L_x^{2,\infty}}^2 \right]^{1/2} \leq C 2^{3k/2} 2^{-3|k-k'|/4} \left\| \tilde{E}_T(\Psi) \right\|_{\tilde{F}(d-2)/2}^2.$$  \hspace{1cm} (5.8)

We will use the following bound: if $k \in \mathbb{Z}$, $k' \in (-\infty, k + 10d] \cap \mathbb{Z}$, and $f \in Z_k$ then

$$\left[ \sum_{n \in \mathcal{E}_{k'}} \mathcal{F}^{-1}(\chi_{k',n}(\xi) \cdot \tilde{f}) \right]_{L_x^{2,\infty}}^2 \leq C 2^{dk/2} \cdot 2^{-d|k-k'|/2}(1 + |k - k'|) \cdot \| f \|_{Z_k}.$$  \hspace{1cm} (5.9)
where $F^{-1} (\tilde{f}) \in \{ F^{-1} (f), \bar{F}^{-1} (f) \}$. For $k - k' \leq C$ this follows directly from (3.12) and the Sobolev imbedding theorem. For $k - k' \geq C$, the bound (5.9) follows by analyzing the cases $f \in X_k$ and $f \in Y_k$ (see Lemma 4.1 in [7] for a similar proof).

The left-hand side of (5.8) is dominated by

$$C \sum_{|k_1 - k| \leq 2} \left[ \sum_{n \in \mathbb{Z}_{\ell'}} \| \tilde{P}_{k,n} P_k (P_{k_1} (\psi) \cdot P_{k_2} (\psi)) \|_{L^2_{\tilde{e}^C}}^2 \right]^{1/2}$$

$$+ C \sum_{|k_1 - k| \leq 2} \sum_{k' \leq k \leq k - 4} \left[ \sum_{n \in \mathbb{Z}_{\ell'}} \| \tilde{P}_{k,n} P_k (P_{k_1} (\psi) \cdot P_{k_2} (\psi)) \|_{L^2_{\tilde{e}^C}}^2 \right]^{1/2}$$

$$+ C \sum_{k_1, k_2 \geq k - 4} \left[ \sum_{n \in \mathbb{Z}_{\ell'}} \| \tilde{P}_{k,n} P_k (P_{k_1} (\psi) \cdot P_{k_2} (\psi)) \|_{L^2_{\tilde{e}^C}}^2 \right]^{1/2}. \quad (5.10)$$

We use the $L^\infty_{x,t}$ estimate (5.9) on the lower frequency term and the localized $L^2_{\tilde{e}^C} \supset$ estimate (3.13) on the higher frequency term. The first sum in (5.10) is dominated by

$$C \sum_{|k_1 - k| \leq 2} \sum_{k' \leq k} \left( 2^{k_1/2} 2^{-3|k - k'|/4} \| \tilde{E}_T (\Psi) \|_{\tilde{L}^2 (d-2)/2} \cdot (2^{k_2} \| \tilde{E}_T (\Psi) \|_{\tilde{L}^2 (d-2)/2}) \right),$$

which suffices for (5.8). The second sum in (5.10) is dominated by

$$C \sum_{|k_1 - k| \leq 2} \sum_{k' \leq k \leq k - 4} \left[ \sum_{n \in \mathbb{Z}_{\ell'}} \| \tilde{P}_{k,n} P_{k_1} (\psi) \|_{L^2_{\tilde{e}^C}}^2 \right]^{1/2} \left[ \sum_{n \in \mathbb{Z}_{\ell'}} \| \tilde{P}_{k,n} P_{k_2} (\psi) \|_{L^\infty}^2 \right]$$

$$\leq C \sum_{|k_1 - k| \leq 2} \sum_{k' \leq k \leq k - 4} \left( 2^{k_1/2} 2^{-7|k - k'|/8} \| \tilde{E}_T (\Psi) \|_{\tilde{L}^2 (d-2)/2} \cdot (2^{k_2} \| k - k' \| \tilde{E}_T (\Psi) \|_{\tilde{L}^2 (d-2)/2}) \right)$$

which suffices for (5.8). The third sum in (5.10) is dominated by

$$C 2^{|k - k'|/2} \sum_{k_1, k_2 \geq k - 4} \sum_{|k_1 - k_2| \leq 10} \left[ \sum_{n \in \mathbb{Z}_{\ell'}} \| \tilde{P}_{k,n} P_{k_1} (\psi) \|_{L^2_{\tilde{e}^C}}^2 \right]^{1/2} \left[ \sum_{n \in \mathbb{Z}_{\ell'}} \| \tilde{P}_{k,n} P_{k_2} (\psi) \|_{L^\infty}^2 \right]^{1/2}$$

$$\leq C 2^{|k - k'|/2} \sum_{k_1, k_2 \geq k - 4} \sum_{|k_1 - k_2| \leq 10} 2^{3k_1/2} \| \tilde{E}_T (\Psi) \|_{\tilde{L}^2 (d-2)/2} \cdot 2^{-(d-1)|k_1 - k'|} (1 + |k_1 - k'|)^2,$$

which suffices for (5.8) since $d \geq 4$. This completes the proof of Lemma 5.1. \hspace{1cm} \Box

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References