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On simplicial and co-simplicial vertices in graphs☆

Chính T. Hoàng[a,1], Stefan Hougardy[b], Frédéric Maffray[c], N.V.R. Mahadev[d]

[a] *Department of Physics and Computer Science, Wilfrid Laurier University, 75 University Ave. W., Waterloo, Ont., Canada N2L 3C5*
[b] *Institut für Informatik, Humboldt-Universität zu Berlin, Unter den Linden 6, Berlin 10099, Germany*
[c] *C.N.R.S., Laboratoire Leibniz—IMAG, 46 Avenue Félix Viallet, Grenoble Cedex 38031, France*
[d] *Department of Computer Science, Fitchburg State College, Fitchburg, MA 01420, USA*

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Abstract

We investigate the class of graphs defined by the property that every induced subgraph has a vertex which is either simplicial (its neighbours form a clique) or co-simplicial (its non-neighbours form an independent set). In particular we give the list of minimal forbidden subgraphs for the subclass of graphs whose vertex-set can be emptied out by first recursively eliminating simplicial vertices and then recursively eliminating co-simplicial vertices.

Keywords: Triangulated graph; Chordal graph; Simplicial vertex

1. Introduction

In a graph G, a vertex x is *simplicial* if its neighbourhood $N(x)$ induces a complete subgraph of G. A graph is *triangulated* if it does not contain as an induced subgraph a chordless cycle of length at least four (a *hole*). A famous theorem of Dirac [3] states that every triangulated graph has a simplicial vertex. Let us also say that a vertex is *co-simplicial* if its non-neighbours form an independent subset of vertices, and that a graph is *co-triangulated* if it does not contain the complement of a chordless cycle

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[1] Supported by NSERC.
E-mail address: choang@wlu.ca (C.T. Hoàng).

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on at least four vertices (an *antihole*). Dirac's theorem says equivalently that every co-triangulated graph has a co-simplicial vertex. Our purpose is to investigate the larger class of graphs, which is called the class of *quasi-triangulated* graphs (QT for short), defined as follows: a graph G is in QT if and only if every induced subgraph H has a vertex which is either simplicial or co-simplicial in H. The problem of characterizing the class QT was raised in [6,8] (where they are called *good*). The reader is referred to [1] for more information on the class QT.

Suppose that G is a graph in QT and has n vertices. Thus, there exists an ordered sequence $\sigma = v_1, v_2, \dots, v_n$ of its vertices such that, for every j, vertex v_j is simplicial or co-simplicial in the induced subgraph $G_j = G[v_j, v_{j+1}, \dots, v_n]$; accordingly we say that v_j is a C-vertex or an S-vertex in σ (some ambiguity may arise as a vertex can be both simplicial and co-simplicial). We call any such σ a *QT elimination sequence* for G. Note that the existence of such a sequence characterizes the class QT; indeed, if H is any induced subgraph of G and j is the smallest index with respect to σ such that $v_j \in H$, then v_j is also a simplicial or co-simplicial vertex of H. From the algorithmic point of view, it is easy to determine if a vertex is simplicial or co-simplicial, thus testing membership in the class QT and finding a QT elimination sequence is a polynomial task.

Given a QT elimination sequence σ, a *switch* in σ is an integer j such that v_j is an S-vertex and v_{j+1} is a C-vertex in σ, or vice versa. A graph in QT may admit many different QT elimination sequences, and they do not necessarily have the same number of switches. Naturally, sequences with the fewest switches are more interesting. Define QT_i as the class of QT graphs which admit a QT elimination sequence with at most i switches. So we have $\mathrm{QT}_0 \subseteq \mathrm{QT}_1 \subseteq \cdots$ (we will see in Theorem 4 that these inclusions are all strict) and $\mathrm{QT} = \bigcup\{\mathrm{QT}_i \mid i \geqslant 0\}$.

Finding a sequence with a minimum number of switches is easy: first remove simplicial vertices as long as one can find any; then remove co-simplicial vertices from the remaining graph as long as one can find any, etc. This is an optimal procedure because if a vertex x is simplicial (resp. co-simplicial) in G then it remains simplicial (resp. co-simplicial) whenever any number of vertices different from x have been removed. This procedure can be applied on any graph $G = (V, E)$. If the procedure stops without eliminating all the vertices, the subgraph induced by the remaining vertices has no simplicial vertex and no co-simplicial vertex, therefore G is not in the class QT. On the other hand, if the procedure succeeds in eliminating all the vertices, then the number of switches in the resulting elimination sequence is certainly the smallest i such that G is in QT_i. Testing if a vertex is simplicial (resp. co-simplicial) can be done in time $\mathrm{O}(|V|^2)$. Thus, the above elimination procedure can be done in time $\mathrm{O}(|V|^4)$ (this was also observed by Voloshin in [9]). Actually, this algorithm can easily be modified to yield an $\mathrm{O}(|V|^3)$ running time as follows. For each pair of neighbours x, y we maintain a count of the number of vertices which are neighbours of x but not y, both for G and its complement $\bar{G}$. If there is an x such that all x, y have count 0 (for G or $\bar{G}$), x goes on a queue for removal as a simplicial/co-simplicial vertex. When a vertex v is removed, for all neighbours x of v and non-neighbours y of v, decrement the value for x, y. The initial counts can be found in $\mathrm{O}(|V|^3)$ time, and each removal takes $\mathrm{O}(|V|^2)$ time, giving an $\mathrm{O}(|V|^3)$ algorithm. This algorithm has been refined by

Hoàng [7] to achieve an O(nm) time bound. It follows that testing membership in the class QT and in each class QT_i is solvable in polynomial time.

Clearly, QT_0 is the class of graphs that are triangulated or co-triangulated. One of our initial questions was whether QT is equal to QT_i for some fixed i (perhaps for $i=1$). It turns out that this is false; the answer is given in Theorem 3.

Since the class QT as well as each class QT_i ($i \geqslant 0$) is hereditary (i.e., every induced subgraph of a graph in the class is also in the class), each such class can be characterized by a family of minimal forbidden induced subgraphs. Any hole or antihole on at least five vertices is such a forbidden induced subgraph for QT and for each class QT_i ($i \geqslant 0$), because it is a graph with no simplicial or co-simplicial vertex, and it is minimal because the removal of any vertex yields a graph in QT_0. Thus one of the main questions is, for each class QT or QT_i ($i \geqslant 0$) to determine the minimal forbidden induced subgraphs other than holes and antiholes. In Section 2, we give a complete characterization of the class QT_1 by the family of all its minimal forbidden induced subgraphs: we actually find that the minimal forbidden subgraphs other than holes and antiholes form a *finite* family. Unfortunately, a similar situation does not hold for the whole family QT (see Theorem 3). Our research leads us to believe that finding the minimal forbidden induced subgraphs for a given class QT_i with $i \geqslant 2$ is very complicated.

Let us recall some classical definitions and results. We say "see" and "miss" instead of "is adjacent to" and "is not adjacent to". The subgraph of a graph G induced by a subset of vertices A is denoted by $G[A]$. A graph is called *weakly triangulated* if it contains no hole or antihole of length at least five. The following lemma will be very useful.

Lemma 1 (Hayward [5]). *Let G be a weakly triangulated graph, and C be any minimal cutset of G. Let $C_1,\ldots,C_t$ be the components of the graph $\bar{G}[C]$. Then for each $j=1,\ldots,t$, each component of $G-C$ contains a vertex that is adjacent to all of C_j.*

Let us also recall more formally the theorem of Dirac.

Theorem 1 (Dirac [3]). *Let G be a triangulated graph. Then either G is a clique or G contains two non-adjacent simplicial vertices.*

2. A characterization of the class QT_1

Consider the following property (P) of a graph G: Every induced subgraph H of G either has a simplicial vertex or is co-triangulated, in other words, the vertices of G can be eliminated by first removing simplicial and then co-simplicial vertices. It is clear that G is in QT_1 if and only if either G or $\bar{G}$ has this property.

Theorem 2. *A graph has property (P) if and only it is weakly triangulated and it does not contain any of the graphs in Figs. 1–3 as induced subgraph.*

The proof of Theorem 2 is given below, after Lemma 5 and its proof. It follows from Theorem 2 that *a graph G is not in QT_1 if and only if either G is not weakly*

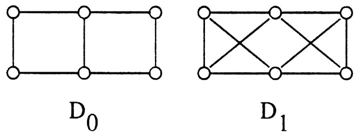

Fig. 1. Minimally bad graphs on six vertices.

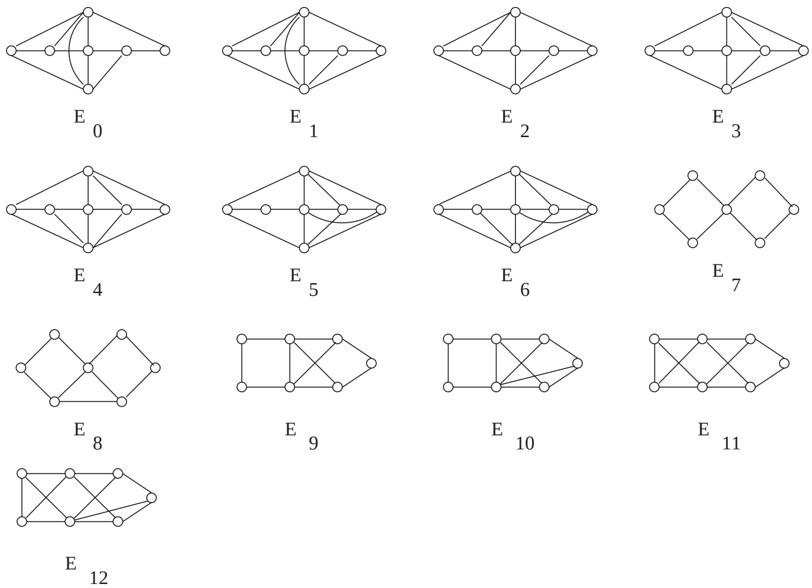

Fig. 2. Minimally bad graphs on seven vertices.

triangulated, *or* G *is weakly triangulated and both* G *and* $\bar{G}$ *contain an induced subgraph from Figs.* 1–3.

Clearly, any graph in QT must be weakly triangulated. Below we will say that a graph G is *bad* if it is weakly triangulated but does not have property (P). So, a minimally non-QT_1 graph is either a hole or antihole of length at least five, or a minimally bad graph or its complement. In other words, in order to prove Theorem 2, it suffices to determine the list of minimally bad graphs. To do this we will first look at the minimal cutsets of a bad graph G.

It is a routine matter to check that the graphs displayed in Figs. 1–3 are minimally bad.

We will often use without further reference the facts expressed in the following remarks, whose proofs are obvious.

Remark 1. If C is a minimal cutset of G, each vertex in C has a neighbour in each component of $G - C$.

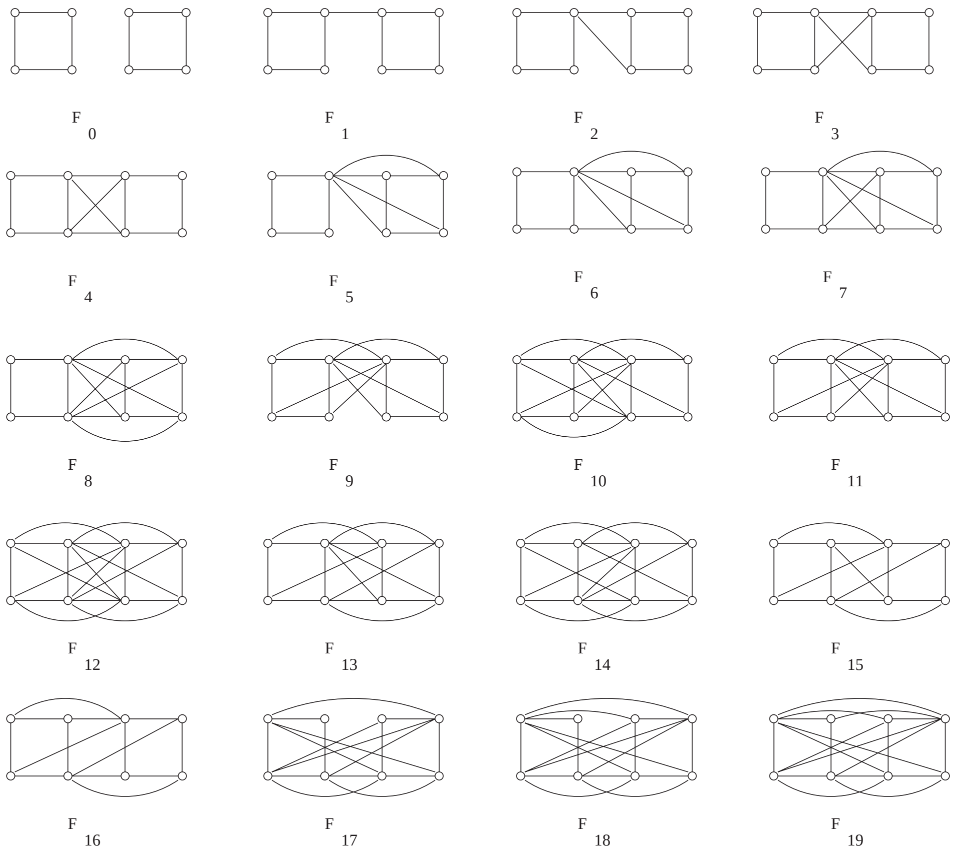

Fig. 3. Minimally bad graphs on eight vertices.

Remark 2. Call a graph H *sufficient* if it contains a $2K_2$ and every vertex of H lies in a square (the hole on four vertices). Then every sufficient graph is bad. Thus, a minimally bad graph cannot contain a proper induced subgraph that is sufficient.

Lemma 2. *If G is a minimally bad graph that has a clique cutset C, then G is one of the graphs $D_0, E_0, E_1, E_7, E_8, E_9, E_{10}, F_0, \ldots, F_{12}$.*

Proof. Without loss of generality, we may assume that C is a minimal cutset of G (possibly $C = \emptyset$). Let A and B be two components of $G - C$. If the subgraph $G[C \cup A]$ contained no square, then it would be triangulated, thus by Theorem 1 there would be a simplicial vertex v of G_A in A, and v would also be a simplicial vertex of G, a contradiction to the minimality of G. So $G[C \cup A]$ contains a square S_A. Similarly, the subgraph $G[C \cup B]$ contains a square S_B. The fact that C is a clique implies that at least two adjacent vertices of the square S_A are in A; likewise, at least two adjacent vertices of S_B are in B. Therefore, the subgraph of G induced by the union of the two squares is sufficient, and so this subgraph is all of G. Write $S_A = 1234$ and $S_B = 5678$, where 13 and 24 are the non-adjacent pairs in S_A, and 57 and 68 are the non-adjacent pairs in S_B. We distinguish between three cases.

Case 1: The square S_A does not intersect C.

Suppose that the square S_B does not intersect C. Then the eight vertices of the two squares induce the graph F_0 and thus $G = F_0$ (and here $C = \emptyset$).

Suppose that the square S_B meets C at exactly one vertex. Thus $|C| \geqslant 1$. This vertex may then see some vertices of S_A. A straightforward case analysis of these unsettled adjacencies shows that G either contains E_7 (contradicting minimality) or is F_0 (contradicting $|C| \geqslant 1$), or is one of F_1, F_2, F_5.

Now suppose that the square S_B meets C at exactly two vertices. Thus $|C| \geqslant 2$. These two vertices may both see some vertices of S_A. A straightforward case analysis of these unsettled adjacencies shows that G either contains one of C_5, $\overline{C_6}$, D_0, E_7, E_8, E_9, E_{10} (contradicting minimality), or is one of F_0, F_1, F_5 (contradicting $|C| \geqslant 2$), or is one of F_3, F_4, F_6, F_7, F_8.

Case 2: The square S_A intersects C in one vertex.

Assume that S_A has vertex $1 \in C$ and vertices $2, 3, 4 \in A$. By Case 1 and by symmetry, we may assume that S_B has at least one vertex in C.

First suppose that S_B has exactly one vertex in C, say $5 \in C$. If $1 = 5$, G is the graph E_7. Assume $1 \neq 5$. Thus $|C| \geqslant 2$. Each of $1, 5$ may see some vertices in the opposite square. Observe that if 5 sees 3 then it must see 2 and 4, for otherwise $G - 4$ or $G - 2$ is sufficient, a contradiction. On the other hand, if 5 misses 3 then it must miss at least one of $2, 4$, for otherwise $G - 1$ is sufficient. Hence, the set $N(5) \cap \{2, 3, 4\}$ is one of $\emptyset$, $\{2\}$ or $\{4\}$, or $\{2, 3, 4\}$. In fact the case $N(5) \cap \{2, 3, 4\} = \emptyset$ is excluded as C is a minimal cutset. Likewise, $N(1) \cap \{6, 7, 8\}$ is one of $\{6\}, \{8\}, \{6, 7, 8\}$. Combining these cases we see that G is either one of F_1, F_2, F_5 (contradicting $|C| \geqslant 2$) or one of F_3, F_6, F_9.

Now, suppose that S_B has two vertices, say $5, 6$, in C; thus $7, 8 \in B$. If $1 = 5$, then 6 can see any of $2, 3, 4$, and consequently G either contains D_0 (contradicting minimality) or is one of E_7, E_8, E_9, E_{10}. Now let us assume that the three vertices $1, 5, 6$ are distinct. By Remark 1, vertex 1 has a neighbour among $7, 8$. If 1 sees exactly one of them, say 1 sees 7 and not 8, it is easy to check that the graph $G - 6$ is sufficient, contradicting the minimality of G. Thus, 1 must see both 7 and 8.

Just as above, observe that if 5 sees 3 then 5 must see 2 and 4, for otherwise $G - 4$ or $G - 2$ is sufficient. On the other hand, if 5 misses 3 then it also misses at least one of $2, 4$, for otherwise $G - 1$ is sufficient. Moreover, by Remark 1, vertex 5 must have a neighbour in $\{2, 3, 4\}$. So, the set $N(5) \cap \{2, 3, 4\}$ is one of $\{2\}$, $\{4\}$, $\{2, 3, 4\}$. The same holds about vertex 6. By symmetry (of the pair $5, 6$ and of the pair $2, 4$), this yields four possibilities: (a) Both $5, 6$ see all of $2, 3, 4$; then G is the graph F_{10}. (b) One of $5, 6$ sees all of $2, 3, 4$, while the other sees 2 and misses 3 and 4; then G is the graph F_{11}. (c) Both $5, 6$ see 2 and miss 3 and 4; then G is F_7. (d) 5 sees 2 and misses $3, 4$, while 6 sees 4 and misses 2 and 3; then G contains C_5 on vertices $5, 2, 3, 4, 6$, a contradiction.

Case 3: The square S_A intersects C in two vertices.

By Cases 1 and 2 and by symmetry, we may also assume that the square S_B intersects C in two vertices. We may assume $S_A \cap C = \{3,4\}$ and $S_B \cap C = \{5,6\}$.

Suppose that $\{3,4\} = \{5,6\}$. Then G is the graph D_0.

Suppose that $|\{3,4\} \cap \{5,6\}| = 1$, say $5 = 3$ and $4 \neq 6$. Thus $|C| \geqslant 3$ and 46 is an edge. Since C is a minimal cutset, 4 must see at least one of 7, 8. Actually vertex 4 must see 8, for otherwise it sees 7 and $G-6$ is sufficient. Likewise, 6 must see 2. If none of 16, 47 are edges, then $G-3$ is the graph D_0, contradicting minimality. If one or two of 16, 47 are edges, G is the graph E_0 or the graph E_1.

Now suppose that $\{3,4\} \cap \{5,6\} = \emptyset$. Since C is a minimal cutset, each of the vertices 5, 6 must have a neighbour in $\{1,2\}$, and each of the vertices 3, 4 must have a neighbour in $\{7,8\}$. If 6 sees 2 and not 1 then $G-3$ is sufficient, a contradiction. So, by symmetry, 6 must see both 1, 2. Similarly, 5 sees both 1, 2, and the two vertices 3, 4 see both 7, 8. Then G is the graph F_{12}. This completes the proof of Lemma 2. □

A *strong cutset* of a graph G is a cutset C of G such that $G-C$ has at least two components of size at least two. We then say that C is a *minimal strong cutset* if it is a strong cutset and it does not strictly contain another strong cutset of G. Note that every bad graph contains a $2K_2$, because a $2K_2$-free bad graph would be co-triangulated, which is impossible. Taking one $2K_2$ in a bad graph and calling C the set consisting of all the other vertices, we see that C is a strong cutset. Therefore we have:

Lemma 3. *Every bad graph has a strong cutset.*

Minimal strong cutsets may be different from minimal cutsets: every minimal strong cutset contains a minimal cutset, but the converse does not necessarily hold; however, strong cutsets have a desirable property expressed in the next lemma. For a cutset C of G, let us say that a component R of $G-C$ is *special* if it has size at least two and every vertex of C has a neighbour in R.

Lemma 4. *For every minimal strong cutset C of a graph G, there exist at least two special components in $G-C$.*

Proof. Let R_1, R_2 be any two components of $G-C$ of size at least two. Suppose indirectly that they are not both special, i.e., some vertex $x \in C$ has no neighbour in one of R_1, R_2, say in R_1. Then consider $C' = C - x$. Observe that R_1 is a connected component of $G-C'$, and that another connected component R_2' of $G-C'$ contains R_2. Thus C' is a strong cutset of G, contradicting the minimality of C. □

Call J a graph with five vertices s,t,u,v,w where su, sw, tu, tw, uv, vw are edges, st, uw, sv are non-edges, and tv is optionally an edge or not. See case (iii) in Fig. 4. We will always write $J = (s,t,u,v,w)$ in this order. As usual, *diamond* denotes the clique on four vertices minus an edge.

Lemma 5. *Let G be a weakly triangulated minimally bad graph with no clique cutset, let C be a minimal strong cutset of G, and let R be any special component of $G-C$.*

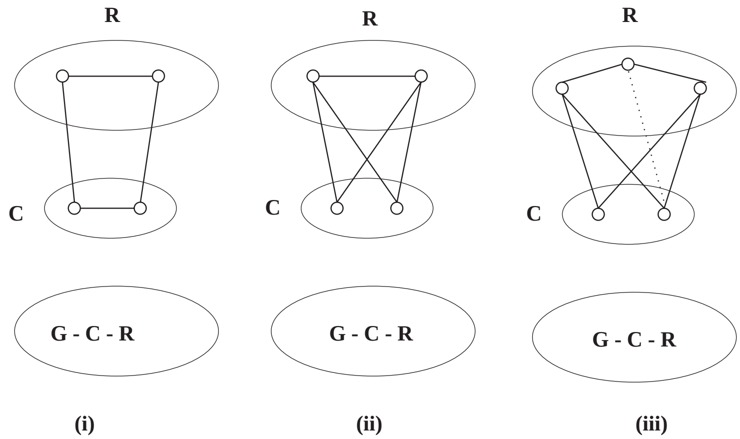

Fig. 4. (The dashed line in the J graph indicates an optional edge.).

Then one of the following three situations must occur:

(i) $R \cup C$ *contains a square that has one edge in* C *and one edge in* R, *or*
(ii) $R \cup C$ *contains a diamond whose two non-adjacent vertices are in* C *and the other two vertices are in* R, *or*
(iii) $R \cup C$ *contains the graph* $J = (s,t,u,v,w)$ *with* s,t in C *and* u,v,w *in* R.

Proof. We first observe that C itself is not a clique, for otherwise any minimal cutset included in C would be a clique cutset of G, contradicting the hypothesis.

Let x be a vertex in R with the most neighbours in C. Let S be a special component of $G - C$ different from R (S exists by the preceding lemma). Let H be the subgraph of G induced by $R \cup C \cup S$. Since R, S are special, C is a minimal cutset of H, and R, S are the two components of $H - C$.

First suppose that x does not see all vertices of C. Since R is connected and special, there exists a path $P = x_0x_1 \cdots x_kz$ from $x = x_0$ to a vertex z in $C - N(x)$, with $x_1, \ldots, x_k \in R$. Let us choose a shortest such path. There exists a vertex $a \in C \cap N(x) - N(x_k)$, for otherwise x_k would have more neighbours than x in C. Let j be the largest subscript such that ax_j is an edge; so $0 \leqslant j < k$. Suppose that az is not an edge. By Lemma 1 applied to the graph H, there is a vertex $b \in S$ adjacent to both a and z. But then $b, a, x_j, \ldots, x_k, z$ induce a hole. Thus, az is an edge. It follows that $j = k - 1$, for otherwise $a, x_j, \ldots, x_k, z$ induce a hole. So we obtain situation (i), with the square formed by a, z, x_k, x_{k-1}.

Now suppose that x sees all vertices of C. The graph $G - x$ must be connected, for otherwise x would be a cutpoint of G. Note that C is a cutset of $G - x$. Let $D \subseteq C$ be a minimal cutset of $G - x$. This D is not a clique, for otherwise $D \cup \{x\}$ would be a clique cutset of G; so there are two non-adjacent vertices $s, t \in D$. Since S is another special component of $G - C$, the subset $S \cup (C - D)$ induces a connected subgraph and thus is included in one component of $G - (D \cup \{x\})$; so there is a component R' of $G - (D \cup \{x\})$ that lies entirely in $R - x$. By Lemma 1, there exists a vertex

$z \in R' \subseteq R - x$ such that zs, zt are edges. Since R is connected there is a shortest path $P = x_0 x_1 \cdots x_k$ from $x = x_0$ to $z = x_k$ in R. As before, note that there is a vertex $b \in G - C - R$ that sees both s, t. If $k \leqslant 2$, it is easily seen that $C \cup R$ contains either a diamond (induced by s, t, x_0, x_1) or the graph J (induced by s, t, x_0, x_1, x_2) as desired for (ii) or (iii). So assume that $k \geqslant 3$. If x_1 sees both s, t then s, t, x_0, x_1 induced the desired diamond for (ii). Thus, we may assume that $x_1 s$ is not an edge. Now, sx_2 must be an edge, for otherwise $s, x_0, x_1, \ldots, x_j$ form a hole, where j is the smallest subscript such that sx_j is an edge ($j > 1$). If tx_2 is an edge, then the vertices s, t, x_0, x_1, x_2 induce the graph J as desired for (iii). Now assume tx_2 is not an edge. Then tx_1 is an edge, for otherwise, there is a hole on $t, x_0, x_1, x_2, x_3 \ldots, x_j$ for some $j \geqslant 3$. Now, there is a C_5 with vertices b, s, x_2, x_1, t, a contradiction. □

Proof of Theorem 2. As we observed above, all graphs in Figs. 1–3 are bad; so we only need prove the "if" part of Theorem 2. Consider a minimally bad graph G. We may assume that G is weakly triangulated, for otherwise G is a hole or anti-hole. By Lemma 2, we may assume that G has no clique cutset. By Lemma 3, G has a minimal strong cutset C. This C is not a clique, for otherwise any minimal cutset included in C would be clique cutset of G. Consider two special components A, B of $G - C$. By the choice of C, C is now a minimal cutset of the induced subgraph $G[C \cup A \cup B]$. By Lemma 5, each of the sets $A \cup C, B \cup C$ must contain one of the graphs described in (i), (ii), (iii). Thus there are six cases to consider.

Case 1: Both $A \cup C, B \cup C$ contain a square as in (i).

Let us assume that there is a square on vertices $1,2,3,4$ with $1,2 \in A$ and $3,4 \in C$, and that there is a square on $5,6,7,8$ with $5,6 \in C$ and $7,8 \in B$. Since the subgraph of G induced by the two squares is sufficient, it follows that the vertex-set of G is $\{1,2,3,4,5,6,7,8\}$.

Subcase 1.1: $\{3,4\} = \{5,6\}$. Here G is the graph D_0.

Subcase 1.2: $3 = 5$ and $4 \neq 6$. Since C is not a clique, 4 misses 6. By Lemma 1, each of A, B contains a vertex adjacent to both $4, 6$. These two vertices can only be 1 and 7. Then 6 must see 2, for otherwise $G - 4$ is sufficient. Likewise, 4 sees 8. Thus, G is the graph E_2.

Subcase 1.3: $\{3,4\} \cap \{5,6\} = \emptyset$. Since C is not a clique cutset, by symmetry we may assume that 3 misses 5. By Lemma 1, there is a vertex in A that sees both $3, 5$; this vertex can only be 2. Then 5 must see 4, for otherwise Lemma 1 is contradicted (as no vertex in A sees $3,4,5$). By symmetry, 3 must see both $6, 8$. Then 5 must see 1, for otherwise $G - 3$ is sufficient, a contradiction. Likewise, 3 must see 7. If 6 misses 4 then, by Lemma 1, 1 must see 6, and 7 must see 4. But then $G - \{2,8\}$ is bad, a contradiction. So 6 sees 4. If 4 sees 7 but not 8, then $G - 6$ is sufficient, a contradiction. If 4 sees 8 but not 7, then $G - 5$ is sufficient. Hence, by Remark 1, 4 sees both $7, 8$. By symmetry, 6 sees both $1, 2$. Now G is the graph F_{14}.

Case 2: $A \cup C$ contains a square as in (i), and $B \cup C$ contains a diamond as in (ii).

Let the square be on vertices $1,2,3,4$ with $1,2 \in A$ and $3,4 \in C$, and where 13 and 24 are the non-edges; let the diamond be on $5,6,7,8$ with $5,6 \in C$ and $7,8 \in B$, where $5,6$ is the non-edge of the diamond. By Lemma 1, there is a vertex $x \in A$ adjacent to both 5, 6. Since the subgraph of G induced by $S = \{1,2,3,4,5,6,7,8,x\}$ is sufficient, G has no other vertex than those in S.

Subcase 2.1: $\{3,4\} \cap \{5,6\} \neq \emptyset$. Under this hypothesis and by symmetry we may assume $3 = 5$ and $4 \neq 6$.

Suppose that 2 sees 6 (thus $x = 2$, for otherwise $G - x$ would be sufficient). Vertex 4 sees 6, for otherwise Lemma 1 is contradicted, as no vertex of A is adjacent to all of 3, 4, 6. Since C is a minimal cutset, 4 must have a neighbour in $\{7,8\}$. If 4 sees exactly one of 7, 8, then G is E_3 or E_4. If 4 sees both 7, 8, then G is E_5 or E_6.

Now, suppose that 2 misses 6. Then 6 misses 1, for otherwise G contains a C_5 on 6, 1, 2, 3, 7. Thus $x \notin \{1,2\}$. But then $G - 8$ is sufficient, a contradiction.

Subcase 2.2: $\{3,4\} \cap \{5,6\} = \emptyset$. Let us first suppose that $x = 2$. So 1 misses 5 or 6, for otherwise $G - \{3,4\}$ would be sufficient. Without loss of generality, we assume that 1 misses 6. If 6 sees 4 then $G - 3$ is bad, a contradiction. Thus 6 misses 4, but then Lemma 1 is contradicted because A contains no vertex adjacent to all vertices in $\{4,5,6\}$. Similarly, we would be led to a contradiction if $x = 1$.

Thus, we have $x \notin \{1,2\}$. It follows that each vertex in $\{1,2\}$ has a non-neighbour in $\{5,6\}$. Vertex 6 must have a neighbour in $\{1,2\}$, for otherwise $G - 7$ is sufficient. Similarly, 5 must have a neighbour in $\{1,2\}$. Without loss of generality, we may assume that 6 sees 2. Then 2 misses 5. Thus, 5 sees 1 and therefore 1 misses 6. But now G contains a C_5 on vertices 1, 2, 6, 7, 5.

Case 3: $A \cup C$ contains a square as in (i), and $B \cup C$ contains the graph J as in (iii).

Let the square be on vertices 1, 2, 3, 4 (13 and 24 are the non-adjacent pairs) with $1,2 \in A$ and $3,4 \in C$, and let the graph J be $J = (5,6,7,8,9)$ with $5,6 \in C$ and $7,8,9 \in B$.

Suppose for a moment that $\{3,4\} \cap \{5,6\} = \emptyset$. By the definition of J, we may assume that 8 misses 6. But then $G - 5$ is sufficient, a contradiction. So we may assume that $\{3,4\} \cap \{4,5\} \neq \emptyset$. Without loss of generality, assume $3 = 5$ (and thus $4 \neq 6$). Vertex 3 must see 8, for otherwise $G - 6$ is sufficient. By the definition of J, 8 misses 6.

Suppose that 2 misses 6. By Lemma 1, there is a vertex $x \in A$ that sees both 3, 6. Note that $x \notin \{1,2\}$. Vertex 6 misses 1, for otherwise G contains a C_5 on vertices 6, 1, 2, 3, 7. But then $G - \{7,8\}$ is sufficient, a contradiction. So we may assume that 2 sees 6.

Since the subgraph G induced by $S = \{1,2,3,4,6,7,8\}$ is sufficient, G has no other vertex than those in S. Vertex 4 must see 6, for otherwise Lemma 1 is contradicted, as no vertex in A sees all of 3, 4, 6. Then 6 sees 1, for otherwise $G - 3$ is sufficient.

Since C is a minimal cutset, 4 must see at least one of 7, 8, 9. If 4 sees 8 then it must see 7 (respectively, 9), for otherwise $G - 9$ (respectively, $G - 7$) is sufficient. Then G is the graph F_{13}. Now, assume that 4 misses 8. If 4 sees both 7, 9 then $G - 6$ is sufficient, a contradiction. Hence, and by symmetry we may assume that 4 sees 7 and misses 9. Thus G is the graph F_{15}.

Case 4: $A \cup C$ contains a diamond as in (ii), and $B \cup C$ contains a diamond as in (ii).

Let the vertices of the diamond in $A \cup C$ be 1, 2, 3, 4 with $1,2 \in A$ and $3,4 \in C$, where 34 is not an edge. Let the vertices of the diamond in $B \cup C$ be 5, 6, 7, 8 with

$5,6 \in C$ and $7,8 \in A$, where 78 is not an edge. By symmetry we can distinguish three subcases.

Subcase 4.1: $\{3,4\} = \{5,6\}$. Here G is the graph D_1.

Subcase 4.2: $3 = 5$ and $4 \neq 6$. By Lemma 1, there are vertices $x \in A, y \in B$ that see all vertices in $\{3,4,6\}$. Since the subgraph of G induced by $S = \{1,2,3,4,6,7,8,9,x,y\}$ is sufficient, G has no other vertex than those in S.

Suppose that $y \notin \{7,8\}$. Since B is connected, y sees one of $7,8$, say $y7$ is an edge. But then $G - 8$ is sufficient. So it must be that $y \in \{7,8\}$. Similarly, $x \in \{1,2\}$. Without loss of generality, we may assume $x = 2$ (i.e., 2 sees 6), and $y = 7$ (i.e., 7 sees 4). Then 6 misses 1, for otherwise $G - 4$ would be sufficient. Likewise 4 misses 8. Then 4 misses 6, for otherwise G contains a C_5 with vertices $4,1,3,8,6$. Now G is the graph E_2.

Subcase 4.3: $\{3,4\} \cap \{5,6\} = \emptyset$. By Lemma 1, there is a vertex $x \in A$ that sees 5 and 6, and there is a vertex $y \in B$ that sees 3 and 4. Since the subgraph of G induced by $S = \{1,2,3,4,5,6,7,8,x,y\}$ is sufficient, G has no other vertex than those in S.

Suppose that $x \notin \{1,2\}$. Since A is connected, x sees 1 (or 2), but then $G - 2$ (or $G - 1$) is sufficient, a contradiction. So we may assume by symmetry that $x = 2$ (i.e., 2 sees $5,6$). Likewise we may assume that $y = 7$ (i.e., 7 sees $3,4$).

Suppose that 4 misses 5. Then 8 misses 4, for otherwise $G-6$ is sufficient. Similarly, 1 misses 5. Suppose 3 sees 6. If 3 sees 8 then 3 sees 5, for otherwise $G-6$ is sufficient. If 3 sees 5 then 3 sees 8, for otherwise $G - 2$ is sufficient. Thus 3 sees 8 if and only if 3 sees 5. Similarly, 6 sees 1 if and only if 6 sees 4. By symmetry, this leads to three possibilities. (a) All of $35, 38, 61, 64$ are edges: then G is the graph F_{14}. (b) None of $35, 38, 61, 64$ are edges: then G is the graph F_{15}. (c) $35, 38$ are edges and $61, 64$ are non-edges (or vice versa): then G is the graph F_{14}. Now, we may assume that 3 misses 6. 1 must miss 6, for otherwise $G - 4$ is sufficient. Similarly, 8 must miss 3. Now the only potential edges are $35, 46$; if both are present in G then there is a C_6 with vertices $1,3,5,8,6,4$; if both are absent then G is the graph F_{16}; in the other two cases, G is the graph F_{15}.

Now, we may assume that 45 is an edge; by symmetry $35, 36$ and 46 are also edges. Then it is straightforward to show that G either contains D_1 (contradicting minimality) or is one of F_{17}, F_{18}, F_{19}.

Case 5: $A \cup C$ contains a diamond as in (ii), and $B \cup C$ contains the graph J as in (iii).

Let the vertices of a diamond in $A \cup C$ be $1,2,3,4$ with $1,2 \in A$ and $3,4 \in C$, where 34 is not an edge. Let the vertices of $J = (5,6,7,8,9)$ in $B \cup C$ be $5,6,7,8,9$ with $5,6 \in C$ and $7,8,9 \in B$.

Subcase 5.1: $\{3,4\} = \{5,6\}$. Here G is the graph E_{11} or E_{12}.

Subcase 5.2: $|\{3,4\} \cap \{5,6\}| = 1$. Let us assume that $4 = 5$ and $3 \neq 6$.

By Lemma 1 there is a vertex $x \in A$ (respectively, $y \in B$) that sees all vertices in $\{3,4,6\}$. The vertex-set of G is $\{1,2,3,4,6,7,8,9,x,y\}$ since these vertices induce a sufficient subgraph.

Suppose $x \notin \{1,2\}$. Since A is connected, x sees 1 or 2; but then $G-2$ (resp. $G-1$) is sufficient, a contradiction. Thus, we may assume by symmetry that $x=2$ (i.e., 2 sees 6).

Suppose $y \notin \{7,9\}$. Since B is connected, y sees at least one vertex in $\{7,8,9\}$. If y sees 9 (or 7) then $G-\{7,8\}$ (or $G-\{8,9\}$) is sufficient. Thus, y misses 7, 9 and sees 8. By definition of J, 8 misses 5 or 6; but then $G-7$ is sufficient. Thus we may assume that $y=7$, i.e, 7 sees 3.

Now vertex 6 misses 1, for otherwise $G-3$ is sufficient. Vertex 4 sees 8, for otherwise $G-6$ is sufficient; thus 8 misses 6. Then 3 must miss 8, for otherwise $G-\{6,9\}$ is D_1. Then 3 misses 9, for otherwise $G-6$ is sufficient. Then 3 misses 6, for otherwise 1, 3, 6, 9, 4 induce a C_5. But then $G-\{1,4\}$ is sufficient (it is a D_0), a contradiction.

Subcase 5.3: The vertices 3, 4, 5, 6 are distinct.

As usual Lemma 1 ensures the existence of a vertex $x \in A$ (respectively, $y \in B$) that sees both 5, 6 (respectively, both 3, 4). As before, one can easily argue that $x=2$. By the definition of J, we may assume that 8 misses 5. But then $G-6$ is sufficient, a contradiction.

Case 6: $A \cup C$ contains the graph J as in (iii), and $B \cup C$ contains the graph J as in (iii).

Let the graph J in $A \cup C$ be $J=(4,5,1,2,3)$ with $1,2,3 \in A$ and $4,5 \in C$. Let the graph J in $B \cup C$ be $J=(6,7,8,9,10)$ with $6,7 \in C$ and $8,9,10 \in B$. As usual, we assume there is a vertex x in A seeing 6, 7 and a vertex y in B seeing 4, 5. For this part, we do not need to show $x \in \{1,2,3\}$ and $y \in \{8,9,10\}$. We only need x and y to show that certain subgraphs are sufficient.

Suppose $\{4,5\}=\{6,7\}$, with $4=6$ and $5=7$. By the definition of J we may assume that 2 misses 5. It follows that 9 sees 5 for otherwise $G-4$ is sufficient. Hence, 9 misses 4 by the definition of J, and thus 2 sees 4, or else $G-5$ is sufficient. So G is the graph F_{16}.

Next, suppose that $|\{4,5\} \cap \{6,7\}|=1$. We may assume $5=6$. Vertex 2 must see 5, for otherwise $G-4$ is sufficient. Then the definition of J implies that 2 misses 4. Similarly, 9 must see 5 and miss 7. But now $G-5$ is sufficient.

Finally suppose that $\{4,5\} \cap \{6,7\}=\emptyset$. By the definition of J, we may assume that 2 misses 5, and that 9 misses 6. Thus $G-\{4,7\}$ is sufficient, a contradiction. □

3. More on the class QT

We present here some further results concerning the graphs in the class QT. First we show that it is not a subclass of weakly triangulated graphs obtained by excluding a finite number of graphs.

For $k \geqslant 3$, we define two graphs Q_k and R_k as follows. To make Q_k, start from a clique on k vertices $v_1,\ldots,v_k$. For each i, add two vertices a_i, b_i and three edges $v_i a_i, a_i b_i, b_i v_{i+1}$ (all subscripts are understood modulo k); also add edges from v_i to all vertices a_j, b_j for $j \notin \{i-1, i, i+1\}$. This yields the graph Q_k.

To make R_k, start from a clique on k vertices $v_1,\dots,v_k$. For each i (again modulo k), add vertices a_i,b_i,c_i and five edges v_ia_i, v_ib_i, a_ib_i, b_ic_i, c_iv_{i+1}; also add edges from v_i to all vertices a_j,b_j for $j \notin \{i-1,i\}$.

Theorem 3. *There exist infinitely many weakly triangulated graphs that are not in the class QT and are minimal with that property.*

Proof. It is a routine matter to check that the graphs Q_k and R_k are weakly triangulated, not in the class QT, and that they are minimal with this property. □

Theorem 4. *For each* $k \geqslant 1$ *there exists a graph in* $QT_k - QT_{k-1}$.

Proof. For $k=1$, the "domino" graph D_0 is in QT_1 but not in QT_0. For $k=2$, the graph F_5 is in QT_2 but not in QT_1. For $k \geqslant 3$, it is a routine matter to check that the graph $Q_k - a_1$ is in QT_{k-2} but not in QT_{k-3}. □

Lemma 6. *Let G be a weakly triangulated graph with no clique cutset. If G is minimal non-QT, then every vertex of G lies in a square.*

After the first version of this paper was written, we learned that Gorgos independently proved Lemma 6. In fact, he used it to prove the following nice characterization of quasi-triangulated graphs which was conjectured by Voloshin.

Theorem 5 (Gorgos [4]). *A graph G is quasi-triangulated if and only if G does not contain an induced subgraph H such that each vertex of H lies in a hole with at least four vertices or its complement.*

Theorem 5 implies Lemma 6. For the sake of completeness, we present our proof of the lemma here.

Proof. If G is not connected then each component D of G must contain a square, for otherwise D contains a simplicial vertex that remains simplicial in G; now the minimality of G implies that G is exactly the union of two disjoint squares and we are done. Thus we may assume that G is connected. Let x be any vertex in G. Suppose that x is a universal vertex (i.e., x sees all of $G-x$). By the minimality of G, there is a simplicial or cosimplicial vertex v in $G-x$; but then it is easy to see that v would be a simplicial or cosimplicial vertex of G, a contradiction. So x is not universal. So $N(x)$ is a cutset, and it contains a minimal cutset C. Note that one component A of $G-C$ contains all of $\{x\} \cup N(x) - C$. By the hypothesis C is not a clique, hence it contains two non-adjacent vertices u,v. Let B be a component of $G-C$ such that $x \notin B$. By Lemma 1 there is a vertex $y \in B$ that sees both u,v. Now x,u,v,y induce the desired square. □

Lemma 7. *If G has no* $3K_2$ *and every two* $2K_2$*'s meet in an edge, then all the* $2K_2$*'s meet in the same edge.*

Proof. Assume if possible that there are three distinct sets of vertices C, D and F, each inducing a $2K_2$ in G such that xz and yw are edges of C, xz and uv are edges of D, and yw and st are edges of F.

Case 1: $\{u,v\} \cap \{y,w\} \neq \emptyset$. Then F must meet D in xz since $uv \neq yw$ (otherwise $C = D$). This implies $F = C$, a contradiction.

Case 2: $\{u,v\} \cap \{y,w\} = \emptyset$ and similarly $\{s,t\} \cap \{x,z\} = \emptyset$. Since F must meet D in an edge, we must have $uv = st$. Thus C,D,F induce a $3K_2$, a contradiction. □

The following lemma is trivial and so we omit the proof.

Lemma 8. *If G has a $3K_2$ and every pair of $2K_2$'s meet in an edge, then G has no other $2K_2$'s than those induced by the $3K_2$.*

Theorem 6. *If G is a weakly triangulated graph such that every pair of squares meet in a non-edge, then G is a QT graph.*

Proof. By induction on the number of vertices. The induction hypothesis allows us to assume that G is minimal non-QT. If G is not connected then each component C of G must contain a square, for otherwise C contains a simplicial vertex which remains simplicial in G. But then G would have two completely disjoint squares, a contradiction. So, we may assume that G is connected.

Now, suppose that G contains a clique cutset C. Let A,B be two components of $G - C$. If $G[A \cup C]$ does not contain a square then by Dirac's theorem, there is a simplicial vertex in A and this vertex remains simplicial in G. Thus $G[A \cup C]$ must contain a square; and similarly, $G[B \cup C]$ must contain a square. Since C is a clique, these two squares cannot meet in a non-edge, a contradiction. We may assume that G contains no clique cutset.

Case 1: G contains no $\overline{3K_2}$. By Lemma 7, all C_4's meet in the same non-edge xy. We may assume that G contains a C_4 $axby$; otherwise G is triangulated and we are done. By Lemma 6, we may assume that each vertex u of G lies in a square $S(u)$. Since $S(u)$ must meet $axby$ at xy, u must see both x,y. But this implies that x and y are cosimplicial in G.

Case 2: G contains a $\overline{3K_2}$. Let F be the set of vertices inducing this $\overline{3K_2}$ in G. By Lemma 6 every vertex in G lie in a square, but by Lemma 8 there is no other square in G, i.e., $G = F$. G is clearly in QT. □

The following theorem was used to prove Theorem 6. It is not needed anymore but it seems to be interesting in its own right, as it has some other consequence below.

Theorem 7. *Every non-empty minimal cutset of a co-triangulated graph contains a co-simplicial vertex.*

Proof. Consider a non-empty minimal cutset C in a co-triangulated graph G. We use induction on the number of vertices in C.

Case 1: $C = \{x\}$. Since at most one component is non-trivial in $G - C$, if x is not co-simplicial, we must have an edge yz in one component of $G-C$ with x not adjacent to both y and z. Now if w is a neighbour of x in any other component, then x, y, z, w induce a $2K_2$ in G, a contradiction.

Case 2: C is the join of two parts, say C_1 and C_2. Since C_1 remains a minimal cutset in $G - C_2$, by induction C_1 contains a co-simplicial vertex in $G - C_2$ which remains co-simplicial in G.

Case 3: $\bar{C}$ is connected. Since G is weakly triangulated, by Lemma 1, each component of $G - C$ has a vertex joined to all the vertices of C. If every component of $G - C$ is trivial, then, by the induction hypothesis C has a co-simplicial vertex, which clearly is also co-simplicial in G. If one component R of $G - C$ is non-trivial, pick a vertex $y \in R$ joined to all of C, and consider the graph $G - y$. If C contains a non-empty minimal cutset of $G - y$, then by the induction hypothesis it also contains a co-simplicial vertex, which is also co-simplicial in G. We may assume now that $G - y$ is disconnected with exactly one non-trivial component. This non-trivial component contains all vertices of $G - R$ and possibly some vertices of R. Let T be the set of all isolated vertices in $G - y$. Now C is a minimal cutset of $G - T$. By the induction hypothesis C contains a co-simplicial vertex, which is also co-simplicial in G. □

A graph is *perfectly orderable* [2] if it admits a linear ordering $v_1 \prec v_2 \prec \cdots \prec v_n$ on its vertices such that, for every induced subgraph H of G, the greedy colouring algorithm applied on H along that ordering produces an optimal colouring of the vertices of H. Such an ordering is called a *perfect ordering*. A *homogeneous set* in a graph G is any subset of vertices S such that every vertex in $G - S$ either sees all vertices of S or misses all vertices of S. Two simple facts are worth recalling:

(1) *If a graph G has a simplicial vertex v and $G - v$ is perfectly orderable*, *then G is perfectly orderable*. Indeed, it suffices to take any perfect ordering of $G - v$ and to add v last to obtain a perfect ordering of G. Likewise, *if a graph G has a co-simplicial vertex v and $G - v$ is perfectly orderable*, *then G is perfectly orderable.* Putting v first and then adding a perfect ordering of $G - v$ yields a perfect ordering of G.

(2) *Let G be a graph that has a homogeneous set S. Suppose that the graph G/S obtained by contracting the set S into one vertex is perfectly orderable*, *and that the induced subgraph $G[S]$ is perfectly orderable. Then G is perfectly orderable.* Indeed, taking any perfect ordering of G/S and replacing the vertex representing S by the vertices of S given in a perfect ordering of $G[S]$ yields a perfect ordering of G.

Theorem 8. *If in every induced subgraph of a graph G each minimal cutset is either a clique or contains a co-simplicial vertex*, *then G is perfectly orderable*.

Proof. We may assume that G is connected and that it has no simplicial or co-simplicial vertex v, for otherwise we can add v to any perfect ordering of $G - v$, respectively, last or first, as in fact (1) above. So every minimal cutset of G is a clique. For any cutset

C, let $f(C)$ denote the smallest size of a component of $G - C$. Picking a minimal cutset C with smallest $f(C)$, we show that the component R of $G - C$ of size $f(C)$ is a homogeneous set; more precisely, every vertex in C is adjacent to every vertex in R. Now, the result follows from the existence of this homogeneous set, from fact (2) above, and from the induction hypothesis. □

The class of graphs described in Theorem 8 contains all triangulated graphs and all co-triangulated graphs. Can they be characterized?

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