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Four new sums of graphs and their Wiener indices

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ABSTRACT

The Wiener index is the sum of distances between all vertex pairs in a connected graph. This notion was motivated by various mathematical properties and chemical applications. In this paper we introduce four new operations on graphs and study the Wiener indices of the resulting graphs.

1. Introduction

The distance between two vertices in a connected graph G is the number of edges in a shortest path between them. This concept has been known for a very long time and recently has received considerable attention as a subject of its own. One of the concepts related to distance in graphs is the Wiener index. It is not only an early index which correlates well with many physico-chemical properties of organic compounds but also a subject that has been studied by many mathematicians and chemists. The Wiener index is the sum of distances between all vertex pairs in a connected graph:

$$W(G) = \frac{1}{2} \sum_{(u,v) \subseteq V(G) \times V(G)} d(u, v|G),$$

where $d(u, v|G)$ is the distance between vertices u and v of graph G, and $V(G)$ is the set of vertices of G. Mathematical properties and chemical applications of the Wiener index have been intensively studied over the past thirty years. For more information about the Wiener index in chemistry and mathematics see [8] and [1–5,7,9,10,12], respectively. Gutman and Yeh examined in [11] operations on a connected graph that have been studied by Weigen Yan et al. in [13]. In this paper we introduce four new operations on graphs and study the Wiener indices of the resulting graphs. At the end we give a new proof of a result of Dobrynin on the Wiener index of hexagonal chains.

2. New sums of graphs

The sum of two connected graphs G_1 and G_2, which is denoted by $G_1 + G_2$, is a graph such that the set of vertices is $V(G_1) \times V(G_2)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $G_1 + G_2$ are adjacent if and only if $[u_1 = v_1$ and $(u_2, v_2) \in E(G_2)]$ or $[u_2 = v_2$ and $(u_1, v_1) \in E(G_1)]$, where $E(G)$ is the set of edges of a graph G. Note that $G_1 + G_2$ has $|V(G_2)|$ copies of G_1, and we may label these copies by vertices of $V(G_2)$. Now two vertices with the same name in different copies are adjacent in $G_1 + G_2$ if and only if their corresponding labels are adjacent in G_2.

We are interested in giving new sums of graphs such that $(E(G_1) \cup V(G_1)) \times V(G_2)$ is the set of vertices. For this purpose we first recall some operations on graphs.

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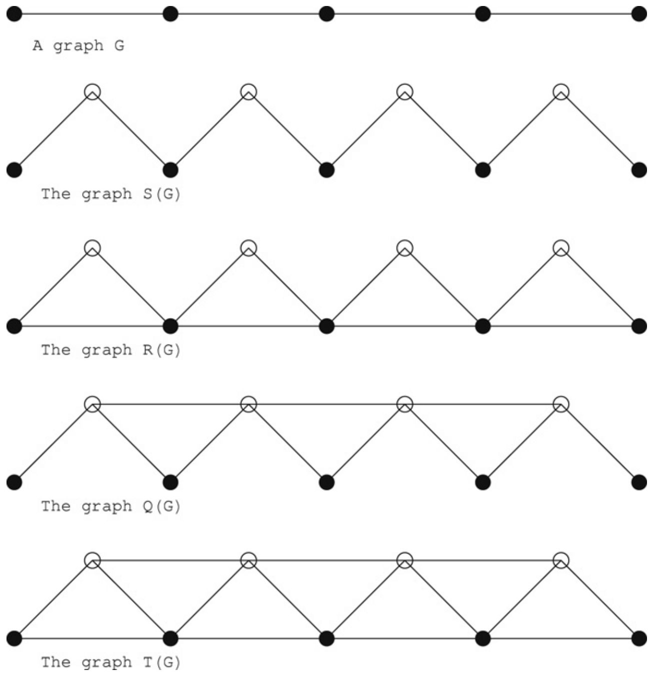

Fig. 1. A graph G and $S(G)$, $R(G)$, $Q(G)$ and $T(G)$.

Definition 2.1. For a connected graph G, define four related graphs as follows (see Fig. 1):

(a) $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2.

(b) $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.

(c) $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.

(d) $T(G)$ has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G.

The graphs $S(G)$ and $T(G)$ are called the subdivision and total graph of G, respectively. For more details on these operations we refer the reader to [3]. Yan, Yang and Yeh in [13] studied the Wiener indices of $S(G)$, $R(G)$ and $Q(G)$. They proved that

$$W(S(G)) = 2W(T(G)) - mn;$$

$$W(R(G)) = W(T(G)) + m(m-1)/2;$$

$$W(Q(G)) = W(T(G)) + n(n-1)/2,$$

where n and m are the numbers of vertices and edges of G, respectively.

Suppose that G_1 and G_2 are two connected graphs. Throughout the paper we denote $V(G_i)$ and $E(G_i)$ by V_i and E_i, $i = 1, 2$, respectively. We consider the following operation on these graphs:

Definition 2.2. Let F be one of the symbols S, R, Q, or T. The F-sum $G_1 +_F G_2$ is a graph with the set of vertices $V(G_1 +_F G_2) = (V_1 \cup E_1) \times V_2$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_F G_2$ are adjacent if and only if $[u_1 = v_1 \in V_1$ and $(u_2, v_2) \in E_2]$ or $[u_2 = v_2$ and $(u_1, v_1) \in E(F(G_1))]$.

Note that $G_1 +_F G_2$ has $|V_2|$ copies of the graph $F(G_1)$, and we may label these copies by vertices of G_2. The vertices in each copy have two situations: The vertices in V_1 (we refer to these vertices as black vertices) and the vertices in E_1 (we refer to these vertices as white vertices). Now we join only black vertices with the same name in $F(G_1)$ in which their corresponding labels are adjacent in G_2. We illustrate this definition in Fig. 2.

3. The Wiener index of F-sums of graphs

Firstly we prove a key lemma on the distances of vertices in $G_1 +_F G_2$. To determine the distance between vertices of the graph $G_1 +_F G_2$ we distinguish the following three cases:

(a) The distance between black vertices and other vertices,
(b) The distance between white vertices in different copies,
(c) The distance between white vertices in the same copy.

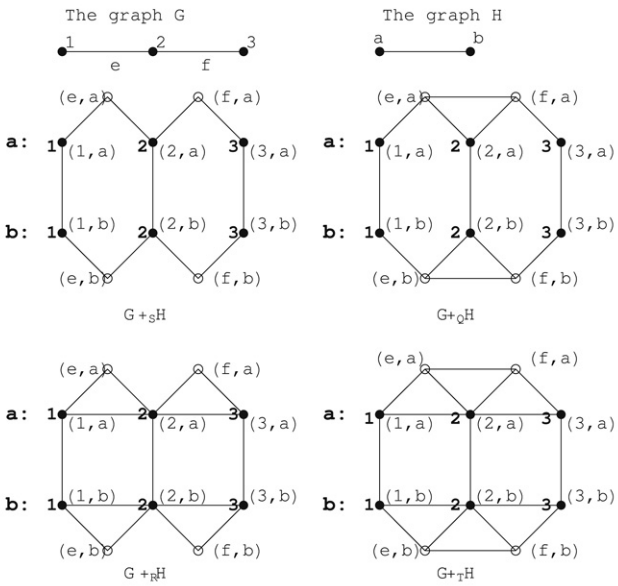

Fig. 2. Graphs G and H and $G +_F H$.

Lemma 3.1. *Let G_1 and G_2 be two connected graphs and $v = (v_1, v_2)$ be a vertex of $G_1 +_F G_2$. Then:*
(a) *If $v_1 \notin E_1$ (that is v is a black vertex), then for all $u = (u_1, u_2) \in V(G_1 +_F G_2)$ we have*

$$d(u, v|G_1 +_F G_2) = d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2).$$

(b) *If $v_1 \in E_1$, then for all $u = (u_1, u_2) \in V(G_1 +_F G_2)$, with $u_2 \neq v_2$, $u_1 = u_1^1 v_1^1 \in E_1$ and $u_1^1, v_1^1 \in V_1$ (that is v and u are white vertices in different copies of $F(G_1)$), we have*

$$d(u, v|G_1 +_F G_2) = 1 + d(u_2, v_2|G_2) + \min\{d(u_1^1, v_1|F(G_1)), d(v_1^1, v_1|F(G_1))\}.$$

(c) *If $v_1 \in E_1$ (that is v and u are white vertices in the same copy), then for all $u = (u_1, u_2) \in V(G_1 +_F G_2)$, where $u_2 = v_2$ and $u_1 \in E_1$, we have*

$$d(u, v|G_1 +_F G_2) = d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) = d(u_1, v_1|F(G_1)).$$

Proof. (a) Since $v_1 \notin E_1$, we have $v_1 \in V_1$. Let

$$P : u = (u_1, u_2) = (p_0^1, p_0^2) \to (p_1^1, p_1^2) \to \cdots \to (p_d^1, p_d^2) = (v_1, v_2) = v$$

be a shortest path of length d between u and v in $G_1 +_F G_2$. Since (p_i^1, p_i^2) and (p_{i+1}^1, p_{i+1}^2) are adjacent in $G_1 +_F G_2$, we have either $[p_i^1 = p_{i+1}^1 \in V_1$ and $(p_i^2, p_{i+1}^2) \in E(G_2)]$ or $[p_i^2 = p_{i+1}^2$ and $(p_i^1, p_{i+1}^1) \in E(F(G_1))]$, for $i = 0, 1, \ldots, d$.

Replacing consecutive vertices of the form $w, w, \ldots, w$ by w in the sequence $u_1 = p_0^1, p_1^1, \ldots, p_1^d = v_1$ of vertices in $F(G_1)$, we obtain a path of length s_1 between u_1 and v_1 in $F(G_1)$. So $s_1 \geq d_1$, where $d_1 = d(u_1, v_1|F(G_1))$. Similarly replacing consecutive vertices of the form $w, w, \ldots, w$ by w in the sequence $u_2 = p_0^2, \ldots, p_d^2 = v_2$ of vertices in G_2, we obtain a path of length s_2 between u_2 and v_2 in G_2. Thus $s_2 \geq d_2$, where $d_2 = d(u_2, v_2|G_2)$. By the definition of the adjacency in $G_1 +_F G_2$ we have $d = s_1 + s_2$. Therefore $d = s_1 + s_2 \geq d_1 + d_2$, and so

$$d(u, v|G_1 +_F G_2) \geq d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2). \quad (1)$$

To prove the reverse inequality in (1), suppose that

$$P_1 : \quad u_1 = q_0^1 \to q_1^1 \to \cdots \to q_{d_1}^1 = v_1$$
$$P_2 : \quad u_2 = r_0^2 \to r_1^2 \to \cdots \to r_{d_2}^2 = v_2$$

are the shortest paths between u_1, v_1 in $F(G_1)$ and u_2, v_2 in G_2, respectively. Using the path P_1 and walking from u_1 in the copy corresponding to $u_2 = r_0^2$ we can reach the vertex v_1 in this copy. Since v_1 is a vertex, using the path P_2 and walking along the copies corresponding to this vertex, we reach v. That is we have the following path between u and v in $G_1 +_F G_2$:

$$u = (u_1, u_2) = (q_0^1, r_0^2) \to (q_1^1, r_0^2) \to \cdots \to (q_{d_1}^1, r_0^2) = (v_1, r_0^2) \to (v_1, r_1^2) \to \cdots \to (v_1, r_{d_2}^2) = (v_1, v_2) = v.$$

The length of this path is $d_1 + d_2$, so that $d(u, v|G_1 +_F G_2) = d \leq d_1 + d_2$ and the equality holds in (1).

(b) Let $d(u_1^1, v_1|F(G_1)) = k_1$, $d(v_1^1, v_1|F(G_1)) = k_2$, $d(u_2, v_2|G_2) = d_2$, and $d = d(u, v|G_1 +_F G_2)$. If $u_1^1 = q_0^1 \to q_1^1 \to \cdots \to q_{k_1}^1 = v_1$ and $v_1^1 = r_0^1 \to r_1^1 \to \cdots \to r_{k_2}^1 = v_1$ are the shortest paths between u_1^1, v_1 and v_1^1, v_1 in $F(G_1)$, respectively, and $u_2 = r_0^2 \to r_1^2 \to \cdots \to r_{d_2}^2 = v_2$ is a shortest path between u_2 and v_2 in G_2, then we can consider the following paths in $G_1 +_F G_2$:

$$u = (u_1, u_2) \to (u_1^1, u_2) \to (u_1^1, r_1^2) \to \cdots \to (u_1^1, r_{d_2}^2) = (q_0^1, v_2) \to (q_1^1, v_2) \to \cdots \to (q_{k_1}^1, v_2) = (v_1, v_2) = v$$

$$u = (u_1, u_2) \to (v_1^1, u_2) \to (v_1^1, r_1^2) \to \cdots \to (v_1^1, r_{d_2}^2) = (r_0^1, v_2) \to (r_1^1, v_2) \to \cdots \to (r_{k_2}^1, v_2) = (v_1, v_2) = v.$$

The length of the first path is $1 + d_2 + k_1$ and so $d \le 1 + d_2 + k_1$. The length of the second path is $1 + d_2 + k_2$. So

$$\begin{aligned} d(u, v|G_1 +_F G_2) = d &\le 1 + d_2 + \min\{k_1, k_2\} \\ &= 1 + d(u_2, v_2|G_2) + \min\{k_1, k_2\}. \end{aligned} \tag{2}$$

To prove the reverse inequality in (2), suppose that

$$P: \quad u = (u_1, u_2) = (p_0^1, p_0^2) \to (p_1^1, p_1^2) \to \cdots \to (p_d^1, p_d^2) = (v_1, v_2) = v$$

is a shortest path between u and v in $G_1 +_F G_2$. Since (u_1, u_2) and (p_1^1, p_1^2) are adjacent in $G_1 +_F G_2$, we have $[(u_2, p_1^2) \in E(G_2),\ p_1^1 \in V_1]$ or $[(p_1^1, u_1) \in F(G_1)\ ,\ p_1^2 = u_2]$. By assumption $u_1 \notin V_1$. So $p_1^2 = u_2$ and p_1^1, u_1 are adjacent in $F(G_1)$. We consider two cases.

Case 1. If $p_1^1 \in V_1$, then by the definition of S, R, T and Q, p_1^1 is one of the end points of $u_1 = u_1^1 v_1^1$. This means that $p_1^1 = u_1^1$ or $p_1^1 = v_1^1$. Suppose that $p_1^1 = u_1^1$ (in the case $p_1^1 = v_1^1$ the argument is similar). Then the (p_1^1, p_1^2)–(v_1, v_2) section of P is a path of length $d - 1$ in $G_1 +_F G_2$.

Replacing consecutive vertices of the form $w, w, \ldots, w$ by w in the sequence $u_1^1 = p_1^1, p_2^1, \ldots, p_d^1 = v_1$ of vertices in $F(G_1)$, we obtain a path of length s_1 between u_1 and v_1 in $F(G_1)$. So $s_1 \ge k_1 = d(u_1^1, v_1|F(G_1))$. Similarly from the sequence $u_2 = p_1^2, p_2^2, \ldots, p_d^2 = v_2$ of vertices in $F(G_1)$, we can obtain a path of length s_1 between u_2 and v_2 in G_2. So $s_2 \ge d_2 = d(u_2, v_2|G_2)$. Hence $d - 1 = s_1 + s_2 \ge k_1 + d_2$ and so

$$\begin{aligned} d \ge 1 + k_1 + d_2 &\ge 1 + d_2 + \min\{k_1, k_2\} \\ &= 1 + d(u_2, v_2|G_2) + \min\{d(u_1^1, v_1|F(G_1)), d(v_1^1, v_1|F(G_1))\}. \end{aligned}$$

Thus in this case the equality holds in (2).

Case 2. If $p_1^1 \in E_1$. Since p_1 and u_1 are adjacent in $F(G_1)$ and $u_2 \in E_1$ (that is (p_1^1, u_2) and (u_1, u_2) are white), by the definition of S and R, they cannot be adjacent in $G_1 +_F G_2$, if $F = S$ or $F = R$. Therefore in this case $F \ne S$ and $F \ne R$.

Now since $u_1^1 v_1^1 = p_1^1$ and u_1 are adjacent, they have a common end point, say u_1^1. Since (u_1^1, u_2) is a black vertex, by case (a), we have

$$\Delta := d((u_1^1, u_2), (v_1, v_2)|G_1 +_F G_2) = d(u_2, v_2|G_2) + d(u_1^1, v_1).$$

By adding the adjacent vertices (u_1, u_2) and (u_1^1, u_2) to the beginning of any path from (u_1^1, u_2) to (v_1, v_1) we obtain a path between (u_1, u_2) and (v_1, v_1). Thus

$$d \le d((u_1^1, u_2), (v_1, v_2)|G_1 +_F G_2) = 1 + \Delta.$$

Since $u_2 = p_0^2 = p_1^2$ and u_1, p_1^1 are adjacent in $F(G_1)$, we can replace (u_1, u_2) by (u_1^1, u_2), in P, and obtain the path

$$(u_1^1, u_2) = (u_1^1, p_0^2) \to (p_1^1, p_1^2) \to \cdots \to (p_d^1, p_d^2) = (v_1, v_2) = v$$

of length d in $G_1 +_F G_2$. Therefore $\Delta \le d$ and so $\Delta \le d \le 1 + \Delta$. Hence $\Delta = d$ or $\Delta + 1 = d$.

If $\Delta + 1 = d$, then equality (2) holds. We prove that the case $\Delta = d$ cannot happen. Suppose, to the contrary, that $\Delta = d$. Let

$$u_1^1 = q_0^1 \to q_1^1 \to \cdots \to q_{k_1}^1 = v_1 \quad \text{and} \quad u_2 = r_0^2 \to r_1^2 \to \cdots \to r_{d_2}^2 = v_2$$

be the shortest paths between u_1^1, v_1 in $F(G_1)$ and u_2, v_2 in G_2, respectively. As in case (a) we can show that the path

$$\begin{aligned} Q: \quad (u_1^1, u_2) &= (u_1^1, r_0^2) \to (u_1^1, r_1^2) \to \cdots \to (u_1^1, r_{d_2}^2) \\ &= (q_0^1, v_2) = (q_0^1, v_2) \to (q_1^1, v_2) \to \cdots \to (q_{k_1}^2, v_2) = (v_1, v_2), \end{aligned}$$

in $G_1 +_F G_2$ is a shortest path between (u_1^1, u_2) and (v_1, v_2). The length of this path is $\Delta = (d_2 - 1) + (k_1 - 1)$.

Let $T_1 := \{u_1^1 = q_0^1, q_1^1, \ldots, q_{k_1}^1 = v_1\}$ and $T_2 := \{u_2 = r_0^2, r_1^2, \ldots, r_{d_2}^2 = v_2\}$. Then the length of the path Q is $d = \Delta = (k_1 - 1) + (d_2 - 1) = (|T_1| - 1) + (|T_2| - 1)$. Now put $\bar{T}_1 := \{u_1^1, p_1^1, p_2^1, \ldots, p_d^1 = v_1\}$ and $\bar{T}_2 := \{u_2 = p_0^2, p_1^2, \ldots, p_d^2 = v_2\}$. Then the length of P is $d = (|\bar{T}_1| - 1) + (|\bar{T}_2| - 1)$. Therefore

$$|T_1| + |T_2| = |\bar{T}_1| + |\bar{T}_2|. \tag{3}$$

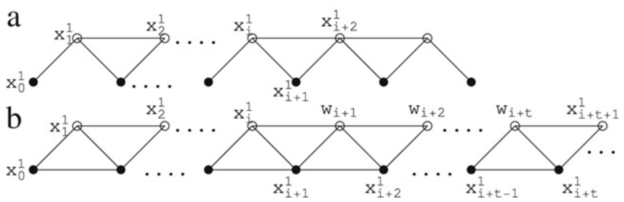

Fig. 3.

Replacing consecutive vertices of the form $w, w, \ldots, w$ by w in the vertices in $\bar{T}_1$, we can obtain a path between u_1^1 and v_1 in $F(G_1)$. So $|\bar{T}_1| - 1 \geq |T_1| - 1$ and $|\bar{T}_1| \geq |T_1| > 0$. By a similar argument we have $|\bar{T}_2| \geq |T_2| > 0$. Therefore by (3) we have $|T_i| = |\bar{T}_i|, i = 1, 2$.

Now since $|\bar{T}_1| = |T_1|$, there exist $x_0^1, x_1^1, \ldots, x_{k_1}^1 \in \bar{T}_1$, such that

$$Q_1: \quad u_1^1 = x_0^1 \to x_1^1 \to \cdots \to x_{k_1}^1 = v_1$$

is a shortest path, between u_1^1 and v_1. Since $u_2 \neq v_2$, not all consecutive vertices in $\bar{T}_1$ are edges of G_1. Suppose i is the least integer such that $x_i^1 \in E_1$ and $x_{i+1}^1 \in V_1$. We consider two cases:

(1) Suppose that $F = Q$. By the definition of Q, x_{i+2}^1 must be an element of E_1. Replacing $x_i^1 \to x_{i+1}^1 \to x_{i+2}^1$ by $x_i^1 \to x_{i+2}^1$ in the path Q_1, we obtain a path from u_1^1 to v_1 whose length is smaller than the length of Q_1 (see Fig. 3(a)), which is a contradiction.

(2) Suppose that $F = T$. Let t be the least integer such that $x_{i+t}^1 \in V_1$ and $x_{i+t+1}^1 \in E_1$. Put $w_{i+j} = x_{i+j}^1 x_{i+j+1}^1, j = 1, 2, \ldots, t$, which are elements of E_1. Replacing $x_i^1 \to x_{i+1}^1 \to \cdots \to x_{i+t}^1 \to x_{i+t+1}^1$ by $x_i^1 \to w_{i+1} \to w_{i+2} \to \cdots \to w_{i+t-1} \to x_{i+t+1}^1$ in the path Q_1, we obtain a path from u_1^1 to v_1 whose length is smaller than the length of Q_1 (see Fig. 3(b)), which is a contradiction.

(c) By the argument given in (a) we can see that

$$\begin{aligned} d &= d((u_1, u_2), (v_1, v_2)|G_1 +_F G_2) \leq d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) \\ &= d(u_1, v_1|F(G_1)). \end{aligned}$$

On the other hand, from every shortest path between (u_1, u_2) and (v_1, v_2) we can find a path between u_1 and v_1 in $F(G_1)$ with length at most d. So $d \geq d(u_1, v_1|F(G_1))$ and this completes the proof. □

Now we explore condition (b) in Lemma 3.1, more precisely. In fact in the following two lemmas we find the distances between vertices of $G_1 +_F G_2$ without the "min" condition, stated in Lemma 3.1(b). At first we consider the case $F = R$ or $F = S$.

Lemma 3.2. *Let G_1 and G_2 be two connected graphs, $u_1, v_1 \in E_1$, $u_2, v_2 \in V_2$ and $F = S$ or R. Then for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V(G_1 +_F G_2)$, with $u_2 \neq v_2$, we have*

$$d(u, v|G_1 +_F G_2) = \begin{cases} 2 + d(u_2, v_2|G_2) & \text{if } u_1 = v_1 \\ d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) & \text{if } u_1 \neq v_1. \end{cases}$$

Proof. Since $u_1 \in E_1$, we have $u_1 = u_1^1 v_1^1$, for some $u_1^1, v_1^1 \in V_1$. First suppose that $u_1 = v_1$. Thus $d(u_1^1, u_1|F(G_1)) = 1 = d(v_1^1, u_1|F(G_1))$ and by Lemma 3.1(b) we have

$$\begin{aligned} d(u, v|G_1 +_F G_2) &= 1 + d(u_2, v_2|G_2) + \min\{d(u_1^1, u_1|F(G_1)), d(v_1^1, u_1|F(G_1))\} \\ &= 2 + d(u_2, v_2|G_2). \end{aligned}$$

Now suppose that $u_1 = u_1^1 v_1^1 \neq v_1$. Let $u_1 = q_0^1 \to q_1^1 \to \cdots \to q_{d_1}^1 = v_1$ be a shortest path between u_1, v_1 in $F(G_1)$. Then since $F = S$ or $F = R$, we have $q_1^1 = u_1^1$ or $q_1^1 = v_1^1$. Therefore

$$\min\{d(u_1^1, v_1|F(G_1)), d(v_1^1, v_1|F(G_1))\} = d(u_1, v_1|F(G_1)) - 1.$$

Hence by Lemma 3.1(b), we have

$$\begin{aligned} d(u, v|G_1 +_F G_2) &= 1 + d(u_2, v_2|G_2) + \min\{d(u_1^1, u_1|F(G_1)), d(v_1^1, u_1|F(G_1))\} \\ &= d(u_2, v_2|G_2) + d(u_1, v_1|F(G_1)). \end{aligned}$$

This completes the proof. □

Now we consider the case $F = Q$ or $F = T$.

Lemma 3.3. *Let G_1 and G_2 be two connected graphs, $u_1, v_1 \in E_1$, $u_2, v_2 \in V_2$ and $F = Q$ or T. Then for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V(G_1 +_F G_2)$ such that $u_2 \neq v_2$ we have*

$$d(u, v|G_1 +_F G_2) = \begin{cases} 2 + d(u_2, v_2|G_2) & \text{if } u_1 = v_1 \\ 1 + d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) & \text{if } u_1 \neq v_1, u_2 \neq v_2. \end{cases}$$

Proof. Suppose that $u_1 = u_1^1 v_1^1$, $u_1^1, v_1^1 \in V_1$. If $u_1 = v_1$, then $d(u_1^1, u_1|F(G_1)) = 1 = d(v_1^1, u_1|F(G_1))$ and by Lemma 3.1(b) we have

$$\begin{aligned} d(u, v|G_1 +_F G_2) &= 1 + d(u_2, v_2|G_2) + \min\{d(u_1^1, u_1|F(G_1)), d(v_1^1, u_1|F(G_1))\} \\ &= 2 + d(u_2, v_2|G_2). \end{aligned}$$

Now suppose that $u_1 \neq v_1$. Let $u_1 = q_0^1 \to q_1^1 \to \cdots \to q_{d_1}^1 = v_1$ be a shortest path between u_1, v_1 in $F(G_1)$. Suppose $d(u_1^1, v_1|F(G_1)) \leq d(v_1^1, v_1|F(G_1))$. Then since $F = Q$ or $F = T$, the path $u_1^1 \to q_1^1 \to \cdots \to q_{d_1}^1 = v_1$ is a shortest path between u_1^1 and v_1. Therefore $\min\{d(u_1^1, v_1|F(G_1)), d(v_1^1, v_1|F(G_1))\} = d(u_1, v_1|F(G_1))$ and hence by Lemma 3.1(b), we have

$$\begin{aligned} d(u, v|G_1 +_F G_2) &= 1 + d(u_2, v_2|G_2) + \min\{d(u_1^1, u_1|F(G_1)), d(v_1^1, u_1|F(G_1))\} \\ &= 1 + d(u_2, v_2|G_2) + d(u_1, v_1|F(G_1)). \end{aligned}$$

This completes the proof. □

Now we are ready to compute the Wiener index of $G_1 +_F G_2$. First we compute the Wiener index of $G_1 +_F G_2$ in terms of Wiener indices of $F(G_1)$ and G_2, where $F = R$ or S.

Theorem 3.4. *Let G_1 and G_2 be two connected graphs and $F = S$ or R. Then*

$$W(G_1 +_F G_2) = |V_2|^2 W(F(G_1)) + (|V_1| + |E_1|)^2 W(G_2) + |E_1||V_2|(|V_2| - 1).$$

Proof. To compute the Wiener index of $G_1 +_F G_2$, we need to compute the sum of distances between vertices u and v. According to the colors of u and v we must consider three cases:

Case 1. Suppose that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are black, that is $u, v \in V_1 \times V_2$. By Lemma 3.1(a),

$$d((u_1, u_2), (v_1, v_2)|G_1 +_F G_2) = d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2).$$

Therefore the summation of distances between black vertices is

$$\begin{aligned} A :&= \frac{1}{2} \sum \{d((u_1, u_2), (v_1, v_2)|G_1 +_F G_2) : (u_1, u_2), (v_1, v_2) \in V_1 \times V_2\} \\ &= \frac{1}{2} \sum_{(u_1,u_2),(v_1,v_2)} d(u_1, v_1|F(G_1)) + \sum_{(u_1,u_2),(v_1,v_2)} d(u_2, v_2|G_1) \\ &= \frac{1}{2} \sum_{u_1,v_1 \in V_1} \sum_{u_2,v_2 \in V_2} d(u_1, v_1|F(G_1)) + \frac{1}{2} \sum_{u_1,v_1 \in V_1} \sum_{u_2,v_2 \in V_2} d(u_2, v_2|G_2) \\ &= \frac{1}{2}|V_2|^2 \sum_{u_1,v_1 \in V_1} d(u_1, v_1|F(G_1)) + |V_1|^2 W(G_2). \end{aligned}$$

Case 2. Suppose that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ have different colors, that is $[u \in E_1 \times V_2$ and $v \in V_1 \times V_2]$ or $[u \in V_1 \times V_2$ and $v \in E_1 \times V_2]$. In this case, by Lemma 3.1(a),

$$d((u_1, u_2), (v_1, v_2)|G_1 +_F G_2) = d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2).$$

Therefore the summation of distances between vertices u and v, where u is black and v is white, is

$$\begin{aligned} &\frac{1}{2} \sum \{d((u_1, u_2), (u_1, v_2)|G_1 +_F G_2) : (u_1, u_2) \in V_1 \times V_2, (v_1, v_2) \in E_1 \times V_2\} \\ &= \frac{1}{2} \sum_{u_2,v_2 \in V_2} \sum_{u_1 \in V_1} \sum_{v_1 \in E_1} d(u_1, v_1|F(G_1)) + \frac{1}{2} \sum_{u_1 \in E_1} \sum_{u_1 \in V_1} \sum_{u_2,v_2 \in V_2} d(u_2, v_2|G_2) \\ &= \frac{1}{2}|V_2|^2 \sum_{v_1 \in E_1} \sum_{u_1 \in V_1} d(u_1, v_1|F(G_1)) + |E_1|V_1||W(G_2). \end{aligned}$$

The summation of distances between vertices with different colors is twice the above quantity, that is

$$B := |V_2|^2 \sum_{v_1 \in E_1} \sum_{u_1 \in V_1} d(u_1, v_1|F(G_1)) + 2|E_1|V_1||W(G_2).$$

Case 3. Suppose that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are white, that is $u \in E_1 \times V_2$ and $v \in E_1 \times V_2$. Let

$$C := \frac{1}{2} \sum \left\{ d((u_1, u_2), (v_1, v_2) | G_1 +_F G_2) : (u_1, u_2), (v_1, v_2) \in E_1 \times V_2 \right\}.$$

We break down this summation into two sums $C = C_1 + C_2$, where

$$C_1 = \frac{1}{2} \sum \{ d((u_1, u_2), (v_1, v_2) | G_1 +_F G_2) : (u_1, u_2), (v_1, v_2) \in E_1 \times V_2, u_1 = v_1, u_2 \neq v_2 \}$$
$$C_2 = \frac{1}{2} \sum \{ d((u_1, u_2), (v_1, v_2) | G_1 +_F G_2) : (u_1, u_2), (v_1, v_2) \in E_1 \times V_2, u_1 \neq v_1 \}.$$

By Lemma 3.2, we have

$$\begin{aligned} C_1 &= \frac{1}{2} \sum_{u_1 \in E_1} \sum_{u_2, v_2 \in V_2; u_2 \neq v_2} 2 + d(u_2, v_2 | G_2) \\ &= \frac{1}{2} \sum_{u_1 \in E_1} \sum_{u_2, v_2 \in V_2; u_2 \neq v_2} 2 + \sum_{u_1 \in E_1} \sum_{u_2, v_2 \in V_2; u_2 \neq v_2} d(u_2, v_2 | G_2) + \frac{1}{2} |V_2|^2 \sum_{u_1, v_1 \in E_1; u_1 \neq v_1} d(u_1, v_1 | F(G_1)) \end{aligned}$$

and

$$\begin{aligned} C_2 &= \frac{1}{2} \sum_{u_2 \in V_2} \sum_{v_2 \in V_2} \sum_{u_1, v_1 \in E_1; u_1 \neq v_1} d(u_1, v_1 | F(G_1)) + \frac{1}{2} \sum_{u_1, v_1 \in E_1; u_1 \neq v_1} \sum_{u_2 \in V_2} \sum_{v_2 \in V_2} d(u_2, v_2 | G_2) \\ &= \left(|E_1|^2 - |E_1| \right) W(G_2). \end{aligned}$$

Using the above result we can compute the Wiener index of $G_1 +_F G_2$:

$$\begin{aligned} W(G_1 +_F G_2) &= A + B + C \\ &= \frac{1}{2} |V_2|^2 \sum_{u_1 \in V_1} \sum_{v_1 \in V_1} d(u_1, v_1 | F(G_1)) + |V_1|^2 W(G_2) \\ &\quad + |V_2|^2 \sum_{v_1 \in E_1} \sum_{u_1 \in V_1} d(u_1, v_1 | F(G_1)) + 2|E_1 |V_1|| W(G_2) \\ &\quad + \frac{1}{2} \sum_{u_1 \in E_1} \sum_{u_2, v_2 \in V_2; u_2 \neq v_2} 2 + \sum_{u_1 \in E_1} \sum_{u_2, v_2 \in V_2; u_2 \neq v_2} d(u_2, v_2 | G_2) \\ &\quad + \frac{1}{2} |V_2|^2 \sum_{u_1, v_1 \in E_1; u_1 \neq v_1} d(u_1, v_1 | F(G_1)) + \left(|E_1|^2 - |E_1| \right) W(G_2) \\ &= |V_2|^2 W(F(G_1)) + (|V_1| + |E_1|)^2 W(G_2) + |E_1| \left(|V_2|^2 - |V_2| \right). \end{aligned}$$

This completes the proof. □

Now we compute the Wiener index of $G_1 +_F G_2$ in terms of Wiener indices of $F(G_1)$ and G_2, where $F = Q$ or T.

Theorem 3.5. *Let G_1 and G_2 be two connected graphs and $F = Q$ or T. Then*

$$W(G_1 +_F G_2) = |V_2|^2 W(F(G_1)) + (|V_1| + |E_1|)^2 W(G_2) + \frac{1}{2} \left(|V_2|^2 - |V_2| \right) \left(|E_1|^2 + |E_1| \right).$$

Proof. Let A, B and C be as in the proof of the Theorem 3.4. The values of A and B do not change here. So we must only compute the value of C. Let

$$C := \frac{1}{2} \sum \left\{ d((u_1, u_2), (v_1, v_2) | G_1 +_F G_2) : (u_1, u_2), (v_1, v_2) \in E_1 \times V_2 \right\}.$$

We break down this summation into two sums $C = C_1 + C_2 + C_3$, where

$$C_1 = \frac{1}{2} \sum \{ d((u_1, u_2), (v_1, v_2) | G_1 +_F G_2) : (u_1, u_2), (v_1, v_2) \in E_1 \times V_2, u_1 = v_1, u_2 \neq v_2 \}$$
$$C_2 = \frac{1}{2} \sum \{ d((u_1, u_2), (v_1, v_2) | G_1 +_F G_2) : (u_1, u_2), (v_1, v_2) \in E_1 \times V_2, u_1 \neq v_1, u_2 = v_2 \}$$
$$C_3 = \frac{1}{2} \sum \{ d((u_1, u_2), (v_1, v_2) | G_1 +_F G_2) : (u_1, u_2), (v_1, v_2) \in E_1 \times V_2, u_1 \neq v_1, u_2 \neq v_2 \}.$$

By Lemma 3.3, we have

$$\begin{aligned}C_1 &= \frac{1}{2}\sum_{u_1\in E_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}(2+d(u_2,v_2|G_2))\\&= \frac{1}{2}\sum_{u_1\in E_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}2+\frac{1}{2}\sum_{u_1\in E_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}d(u_2,v_2|G_2)\\&= |E_1|\left(|V_2|^2-|V_2|\right)+2|E_1|W(G_2).\end{aligned}$$

Also by Lemma 3.3

$$\begin{aligned}C_2 &= \sum_{u_1,v_1\in E_1;u_1\neq v_1}\sum_{u_2=v_2\in V_2}d(u_1,v_1|F(G_1))\\&= |V_2|\sum_{u_1,v_1\in E_1;u_1\neq v_1}d(u_1,v_1|F(G_1)),\end{aligned}$$

and

$$\begin{aligned}C_3 &= \sum_{u_1,v_1\in E_1;u_1\neq v_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}(1+d(u_1,v_1|F(G_1))+d(u_2,v_2|G_2))\\&= \sum_{u_1,v_1\in E_1;u_1\neq v_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}1+\sum_{u_2,v_2\in V_2;u_2\neq v_2}\sum_{u_1,v_1\in E_1;u_1\neq v_1}d(u_1,v_1|F(G_1))\\&\quad+\sum_{u_1,v_1\in E_1;u_1\neq v_1}\sum_{u_2,v_2\in V_2;u_2\neq v_2}d(u_2,v_2|G_2)\\&= \left(|E_1|^2-|E_1|\right)\left(|V_2|^2-|V_2|\right)+\left(|V_2|^2-|V_2|\right)\sum_{u_1,v_1\in E_1;u_1\neq v_1}d(u_1,v_1|F(G_1))+2\left(|E_1|^2-|E_1|\right)W(G_2).\end{aligned}$$

Therefore

$$\begin{aligned}C &= C_1+C_2+C_3\\&= |E_1||V_2|^2+|E_1|^2|V_1|^2-|E_1|^2|V_2|-|E_1||V_1|^2+2W(G_2)|E_1|^2+|V_2|^2\sum_{u_1,v_1\in E_1;u_1\neq v_1}d(u_1,v_1|F(G_1)).\end{aligned}$$

Now we can compute the Wiener index of $G_1+_F G_2$:

$$\begin{aligned}W(G_1+_F G_2) &= A+B+C\\&= \frac{|V_2|^2}{2}\left(\sum_{u_1\in V_1}\sum_{v_1\in V_1}d(u_1,v_1|F(G_1))+\sum_{u_1\in E_1}\sum_{u_1\in V_1}d(u_1,u_1|F(G_1))+\sum_{u_1,v_1\in E_1;u_1\neq v_1}d(u_1,v_1|F(G_1))\right)\\&\quad+\left(|V_1|^2+2|V_1||E_1|+|E_1|^2\right)W(G_2)+|E_1|(|V_2|^2-|V_2|)+\frac{1}{2}\left(|E_1|^2-|E_1|\right)\left(|V_2|^2-|V_2|\right)\\&= |V_2|^2W(F(G_1))+(|V_1|+|E_1|)^2W(G_2)+\frac{1}{2}\left(|V_2|^2-|V_2|\right)\left(|E_1|^2+|E_1|\right).\end{aligned}$$

This completes the proof. □

4. Corollaries and examples

We recall the following result of Yan et al. (see Corollary 4.2. in [13]).

Lemma 4.1. *For a tree G with n vertices,*

$$\begin{aligned}&W(S(G))=8W(G)-2n(n-1);\\&W(Q(G))=4W(G);\\&W(R(G))=4W(G)-n+1;\\&W(T(G))=4W(G)-\frac{n(n-1)}{2}.\end{aligned}$$

Let P_n denote a path with n vertices. Then $W(P_n)=\frac{(n+1)n(n-1)}{6}$, and so by the above one can see that

$$W(S(P_n))=\frac{2n(2n-1)(n-1)}{3},$$

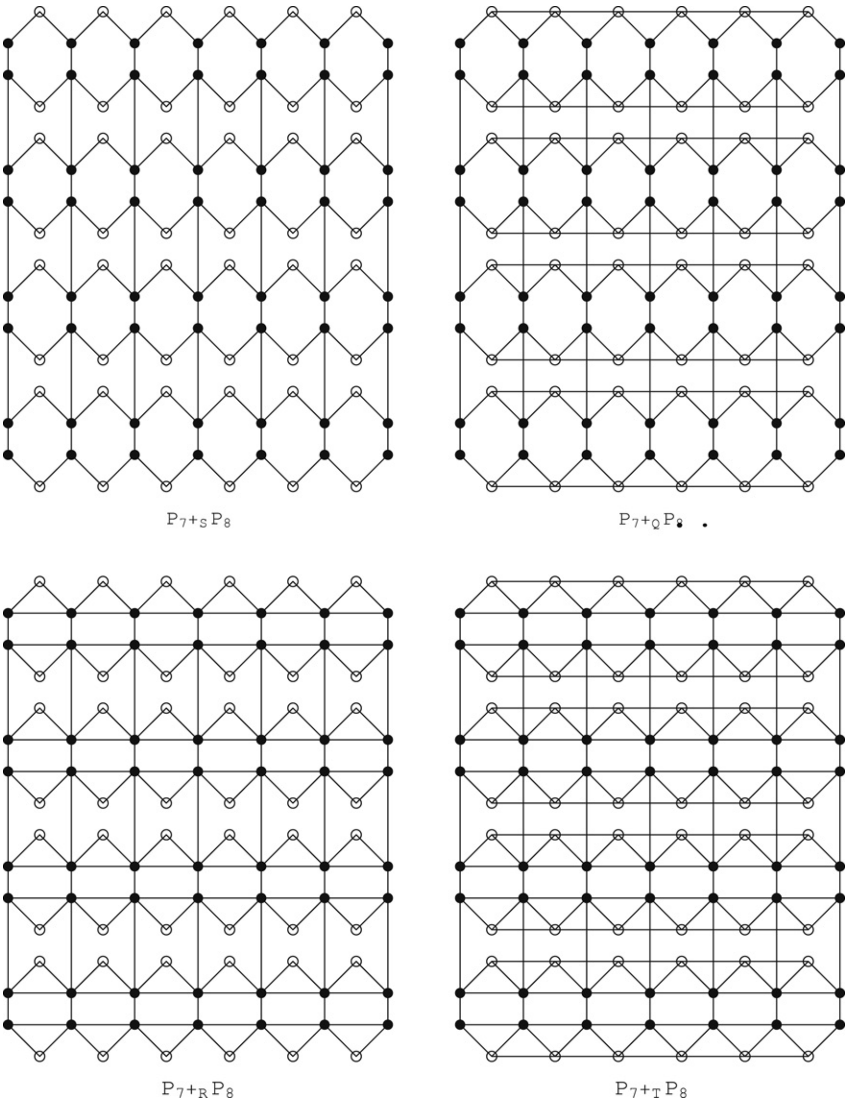

Fig. 4. The graphs $P_7 +_F P_8$.

$$W(Q(P_n)) = \frac{2(n+1)n(n-1)}{3},$$
$$W(R(P_n)) = \frac{(n-1)(2n^2+2n-3)}{3},$$
$$W(T(P_n)) = \frac{n(4n+1)(n-1)}{6}.$$

Hence by Theorems 3.4 and 3.5 we have:

Corollary 4.2. *Let $n > 1$ and $m > 1$ be two integers. Then*

$$\begin{aligned}
W(P_n +_S P_m) &= \frac{1}{6}m(8mn^3 - 12mn^2 + 10mn + 4m^2n^2 - 4m^2n + m^2 - 4n^2 - 2n + 5 - 6m),\\
W(P_n +_Q P_m) &= \frac{1}{6}m(4mn^3 - 7nm + 4m^2n^2 - 7n^2 - 4m^2n + 7n + m^2 + 3n^2m - 1),\\
W(P_n +_R P_m) &= \frac{1}{6}m(4mn^3 - 4mn + 4m^2n^2 - 4m^2n + m^2 - 4n^2 - 2n + 5),\\
W(P_n +_T P_m) &= \frac{1}{6}m(mn^3 - 4mn + 4m^2n^2 - 7n^2 - 4m^2n + 7n + m^2 - 1).
\end{aligned} \tag{4}$$

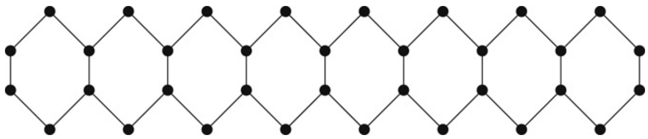

Fig. 5. The chain L_n with $n = 8$.

As a corollary of our results we compute the Wiener index of hexagonal chains (see Fig. 5). This formula was already obtained by Dobrynin [6].

Corollary 4.3. *Let n be an integer. Then the Wiener index of hexagonal chains with n hexagonal, L_n, is*

$$W(L_n) = \frac{16n^3 + 36n^2 + 26n + 1}{3}.$$

Proof. It is easy to see that $L_n = P_{n+1} +_S P_2$. So by Corollary 4.2(4) (See Fig. 4) we have

$$W(L_n) = P_{n+1} +_S P_2 = \frac{16n^3 + 36n^2 + 26n + 1}{3},$$

completing the proof. □

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