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An effective model for Lipschitz wrinkled arches

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Abstract

Within the framework of the Koiter's linear elastic shell theory, we study the limit model of a Lipschitz curved arch whose mid-surface is periodically waved. The magnitude and the period of the wavings are of the same order. To achieve the asymptotic analysis, we consider a mixed formulation, for which we perform a two-scale homogenization technique. We prove the convergence of the displacements, the rotation of the normal, and the membrane strain. From the limit formulation, we derive an effective model for curved critically wrinkled arches. It introduces two membrane strain functions—instead of one in the classical case—and exhibits a corrector membrane term to the coupling between the rotation of the normal and the membrane strain.

1. Introduction

The aim of the present paper is the introduction, with a rigorous mathematical analysis, of an effective model for critically wrinkled arch structures of Lipschitz mid-curve.

In many industrial areas such as automotive or aerospace, elastic shell structures play a central role. In a few words, a shell is a three-dimensional structure of small thickness. The importance of the potential applications, as well as an original and exciting mathematical modelling, combining differential geometry, and continuum mechanics has led to the emergence of a fast growing discipline, the *shell theory*. A huge amount of literature is nowadays dedicated to the modelling, mathematical and numerical analysis, optimal design, and active control of shells. Among many others, starting from the seminal works of Koiter [1], some recent references are [2–8].

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Generally, classical shells are considered with a smooth mid-surface and a bounded slowly varying curvature. Some authors have investigated the case of rapidly oscillating thickness, e.g., Kohn and Vogelius for plates in [9]. In the cited reference, the authors obtained a model of plate for a critical rate of oscillations, precisely when the magnitude and period of these are comparable.

In the present paper, we study the case where the mid-surface of the shell is waved instead of its thickness. To our knowledge, only very few authors have investigated this approach. In the situation where the magnitude is one order or more smaller than the period, the so-called moderately and slightly wrinkled cases, we refer to the works of [10,11]. A closely related work for smooth wrinkled rods has been studied in [12]. In the cited paper, the author uses the two-scale convergence to derive and justify an effective model for arches of fourth order continuously differentiable mid-surface. Corrector results are also proved. In the present paper, the mid-surface of the arch is required to be only Lipschitz continuous, but we point out that, restricted to C^4 smooth arches, the two models do coincide.

We consider one-dimensional shell structure, that is an elastic arch. The mid-surface of the arch is waved periodically, and the magnitude and period are of the same order. We justify the need for a mixed formulation, necessary to go further in the asymptotic analysis of the waved arch. Then, to achieve the asymptotic analysis, we use the two-scale homogenization method. The mixed formulation for the arches has been introduced by [13]. For a general introduction to the mixed formulation of variational problems, we refer to [14]. The two-scale homogenization technique, introduced by Nguetseng [15] and Allaire [16] is a powerful tool to deal with periodic homogenization. We refer to these papers for the definition and an extensive study of the properties of the two-scale convergence.

2. Classical modelling of an elastic arch

An arch structure is an infinite three-dimensional cylindrical body of small thickness. We denote by L its width at the ground. Then, its geometrical description is the following.

Let $\phi : [0, L] \to \mathbb{R}$ be a function such that $\phi(0) = \phi(L) = 0$. The function ϕ is assumed to have bounded derivatives up to the third order, i.e., $\phi \in W^{3,\infty}([0, L])$.

The *surface* ω of the arch is defined by

$$\omega = \left\{(x, y, z) \in \mathbb{R}^3 \text{ such that } x \in]0, L[,\ z = \phi(x),\ y \in \mathbb{R}\right\}.$$

Let now e be a small positive parameter (the thickness). Then, the three-dimensional arch structure Ω_e is defined by

$$\Omega_e = \left\{M \in \mathbb{R}^3,\ M = m + t \cdot \vec{n}(m),\ \text{where } m \in \omega \text{ and } t \in]-e/2, +e/2[\right\},$$

where $\vec{n}(m)$ denotes the unit normal vector to ω. The thickness parameter e is assumed to be small enough, compared to the curvature $1/R$ of ω, so that any point of Ω_e belongs to one and only one normal to ω. The relative ratio e/R is sometimes used as a parameter to classify shells as thin, shallow, or thick [5].

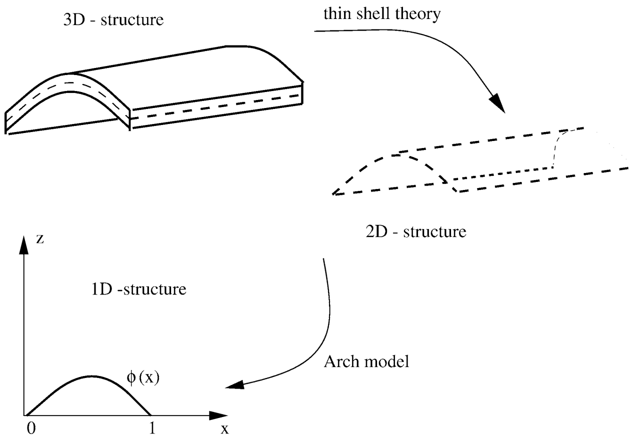

Fig. 1. Description of the arch geometry.

The arch is now loaded, with a load assumed to be *invariant* with respect to the cylinder axis (the direction *Oy* for instance). From the Kirchoff–Love thin shell theory [17] within the linear elasticity framework, the problem reduces to *one-dimensional* problem, set over the generic curve $z = \phi(x)$ (Fig. 1).

In the following, some definitions needed for the statement of the arch equations are given.

- *The local basis* $(\vec{t}(m), \vec{n}(m))$ at a given point $m \in \omega$ of coordinates $(x, \phi(x))$ is

$$\vec{t}(m) = \vec{t}(x) = \begin{pmatrix} \frac{1}{S(x)} \\ \frac{\phi'(x)}{S(x)} \end{pmatrix}, \qquad \vec{n}(m) = \vec{n}(x) = \begin{pmatrix} \frac{-\phi'(x)}{S(x)} \\ \frac{1}{S(x)} \end{pmatrix},$$

  where $\vec{t}(x), \vec{n}(x)$ are, respectively, the unit tangent and normal vectors at the point x, $\phi' = d\phi/dx$ is the derivative of ϕ with respect to the space variable x, and $S(x) = \sqrt{1 + \phi'(x)^2}$.
- *The local displacement vector* $\vec{u}(m)$ of a point m is given by

$$\vec{u}(m) = \vec{u}(x) = u_t(x)\vec{t}(x) + u_n(x)\vec{n}(x),$$

  where u_t and u_n are, respectively, the tangent and normal displacements. From now on, the local displacement *variable* u will be denoted by $u = (u_t, u_n)$.

Let $\Omega =]0, L[$ and denote by V the space of admissible displacements

$$V = H_0^1(\Omega) \times H_0^2(\Omega) \quad \text{arch clamped at both ends,} \tag{1}$$

$$V = H_0^1(\Omega) \times \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \quad \text{arch simply supported at both ends,} \tag{2}$$

where $H_0^1(\Omega)$ and $H_0^2(\Omega)$ are the usual Sobolev spaces.

The elastic energy functional is defined by

$$a(u, v) = \int_0^L \big[Ee\Gamma(u)\Gamma(v) + EMK(u)K(v)\big]S(x)\,dx \quad \text{for all } u, v \in V, \tag{3}$$

where E is the Young modulus, e the constant thickness, and M the second moment of area of the cross-section,

$$\Gamma(v) = \frac{1}{S}v_t' + \frac{1}{R}v_n \quad \text{is the membrane strain,}$$
$$K(v) = \frac{1}{S}\theta'(v) \quad \text{is the bending strain,}$$
$$\theta(v) = -\frac{1}{S}v_n' + \frac{1}{R}v_t \quad \text{is the rotation of the normal,}$$
$$\frac{1}{R} = -\frac{\phi''}{S^3} \quad \text{is the curvature.} \tag{4}$$

The mechanical stress distribution is given by

$$\sigma(v)(x, t) = E\big(\Gamma(v)(x) + tK(v)(x)\big), \quad x \in [0, L],\ t \in [-e/2, +e/2]. \tag{5}$$

In order to give a sense to the elastic energy functional, the derivatives of ϕ up to the third order (appearing in the term $K(v)$) must be bounded, whence the assumption that $\phi \in W^{3,\infty}(\Omega)$.

Now, if we denote by $f = (f_t, f_n)$ the density of the load, then the *equilibrium equation* is given in its variational form by

$$\text{find } u \in V \quad \text{such that } a(u, v) = L(v) \quad \text{for all } v \in V, \tag{6}$$

where the compliance $L(v)$ is generally of the form

$$L(v) = \int_0^L (\vec{f} \cdot \vec{v})S(x)\,dx.$$

It is proved in [18] that the symmetric bilinear mapping $a(\cdot, \cdot)$ is continuous, V-elliptic. Then, assumed that $f \in V'$, the dual space of V, there exists one and only one solution $u \in V$ satisfying Eq. (6).

3. The arch is waved. The first analysis

We consider a plane beam, seen as a particular arch with a mid-surface given by $\phi_p = 0$. The plane mid-surface is periodically waved into a function

$$\phi_\epsilon(x) = \epsilon^r \phi(x/\epsilon), \quad x \in \Omega.$$

The period of the waving is given by the real positive number ϵ which is intended to go to zero. The amplitude is represented by ϵ^r, the positive number r denoting the relative period/amplitude rate.

If we denote by $Y =]0, 1[$ the usual periodic unit-cell, then the function ϕ is a Y-periodic function which is smooth enough to yield a mid-surface ϕ_ϵ of global $W^{3,\infty}(\Omega)$ regularity.

From now on, the useful notation $\dot{\phi}$ stands for the derivative of the function ϕ with respect to the microscopic variable $y = x/\epsilon$.

Now, we have a curved arch whose geometric description strongly depends on ϵ,

$$S_\epsilon(x) = \sqrt{1 + \left(\epsilon^{r-1}\dot{\phi}(y)\right)^2}, \tag{7}$$

$$\frac{1}{R_\epsilon} = -\epsilon^{r-2}\frac{\ddot{\phi}}{S_\epsilon^3}, \tag{8}$$

$$\left(\frac{1}{R_\epsilon}\right)' = -\epsilon^{r-3}\frac{\dddot{\phi}}{S_\epsilon^3} + \epsilon^{r-2}\cdots. \tag{9}$$

The membrane strain $\Gamma(v)$, the rotation of the normal $\theta(v)$, and the bending strain $K(v)$ also depend on ϵ, and so is the solution to the waved arch equations (6), which we denote by u_ϵ. Our main goal is to study the convergence of the sequence of displacements (u_ϵ) when ϵ goes to zero and to state the limit or effective equation satisfied by the limit displacement. We are particularly interested in the cases where effective equations still model (waved) shells.

From a simple look at the leading terms in (7)–(9) one naturally expects the following classification:

(a) $0 \leqslant r < 1$: one has $S_\epsilon \to +\infty$ as $\epsilon \to 0$. Here, we intend to use an infinite length of material. In the limit case $r = 0$, one expects a two-dimensional laminated composite behavior. The shell theory is no more valid.
(b) $1 \leqslant r < 2$: one has $1/R_\epsilon \to +\infty$ as $\epsilon \to 0$. In this case, a Budiansky–Sanders limit model seems out of reach. However, at the rate $r = 1$ numerical experiments exhibit non-negligible effects: the plane beam displacement is affected by the waving at a macroscopic scale.
(c) $2 \leqslant r < 3$: one has $(1/R_\epsilon)' \to +\infty$ as $\epsilon \to 0$. At the rate $r = 2$ numerical experiments show only negligible first order effects.
(d) $3 < r$: one has a strong convergence to zero of the sequence (ϕ_ϵ) in the $W^{3,\infty}(\Omega)$ norm. Since the displacement solution is a smooth function of the shape, see [8], for instance, we get a strong convergence (in the H^1 norm of displacements) of the waved model to the simple plane beam.

The first case (a) is out of the scope of the present paper, which focuses on situations where the limit model is a shell one. The last case (d) is in contrast trivial since the displacements are infinitely differentiable with respect to the arch shapes. Considering—in Ref. [8]—the equation satisfied by the derivative of the displacements with respect to the mid-surface at the point $\phi_p = 0$ which is a plane beam, it is easy to show that this derivative is itself identically equal to zero. Hence, we get a direct proof of the following first order expansion:

$$u(\phi_\epsilon) = u(\phi_p) + o(\epsilon^{r-3}). \tag{10}$$

The expansion above implies that we have a strong convergence of the local waved arch's displacements to the plane beam ones.

We shall see in the next section that both the cases (c) and (b) with $1 < r < 2$ also fit in this situation.

Thus the case $r = 1$ could be seen legitimately as a *critical* waving rate, and all the mathematical analysis done in Sections 4.2 and 5 is related to this critical case.

Now, the numerical tests are clearly in contradiction with the behavior (i.e., divergence to infinity) of the main geometric component in shell theory, namely the curvature and its derivative. This suggests that the classical arch model is not adequate to an asymptotic analysis.

One should *relax* the dependence on the curvature, and get rid of those oscillations only due to the representation of the displacements in the local basis, which is itself rapidly varying.

This is exactly what the mixed formulation presented in the next section is dedicated to.

4. Two-scale asymptotic analysis via a mixed formulation

In the present section, we recall a mixed formulation framework for elastic arches, introduced by Lods [13], on which we perform an asymptotic analysis of the mixed formulation for waved arches by means of the two-scale homogenization technique.

4.1. Recall of the mixed formulation for elastic arches

We start by remarking that any virtual displacement *vector* $\vec{v}$ over a generic arch structure $\psi \in W^{3,\infty}(\Omega)$ can be written in the local basis of tangent-normal unit vectors $(\vec{t}(\psi), \vec{n}(\psi))$ as well as in the global $(\vec{e}_1, \vec{e}_2)$ one,

$$\vec{v} = U_1(\psi, v)\vec{e}_1 + U_2(\psi, v)\vec{e}_2 = v_t\vec{t}(\psi) + v_n\vec{n}(\psi). \tag{11}$$

The key-point of the mixed formulation is the following identity, which *eliminates the curvature term*. It relates the rotation of the normal $\theta(\psi, v)$ and the membrane strain $\Gamma(\psi, v)$ given by the formulae (4) to the global components $(U_1(\psi, v), U_2(\psi, v))$ of the displacement.

Lemma 4.1. *Using the notations above, we have the following*:

$$\begin{aligned}
\theta(\psi, v) &= \frac{1}{S(\psi)^2}\big(\psi' U_1'(\psi, v) - U_2'(\psi, v)\big),\\
\Gamma(\psi, v) &= \frac{1}{S(\psi)^2}\big(U_1'(\psi, v) + \psi' U_2'(\psi, v)\big),
\end{aligned} \tag{12}$$

or, in an equivalent form:

$$\begin{aligned}
U_1'(\psi, v) &= \psi'\theta(\psi, v) + \Gamma(\psi, v),\\
U_2'(\psi, v) &= -\theta(\psi, v) + \psi'\Gamma(\psi, v).
\end{aligned} \tag{13}$$

The equalities hold in $L^2(\Omega)$.

See [13] for the proof.

In the following, we introduce or recall some useful notations and functional spaces

$$v_m = (U_1, U_2, \theta, \mu) \in V_m, \quad \text{where } V_m = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega),$$
$$q_m = (q_1, q_2) \in Q_m, \quad \text{where } Q_m = L^2(\Omega) \times L^2(\Omega). \tag{14}$$

Next, we define the bilinear mappings

$$b_m(\psi; \cdot, \cdot) : V_m \times Q_m \to \mathbb{R},$$
$$b_m(\psi; v_m, q_m) = \int_\Omega (U_1' - \psi'\theta - \mu)q_1 + (U_2' + \theta - \psi'\mu)q_2 \, dx,$$

and (with obvious notations)

$$a_m(\psi; \cdot, \cdot) : V_m \times V_m \to \mathbb{R},$$
$$a_m(\psi; v_m^1, v_m^2) = Ee \int_\Omega \mu^1 \mu^2 S(\psi)\, dx + EM \int_\Omega \frac{1}{S(\psi)} \theta^{1'} \theta^{2'} \, dx.$$

The continuous bilinear mapping b_m expresses via a duality viewpoint that relations (13) are seen as constraints, while a_m is simply a reformulation of the elastic energy of the arch formerly given by (3).

The right-hand side modeling the external forces is written (in the global coordinates system) as

$$L_m(\psi; v_m) = \int_\Omega (f_1 U_1 + f_2 U_2) S(\psi)\, dx. \tag{15}$$

Now, we are ready to set up the mixed formulation.

Find $(u_m, p_m) \in V_m \times Q_m$ such that

$$\begin{cases} \forall v_m \in V_m, & a_m(\psi; u_m, v_m) + b_m(\psi; v_m, p_m) = L_m(\psi; v_m), \\ \forall q_m \in Q_m, & b_m(\psi; u_m, q_m) = 0. \end{cases} \tag{16}$$

The existence and uniqueness of $(u_m, p_m) \in V_m \times Q_m$ solution to the mixed problem above is proved in Lods [13] by application of the Brezzi's theorem [14]. To this end, the following assumptions, also known as the BBL conditions, are shown to hold:

(Ha) The continuous bilinear mapping $a_m(\psi; \cdot, \cdot)$ is elliptic on the kernel of b_m, that is the space

$$V_m^\psi = \{ v_m \in V_m \text{ such that } \forall q_m \in Q_m,\ b_m(\psi; v_m, q_m) = 0 \}. \tag{17}$$

(Hb) The continuous bilinear mapping $b_m(\psi; \cdot, \cdot)$ satisfies the condition

$$\inf_{\substack{q_m \in Q_m \\ \|q_m\|=1}} \sup_{\substack{v_m \in V_m \\ \|v_m\|=1}} b_m(\psi; v_m, q_m) > 0.$$

The equivalence of two problems (16) and (6) holds when the mid-surface $\psi \in W^{3,\infty}(\Omega)$. In the case of Lipschitzian arches, i.e., $\psi \in W^{1,\infty}(\Omega)$, the mixed formulation yields a *generalized* model for arch structures.

From now on, we shall consider exclusively the generalized Lipschitzian arch model. We shall omit the subscript “m” standing for “mixed” in the present section.

In the next section, we use two properties (Ha) and (Hb) to get a priori estimates of the mixed solution for the waved arch. These estimates are used as a preamble to the two-scale homogenization technique. Then, we derive a limit mixed problem for which we prove that corresponding (Ha) and (Hb) hold.

4.2. A two-scale limit for the mixed problem

First, we recall a few results from the two-scale homogenization [16].

We denote by $\mathcal{C}^\infty_\#(Y)$ the space of infinitely differentiable functions in $\mathbb{R}$ which are Y-periodic. The space $\mathcal{D}(\Omega; \mathcal{C}^\infty_\#(Y))$ denotes the space of infinitely differentiable functions of compact support in Ω with values in $\mathcal{C}^\infty_\#(Y)$.

Definition 4.1. *A sequence (u_ϵ) of $L^2(\Omega)$ is said to two-scale converge if there exists a function $u_0(x; y) \in L^2(\Omega \times Y)$ such that*

$$\lim_{\epsilon \to 0} \int_\Omega u_\epsilon(x) v(x; x/\epsilon)\, dx = \int_{\Omega \times Y} u_0(x; y) v(x; y)\, dx\, dy \tag{18}$$

for any $v(x; y) \in \mathcal{D}(\Omega; \mathcal{C}^\infty_\#(Y))$.

We shall denote by $u_\epsilon \rightharpoonup\!\!\rightharpoonup u_0$ when u_ϵ two-scale converges to u_0.

We shall also use the standard notation $\langle v \rangle = \int_Y v(x; y)\, dy$ which stands for the mean-value of a $Y-$periodic function v.

We have the following:

(P1) Up to a subsequence, bounded sequences of $L^2(\Omega)$ two-scale converge;
(P2) If $u_\epsilon \rightharpoonup\!\!\rightharpoonup u_0$ in $L^2(\Omega \times Y)$ then $u_\epsilon \rightharpoonup \langle u_0 \rangle$ in $L^2(\Omega)$ weakly;
(P3) Up to a subsequence, bounded sequences (u_ϵ) of $H^1(\Omega)$ two-scale converge: there exist $u \in H^1(\Omega)$ and $u_1 \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ such that $u_\epsilon \rightharpoonup u$ in $H^1(\Omega)$ weakly and $u'_\epsilon \rightharpoonup\!\!\rightharpoonup u' + \dot{u}_1$.

Now, let us first rapidly conclude in the case where the wavings are of the form

$$\phi_\epsilon(x) = \phi_0(x) + \epsilon^r \phi(x/\epsilon), \quad r > 1.$$

We shall denote by (u^ϵ, p^ϵ) and (u^0, p^0) the respective solutions in $V \times Q$ of the mixed problem (16) set for $\psi = \phi_\epsilon$ and for $\psi = \phi_0$ (i.e., the non-waved arch).

It is then proved in [19] that under the assumptions

$$\phi_\epsilon, \phi_0 \in W^{1,\infty}(\Omega),$$

$$\|\phi_\epsilon\|_{W^{1,\infty}} \text{ is uniformly bounded w.r.t. } \epsilon,$$

$$\|\phi_\epsilon - \phi_0\|_{H^1(\Omega)} \to 0 \quad \text{with } \epsilon \to 0,$$
$$L(\phi_\epsilon; \cdot) \to L(\phi_0; \cdot) \quad \text{with } \epsilon \to 0 \text{ in the dual space } V', \tag{19}$$

one has the strong convergences

$$u^\epsilon \to u^0 \quad \text{in } V, \tag{20}$$
$$p^\epsilon \to p^0 \quad \text{in } Q. \tag{21}$$

The latter assumptions obviously hold for our sequence of periodic functions $\phi_\epsilon(x)$ with $r > 1$. Thus, the limit model is simply the plane beam one. This result is an evidence which corroborates the criticality of the case $r = 1$.

From now on, we consider the waved mid-surfaces described by functions

$$\phi_\epsilon(x) = \phi_0(x) + \epsilon\phi(x/\epsilon),$$

where $\epsilon > 0$ is the period as well as the magnitude of the waving, $x \in \Omega$ is a macroscopic space variable. The function $\phi_0 \in W^{1,\infty}(\Omega)$ describes the mid-curve of the arch before it is waved.

The function ϕ belongs to a set Λ defined by

$$\Lambda = \{\psi \in W^{1,\infty}(Y),\ \psi \text{ is } Y\text{-periodic}, \psi(0) = \psi(1)\}. \tag{22}$$

Thanks to the definition of Λ, the functions ϕ_ϵ belong to the space $W^{1,\infty}(\Omega)$ and are admissible generalized arch mid-surfaces.

The mechanical unknowns which describe the behavior of the loaded waving elastic arch are now the mixed variables

$$u^\epsilon = (U_1^\epsilon, U_2^\epsilon, \theta^\epsilon, \mu^\epsilon) \in V, \qquad p^\epsilon = (p_1^\epsilon, p_2^\epsilon) \in Q,$$

solution to the mixed problem

$$\begin{cases} \forall v = (U_1, U_2, \theta, \mu) \in V, \\ Ee\int_\Omega \mu^\epsilon \mu S(\phi_\epsilon)\,dx + EM\int_\Omega \frac{1}{S(\phi_\epsilon)}(\theta^\epsilon)'\theta'\,dx \\ \qquad + \int_\Omega (U_1' - (\phi_\epsilon)'\theta - \mu)p_1^\epsilon + (U_2' + \theta - (\phi_\epsilon)'\mu)p_2^\epsilon\,dx \\ \quad = \int_\Omega (f_1^\epsilon U_1 + f_2^\epsilon U_2)S(\phi_\epsilon)\,dx, \\ \forall q = (q_1, q_2) \in Q, \\ \int_\Omega ((U_1^\epsilon)' - (\phi_\epsilon)'\theta^\epsilon - \mu^\epsilon)q_1 + ((U_2^\epsilon)' + \theta^\epsilon - (\phi_\epsilon)'\mu^\epsilon)q_2\,dx = 0. \end{cases} \tag{23}$$

For the waved arch structures, it is natural to assume that the external forces $f^\epsilon = (f_1^\epsilon, f_2^\epsilon)$ are periodic. For instance, this is the case of the pressure, self-weight and snow loadings which are common loadings for arch structures.

We shall assume that the loading is of the form $f^\epsilon(x) = f(x; x/\epsilon)$. The function $f(x; y)$ belongs to the space $L^2(\Omega; C_\#(Y))$ of measurable and square integrable functions, with values in the space of continuous Y-periodic functions.

For such functions f^ϵ in $L^2(\Omega; C_\#(Y))$, one has $\|f^\epsilon(x)\|_{L^2(\Omega)} \leqslant \|f(x; y)\|_{L^2(\Omega\times Y)}$.

We recall that by convention $\dot\phi(x; y)$ denotes the derivative of a function $\phi(x; y)$ with respect to the microscopic variable $y \in Y$. We shall also denote by S the function

$$S = S(x, y) = \sqrt{1 + (\phi_0')^2(x) + (\dot\phi)^2(y)}.$$

Now, we are ready to state the following convergence theorem.

Theorem 4.1. *Let $u^\epsilon = (U_1^\epsilon, U_2^\epsilon, \theta^\epsilon, \mu^\epsilon) \in V$ and $p^\epsilon = (p_1^\epsilon, p_2^\epsilon) \in Q$ be the unique solutions to the waved arch problem* (23)*. Then, we have*

(i) *There exist* (*unique*) *functions $U_1^0, U_2^0, \theta^0 \in H_0^1(\Omega)$, $\mu^0 \in L^2(\Omega \times Y)$, and $U_{1c}, U_{2c}, \theta_c \in L^2(\Omega; H_\#^1(Y)/\mathbb{R})$ such that the functions U_1^ϵ, U_2^ϵ, and θ^ϵ weakly converge in $H_0^1(\Omega)$, respectively, to U_1^0, U_2^0, θ^0, and*

$$\begin{cases} (U_1^\epsilon)' \rightharpoonup\rightharpoonup (U_1^0)' + \dot{U}_{1c}, \\ (U_2^\epsilon)' \rightharpoonup\rightharpoonup (U_2^0)' + \dot{U}_{2c}, \\ (\theta^\epsilon)' \rightharpoonup\rightharpoonup (\theta^0)' + \dot{\theta}_c, \\ \mu^\epsilon \quad \rightharpoonup\rightharpoonup \mu^0. \end{cases} \tag{24}$$

Moreover, the function μ^ϵ weakly converges in $L^2(\Omega)$ to $(U_1^0)' - \phi_0'\theta^0$.

(ii) *There exists a unique function $p^0 \in L^2(\Omega \times Y)^2$ such that $p^\epsilon \rightharpoonup\rightharpoonup p^0$.*

(iii) *The functions $U_1^0, U_2^0, \theta^0 \in H_0^1(\Omega)$, $\mu^0 \in L^2(\Omega \times Y)$, $U_{1c}, U_{2c}, \theta_c \in L^2(\Omega; H_\#^1(Y)/\mathbb{R})$, and $p^0 = (p_1^0, p_2^0) \in L^2(\Omega \times Y)^2$ are solution to the well-posed limit mixed formulation*

$$\begin{cases} \forall U_1, U_2, \theta \in H_0^1(\Omega),\ V_1, W_1, \theta_1 \in L^2(\Omega; H_\#^1(Y)/\mathbb{R}),\ \mu \in L^2(\Omega \times Y), \\ Ee \int_{\Omega\times Y} \mu^0 \mu S\, dx\, dy + EM \int_{\Omega\times Y} \frac{1}{S}[(\theta^0)' + \dot{\theta}_c)(\theta' + \dot{\theta}_1)]\, dx\, dy \\ \qquad + \int_{\Omega\times Y} (U_1' + \dot{V}_1 - (\phi_0' + \dot{\phi})\theta - \mu) p_1^0 \\ \qquad + (U_2' + \dot{W}_1 + \theta - (\phi_0' + \dot{\phi})\mu) p_2^0\, dx\, dy \\ \quad = \int_{\Omega\times Y} (f_1(x;y)U_1 + f_2(x;y)U_2) S\, dx\, dy, \\ \forall q_1, q_2 \in L^2(\Omega \times Y), \\ \int_{\Omega\times Y} ((U_1^0)' + \dot{U}_{1c} - (\phi_0' + \dot{\phi})\theta^0 - \mu^0) q_1 \\ \qquad + ((U_2^0)' + \dot{U}_{2c} + \theta^0 - (\phi_0' + \dot{\phi})\mu^0) q_2\, dx\, dy = 0. \end{cases} \tag{25}$$

Proof. *The sequence (u^ϵ) is uniformly bounded w.r.t. ϵ in V so that it two-scale converges.*

Zeine has proved in [20] that the continuous bilinear mappings $a(\phi_\epsilon; \cdot, \cdot)$ are uniformly elliptic with respect to the parameter $\epsilon > 0$ over the spaces V^{ϕ_ϵ} defined by (17), provided that one has a uniform bound: $\|\phi_\epsilon\|_{1,\infty} \leqslant C$. In our case, we have $\|\phi_\epsilon\|_{1,\infty} \leqslant \|\phi_0\|_{1,\infty} + \epsilon\|\phi\|_\infty + \|\dot{\phi}\|_\infty$ which ensures the needed uniform upper-bound.

From other part, since $a(\phi_\epsilon; \cdot, \cdot)$ depends on ϕ_ϵ through only its first derivative, the bilinear mapping is also uniformly continuous. We conclude by the classical arguments of a priori estimates for elliptic problems that $\|u^\epsilon\| \leqslant C\|f^\epsilon(x)\|_{L^2(\Omega)} \leqslant C\|f(x;y)\|_{L^2(\Omega\times Y)}$ uniformly.

Since

$$\|u^\epsilon\|^2 = \|U_{1\epsilon}\|_{H_0^1}^2 + \|U_{2\epsilon}\|_{H_0^1}^2 + \|\theta_\epsilon\|_{H_0^1}^2 + \|\mu_\epsilon\|_{L^2}^2,$$

we apply the two-scale compactness result (P3) to get the weak convergence of the functions in H_0^1 and the two-scale convergence in $L^2(\Omega \times Y)$ of the derivatives.

The last point in assertion (i) comes from the remark that since u^ϵ belongs to the space V^{ϕ_ϵ} we have: $\mu_\epsilon = (U_{1\epsilon})' - (\phi_\epsilon)'\theta_\epsilon$. Using property (P2) and noticing that $(\phi_\epsilon)' \rightharpoonup (\phi_0)' + \langle\dot{\phi}\rangle = (\phi_0)'$, we have

$$\mu_\epsilon \rightharpoonup \big\langle \big(U_1^0\big)' + \dot{U}_{1c} - (\phi_0' + \dot{\phi})\theta_0 \big\rangle = \big(U_1^0\big)' - \phi_0'\theta_0.$$

The sequence (p^ϵ) is uniformly bounded w.r.t. ϵ in Q so that it two-scale converges.

It is proved in [13] that the bilinear mapping $b(\phi_\epsilon;\cdot,\cdot)$ enjoys the following property: There exists a positive constant C such that, for any given $q \in Q$, there exists a function $v \in V$ such that

$$b(\phi_\epsilon; v, q) = \|q\|^2 \quad \text{and} \quad \|v\| \leqslant C\big(\|\phi_\epsilon\|_{1,\infty} + 1\big)\|q\|, \tag{26}$$

the constant $C > 0$ being independent of ϕ_ϵ. We shall denote by w^ϵ the corresponding function obtained thanks to the property above when we set $q = p^\epsilon$.

Now, from Eq. (23) we have

$$\|p^\epsilon\|^2 = b(\phi_\epsilon; w^\epsilon, p^\epsilon) = -a(\phi_\epsilon; u^\epsilon, w^\epsilon) + L(\phi_\epsilon; w^\epsilon).$$

Then, using the uniform continuity of $a(\phi_\epsilon;\cdot,\cdot)$ and $L(\phi_\epsilon;\cdot)$ with respect to ϵ we get

$$\|p^\epsilon\|^2 \leqslant \big\{C(\phi)\|u^\epsilon\| + \|f\|_{L^2(\Omega\times Y)}\big\}\|w^\epsilon\|.$$

We replace now $\|w^\epsilon\|$ by its upper-bound given by (26) and simplify the inequality above by $\|p^\epsilon\|$. The proof ends by remarking that from above, $\|u^\epsilon\|$ is itself uniformly bounded.

Since the sequence (p^ϵ) is bounded uniformly with respect to ϵ, there exists a subsequence which two-scale converges to a limit $p^0 \in L^2(\Omega \times Y)^2$. The convergence of the whole sequence comes from the uniqueness of the limit, and is proved below.

We pass to the two-scale limit in the mixed equation (23).

First, we choose test functions of the form

$$\begin{aligned} &v = \big(U_1(x) + \epsilon V_1(x; x/\epsilon); U_2(x) + \epsilon W_1(x; x/\epsilon); \theta(x) + \epsilon\theta_1(x; x/\epsilon); \mu(x; x/\epsilon)\big),\\ &\quad U_1, U_2, \theta \in \mathcal{D}(\Omega),\ \mu, V_1, W_1, \theta_1 \in \mathcal{D}\big(\Omega; C_\#^\infty(Y)\big),\\ &q = \big(q_1(x; x/\epsilon), q_2(x; x/\epsilon)\big), \quad q_1, q_2 \in \mathcal{D}\big(\Omega; C_\#^\infty(Y)\big). \end{aligned} \tag{27}$$

(Here, the usual notation $\mathcal{D}$ stands for the space of infinitely differentiable functions with compact support, and a standard density argument of such spaces in L^2 and H_0^1 is used.)

Then, applying the definition of the two-scale convergence, we can pass to the limit in ϵ in each of the terms of Eq. (23).

As an illustrating example, considering the test function

$$w(x; y) = \frac{1}{S}\big(\theta'(x) + \dot{\theta}_1(x; y)\big),$$

we get

$$\int_\Omega (\theta^\epsilon)' w(x; x/\epsilon)\,dx = \int_\Omega \frac{1}{S(\phi_\epsilon)}(\theta^\epsilon)'\big(\theta + \epsilon\theta_1(x; x/\epsilon)\big)'\,dx + O(\epsilon) \tag{28}$$

so that

$$\int_\Omega (\theta^\epsilon)' w(x; x/\epsilon)\,dx \to \int_{\Omega\times Y} \frac{1}{S}\big[\big((\theta^0)' + \dot{\theta}_c\big)(\theta' + \dot{\theta}_1)\big]\,dx\,dy$$

for the bending term, and

$$\int_{\Omega} \big((U_1^\epsilon)' - (\phi_\epsilon)'\theta^\epsilon - \mu^\epsilon\big) q_1\,dx \to \int_{\Omega\times Y} \big((U_1^0)' + \dot U_{1c} - (\phi_0' + \dot\phi)\theta^0 - \mu^0\big) q_1\,dx\,dy$$

for the first term of the duality functional.

The limit mixed formulation given by (25) is then straightforward.

The limit mixed formulation is well-posed.

In order to make the expository as clear as possible, we again introduce adapted notations and functional spaces

$$\begin{aligned} &v^0 = (U_1, U_2, \theta, \mu) \in V^0, \quad \text{where } V^0 = H_0^1(\Omega)^3 \times L^2(\Omega\times Y),\\ &v^c = (V_1, W_1, \theta_1) \in V^c, \quad \text{where } V^c = L^2\big(\Omega; H^1_\#(Y)/\mathbb{R}\big)^3,\\ &v^H = (v^0, v^c) \in V^H, \quad \text{where } V^H = V^0 \times V^c,\\ &q^H = (q_1, q_2) \in Q^H, \quad \text{where } Q^H = L^2(\Omega\times Y) \times L^2(\Omega\times Y). \end{aligned} \tag{29}$$

The space V^H is endowed with the norm

$$\begin{aligned} \|v^H\|^2 = {}& \|U_1'\|^2_{L^2(\Omega)} + \|U_2'\|^2_{L^2(\Omega)} + \|\theta'\|^2_{L^2(\Omega)} + \|\mu\|^2_{L^2(\Omega\times Y)}\\ &+ \|\dot V_1\|^2_{L^2(\Omega\times Y)} + \|\dot W_1\|^2_{L^2(\Omega\times Y)} + \|\dot\theta_1\|^2_{L^2(\Omega\times Y)}, \end{aligned} \tag{30}$$

while the space Q^H is endowed with its natural L^2 norm.

The limit bilinear mappings are defined by

$$\begin{aligned} &b^H(\phi;\cdot,\cdot): V^H \times Q^H \to \mathbb{R},\\ &b^H(\phi; v^H, q^H) = \int_{\Omega\times Y} \big(U_1' + \dot V_1 - (\phi_0' + \dot\phi)\theta - \mu\big) q_1\\ &\qquad + \big(U_2' + \dot W_1 + \theta - (\phi_0' + \dot\phi)\mu\big) q_2\,dx\,dy, \end{aligned} \tag{31}$$

and (with obvious notations)

$$\begin{aligned} &a^H(\phi;\cdot,\cdot): V^H \times V^H \to \mathbb{R},\\ &a^H\big(\phi; (v^H)^1, (v^H)^2\big) = Ee \int_{\Omega\times Y} \mu^1(x;y)\mu^2(x;y) S\,dx\,dy\\ &\qquad + EM \int_{\Omega\times Y} \frac{1}{S}\big[\big((\theta^1)' + \dot\theta_c^1\big)\big((\theta^2)' + \dot\theta_1^2\big)\big]\,dx\,dy. \end{aligned} \tag{32}$$

We shall also need to define the kernel of b^H by

$$V^{H,\phi} = \big\{v^H \in V^H \text{ such that } \forall q^H \in Q^H,\ b^H(\phi; v^H, q^H) = 0\big\}. \tag{33}$$

The limit right-hand side is easily obtained as being

$$L^H(\phi; v^H) = \int_{\Omega\times Y} \big(f_1(x;y)U_1 + f_2(x;y)U_2\big) S\,dx\,dy. \tag{34}$$

We denote by $u^H = (U_1^0, U_2^0, \theta^0, \mu^0; U_{1c}, U_{2c}, \theta_c)$ and $p^H = p^0 = (p_1^0, p_2^0)$.

Then the limit equation reads in the classical mixed formulation: Find $(u^H, p^H) \in V^H \times Q^H$ such that

$$\begin{cases} \forall v^H \in V^H, & a^H(\phi; u^H, v^H) + b^H(\phi; v^H, p^H) = L^H(\phi; v^H), \\ \forall q^H \in Q^H, & b^H(\phi; u^H, q^H) = 0. \end{cases} \tag{35}$$

The continuity of the mappings $a^H(\phi; \cdot, \cdot)$, $L^H(\phi; \cdot)$, and $b^H(\phi; \cdot, \cdot)$ over their respective spaces is straightforward. In order to prove that the problem (35) above is well-posed, it is enough to prove that the following BBL conditions hold:

(a) Ellipticity of the limit mixed energy $a^H(\phi; \cdot, \cdot)$ over the space $V^{H,\phi}$;
(b) The inf-sup condition for the limit bilinear mapping $b^H(\phi; \cdot, \cdot)$.

We shall use the generic element $v^H = (U_1, U_2, \theta, \mu; V_1, W_1, \theta_1)$ of V^H.

(a) The continuous bilinear mapping $a^H(\phi; \cdot, \cdot)$ is elliptic over the space $V^{H,\phi}$ defined by (33). First, remark that since θ_1 is Y-periodic, one has immediately

$$a^H(\phi; v^H, v^H) \geqslant A \int_{\Omega \times Y} \mu^2 \, dx \, dy + B \int_{\Omega \times Y} (\theta' + \dot{\theta}_1)^2 \, dx \, dy \tag{36}$$

$$\geqslant A \int_{\Omega \times Y} \mu^2 \, dx \, dy + B \int_{\Omega} (\theta')^2 dx + B \int_{\Omega \times Y} (\dot{\theta}_1)^2 \, dx \, dy. \tag{37}$$

Secondly, since the function v^H belongs to the space $V^{H,\phi}$, we have

$$(U_1)' + \dot{V}_1 = (\phi_0' + \dot{\phi})\theta + \mu, \qquad (U_2)' + \dot{W}_1 = -\theta + (\phi_0' + \dot{\phi})\mu. \tag{38}$$

Now, using the identities above, the Poincaré inequality for θ and the Y-periodicity of V_1 and W_1 it is an easy exercise, left to the reader, to derive the ellipticity of $a^H(\phi; \cdot, \cdot)$ in the (induced) norm of $V^{H,\phi}$.

(b) The inf-sup condition. A classical method to prove the inf-sup condition (Hb) is to explicitly construct for any given $q^H \in Q^H$, a function $v^H \in V^H$ such that

$$b^H(\phi; v^H, q^H) = \|q^H\|^2 \quad \text{and} \quad \|v^H\| \leqslant C\|q^H\|, \tag{39}$$

the constant $C > 0$ being independent of q^H.

Given any arbitrary function $q^H = (q_1, q_2)$ in Q^H, one has to yield a function $v^H \in V^H$ such that

$$\begin{aligned} q_1(x; y) &= (U_1)' + \dot{V}_1 - (\phi_0' + \dot{\phi})\theta - \mu, \\ q_2(x; y) &= (U_2)' + \dot{W}_1 + \theta - (\phi_0' + \dot{\phi})\mu. \end{aligned} \tag{40}$$

One could easily check that the following candidates work:

$$\mu(x; y) = -\int_{\Omega \times Y} q_1(x; y) \, dx \, dy,$$

$$\theta(x) = 4\left(\frac{1}{2} - \left|\frac{1}{2} - x\right|\right) \int_{\Omega \times Y} q_2(x; y) \, dx \, dy,$$

$$U_1(x) = \int_0^x \big(\langle q_1(s;\cdot)\rangle + \langle \mu(s;\cdot)\rangle\big)\,ds,$$
$$U_2(x) = \int_0^x \big(\langle q_2(s;\cdot)\rangle - \theta(s)\big)\,ds, \tag{41}$$

and,

$$V_1(x;y) = \int_0^y \big(q_1(x;t) + \mu(x;t) + (\phi_0' + \dot\phi)(t)\theta(x) - U_1'(x)\big)\,dt + \text{constant},$$
$$W_1(x;y) = \int_0^y \big(q_2(x;t) - \theta(x) + (\phi_0' + \dot\phi)(t)\mu(x;t) - U_2'(x)\big)\,dt + \text{constant}. \tag{42}$$

It is straightforward from this explicit construction that the upper-bound required in (39) is fulfilled. Moreover, the constant C can be chosen independent of the parameter ϕ. □

We have then established the existence and uniqueness of the limits u^H and p^H solutions to the limit mixed problem (35). As a consequence, we also have proved the convergence of the whole sequences (u^ϵ) and (p^ϵ).

5. An effective model for Lipschitz waved arches

The limit mixed formulation obtained in the previous section has the advantage to precisely describe the two scales of behavior, the macroscopic and the microscopic (also called hidden scale) one. For numerical purpose nevertheless, this advantage becomes a drawback, since it implies a dramatical increasing in the complexity of the calculations.

Mainly for this reason, computational mechanicians are always interested in models where one can get rid of the microscopic variable and functions (e.g., first order correctors). When possible, one tries to obtain a so-called effective or homogenized model which is set in the macroscopic variable/functions only.

In the sequel, we build in three steps such an effective model for the present case of periodically waved arches.

First step. In the limit equation (25), we make $U_1 = U_2 = \theta = \mu = 0$ and $\theta_1 = 0$. We obtain that for all $V_1, W_1 \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$,

$$\int_{\Omega\times Y} \big(\dot V_1 p_1^0 + \dot W_1 p_2^0\big)\,dx\,dy = 0. \tag{43}$$

A simple integration by parts yields that the function p^0 does not depend on the microscopic variable y,

$$p_1^0(x;y) = p_1^0(x), \qquad p_2^0(x;y) = p_2^0(x).$$

Thus, thanks to the periodicity of the functions V_1, W_1, and ϕ, the dual term $b^H(\phi; v^H, p^H)$ reduces to

$$b^H(\phi; v^H, p^H) = \int_\Omega \big((U_1)' - \langle\mu\rangle\big) p_1^0 + \big((U_2)' + \theta - \big\langle(\phi_0' + \dot\phi)\mu\big\rangle\big) p_2^0 \, dx. \tag{44}$$

Then we consider test functions q^H which themselves do not depend on the variable y. Hence the dual equation in (25) reads

$$\forall q_1, q_2 \in L^2(\Omega),$$
$$\int_\Omega \big(\big(U_1^0\big)' - \langle\mu^0\rangle\big) q_1 + \big(\big(U_2^0\big)' + \theta^0 - \big\langle(\phi_0' + \dot\phi)\mu^0\big\rangle\big) q_2 \, dx = 0. \tag{45}$$

Remark that V_1, W_1 as well as U_{1c}, U_{2c} have completely disappeared from Eqs. (44) and (45).

Second step. Now, we focus our attention on the bending term, namely,

$$\mathcal{I}_{\text{bending}} = \int_{\Omega\times Y} \frac{1}{S}\big[(\theta^0)' + \dot\theta_c)(\theta' + \dot\theta_1)\big] \, dx \, dy. \tag{46}$$

First, by setting $U_1 = U_2 = \theta = \mu = 0$ and $V_1 = W_1 = 0$ in (25), we derive the equation

$$\begin{cases} \frac{d}{dy}\big(\frac{1}{S}\{(\theta^0)' + \dot\theta_c\}\big) = 0 \quad \text{in } \Omega \times Y, \\ y \to \theta_c(x; y) \text{ is } Y\text{-periodic.} \end{cases} \tag{47}$$

Now, we have to handle a classical homogenized equation for which the *cell equations* technique can be used.

One defines the function $w_\theta \in H^1_\#(Y)/\mathbb{R}$ by

$$\begin{cases} \frac{d}{dy}\big(\frac{1}{S}\{1 + \dot w_\theta\}\big) = 0 \text{ in } Y, \\ y \to w_\theta(y) \text{ is } Y\text{-periodic.} \end{cases} \tag{48}$$

Then, one can easily show that $\theta_c(x; y) = (\theta^0)'(x) w_\theta(y)$. Then, setting $\theta_1(x; y) = \theta'(x) z_\theta(y)$, where $z_\theta \in H^1_\#(Y)/\mathbb{R}$, one gets

$$\mathcal{I}_{\text{bending}} = \int_\Omega (\theta^0)'\theta' \int_Y \frac{1}{S}(1 + \dot w_\theta) \, dy \, dx. \tag{49}$$

It is also easy to get from Eq. (48) that

$$\int_Y \frac{1}{S}(1 + \dot w_\theta) dy = \frac{1}{\langle S\rangle},$$

which reduces the term $\mathcal{I}_{\text{bending}}$ to

$$\mathcal{I}_{\text{bending}} = \int_\Omega \frac{1}{\langle S\rangle}(\theta^0)'\theta' \, dx. \tag{50}$$

The (nearly) effective equation for the waved arch model can then be stated as follows: Find $U_1^0, U_2^0, \theta^0 \in H_0^1(\Omega)$, $\mu^0 \in L^2(\Omega \times Y)$, and $p^0 = (p_1^0, p_2^0) \in L^2(\Omega) \times L^2(\Omega)$ such that

$$\begin{cases} \forall U_1, U_2, \theta \in H_0^1(\Omega),\ \forall \mu \in L^2(\Omega \times Y), \\ Ee\int_{\Omega\times Y} \mu^0 \mu S\,dx\,dy + EM\int_\Omega \frac{1}{\langle S\rangle}\int_\Omega (\theta^0)'\theta'\,dx \\ \quad + \int_\Omega ((U_1)' - \langle\mu\rangle)p_1^0 + ((U_2)' + \theta - \langle(\phi_0' + \dot\phi)\mu\rangle)p_2^0\,dx \\ \quad = \int_\Omega (\langle f_1 S\rangle U_1 + \langle f_2 S\rangle U_2)\,dx, \\ \forall (q_1, q_2) \in L^2(\Omega) \times L^2(\Omega), \\ \int_\Omega ((U_1^0)' - \langle\mu^0\rangle)q_1 + ((U_2^0)' + \theta^0 - \langle(\phi_0' + \dot\phi)\mu^0\rangle)q_2\,dx = 0. \end{cases} \tag{51}$$

The Brezzi conditions for this mixed formulation are fulfilled. The proof is slightly the same than the one of the limit problem (35). Hence, $(U_1^0, U_2^0, \theta^0; \mu^0; p^0)$ is the unique mixed solution of both Eqs. (35) and (51).

Third step. Notice that (51) is only a semi-effective mixed formulation because $\mu_0(x; y)$ shows. As a matter of fact, one cannot expect that the mean-value $\langle\mu^0\rangle$ is the effective unknown for membrane strain, since in the problem above $\langle(\phi_0' + \dot\phi)\mu^0\rangle$ cannot be expressed as a linear function of the latter.

So, in order to go on in the homogenization process, we set $U_1 = U_2 = \theta = 0$ in Eq. (51), which reduces to

$$\begin{aligned} &\forall \mu \in L^2(\Omega \times Y), \\ &Ee\int_{\Omega\times Y} \mu^0 \mu S\,dx\,dy = \int_{\Omega\times Y} \big(p_1^0 + (\phi_0' + \dot\phi)p_2^0\big)\mu\,dx\,dy. \end{aligned} \tag{52}$$

This equality in $L^2(\Omega \times Y)$ proves that μ^0 can be written as

$$Ee\mu^0(x; y)S = p_1^0(x) + \big(\phi_0'(x) + \dot\phi(y)\big)p_2^0(x).$$

It is then legitimate to take test functions $\mu \in L^2(\Omega \times Y)$ of the same form

$$\mu(x; y) = \frac{1}{S}\mu_1(x) + \frac{(\phi_0'(x) + \dot\phi(y))}{S}\mu_2(x)$$

with μ_1, μ_2 generic elements of the space $L^2(\Omega)$.

Thus, the homogenized membrane strain μ^0 is uniquely described by the pair $(\mu_1^0, \mu_2^0) \in L^2(\Omega) \times L^2(\Omega)$ such that

$$\mu^0(x; y) = \frac{1}{S}\mu_1^0(x) + \frac{(\phi_0'(x) + \dot\phi(y))}{S}\mu_2^0(x).$$

Finally, using these new expressions for μ and μ^0, we put them in the mixed formulation (51) in order to get, this time, a completely effective equation.

We have the following result.

Theorem 5.1. *The global displacements $(U_1^\epsilon, U_2^\epsilon) \in H_0^1(\Omega)$, the rotation of the normal $\theta^\epsilon \in H_0^1(\Omega)$, the membrane strain $\mu^\epsilon \in L^2(\Omega)$, and the Lagrange multipliers $(p_1^\epsilon, p_2^\epsilon) \in$*

$L^2(\Omega)\times L^2(\Omega)$ *which are the solution to the mixed waved arch problem* (23) *weakly converge in their spaces, respectively, to*

$$\left(U_1^0,U_2^0\right)\in H_0^1(\Omega),\quad \theta^0\in H_0^1(\Omega),\quad \mu^E=\left(\left\langle\frac{1}{S}\right\rangle\mu_1^0+\left\langle\frac{(\phi_0'+\dot\phi)}{S}\right\rangle\mu_2^0\right)\in L^2(\Omega),$$

and

$$\left(p_1^0,p_2^0\right)\in L^2(\Omega)\times L^2(\Omega).$$

The limit, or effective, functions above are the unique solution of the following effective mixed formulation: *Find* $U_1^0,U_2^0,\theta^0\in H_0^1(\Omega)$, $\boldsymbol{\mu^0}=(\mu_1^0,\mu_2^0)^T\in L^2(\Omega)^2$, *and* $p^0=(p_1^0,p_2^0)\in L^2(\Omega)^2$, *such that*

$$\begin{cases}\forall U_1,U_2,\theta\in H_0^1(\Omega),\ \forall\boldsymbol{\mu}=(\mu_1,\mu_2)^T\in L^2(\Omega)^2,\\ Ee\int_\Omega(A^E\boldsymbol{\mu^0})\cdot\boldsymbol{\mu}\,dx+EM\int_\Omega B^E(\theta^0)'\theta'\,dx\\ \qquad+\int_\Omega((U_1)'-(\phi_0)'\theta-A_1^E\cdot\boldsymbol{\mu})p_1^0+((U_2)'+\theta-A_2^E\cdot\boldsymbol{\mu})p_2^0dx\\ \quad=\int_\Omega(\langle f_1S\rangle U_1+\langle f_2S\rangle U_2)\,dx,\\ \forall(q_1,q_2)\in L^2(\Omega)\times L^2(\Omega),\\ \int_\Omega((U_1^0)'-(\phi_0)'\theta^0-A_1^E\cdot\boldsymbol{\mu^0})q_1+((U_2^0)'+\theta^0-A_2^E\cdot\boldsymbol{\mu^0})q_2\,dx=0.\end{cases}\tag{53}$$

The effective material properties are given by

$$A^E=\begin{pmatrix}\left\langle\frac{1}{S}\right\rangle & \left\langle\frac{(\phi_0'+\dot\phi)}{S}\right\rangle\\ \left\langle\frac{(\phi_0)'+\dot\phi}{S}\right\rangle & \left\langle\frac{((\phi_0)'+\dot\phi)^2}{S}\right\rangle\end{pmatrix},\qquad B^E=\frac{1}{\langle S\rangle},\tag{54}$$

where

$$S=S(x,y)=\sqrt{1+(\phi_0')^2(x)+(\dot\phi)^2(y)}$$

and the brackets denote the meanvalue taken for $y\in Y$. *The dot* $\cdot$ *denotes the canonical scalar product in* $\mathbb{R}^2$ *and* A_1^E, A_2^E *are the first and second columns of the symmetric positive matrix* A^E, *which is always definite except for the trivial case of the non-waved curved arch.*

Proof. We already know that the candidates U_1^0,U_2^0,θ^0,p^0, and μ_1^0,μ_2^0 (through the function $\mu^0=(1/S)\mu_1^0+((\phi_0'+\dot\phi)/S)\mu_2^0$) are the unique functions which satisfy Eq. (51). It is then sufficient to prove that the mixed formulation (53) has a unique solution, or in other words, that it fulfills the BBL conditions. If so, we can conclude that two problems (51) and (53) *are equivalent*. Theorem 4.1 completes the proof.

Now, we claim that Brezzi conditions hold for the mixed formulation (53) above. Indeed, the continuity of the involved bilinear (and linear) forms is straightforward. The inf-sup condition for the dual bilinear mapping is also fulfilled. The proof is done by exhibiting candidates that fulfill property (39)—updated for our mixed problem. It can be easily shown that such candidates exist, using the same techniques as those of (41) and (42). As a hint, one should seek for candidates μ_1^0,μ_2^0 which are constant, solution to the simple 2×2 linear system $A^E\boldsymbol{\mu^0}=(\int_\Omega q_1(x)\,dx,0)^T$.

It remains to prove the ellipticity condition (over the relevant space, roughly speaking the kernel space of the dual mapping). This is also straightforward as soon as we can state that the matrix A^E is symmetric positive definite. This property of A^E is obtained through the simple Cauchy–Schwarz inequality

$$\left(\int_Y \frac{(\phi_0' + \dot{\phi})}{\sqrt{S}} \frac{1}{\sqrt{S}}\right)^2 \leqslant \int_Y \frac{1}{S} \int_Y \frac{((\phi_0' + \dot{\phi}))^2}{S},$$

yielding that the matrix A^E has a positive determinant, which is equal to zero if and only if the waving ϕ is itself equal to zero (thanks to the periodicity condition $\phi(0) = \phi(1)$). □

Remark 5.1. For the plane arch, the membrane strain and rotation of the normal are given by

$$\mu^P = \left(U_1^P\right)', \qquad \theta^P = -\left(U_2^P\right)', \tag{55}$$

while we have shown that for the limit model of the waved arch, one has

$$\mu^E = \left(U_1^0\right)', \qquad \theta^0 = -\left(U_2^0\right)' + A_2^E \cdot \boldsymbol{\mu}^{\mathbf{0}}. \tag{56}$$

This coupling between the rotation and the membrane strain shows that the limit structure is not simply a *plane* beam with new effective mechanical constants (as comes from the homogenization of a plane beam with periodic thickness). Notice that, contrarily to the waved plane arch case, the coupling between bending and membrane effects is already present for curved arches.

Also remark that when the waving ϕ in the formulation above reduces to zero, there is no need for a couple of unknowns (μ_1^0, μ_2^0). One has $A_2^E \cdot \boldsymbol{\mu}^{\mathbf{0}} = \phi_0' \mu^E$, and μ^E must be taken as the—classical membrane strain—unknown.

As a conclusion, we emphasize that Theorem 5.1 introduces a new elastic arch model, of Lipschitz effective mid-surface, showing a corrector term to the coupling between bending and membrane effects. The corrector, which depends on the shape of the waving could be used in view of, e.g., structural optimal design.

It is still well suited to numerical implementation, using classical mixed finite element methods, like the one presented in [13] where the displacements are approximated by (P1) polynomials, the membrane strain and the Lagrange multipliers by piecewise constant polynomials and the rotation of the normal by (P3) Lagrange–Hermite polynomials. However, one should be careful when developing finite element methods for this model. It is of course a shell of parabolic type, which still exhibits inextensional fields which are known to be responsible for numerical locking phenomena.

A possible development is the extension of the critical wrinkling to the general thin shells. To this end, for standard mixed formulations, we unfortunately cannot get rid of the curvature. But a similar study to ours should be possible for the case of axisymmetric models, an important class of the hyperbolic shells [21].

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